Treatment of centrifugal elastic stresses in nuclear rotation

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A treatment of elastic stresses in nuclear rotation with the nucleus considered as an incompressible elastic body with a stationary spherical core surrounded by a circulating mass is presented. Centrifugal stresses acting in directions perpendicular to the axis of rotation introduce variations in the moment of inertia. The variable moment of inertia model, so successful in predicting the rotational spectrum of even-even nuclei, is explained.

[NUCLEAR STRUCTURE Explanation of rotational energy levels for even-even nuclei using elastic stress analysis.]

I. INTRODUCTION

Since the pioneering work of Bohr and Mottelson,[1] several advancements have been made in describing various empirical observations regarding the rotational levels in nuclei. Phenomenological and microscopic theories have been put forward by a number of authors.[2-16]

At the same time that Mariscotti, Goldhaber, and Buck[16] were pointing out the inadequacy of the liquid drop model in predicting rotational energy levels, some evidences were found for a solidlike behavior of the nucleus. For instance, for a Fermi system ^{3}He, Corruccini *et al.*[17] predict transverse waves. As we know, this can exist only in mediums with some rigidity. Later, using the RPA (random phase approximation), Bertsch[18] found that the macroscopic equations for the vibration of nuclei give rise to vibrational modes similar to those of an elastic solid. Also, as Bertsch pointed out, the fluid model predicts mass dependence of vibration frequency to be of the order of $A^{-1/2}$ where A is the mass number. However, empirically, an $A^{-1/3}$ dependency was found.

Since the work of Mariscotti, Goldhaber, and Buck which suggested the variable model of inertia (VMI) model, we have been studying[19] the possibility of interpreting the levels as quantized states of an elastic solid body. This paper contains the summary of the preliminary efforts. The VMI model is explained in a simple manner by considering small displacements of the elastic nuclear matter. It is shown that this macroscopic treatment of the nucleus as an elastic solid predicts with reasonable accuracy the energy levels belonging to the ground-state rotational band of well-deformed even-even nuclei and also accounts for the ground-state quadrupole moments. Although the predictions of the model are quite satisfactory, it should be emphasized that the model is a classical one for which a microscopic justification is still lacking.

Recently, great efforts have been exerted by many authors in searching for a microscopic explanation of the rotational bands. These models have dealt with microscopic methods such as the RPA, Hartree-Fock, and Hartree-Bogoliubov approximations where elementary interactions are used. There the pairing effect plays an important role. Eventually any macroscopic model such as the one presented here should be justified by a microscopic approach similar to those mentioned and where the Coulomb and pairing interactions should be taken into account.

The pairing effect, however, is somehow implicit in the model by the hypothesis that the rotational axis is perpendicular to the symmetry axis. In fact, if we assume a decoupling between the collective and intrinsic motion, by consideration of symmetry the projection K of the total angular momentum on the symmetry axis is equal to Ω, the projection of the intrinsic angular moment on the same axis. In the lowest state the strong pairing force binds the particles in pairs with the same angular momentum but opposite magnetic quantum number. As a result $\Omega = 0$, therefore $K = 0$ and the rotation is performed around an axis perpendicular to the symmetry axis.

In the present work it is assumed that there is an inner spherical core which does not contribute to the rotation. This inner core may be visualized as a group of closed shells which does not contribute to the collective rotation. Its existence, however, is only justified in the model by the need for dealing with small moments of inertia and quadrupole moments that agree with the experimental data.

Also, the model allows changes in shape only under the dynamic action of the centrifugal force. Thus, the initial shape implies the analysis of a Hamiltonian which should contain somehow the elementary interaction between nucleons.

Finally, regarding surface energy, it should be pointed out that the boundary condition, namely, that the normal tension at the surface is zero, does not take into account the surface energy. This concept implies considerations of heterogeneity at the surface which are completely absent in our model. Within the confine of such a simple model, we wish to explore the elastic response of a nucleus in rotational motion so as to gain some preliminary insight in the elasticity of nuclear matter.

II. THE MODEL

Each rotating nucleus is assumed to have an ellipsoidal shape with a spherical core which does not contribute to the rotational energy. However, the outer mass is under rotation and subjected to stresses due to the effect of the centrifugal force. In agreement with the decoupling assumption between intrinsic and collective motion the ground-state ($K=0$) rotational band is built up by rotating the nucleus around an axis perpendicular to the symmetry axis. The resulting forces are not necessarily harmonic.[20] A realistic density distribution for nuclear matter is employed in the calculations, so that for each nucleus the average density of the rotating mass in relation to that of the nonrotating core is dependent upon the average radius of the nucleus and the radius of the core. The volume of the core and that of the rotating shell are kept unvaried when angular momentum is changed. It is assumed that the charge distribution is proportional to the distribution of mass.

We write the total energy in the standard form as

$$E=\tfrac{1}{2}I\,\omega^2+U. \tag{1}$$

The potential energy of deformation U is a function of the angular momentum J, the elasticity constant ξ, the ground-state dimensions, and the density distribution of the nucleus. According to the theory of elasticity applied to solids[21] this potential is given by

$$U=\int\int\int\left[\frac{1}{2\xi}(\sigma_x^2+\sigma_y^2+\sigma_z^2)-\frac{\mu}{\xi}(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x)+\frac{1+\mu}{\xi}(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2)\right]dx\,dy\,dz\,, \tag{2}$$

where σ_x, σ_y, and σ_z are the normal stresses and τ_{xy}, τ_{yz}, and τ_{xz} represent the shears acting on an element of the body. The Poisson coefficient of the body is represented by μ. In our treatment we take $\mu=0.5$, so that the nucleus is considered to be incompressible. Different values of μ will allow for deformations which do not preserve volume.

To determine the potential U the following differential equations which express the equilibrium of an element of matter and the compatibility equations are solved.[21] Equilibrium equations:

$$\begin{aligned}&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+F_x=0\,,\\&\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+\frac{\partial\tau_{yx}}{\partial x}+F_y=0\,,\\&\frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+F_z=0\,.\end{aligned} \tag{3}$$

Compatibility equations:

$$\begin{aligned}&\nabla^2\sigma_x+\frac{1}{1+\mu}\frac{\partial^2\theta}{\partial x^2}=-\frac{\mu}{1+\mu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)-2\frac{\partial F_x}{\partial x}\,,\\&\nabla^2\tau_{yz}+\frac{1}{1+\mu}\frac{\partial^2\theta}{\partial y\partial z}=-\left(\frac{\partial F_z}{\partial y}+\frac{\partial F_y}{\partial z}\right)\,,\end{aligned} \tag{4}$$

where

$$\theta=\sigma_x+\sigma_y+\sigma_z\,. \tag{5}$$

In these equations F_x, F_y, and F_z are the components of the centrifugal force in the x, y, and z directions. For the ellipsoid rotating around the z axis, we have $F_x=\rho\omega^2x$, $F_y=\rho\omega^2y$, and $F_z=0$.

By symmetry considerations we have $\tau_{xy}=\tau_{yx}$, $\tau_{xz}=\tau_{zx}$, and $\tau_{yz}=\tau_{zy}$. The boundary condition of the problem requires that the normal components of the tension at the body surface be zero.

Solving the differential equations it is found that[22]

$$\begin{aligned}&\sigma_x=\tfrac{1}{2}\rho\omega^2a^2\left[A_1\left(1-\frac{x^2}{a^2}\right)-B_1\frac{z^2}{c^2}-C_1\frac{y^2}{b^2}\right],\\&\sigma_y=\tfrac{1}{2}\rho\omega^2b^2\left[A_2\left(1-\frac{y^2}{b^2}\right)-B_2\frac{x^2}{a^2}-C_2\frac{z^2}{c^2}\right],\\&\sigma_z=\tfrac{1}{2}\rho\omega^2c^2\left[A_3\left(1-\frac{z^2}{c^2}\right)-B_3\frac{y^2}{b^2}-C_3\frac{x^2}{a^2}\right],\end{aligned} \tag{6}$$

where

$$A_1 = 1 + M + N, \quad A_2 = 1 + L + N, \quad A_3 = L + M,$$
$$B_1 = 1 + 3M + N, \quad B_2 = 1 + L + 3N, \quad B_3 = 3L + M, \quad (7)$$
$$C_1 = 1 + M + 3N, \quad C_2 = 1 + 3L + N, \quad C_3 = L + 3M,$$

and

$$\tau_{xy} = \rho\omega^2 N xy\,,$$
$$\tau_{yz} = \rho\omega^2 L\, yz\,, \quad (8)$$
$$\tau_{zx} = \rho\omega^2 M zx\,,$$

where ρ and ω represent the mass density and the angular speed of the rotation, respectively. The terms L, M, and N are functions of μ, ξ, and the ground-state nuclear dimensions of the ellipsoid: a, b, and c. They are shown explicitly in Appendix A.

Under the action of the stresses and shears, each point (x, y, z) inside the body is displaced to (X, Y, Z) where

$$X = x + x\frac{\rho\omega^2}{2\xi}(\alpha_0 + \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2)\,, \quad (9)$$

and so on. The coefficients α_i are functions of μ, a, b, and c (see Appendix A). The moment of inertia I and the quadrupole moment for any rotational state may be calculated using the new coordinates.

After substituting Eqs. (6) and (8) in Eq. (2) and performing some straightforward integration, the potential energy term may be written in the form

$$U = \frac{\rho^2\omega^4}{\xi} f(a, b, c, \mu)\,. \quad (10)$$

The function $f(a,b,c,\mu)$ depends only on the geometry through the semiaxes a, b, and c and on the Poisson coefficient μ, and it is given explicitly in Appendix A. The moment of inertia I is given by

$$I = \rho \int\int\int (X^2 + Y^2)(J)\, dx\, dy\, dz\,, \quad (11)$$

where (J) stands for the Jacobian of the transformation from (x, y, z) to (X, Y, Z). Using Eq. (9), the moment of inertia I is found to be a polynomial of fifth degree in ω^2, the square of the rotational speed;

$$I = I_0 + \frac{a_1\omega^2}{2\xi} + \frac{a_2\omega^4}{4\xi^2} + \frac{a_3\omega^6}{8\xi^3} + \frac{a_4\omega^8}{16\xi^4} + \frac{a_5\omega^{10}}{32\xi^5}\,. \quad (12)$$

The coefficients a_i are deduced after integrating Eq. (11). They are functions of a, b, c, and μ (they are not given explicitly in this paper because of their length).

When the Poisson coefficient μ is $\frac{1}{2}$, I becomes a polynomial of second degree. In fact, for $\mu = \frac{1}{2}$ there is conservation of volume and consequently the Jacobian is equal to one.

As usual, we quantize the angular velocity by

$$\omega = [J(J+1)]^{1/2}\hbar/I_J\,, \quad (13)$$

where I_J is the moment of inertia for the level of angular momentum J.

The total energy is then written as

$$E = \frac{J(J+1)}{2I_J}\hbar^2 + \frac{\rho^2[J(J+1)]^2}{\xi I_J^4}\hbar^4 f(a, b, c, \mu)\,. \quad (14)$$

The fact that the inner core does not contribute to the energy was taken into account in the calculations by subtracting the deformation energy of the core from that of the whole ellipsoid. This approximation is hard to justify since the actual boundary conditions between the inner core and the shell are unknown. However, the general conclusions of this work as well as the form of the general equations are not critically dependent on this approximation.

In our analysis the nuclear density function is assumed to be

$$\rho = \frac{\rho_0}{1 + \exp[(r - R)/a_0]}\,. \quad (15)$$

Two experimental parameters, Q_0 the ground-state quadrupole moment and E_2 the energy of the first excited state, are used to evaluate the ground-state deformation parameter and the elasticity constant ξ. Solutions are sought for various values of the mean radius of the core and the one yielding the minimum percentage deviation from the experimental value E_4 is accepted. For the calculation of the quadrupole moment we divide the nucleus into thin concentric shells of charge. The inner shell is just outside the core and therefore has a near-spherical shape. The outer shell corresponds to the surface of the nucleus and has an eccentricity ϵ_0. The intermediate shells have eccentricities between zero and ϵ_0. This variable eccentricity treatment allowed us to account for the ground-state quadrupole moment as well as for the energy levels. The model has been able to obtain fits for the experimental rotational levels of some nuclei in the $Z = 50-80$ region (see Table I). In agreement with Baranger and Kumar,[23] these nuclei were assumed to be prolate in the ground-state. The variable eccentricity concept was used previously by Brueckner and his co-workers[24] in the energy-density theory of nuclei and still earlier by Carlson[25] who studied the mass-charge distribution of homotetic ellipsoidal shells. In their work, using this model, Brueckner *et al.* found reasonable results for the mass-density function of some nuclei.

TABLE I. Experimental energy E(exp) in keV. Predicted energy E in keV and the moment of inertia I in keV^{-1} for the different angular momenta J.

J	2	4	6	8	10	12	14	16
^{154}Gd								
E(exp)	123.14	371.18	711.9	1144.7	1637.5	2184.1	2777.1	3404.7
E	···	374.63	711.2	1144.7	1639.3	2184.6	2771.8	3394.6
I	0.025 26	0.028 88	0.032 88	0.034 68	0.037 66	0.040 64	0.043 59	0.046 50
^{160}Dy								
E(exp)	86.79	283.8	581.2	967.2	1428.9	1951.7		
E	···	284.1	581.3	966.2	1426.6	1952.1		
I	0.034 78	0.035 86	0.037 38	0.039 19	0.041 19	0.043 29		
^{162}Dy								
E(exp)	80.66	265.7	548.6	921.2	1374.9	1901.1		
E	···	265.8	548.9	921.4	1374.3	1898.8		
I	0.037 34	0.038 07	0.039 15	0.040 48	0.041 99	0.043 64		
^{164}Dy								
E(exp)	73.39	242.2	501.3	839.3				
E	···	242.0	500.0	839.9				
I	0.041 02	0.041 80	0.042 93	0.044 34				
^{176}Hf								
E(exp)	88.35	290.15	596.85	997.85	1481.2	2034.45	2646.35	
E	···	290.25	596.90	997.85	1480.4	2035.96	2655.14	
I	0.034 12	0.034 97	0.036 18	0.037 65	0.039 30	0.041 07	0.042 90	
^{178}Hf								
E(exp)	93.181	306.63	632.19	1058.6				
E	···	306.66	632.17	1059.2				
I	0.032 21	0.032 98	0.033 94	0.035 13				
^{180}Hf								
E(exp)	93.308	308.55	640.82	1083.9				
E	···	308.67	640.88	1082.8				
I	0.032 21	0.032 60	0.033 17	0.033 90				
^{232}U								
E (exp)	47.6	156.6	321.0					
E	···	156.4	321.9					
I	0.063 33	0.064 87	0.067 08					
^{234}U								
E(exp)	43.5	143.5	296.6	499.0				
E	···	143.5	296.8	499.2				
I	0.069 22	0.070 49	0.072 35	0.074 66				
^{236}U								
E(exp)	45.28	148.7	312.0					
E	···	149.7	310.3					
I	0.066 44	0.067 42	0.068 86					
^{238}U								
E(exp)	44.7	148.0	309.0	523.0	787.0	1100.0		
E	···	148.2	309.0	523.0	788.5	1101.4		
I	0.067 24	0.067 89	0.068 88	0.070 16	0.071 68	0.073 40		

III. RESULTS AND CONCLUSIONS

The quadrupole moment was found to increase slowly with J, 1 or 2% between consecutive levels. The moment of inertia I is found to have a significantly greater role in the potential energy function.

For each nucleus to which the model is applied, the number of nucleons taking part in the rotational movement was estimated. It was found that this number is approximately consistent with what is expected from shell-model considerations. The U and Hf isotopes are particularly interesting since the number of nucleons of the shell happens to be equal to the number of nucleons outside closed

TABLE II. Parameters corresponding to the predicted energies shown in Table I. The mean core radius R_c in fm, the elasticity constant ξ in keV/fm^3, the ground-state moment of inertia I_0 in keV^{-1}, and the number of nucleons N which participate in the rotation. For ^{154}Gd there are two cases according to the discussion presented in the text.

Isotope	R_c (fm)	ξ (keV/fm^3)	a (fm)	b (fm)	I_0 (keV^{-1})	N
^{154}Gd						
(a)	5.558	3.86	5.86	7.76	0.02565	31.1
(b)	5.457	7.18	5.86	7.76	0.02565	35.6
^{160}Dy	5.285	14.47	5.89	7.96	0.03428	48.5
^{162}Dy	5.238	21.04	5.90	8.03	0.03700	52.5
^{164}Dy	5.144	18.58	5.94	8.04	0.04068	58.9
^{176}Hf	5.640	18.87	6.14	8.06	0.03373	44.0
^{178}Hf	5.632	21.53	6.40	7.51	0.03201	46.0
^{180}Hf	5.624	36.02	6.55	7.25	0.03204	48.0
^{232}U	6.027	7.40	6.65	9.06	0.06263	68.0
^{234}U	6.022	9.54	6.53	9.49	0.06865	70.0
^{236}U	6.031	11.55	6.71	9.05	0.06601	72.0
^{238}U	6.013	16.80	6.76	9.00	0.06695	74.0

shells, 82-82 and 50-82, respectively. It was found that the calculation is not very sensitive to variations, of the order of 5 to 10% in the nucleon number. Such freedom is not surprising considering the approximations involved such as in the density averaging and the exclusion of microscopic effects from the model. ^{154}Gd is also an interesting case. It was possible to fit its energy levels up to $J=8$ with the core radius equal to 5.558 fm. To fit levels from $J=8$ to $J=16$ it was necessary to reduce the core and so allow for four particles to come out of the inner core and join the rotational motion.

In Table II are given, for each nucleus, the parameters ξ in keV/fm^3, the ground-state moment of inertia I_0 in keV^{-1}, the number of nucleons that participate in the rotation and, finally, the minor and semimajor axes a and b of the prolate ellipsoid.

Neither the VMI nor this model predicts smooth varying values of ξ as a function of A. This is somehow expected because of the shell-model dependency of these parameters. The value of ξ corresponds to the potential energy per unit volume. It would be interesting to obtain it from a microscopic theory. In this sense some calculations were carried out using the results on nuclear matter obtained from the Thomas-Fermi theory of nuclei by Brueckner, *et al.*[24] and also by Bethe.[26] Unfortunately, the uncertainties on the tail of the density distribution that this theory makes use of did not allow for a reliable comparison between the macroscopic and microscopic values of ξ.

We may say that the theory of linear elasticity of solids predicts well the quadrupole moment and the rotational energy levels of those well-deformed nuclei which suffer small displacements during the centrifugal stretching. For these cases the model gives a simple physical explanation of the VMI model. In fact, when the displacements are small and the theory of elasticity holds well, the coefficient of ω^4 in the polynomial of second degree in ω^2 which expresses the moment of inertia is small. Thus, for these cases, we may write the moment of inertia as

$$I=I_0+\alpha\omega^2 . \tag{16}$$

This equation, plus the one which gives the potential energy [Eq. (10)], allows us to write

$$U=\tfrac{1}{2}B(I-I_0)^2 , \tag{17}$$

where

$$B=\frac{2\rho^2}{\alpha^2\xi}f(a,b,c,\mu) . \tag{18}$$

This is precisely the potential energy used by the VMI model. If the displacements are not small, however, the linear theory of elasticity starts failing and strains which are not necessarily proportional to the stresses may have to be considered. And so one reaches the domain of the elastoviscous fluids. We point out that in the case that some compressibility must be included, the expression for the moment of inertia is given by a polynomial of fifth degree in ω^2 and the previous approximations which lead to the VMI-model equations may not be correct.

Also, the Harris formulation[27] based on the cranking model of Inglis[28] includes Eqs. (10) and (12) which result from the theory of elasticity. Harris expands the energy and the angular momentum in ω^2, that is,

$$E=\tfrac{1}{2}\omega^2(I_0'+3C'\omega^2+5D'\omega^4+7F'\omega^6+\cdots) \tag{19}$$

and

$$[J(J+1)]^{1/2}=\omega(I_0'+2C'\omega^2+3D'\omega^4+4F'\omega^6+\cdots) . \tag{20}$$

Using Eq. (13), we write these previous equations as

$$E=\tfrac{1}{2}I'^2\omega^2+\tfrac{1}{2}C'\omega^4+4F'\omega^6+\cdots \tag{21}$$

and

$$I'-I_0'=2C'\omega^2+3D'\omega^4+\cdots . \tag{22}$$

Thus, the theory of elasticity [Eqs. (10) and (12)], predicts only two terms in the energy E and five terms in the equation for the moment of inertia of the Harris formulation.

APPENDIX A

Let Π denote the determinant

$$\begin{vmatrix} 3b^4+2b^2c^2+3c^4 & c^4-\mu(b^2c^2+c^2a^2+3a^2b^2) & b^4-\mu(b^2c^2+3c^2a^2+a^2b^2) \\ c^4-\mu(b^2c^2+c^2a^2+3a^2b^2) & 3c^4+2c^2a^2+3a^4 & a^4-\mu(3b^2c^2+c^2a^2+a^2b^2) \\ b^4-\mu(b^2c^2+3c^2a^2+a^2b^2) & a^4-\mu(3b^2c^2+c^2a^2+a^2b^2) & 3a^4+2a^2b^2+3b^4 \end{vmatrix} \tag{A1}$$

and Π_{11}, Π_{12}, $\Pi_{13}, \ldots$, its minors, where $\Pi_{ij}=\Pi_{ji}$. The constants L, M, and N [see Eq. (7)] are then

$$\begin{aligned} L&=\{a^2[\mu(b^2+c^2)\Pi_{11}+(\mu c^2-a^2)\Pi_{12}+(\mu b^2-a^2)\Pi_{13}]+b^2[(\mu c^2-b^2)\Pi_{11}+\mu(c^2+a^2)\Pi_{12}+(\mu a^2-b^2)\Pi_{13}]\}/\Pi\ , \\ N&=\{a^2[\mu(b^2+c^2)\Pi_{31}+(\mu c^2-a^2)\Pi_{32}+(\mu b^2-a^2)\Pi_{33}]+b^2[(\mu c^2-b^2)\Pi_{31}+\mu(c^2+a^2)\Pi_{32}+(\mu a^2-b^2)\Pi_{33}]\}/\Pi\ , \\ M&=\{a^2[\mu(b^2+c^2)\Pi_{21}+(\mu c^2-a^2)\Pi_{22}+(\mu b^2-a^2)\Pi_{23}]+b^2[(\mu c^2-b^2)\Pi_{21}+\mu(c^2+a^2)\Pi_{22}+(\mu a^2-b^2)\Pi_{23}]\}/\Pi\ . \end{aligned} \tag{A2}$$

For the calculation of the displacements $(X-x)$, $(Y-y)$, and $(Z-z)$ [see Eq. (9)] the constants α_0, α_1, α_2, and α_3 are given by

$$\begin{aligned} &\alpha_0=A_1a^2-\mu A_2b^2-\mu A_3c^2, \quad \alpha_1=-\frac{1}{3a^2}[A_1a^2-\mu B_2b^2-\mu C_3c^2]\ , \\ &\alpha_2=-\frac{1}{b^2}[C_1a^2-\mu A_2b^2-\mu B_3c^2], \quad \alpha_3=-\frac{1}{c^2}[B_1a^2-\mu C_2b^2-\mu A_3c^2]\ , \end{aligned} \tag{A3}$$

where A_1, A_2, A_3, etc., are given in Eq. (7). Similar expressions may be written for the coefficients of the displacements along y and z directions.

The function $f(a, b, c, \mu)$ presented in Eq. (10) of the text is given by

$$\begin{aligned} f(a,b,c,\mu)=\pi abc\{&[\tfrac{1}{6}S_0+\tfrac{1}{30}(S_1+S_2+S_3)+\tfrac{1}{210}(S_4+S_5+S_6)+\tfrac{1}{70}(S_7+S_8+S_9)] \\ &-\mu[\tfrac{1}{3}K_0-\tfrac{1}{15}(K_1+K_2+K_3)+\tfrac{1}{105}(K_{12}+K_{23}+K_{31})+\tfrac{1}{35}(L_1+L_2+L_3)] \\ &-\tfrac{4}{105}(1+\mu)(a^2b^2N^2+b^2c^2L^2+c^2a^2M^2)\}\ , \end{aligned} \tag{A4}$$

where

$$\begin{aligned} &S_0=a^4A_1{}^2+b^4A_2{}^2+c^4A_3{}^2\ , \quad S_1=-2(a^4A_1{}^2+b^4A_2B_2+c^4A_3C_3), \quad S_2=-2(a^4A_1C_1+b^4A_2{}^2+c^4A_3B_3)\ , \\ &S_3=-2(a^4A_1B_1+b^4A_2C_2+c^4A_3{}^2), \quad S_4=2(a^4A_1C_1+b^4A_2B_2+c^4B_3C_3), \quad S_5=2(a^4B_1C_1+b^4A_2C_2+c^4A_3B_3)\ , \\ &S_6=2(a^4A_1B_1+b^4B_2C_2+c^4A_3C_3), \quad S_7=a^4A_1{}^2+b^4B_2{}^2+c^4C_3{}^2, \quad S_8=a^4C_1{}^2+b^4A_2{}^2+c^4B_3{}^2, \\ &S_9=a^4B_1{}^2+b^4C_2{}^2+c^4A_3{}^2 \end{aligned} \tag{A5}$$

and

$$\begin{aligned} K_0&=a^2b^2A_1A_2+b^2c^2A_2A_3+c^2a^2A_3A_1\ , \\ K_1&=(A_1B_2+A_1A_2)a^2b^2+(A_2C_3+B_2A_3)b^2c^2+(A_3A_1+C_3A_1)c^2a^2\ , \\ K_2&=(A_1A_2+C_1A_2)a^2b^2+(A_2B_3+A_2A_3)b^2c^2+(A_3C_1+B_3A_1)c^2a^2\ , \\ K_3&=(A_1C_2+B_1A_2)a^2b^2+(A_2A_3+C_2A_3)b^2c^2+(A_3B_1+A_3A_1)c^2a^2\ , \\ K_{12}&=(C_1B_2+A_1A_2)a^2b^2+(A_2C_3+B_2B_3)b^2c^2+(B_3A_1+C_3C_1)c^2a^2\ , \\ K_{23}&=(B_1A_2+C_1C_2)a^2b^2+(C_2B_3+A_2A_3)b^2c^2+(A_3C_1+B_3B_1)c^2a^2\ , \\ K_{31}&=(A_1C_2+B_1B_2)a^2b^2+(B_2A_3+C_2C_3)b^2c^2+(C_3B_1+A_3A_1)c^2a^2\ , \end{aligned} \tag{A6}$$

and

$$\begin{aligned} &L_1=A_1B_2a^2b^2+B_2C_3b^2c^2+C_3A_1c^2a^2, \quad L_2=C_1A_2a^2b^2+A_2B_3b^2c^2+B_3C_1c^2a^2, \\ &L_3=B_1C_2a^2b^2+C_2A_3b^2c^2+A_3B_1c^2a^2\ . \end{aligned} \tag{A7}$$

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