Distribution of electrical charge in the nucleus of ^{181}Ta

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The K and L transitions of muonic atoms of ^{181}Ta were measured. A comparison of the experimental values with calculated values was made. A modified Fermi distribution of charge of three parameters was assumed for the calculation. The calculations include corrections for nuclear polarization and extended space of the Dirac hydrogenlike atom. A range of values for the half radius, the surface thickness, and the intrinsic quadrupole moment was obtained. A description of the analysis is included.

[NUCLEAR STRUCTURE Energies of γ rays of K and L transitions of muonic ^{181}Ta measured. Data fitted by three parameter modified Fermi distribution of charge.]

INTRODUCTION

The energies of the K and L x rays of muonic atoms of ^{181}Ta were measured and the results compared to theoretically calculated values. Such a comparison yields quantitative information about the values of the parameters used to describe the radial distribution and shape of the electric charge in the nucleus.

As in the past, the methods of Wilets[1] and Jacobsohn[2] provided the basis for the initial procedures for the theoretical calculation. Additional corrections, suggested by Chen,[3,4] were found necessary to obtain a fit to the experimental data.

Following Wilets and the independent work of Jacobsohn, we start with a physical system described by a Hamiltonian

$$H = H_N + H_\mu + V , \tag{1}$$

where H_N is a nuclear Hamiltonian assumed to describe the lower energy levels of the deformed nucleus in a rotational band; H_μ represents the Dirac Hamiltonian, describing the muon in an attractive potential provided by the monopole term of the electrostatic interaction between the muon and the charge of the nucleus; and V is the electrostatic interaction with the monopole term subtracted.

The matrix elements of the Hamiltonian operator can be conveniently expressed in a basis provided by a linear combination of products of the muonic state vectors and the nuclear state vectors:

$$|\lambda I J F m(F)\rangle = \sum_{m(I),m(J)} (I J m(I) m(J) | F m(F)) \times |n \kappa J m(J)\rangle |\nu I K m(I)\rangle . \tag{2}$$

$I, m(I), K$ are nuclear quantum numbers describing the total angular momentum of the nuclear state, its projection on the z axis of the laboratory, and the projection on the nuclear symmetry axis, respectively. J and $m(J)$ are corresponding quantum numbers for the hydrogenlike muonic atom. The quantum number which uniquely denotes the states in the Dirac solutions of the hydrogen problem is κ. We use λ to denote all other quantum numbers needed to describe the total physical system. Similarly, n and ν denote all other quantum numbers needed to describe the muonic state and the nuclear state, respectively. The total angular momentum of the physical system is characterized by the quantum numbers $F, m(F)$. The interaction term without the monopole term has the following form:

$$V = \sum_p \sum_{q=-l}^{l} \sum_{l=1}^{\infty} \frac{4\pi(-1)^q(-e^2)}{2l+1} \left(\frac{(r<)^l}{(r>)^{l+1}} \right) \times Y_l^{-q}(\theta_\mu, \phi_\mu) Y_l^q(\theta_p, \phi_p) . \tag{3}$$

The sum over p is the sum over all the protons of the nucleus. $r<$ stands for the smaller of the magnitudes of the radius vector to the muon or the radius vector to the pth proton. $r>$ stands for the larger of the two quantities. By (θ_μ, ϕ_μ), we mean the angular variables of the muon in the coordinate

system fixed to the nucleus. Similarly, the angular variables of the pth proton are given by (θ_p, ϕ_p).

The wave functions of the nucleus corresponding to $|\nu I K m(I)\rangle$ are taken from the rotational model. The muonic wave functions are provided by solutions of the Dirac equation. In these differential equations the expectation value of the monopole term of the electrostatic interaction in the nuclear ground state is used as the attractive potential. A modified Fermi distribution of three parameters is used to describe the shape and radial distribution of the charge density which appears in this term:

$$\rho(r,\theta)=\rho(0)[1+\exp(\{r[1+\beta P_2(\cos\theta)]-c\}4\ln 3/t)]^{-1}, \tag{4}$$

where $\rho(0)$ is approximately the density at the center of the nucleus, β is a deformation parameter; c is the parameter which measures the half radius, i.e., the value of the radius at $P_2(\cos\theta)=0$, for which the density is decreased to one-half of $\rho(0)$; t is the surface thickness and is the interval in which the density falls from 90% to 10% of $\rho(0)$. It is customary to introduce c_0 related by c by

$$c=c_0 A^{1/3}. \tag{5}$$

The interaction term, V of Eq. (3) is not diagonal in the basis given by Eq. (2). We calculate the matrix elements of V in that basis and diagonalize the ensuing matrix to get the eigenvalues of H of Eq. (1). Since V does not mix states characterized by different eigenvalues of F, the matrix of V reduces to submatrices corresponding to specific values of F and $m(F)$. There is still a practical difficulty in the calculation because infinitely many states of the muon and the nucleus can be combined to give a particular value of F, the total angular momentum. A compromise can be made by selecting states of the muonic atoms which have the same principal quantum number which will combine with one of the first five nuclear levels of ^{181}Ta to give a particular value of F. The elements of the submatrix of H corresponding to a particular F are then calculated for a limited number of states of Eq. (2). This limited number of states will hereafter be said to comprise the model space. One justifies the procedure outlined above by noting that muonic states of different principal quantum number are far removed in energy and therefore, make negligible contribution.

Note that the term for $l=1$, in the expression for the interaction V in Eq. (3), will not make any contribution for it will not connect nuclear states of the same parity and similarly for the term with $l=3$. The higher terms, leading to small contributions compared to the quadrupole term, are neglected. The quadrupole part of V which we designate by HQ has matrix elements in the basis of Eq. (2):

$$(\lambda I J F m(F)|HQ|\lambda' I' J' F m(F))=(-1)^{I'+J-F}[20\pi(2I'+1)(2J+1)]^{1/2}[(J||Y_2||J')W(IJI'J';F2)(2I'0K|IK)\alpha(JJ')], \tag{6}$$

where we have specialized to deformed nuclei described by the rotational model. $W(IJI'J';F2)$ is the Racah coefficient. The reduced matrix element $(J||Y_2||J')$ is defined by

$$(2J'\,q\,m(J')|Jm(J))(J||Y_2||J')=(n\kappa Jm(J)|Y_2^q|n'\kappa'J'm(J')). \tag{6a}$$

$\alpha(JJ')$ contains the dependence on the intrinsic quadrupole moment and the penetrability factor

$$\alpha(JJ')=\frac{-e^2}{5}\int_0^\infty R(n\kappa;r_\mu)R(n'\kappa';r_\mu)Q(r_\mu)r_\mu^2 dr_\mu, \tag{7}$$

where $R(n\kappa;r_\mu)$ is the radial part of the wave function of the muonic atom in a state with principal quantum number n, specified by the quantum number κ, and

$$Q(r_\mu)=\int_0^\pi\int_0^{2\pi}\int_0^{r_\mu}\rho(r,\theta)P_2(\cos\theta)\frac{r^2}{(r_\mu)^3}d^3r+\int_0^\pi\int_0^{2\pi}\int_{r_\mu}^{\infty}\rho(r,\theta)P_2(\cos\theta)\frac{(r_\mu)^2}{r^3}d^3r. \tag{8}$$

In this calculation of the matrix elements of Eq. (6), the states used are restricted to the model space. For example, when we treat the $n=2$ states of the muonic atom we diagonalize a submatrix for a given value of F by considering only those off-diagonal terms which connect the $2p$ states and have nonzero value for the matrix elements for the quadrupole operator connecting pairs of the first five low-lying nuclear levels.

For the $n=3$ states of the muonic atom we consider the submatrix involving the $3d$ levels and the $3s$ levels and suitable pairs of the same nuclear levels. This procedure neglects quadrupole interactions connecting the $2p$ states with all other p states and f states. It neglects off-diagonal elements connecting the $3d$ states with all other d states, s states, and g states. It further ignores all connections between low-lying nuclear states and all other nuclear states of the same parity connected by the operator $Y_2^q(\theta,\phi)$.

The sum total of these neglected quantities leads to finite corrections to the calculations of Eq. (6).

These corrections are sizeable enough to be detected by currently available experimental techniques. The rest of this paper will be devoted to the description of the calculation of these corrections and to a description of the analysis of the experimental data. The final analysis will consist of a comparison of the calculations with the results of the experiment and a determination of values of the parameters which described the distribution of charge in the nucleus of ^{181}Ta.

EXTENSION BEYOND MODEL SPACE

The calculation of the matrix elements of the electrostatic interaction minus the monopole term requires the evaluation of quantities of the type $\langle v\rangle$ defined below for given value of F:

$$\langle v\rangle = \langle \nu I K m(I)\,|\,\langle n\,\kappa J m(J)\,|\,[\langle r\rangle^{l}/\langle r\rangle^{l+1}]Y_i^{-q}(\Omega_\mu)\,|\,n'\,\kappa' J' m(J')\rangle Y_i^{q}(\Omega_\rho)\,|\,\nu' I' K m(I')\rangle\,. \tag{9}$$

As given by Eq. (6) the elements of the interaction matrix are calculated by restricting the nuclear levels to the first five levels of the ground state rotational band of ^{181}Ta and the muonic levels of the muonic atom with the same principal quantum number. For example, the submatrix for $F=4$ for the $2p$ levels will be built up from the product states of

$$|\nu\tfrac{7}{2}\tfrac{7}{2}m(\tfrac{7}{2})\rangle\,|2\,1\,\tfrac{1}{2}\,m(\tfrac{1}{2})\rangle$$
$$|\nu\tfrac{7}{2}\tfrac{7}{2}m(\tfrac{7}{2})\rangle\,|2\,1\,\tfrac{3}{2}\,m(\tfrac{3}{2})\rangle$$
$$|\nu'\tfrac{9}{2}\tfrac{7}{2}m(\tfrac{9}{2})\rangle\,|2\,1\,\tfrac{1}{2}\,m(\tfrac{1}{2})\rangle$$
$$|\nu'\tfrac{9}{2}\tfrac{7}{2}m(\tfrac{9}{2})\rangle\,|2\,1\,\tfrac{3}{2}\,m(\tfrac{3}{2})\rangle$$
$$|\nu''\tfrac{11}{2}\tfrac{7}{2}m(\tfrac{11}{2})\rangle\,|2\,1\,\tfrac{3}{2}\,m(\tfrac{3}{2})\rangle$$

and will be 5×5 matrix. This set of states form the model space for the $2p$ splitting for $F=4$.

Furthermore, the matrix elements of Eq. (9) have nonzero values connected to states outside of the model space and we are interested in calculating the corrections necessary to account for the neglect of these states beyond the model space. This also suggests that terms in the electrostatic interaction other than the quadrupole term will connect to states outside of the model space and the contributions of those will also be calculated.

The projection operator formalism introduced by Lowdin[5] provides a convenient method for modifying the total matrix of the energy operator so that the modified energy operator can be evaluated in the model space. We describe this method in the following.

We introduce the operator:

$$Q = \sum_m{}' |m\rangle\langle m|\,, \tag{10}$$

where the sum over m stands for the sum over the model space and $|m\rangle$ represents a typical model space ket. The projection operator P is introduced

$$P = 1 - Q = \sum_b{}'' |b\rangle\langle b|\,, \tag{11}$$

where the sum over b represents summation over states external to the model space. These operators have the following set of properties:

$$Q^2 = Q\,, \tag{12a}$$
$$Q^\dagger = Q\,, \tag{12b}$$
$$P^2 = P\,, \tag{12c}$$
$$P^\dagger = P\,, \tag{12d}$$
$$QP = PQ = 0\,. \tag{12e}$$

Starting with

$$H\,|\gamma F m(F)\rangle = E\,|\gamma F m(F)\rangle\,, \tag{13}$$

where the $|\gamma F m(F)\rangle$ is the state vector representing the complete physical system and is a linear combination of state vectors given by Eq. (2). F is the quantum number for the total angular momentum and γ stands for all other quantum numbers needed to specify the system, E is the energy eigenvalue of the total system, and H is the operator given in Eq. (1). Combining Eqs. (11) and (13) we obtain

$$PH(P+Q)\,|\gamma F m(F)\rangle = PE(P+Q)\,|\gamma F m(F)\rangle$$

from which we find

$$PHQ\,|\gamma F m(F)\rangle = P(E-H)P\,|\gamma F m(F)\rangle$$
$$= [Q\delta + P(E-H)P]P\,|\gamma F m(F)\rangle\,,$$

where δ is an arbitrary constant. Multiplying on the left by the inverse of the operator in the bracket one sets

$$P[Q\delta + P(E-H)P]^{-1}PHQ\,|\gamma F m(F)\rangle = P\,|\gamma F m(F)\rangle\,. \tag{14}$$

From Eqs. (11) and (13) one can also obtain

$$QH[Q+P]\,|\gamma F m(F)\rangle = QE\,|\gamma F m(F)\rangle\,.$$

We substitute for $P\,|\gamma F m(F)\rangle$ the expression given in Eqs. (14); we get

$$[QHTHQ + QHQ]Q\,|\gamma F m(F)\rangle = EQ\,|\gamma F m(F)\rangle\,, \tag{15}$$

where

$$T = P[Q\delta + P(E-H)P]^{-1}P \tag{16}$$

and the following properties of T obtain

$$\begin{aligned} TP &= PT\,, \\ QT &= TQ = 0\,. \end{aligned} \tag{17}$$

Upon substituting in Eq. (15) for H, the expression obtained from Eq. (1) and taking the scalar product with $|m'\rangle$ which represents a state in the model space, one gets

$$\sum_m \langle m'|H + VTV|m\rangle\langle m|\gamma F m(F)\rangle = \sum_m E\langle m|\gamma F m(F)\rangle\delta_{mm'}\,, \tag{18}$$

where the sum over m is over the model space only.

We see that the eigenvalue of the total energy E can be obtained by diagonalizing a matrix of finite dimensions of the operator $(H+VTV)$ in the basis of the state vectors of the model space.

To get an expression for T suitable for computation one proceeds to rewrite it as

$$\begin{aligned} T &= P\{[Q\delta + P(E-H)P]^{-1} \\ &\quad + [Q\delta + P(E-H)P]^{-1}PVPT\}P\,, \\ &= T_0 + T_0VT\,, \end{aligned} \tag{19}$$

where

$$\begin{aligned} T_0 &\equiv P[Q\delta + P(E-H_0)P]^{-1}P\,, \\ H_0 &= H_N + H_\mu\,, \end{aligned} \tag{20}$$

and we have used the operator identity

$$(A-B)^{-1} = A^{-1} + A^{-1}B(A-B)^{-1}$$

with the identification

$$A = Q + P(E-H_0)P$$

and

$$B = PVP\,.$$

We make the approximation substituting T_0 for T in Eq. (19). The justification for this approximation is shown in Appendix A. Equation (20) can be rewritten:

$$T_0 = P[Q/\delta + P(E-H_0)^{-1}P]P \tag{21}$$

which can be verified by multiplying $Q\delta + P(E-H_0)P$ by the quantity in the brackets in Eq. (21) and obtaining the identity operator. Finally one simplifies T_0:

$$T_0 = P(E-H_0)^{-1}P\,, \tag{22}$$

where we remind the reader that

$$P = \sum_b{}'' |b\rangle\langle b| \tag{11}$$

and the sum b is over states external to the model space.

Since $H_N + H_\mu$ is diagonal in the basis selected, the problem reduces to the computation of the matrix element $\langle m'|[V + VP(E-H_0)^{-1}PV]|m\rangle$ in the model space. The elements $\langle m'|V|m\rangle$ are given in Eq. (6) and we shall concentrate for the present on $\langle m'|VP(E-H_0)^{-1}PV|m\rangle$:

$$\langle m'|VP(E-H_0)^{-1}PV|m\rangle = \sum_b{}''\langle m'|V|b\rangle\langle b|(E-H_0)^{-1}V|m\rangle\,. \tag{23}$$

By $|m\rangle$ and $|m'\rangle$ we mean state vectors of the type given by Eq. (2) restricted to the model space. The sum over $|b\rangle$ is the sum over states outside the model space, which are also represented by state vectors of the type given in Eq. (2):

$$|m\rangle = |\lambda I J F m(F)\rangle = \sum_{m(I)} (I J m(I) m(J)|F m(F))|n\kappa J m(J)\rangle|\nu I K m(I)\rangle\,, \tag{2'}$$

$$|m'\rangle = |\lambda' I' J' F m(F)\rangle = \sum_{m(I')} (I' J' m(I') m(J')|F m(F))|n'\kappa' J' m(J')\rangle|\nu' I' K m(I')\rangle\,, \tag{2''}$$

$$|b\rangle = |\lambda'' I'' F m(F)\rangle = \sum_{m(I'')} (I'' J'' m(I'') m(J'')|F m(F))|n''\kappa'' J'' m(J'')\rangle|\nu'' I'' K m(I'')\rangle\,. \tag{2'''}$$

In Eq. (23) the sum over the outer states $|b\rangle$ is broken up into several parts as follows:

$$\begin{aligned} \sum_b |b\rangle\langle b| &= \sum(|\mu^0N^0\rangle\langle\mu^0N^0| + |\mu^iN^0\rangle\langle\mu^iN^0| + |\mu^0N^i\rangle\langle\mu^0N^i|) \\ &= \sum[(|\mu^0N^0\rangle\langle\mu^0N^0| + |\mu^iN^0\rangle\langle\mu^iN^0| + |\mu^0N^i\rangle\langle\mu^0N^i| + |\mu^iN^i\rangle\langle\mu^iN^i| - |\mu^0N^i\rangle\langle\mu^0N^i| - |\mu^iN^i\rangle\langle\mu^iN^i|) \\ &\quad + (|\mu^0N^i\rangle\langle\mu^0N^i| + |\mu^iN^i\rangle\langle\mu^iN^i| - |\mu^iN^i\rangle\langle\mu^iN^i|)]\,, \end{aligned} \tag{24}$$

where $\sum|\mu^0N^0\rangle\langle\mu^0N^0|$ represents a sum over all states which originate from muonic states outside of the model space and from nuclear states outside of the model space and $\sum|\mu^iN^0\rangle\langle\mu^iN^0|$ represents a sum over states which originate from muonic states inside the model space and nuclear states outside of the model space and so forth.

In Eq. (23) the term in the sum over the outer states corresponding to the sums of the first bracket on the right hand side of Eq. (24) is designated the nuclear polarization correction. The partial sums of Eq. (23) corresponding to the second bracket of Eq. (24) are called the extended space correction. We will treat these corrections separately.

In order to consider the contribution of the nuclear polarization to the sum over the outer states in Eq. (23), we approximate the total energy E in Eq. (23) with $E(m)$:

$$E(m)=E(\nu' I' K)+E(n' \kappa')\,,$$

where $E(\nu' I' K)$ is the energy of the nuclear level with respect to the ground state of the nucleus. The level designated by the quantum numbers $[\nu',I',K,m(I')]$ is, of course, in the nuclear part of the model space. $E(n' \kappa')$ is the binding energy of the level of the muonic atom designated by the quantum numbers $[n', \kappa', J', m(J')]$ and is an energy level corresponding to a state in the model space.

The operator $[E-H_0]^{-1}$ in Eq. (23) is written as $[E(m)-E_N(b)-H_\mu]^{-1}$ since the state $|b\rangle$ contains a nuclear part which is an eigenstate of H_N with eigenvalue $E_N(b)$.

The sum over states $|b\rangle$ in Eq. (23) is written in terms of the bracket indicated on the right hand side of Eq. (24). The sums of the first bracket, the nuclear polarization term, will now be considered.

The matrix elements corresponding to the nuclear polarization become

$$\langle n|NP|m\rangle=\sum_c\langle n|V|c\rangle\langle c|[E(m)-E_N(c)-H_\mu]^{-1}V|m\rangle-\sum_{N^i}\langle n|V|N^i\rangle\langle N^i|[E(m)-E_N(N^i)-H_\mu]^{-1}V|m\rangle\,, \tag{25}$$

where the sum over N^i in the second term is over nuclear states in the model space only, and where we have made use of the completeness of the muonic state vectors. The sum over $|c\rangle$ is over the complete set of states describing the physical system. To use completeness in the first term of the right hand side of Eq. (25), we can introduce an $\bar{E}$ for which the following is true:

$$\begin{aligned}\sum_c\langle n|V|c\rangle\langle c|[E(m)-E_N(c)-H_\mu]^{-1}V|m\rangle&=\sum_c\langle n|V|c\rangle\langle c|[\bar{E}+E(n' \kappa')-H_\mu]^{-1}V|m\rangle\\&=\langle n|V[\bar{E}+E(n' \kappa')-H_\mu]^{-1}V|m\rangle\,.\end{aligned} \tag{26}$$

Since $\bar{E}$ has been introduced to remove the dependence of the energy term on $|c\rangle$, one can invoke completeness in the first term on the right hand side of Eq. (25). The nuclear polarization correction to the (n,m)th element becomes:

$$\langle n|NP|m\rangle=\langle n|V[\bar{E}+E(n' \kappa')-H_\mu]^{-1}V|m\rangle-\sum_{N^i}\langle n|V|N^i\rangle\langle N^i[E(m)-E_N(N^i)-H_\mu]^{-1}V|m\rangle\,. \tag{27}$$

The matrix elements of Eq. (27) are calculated by use of the reference spectrum method[6] after the nuclear contributions are separated. The state vectors $|n\rangle$ and $|m\rangle$ are given by Eqs. (2′) and (2″). The nuclear wave functions for nuclei with odd A are taken to be:

$$|\nu I K m(I)\rangle=[(2I+1)/16\pi^2]^{1/2}[D^I_{M\,(I)K}\chi(K)+(-1)^{I-K}D^I_{m\,(I)-K}\chi(K)]\,. \tag{28}$$

The phase relations are defined by Rogers.[7] $\chi(K)$ and $\chi(-K)$ are intrinsic functions of the coordinates of all the protons:

$$\chi(K)=(K;\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_p,\ldots\vec{r}_z)\,. \tag{29}$$

The $D^I_{m\,(I)K}$ are the rotational functions which have Euler angular variables for their argument. The Euler angles specify the orientation of the nucleus with respect to a coordinate system fixed in the laboratory.

We will concentrate on the first term of the right hand side of Eq. (27). After integrating over the Euler angle variables we get that term

$$\begin{aligned}\langle n|NP1|m\rangle&=\langle n|V[\bar{E}+E(n' \kappa')-H_\mu]^{-1}V|m\rangle\\&=\sum_{\substack{m(I),m(I')\\m(J),m(J')}}\{A[II'JJ'Fm(F);m(I)m(I')m(J)m(J')]B[II'Knn'JJ';m(I)m(I')m(J)m(J')]\}\,,\end{aligned} \tag{30}$$

where

$$A[II'JJ'Fm(F);m(I)m(I')\,m(J)m(J')]=8\pi^2e^4\left(\frac{2I'+1}{2I+1}\right)^{1/2}[(IJm(I)m(J)|Fm(F))(I'J'm(I')m(J')|Fm(F))], \tag{30a}$$

$$B[II'Knn'JJ';m(I)m(I')m(J)m(J')]$$
$$=\sum_L\sum_{l,l'}\sum_{m,m'}\{C[II'Knn'JJ';Lll'mm';m(I)m(I')m(J)m(J')]\}$$
$$\times[(l'I'm'K|LM)(lLmM|IK)+(-1)^{I'-K}(l'I'm'-K|LM_1)(lLmM_1|IK)$$
$$+(-1)^{I-K}(l'I'm'K|LM)(lLmM|I-K)+(-1)^{I+I'-2K}(l'I'm'-K|LM_1)(lLmM_1|I-K)], \tag{30b}$$

$$C[II'Knn'JJ';Lll'mm';m(I)m(I')m(J)m(J')]$$
$$=[(2l+1)(2l'+1)]^{-1}\sum_{q,q'}[(-1)^{q+q'}(lLqQ|Im(I))l'I'q'm(I')|LQ)]$$
$$\times[\langle n\kappa Jm(J)|D(r_\mu K;ll'qq'mm')|n'\kappa'J'm(J')\rangle], \tag{30c}$$

and

$$D(r_\mu K;ll'qq'mm')$$
$$=\sum_{p,p'}\int[|\chi(K)|^2Y_l^m(\Omega_p)T(r_\mu,r_p,l)U(r_\mu,ll'qq')T(r_\mu,r_{p'},l')Y_{l'}^{m'}(\Omega_{p'})]\,d^3r_1d^3r_2\cdots d^3r_p\cdots d^3r_{p'}\cdots d^3r_z, \tag{30d}$$

$$T(r_\mu,r_p,l)=[(r<)^l/(r>)^{l+1}]_p, \tag{30e}$$

$$U(r_\mu,ll'qq')=Y_l^{-q}(\Omega_\mu)[\bar{E}+E(n'\kappa')-H_\mu]^{-1}Y_{l'}^{-q'}(\Omega_\mu). \tag{30f}$$

Quantities of the type $(abcd|ef)$ are the vector addition coefficients. The sums over p and p' are sums over the protons of the nucleus. Q and M will only have those values for which the terms in Eq. (30) are nonzero. The sum over L will contain only those terms for which L satisfies the triangular inequalities with l and I and with l' and I'.

We will now concentrate on $D(r_\mu K;ll'qq'mm')$ defined in Eq. (30d). The double sum over all protons indicated by the sums over p and p' can be broken up into three parts. One part comes from those terms for which $p=p'$. The other two parts come from the terms in the sum for which $p\neq p'$:

$$D(r_\mu K;ll'qq'mm')=N(r_\mu K;ll'qq'mm')+V(r_\mu K;ll'qq'mm')+W(r_\mu K;ll'qq'mm'), \tag{31}$$

where

$$N(r_\mu K;ll'qq'mm')=\int[\rho(\vec{r})Y_l^m(\Omega)T(r_\mu,r,l)][U(r_\mu,ll'qq')T(r_\mu,r,l')Y_{l'}^{m'}(\Omega)]d^3r, \tag{31a}$$

where for the density of charged particles, $\rho(\vec{r})$, we will use the modified Fermi Distribution function normalized to Z charged particles:

$$\rho(\vec{r})=\rho(0)/[1+\exp(\{r[1+\beta P_2(\cos\theta)]-c\}4\ln 3/t)],$$
$$\rho(0)=Z/\int d^3r[1+\exp(\{r[1+\beta P_2(\cos\theta)]-c\}4\ln 3/t)]^{-1}. \tag{31b}$$

The direct term is

$$V(r_\mu K;ll'qq'mm')\equiv\int\rho(\vec{r})Y_l^m(\Omega)T(r_\mu,r,l)d^3r\int[\rho(\vec{r}')Y_{l'}^{m'}(\Omega')][U(r_\mu,ll'qq')T(r_\mu,r',l')]d^3r'. \tag{31c}$$

The exchange term is

$$W(r_\mu K;ll'qq'mm')=-\sum_{h,h'}\int d^3r\,S_h^*(\vec{r})S_{h'}(\vec{r})Y_l^m(\Omega)T(r_\mu,r,l)\int d^3r'\,S_{h'}^*(\vec{r}')S_h(r')Y_{l'}^{m'}(\Omega')U(r_\mu,ll'qq')T(r_\mu,r',l'), \tag{31d}$$

where $S_h(\vec{r})$ denotes the single particle function of a three-dimensional isotropic harmonic oscillator. The state h is specified by spin as well as spatial coordinates. The sums over h and h' are over all states filled by the Z protons of the nucleus under consideration. In this case Z is 73 which specifies the nuclide ^{181}Ta.

The single particle functions take the form:

$$S_h(\vec{r}) = s_{\eta\lambda}(\gamma r^2) \sum_{m(\frac{1}{2})} (\lambda\, \tfrac{1}{2}\, m(\lambda)\, m(\tfrac{1}{2}) \mid \xi\, m(\xi)) \times Y_\lambda^{m(\lambda)}(\Omega)\chi[\tfrac{1}{2}, m(\tfrac{1}{2})], \quad (31e)$$

where $\chi[\frac{1}{2}, m(\frac{1}{2})]$ is the spin wave function for a particle with spin $\frac{1}{2}$ and projection of spin $m(\frac{1}{2})$. The orbital angular momentum is λ and the total angular momentum is ξ. $s_{\eta\lambda}(\gamma r^2)$ is the Laguerre polynomial with appropriate normalization. γ is a measure of harmonic oscillator strength. It is determined by calculating the second moment of the charge distribution with the single particle functions and by fitting it to the second moment calculated from the Fermi distribution. We shall write

$$s(\eta\lambda, r) \equiv s_{\eta\lambda}(\gamma r^2).$$

We have taken, for purposes of calculating the nuclear polarization correction, an antisymmetric combination of the single particle functions:

$$\chi(K; \vec{r}_1, \vec{r}_2, \ldots \vec{r}_p, \ldots \vec{r}_z) = (N!)^{-1/2} \sum_{\pi(h)} (-1)^{\pi(h)} S_{h_1}(\vec{r}_1)\cdots S_{h_p}(\vec{r}_p)\cdots \times S_{h_{p'}}(\vec{r}_{p'})\cdots S_{h_z}(\vec{r}_z). \quad (31f)$$

The sum is made over all permutations of the integers $h_1, \ldots, h_z$ with respect to $1, 2, \ldots, z$. The factor $(-1)^{\pi(h)}$ is -1 for odd permutations and is $+1$ for even permutations.

Since the results of this calculation will lead to a correction of about one part in a thousand, we use the spherical average for $\rho(\vec{r})$ in the calculation of N, V, and W of Eq. (31). This leads to

$$V(r_\mu K; ll'qq'mm') = 0.$$

In the expression for N, Eq. (31a), the use of spherical symmetry implies

$$\begin{aligned} m + m' &= 0, \\ l &= l'. \end{aligned} \quad (32)$$

Invoking these conditions and integrating over nuclear angular variables we get

$$N(r_\mu k; ll' = lqq'mm' = -m) = \left(\int_0^{r_\mu} \rho_0(r) r^{2l+2}\, r_\mu^{-l-1} U(r_\mu, ll'qq') r_\mu^{-l-1} dr + \int_{r_\mu}^{\infty} \rho_0(r) r^{-2l} r_\mu^{l} U(r_\mu, ll'qq') r_\mu^{l} dr \right) (-1)^m, \quad (33)$$

where $\rho_0(r)$ is the spherical average of $\rho(\vec{r})$

$$\rho_0(r) = \frac{1}{2} \int_{-1}^{1} \rho(\vec{r}) d(\cos\theta). \quad (33a)$$

We next calculate $W(r_\mu K; ll'qq'mm')$, the exchange term in Eq. (31d). The integration over the angular variables and the summations over single particle states will give nonzero terms only when the conditions stated in Eq. (32) are satisfied.

We get in fact

$$W(r_\mu K; l\,l' = l\,qq'\,mm' = -m) = -2 \sum_{\eta,\eta'} \sum_{\lambda,\lambda'} (-1)^m (4\pi)^{-1} (2l+1)^{-1} [(2\lambda+1)(2\lambda'+1)] [(\lambda\lambda' 00 \mid l0)]^2 \delta_{ll'} \delta_{m-m'} \times [a(\eta\eta' l; r_\mu) r_\mu^{-l-1} U(r_\mu, ll\,qq') r_\mu^{-l-1} + b(\eta\eta' l; r_\mu) r_\mu^{-l-1} U(r_\mu, ll\,qq') r_\mu^{l} + c(\eta\eta' l; r_\mu) r_\mu^{l} U(r_\mu, ll\,qq') r_\mu^{-l-1} + d(\eta\eta' l; r_\mu) r_\mu^{l} U(r_\mu, ll\,qq') r_\mu^{l}], \quad (34)$$

where

$$\begin{aligned} a(\eta\eta' l; r_\mu) &\equiv \int_0^{r_\mu} s^*(\eta\lambda r) s(\eta'\lambda' r) r^{l+2} dr \int_0^{r_\mu} s^*(\eta'\lambda' r') s(\eta\lambda r') r'^{\,l+2} dr', \\ b(\eta\eta' l; r_\mu) &\equiv \int_0^{r_\mu} s^*(\eta\lambda r) s(\eta'\lambda' r') r^{l+2} dr \int_{r_\mu}^{\infty} s^*(\eta'\lambda' r') s(\eta\lambda r') (r')^{-l+1} dr', \\ c(\eta\eta' l; r_\mu) &\equiv \int_{r_\mu}^{\infty} s^*(\eta\lambda r) s(\eta'\lambda' r) r^{-l+1} dr \int_0^{r_\mu} s^*(\eta'\lambda' r') s(\eta\lambda r') (r')^{l+2} dr', \\ d(\eta\eta' l; r_\mu) &\equiv \int_{r_\mu}^{\infty} s^*(\eta\lambda r) s(\eta'\lambda' r) r^{-l+1} dr \int_{r_\mu}^{\infty} s^*(\eta'\lambda' r') s(\eta\lambda r') (r')^{-l+1} dr'. \end{aligned} \quad (35)$$

We next treat the muonic part of the matrix element $\langle n|NP1|m\rangle$. From Eq. (30c) we have

$$\langle n\kappa Jm(J)|D(r_\mu K;ll'qq'mm')|n'\kappa'J'm(J')\rangle = (N1)+(N2)-2\Big\{\sum_{\eta\eta'}\sum_{\eta\eta'}(-1)^m[(4\pi)(2l+1)]^{-1}(2\lambda+1)(2\lambda'+1)[(\lambda\lambda'00|l0)]^2 \times\delta_{ll'}\delta_{m-m'}[(N1W1)+(N2W2)+(W2)+(W3)]\Big\}, \tag{36}$$

where

$$(N1)\equiv\langle n\kappa Jm(J)|n(r_\mu)r_\mu^{-l-1}U(r_\mu,ll'qq')r_\mu^{-l-1}|n'\kappa'J'm(J')\rangle, \qquad n(r_\mu)\equiv\int_0^{r_\mu}\rho_0(r)r^{2l+2}dr,$$

$$N(2)\equiv\langle n\kappa Jm(J)|m(r_\mu)r_\mu^{l}U(r_\mu,ll'qq')r_\mu^{l}|n'\kappa'J'm(J')\rangle, \qquad m(r_\mu)=\int_{r_\mu}^{\infty}\rho_0(r)r^{-2l}dr,$$

$$(N1W1)\equiv\langle n\kappa Jm(J)|a(\eta\eta'l;r_\mu)r_\mu^{-l-1}U(r_\mu;llqq')r_\mu^{-l-1}|n'\kappa'J'm(J')\rangle,$$

$$(N2W4)\equiv\langle n\kappa Jm(J)|b(\eta\eta'l;r_\mu)r_\mu^{-l-1}U(r_\mu,llqq')r_\mu^{l}|n'\kappa'J'm(J')\rangle,$$

$$(W2)\equiv\langle n\kappa Jm(J)|c(\eta\eta'l;r_\mu)r_\mu^{l}U(r_\mu,llqq')r_\mu^{-l-1}|n'\kappa'J'm(J')\rangle,$$

$$(W3)\equiv\langle n\kappa Jm(J)|d(\eta\eta'l;r_\mu)r_\mu^{l}U(r_\mu,llqq')r_\mu^{l}|n'\kappa'J'm(J')\rangle.$$

The operator $U(r_\mu,llqq')$ was defined in Eq. (30e).

Let

$$|\chi(1)\rangle=[\bar{E}+E(n'\kappa')-H_\mu]^{-1}Y_l^{-q'}(\Omega_\mu)r_\mu^{-l-1}|n'\kappa'J'm(J')\rangle$$

and (37a)

$$|\chi(2)\rangle=[\bar{E}+E(n'\kappa')-H_\mu]^{-1}Y_l^{-q'}(\Omega_\mu)r_\mu^{l}|n'\kappa'J'm(J')\rangle$$

or

$$Y_l^{-q'}(\Omega_\mu)r_\mu^{-l-1}|n'\kappa'J'm(J')\rangle=[\bar{E}+E(n'\kappa')-H_\mu]|\chi(1)\rangle,$$

$$Y_l^{-q'}(\Omega_\mu)r_\mu^{l}|n'\kappa'J'm(J')\rangle=[\bar{E}+E(n'\kappa')-H_\mu]|\chi(2)\rangle. \tag{37b}$$

$|\chi(1)\rangle$ and $|\chi(2)\rangle$ are determined as solutions of Eq. (37a). The quantities of Eq. (36) are then calculated:

$$\begin{aligned}
&(N1)=\langle n\kappa Jm(J)|n(r_\mu)r_\mu^{-l-1}Y_l^{-q}(\Omega_\mu)|\chi(1)\rangle,\\
&(N2)=\langle n\kappa Jm(J)|m(r_\mu)r_\mu^{l}Y_l^{-q}(\Omega_\mu)|\chi(2)\rangle,\\
&(N1\,W1)=\langle n\kappa Jm(J)|a(\eta\eta'l;r_\mu)r_\mu^{-l-1}Y_l^{-q}(\Omega_\mu)|\chi(1)\rangle,\\
&(N2W4)=\langle n\kappa Jm(J)|b(\eta\eta'l;r_\mu)r_\mu^{-l-1}Y_l^{-q}(\Omega_\mu)|\chi(2)\rangle,\\
&(W2)=\langle n\kappa Jm(J)|c(\eta\eta'l;r_\mu)r_\mu^{l}Y_l^{-q}(\Omega_\mu)|\chi(1)\rangle,\\
&(W3)=\langle n\kappa Jm(J)|d(\eta\eta'l;r_\mu)r_\mu^{l}Y_l^{-q}(\Omega_\mu)|\chi(2)\rangle.
\end{aligned} \tag{38}$$

This completes the calculation of $\langle n\kappa Jm(J)|D(r_\mu K;ll'qq'mm')|n'\kappa'J'm(J')\rangle$ of Eq. (30c).

We will devote some space to the solutions of Eq. (37a) for $|\chi(1)\rangle$. The solution for $|\chi(2)\rangle$ will follow similarly.

The $|n'\kappa'J'\rangle$ are the solutions of the Dirac equation and take the form of bispinors. Given by Sakurai[8] these are:

$$|n'\kappa'J'm(J')\rangle=\begin{pmatrix} g(n'\kappa';r_\mu)Y(J',l'(A);m(J')) \\ if(n'\kappa';r_\mu)Y(J',l'(B);m(J'))\end{pmatrix}, \tag{39}$$

where the angular dependence is contained in the two component quantities, $Y(J,l;m(J))$.

We have

$$Y(J,l;m(J))=\{[l+m(J)+\tfrac{1}{2}]/(2l+1)\}^{1/2}Y_l^{m(J)-1/2}\begin{pmatrix}1\\0\end{pmatrix}+\{[l-m(J)+\tfrac{1}{2}]/(2l+1)\}^{1/2}Y_l^{m(J)+1/2}\begin{pmatrix}0\\1\end{pmatrix},$$

$$\text{for } J=l+\tfrac{1}{2},$$

$$Y(J,l;m(J))=-\{[l-m(J)+\tfrac{1}{2}]/(2l+1)\}^{1/2}Y_l^{m(J)-1/2}\begin{pmatrix}1\\0\end{pmatrix}+\{[l+m(J)+\tfrac{1}{2}]/(2l+1)\}^{1/2}Y_l^{m(J)+1/2}\begin{pmatrix}0\\1\end{pmatrix},\tag{39a}$$

$$\text{for } J=l-\tfrac{1}{2}.$$

When $\kappa=J+\frac{1}{2}$,

$$l(A)=J+\tfrac{1}{2},\quad l(B)=J-\tfrac{1}{2}\tag{39b}$$

and when $\kappa=-(J+\frac{1}{2})$,

$$l(A)=J-\tfrac{1}{2},\quad l(B)=J+\tfrac{1}{2}.\tag{39c}$$

We expand $|\chi(1)\rangle$ in a set of solutions of the Dirac equation:

$$|\chi(1)\rangle=\begin{pmatrix}\chi(1A)\\\chi(1B)\end{pmatrix}=\sum_{n'',\kappa'',m''}z(n'',\kappa'',m'')\begin{pmatrix}g(n'',\kappa'';r_\mu)Y(J'',l''(A);m'')\\if(n'',\kappa'';r_\mu)Y(J'',l''(B);m'')\end{pmatrix}\tag{40}$$

and substitute this expansion for $|\chi(1)\rangle$ in the first of Eqs. (37a). H_μ is the Dirac Hamiltonian:

$$H_\mu=c\vec{\alpha}\cdot\vec{p}+\beta m_\mu c^2+V(r).$$

m_μ is the reduced mass of the muon. For $V(r)$ we use the monopole term of the electrostatic interaction corrected for the vacuum polarization averaged over the nuclear ground state:

$$V(r_\mu)=V_0(r_\mu)\left[1-\frac{2\alpha}{3\pi}\left(l_n\frac{1.781\,07(r_\mu^2+t^2)^{1/2}}{\lambda}+\frac{5}{6}-\frac{3}{4}\frac{\pi}{\lambda}(r_\mu^2+\sigma^2)^{1/2}+\frac{3\lambda^{-2}}{2}[r_\mu^2+(\overline{r^2})]\right.\right.$$
$$\left.\left.-\frac{r_\mu^2[\frac{1}{2}t^2+\frac{1}{6}(\overline{r^2})]}{[r_\mu^4+(t^2)(\overline{r^2})]}-\frac{\pi}{3\lambda^3}[r_\mu^3+(\overline{r^2})^{3/2}]\right)\right],\tag{41}$$

where $V_0(r_\mu)$ is the monopole term of the interaction potential Eq. (3) and is given by

$$V_0(r_\mu)=-e\int_0^{r_\mu}\int\rho(\vec{r}')(r_\mu)^{-1}d^3r'-e\int_{r_\mu}^{\infty}\int\rho(\vec{r}')(r')^{-1}d^3r'$$

and where α is the fine structure constant and

$$(\overline{r^2})=\int\rho(\vec{r})r^2d^3r\Big/\int\rho(\vec{r})d^3r,$$

$$t=\exp\left[\int\rho(\vec{r})r^{-1}\ln(\vec{r})d^3r\Big/\int\rho(\vec{r})r^{-1}d^3r\right],$$

$$\sigma=\int\rho(\vec{r})d^3r\Big/\int\rho(\vec{r})r^{-1}d^3r,$$

and λ is the Compton wavelength of the electron.

By taking the scalar product of the first equation of the upper component of Eq. (37) with $Y(\bar{J},\bar{l}(A);\bar{m})$ and then taking the scalar product with the equation for the lower component with $Y(\bar{J},\bar{l}(B);\bar{m})$ we get the coupled differential equations satisfied by the radial parts of $|\chi(1)\rangle$:

$$\left[\bar{E}+E(n'\kappa')-mc^2-V(r_\mu)\right]g_1(\bar{\kappa};r_\mu)+c\hbar\left[\frac{d}{dr_\mu}+\frac{(1-\bar{\kappa})}{r_\mu}\right]f_1(\bar{\kappa};r_\mu)=(\bar{J}\,||\,l\,||\,J')r_\mu^{-l-1}g(n'\kappa';r_\mu),$$

$$\left[\bar{E}+E(n'\kappa')+mc^2-V(r_\mu)\right]f_1(\bar{\kappa};r_\mu)-c\hbar\left[\frac{d}{dr_\mu}+\frac{(1+\bar{\kappa})}{r_\mu}\right]g_1(\bar{\kappa};r_\mu)=(\bar{J}\,\|\,l\,\|\,J')r_\mu^{-l-1}f(n'\kappa';r_\mu),\tag{42}$$

where

$$(\bar{J}\,\|\,l\,\|\,J') \equiv [(lJ' - q'\,m(J')\,|\,\bar{J}\,\bar{m})]^{-1}\,\langle Y(\bar{J}\,l(A);\,\bar{m})\,|\,Y_l^{q'}(\Omega_\mu)\,|\,Y(J',\,l'(A);\,m(J'))\rangle\,,$$

$$g_1(\bar{\kappa};r_\mu) = \gamma_1(\bar{\kappa}\,\bar{m};\,r_\mu)[(lJ' - q'\,m(J')\,|\,\bar{J}\,\bar{m})]^{-1}\,,$$

$$\gamma_1(\bar{\kappa}\,\bar{m};r_\mu) = \sum_{\bar{\bar{n}}} z(\bar{\bar{n}},\,\bar{\kappa},\,\bar{m})\,g(\bar{\bar{n}}\,\bar{\kappa};\,r_\mu)\,.$$

Similar definitions apply for $f_1(\bar{\kappa},r_\mu)$ and $\phi_1(\bar{\kappa}\,\bar{m};\,r_\mu)$. Equation (40) has the following appearance:

$$|\chi(1)\rangle = \begin{pmatrix}\chi(1A)\\ \chi(1B)\end{pmatrix} = \sum_{\kappa'',\,m''} \begin{pmatrix}\gamma_1(\kappa''\,m'';\,r_\mu)Y(J'',\,l''(A);\,m'')\\ i\,\phi_1(\kappa''\,m'';\,r_\mu)Y(J'',\,l''(B);\,m'')\end{pmatrix}.$$

A similar procedure is used to calculate the bispinor $|\chi(2)\rangle$.

There is a second term of $\langle n\,|\,NP\,|\,m\rangle$ on the right hand side of Eq. (27) which now must receive consideration. This term is given by

$$\langle n\,|NP2\,|\,m\rangle = \sum_{N^i}\langle n\,|\,V\,|N^i\rangle\langle N^i\,|\,[E(m) - E_N(N^i) - H_\mu]^{-1}V\,|m\rangle\,. \tag{43}$$

N^i represents the nuclear state functions for states in the model space. $E(N^i)$ is the energy of the nuclear level with respect to the nuclear ground state. Consider $\langle \nu I K\,m(I)\,|\,V\,|N^i\rangle$, the nuclear part of the first factor of the right hand side of Eq. (43):

$$\langle \nu I K\,m(I)\,|\,V\,|N^i\rangle = \sum_{l,\,q} M\big(II(\alpha)\,m(I)\,m(I(\alpha))K\big)\int \rho(\vec{r})\,Y_l^q(\Omega)\,T(r_\mu,\,r,\,\Omega)\,d^3\vec{r}\,, \tag{44}$$

where

$$M\big(II(\alpha)\,m(I)\,m(I(\alpha))K\big) = \{[(-2\pi e^2)/(2l+1)]\}\{[2I(\alpha)+1]/(2I+1)\}^{1/2}\delta\big(m(I),\,m(I(\alpha))\big)\,[1+(-1)^{I+I(\alpha)-2K}]\,.$$

The use of the approximation of spherical symmetry for the density of charged particles will make the integral of Eq. (44) equal to zero for $l \geq 1$. Therefore,

$$\langle n\,|NP2\,|\,m\rangle = 0\,.$$

The total nuclear polarization expression now becomes:

$$\begin{aligned}
&\langle n\,|NP\,|\,m\rangle \\
&= 16\pi^2 e^4 \sum_{l,\,\bar{\kappa},\,L} (-1)^{L-\bar{J}}\,\frac{[(2L+1)^2(2J+1)(2\bar{J}+1)]^{1/2}}{(2l+1)^2}\,(J\,\|\,Y_l\,\|\,\bar{J})\,[W(L\,\bar{J}\,I\,J;\,Fl)W(L\,\bar{J}\,I\,J';\,Fl)] \\
&\quad\times\Bigg\{\Bigg(\int_0^\infty r_\mu^2 dr_\mu[\,g(n\,\kappa;\,r_\mu)\,g_1(\bar{\kappa};\,r_\mu) + f(n\,\kappa;\,r_\mu)\,f_1(\bar{\kappa};\,r_\mu)]\,r_\mu^{-l-1}\int_0^{r_\mu}\rho_0(r)\,r^{2l+2}\,dr\Bigg) \\
&\quad+\Bigg(\int_0^\infty r_\mu^2 dr_\mu[\,g(n\,\kappa;\,r_\mu)\,g_2(\bar{\kappa};\,r_\mu) + f(n\,\kappa;\,r_\mu)\,f_2(\bar{\kappa};\,r_\mu)]\,r_\mu^{\,l}\int_{r_\mu}^\infty \rho_0(r)\,r^{-2l}\,dr\Bigg) \\
&\quad- 2\sum_{\eta\eta'}\sum_{\xi\xi'}\frac{(2\xi+1)(2\xi'+1)}{4\pi(2l+1)}\,(\xi\,\xi'\,00\,|\,l0)^2 \\
&\quad\times\Bigg[\Bigg(\int_0^\infty dr_\mu r_\mu^2[\,g(n\,\kappa;\,r_\mu)\,g_1(\bar{\kappa};\,r_\mu) + f(n\,\kappa;\,r_\mu)\,f_1(\bar{\kappa};\,r_\mu)]\,[a(\eta\eta'\,l;\,r_\mu)\,r_\mu^{l-1} + c(\eta\eta'\,l;\,r_\mu)\,r_\mu^{l}]\Bigg) \\
&\quad+\Bigg(\int_0^\infty dr_\mu r_\mu^2[\,g(n\,\kappa;\,r_\mu)\,g_2(\bar{\kappa};\,r_\mu) + f(n\,\kappa;\,r_\mu)\,f_2(\bar{\kappa};\,r_\mu)][b(\eta\eta'\,l;\,r_\mu)\,r_\mu^{-l-1} + d(\eta\eta'\,l;\,r_\mu)\,r_\mu^{l}]\Bigg)\Bigg]\Bigg\},
\end{aligned} \tag{45}$$

where L is summed from $|l-I|$ to $(l+I)$.

Extended space correction

Let us now turn to that part of the sum over the outer states in Eq. (23) which corresponds to the extended space corrections. We had

$$\langle n|ES|m\rangle=\sum_{N^i}\langle n|V|N^i\rangle\langle N^i|[E(m)-E_N(N^i)-H_\mu]^{-1}V|m\rangle-\sum_{\mu^i,N^i}\langle n|V|u^iN^i\rangle\langle u^iN^i|[E(m)-E_N(N^i)-H_\mu]^{-1}V|m\rangle. \tag{46}$$

Completeness of the muonic states in the second bracket of Eq. (24) has been invoked. Consider the first term on the right hand side of Eq. (46):

$$\begin{aligned}\langle n|ES1|m\rangle=\sum_{\substack{m(J)\\m(J')}}\sum_{\substack{m(I)\\m(I')}}\sum_{\substack{q\\q'}}\sum_{\substack{I(\alpha)\\m(I(\alpha))}}\sum_{\substack{p\\p'}}(-1)^{q+q'}\left[\frac{4\pi}{5}(-e^2)\right]^2(IJm(I)m(J)|Fm(F))\\ \times(I'J'm(I')m(J')|Fm(F))\langle n\kappa Jm(J)|\mathfrak{R}(Im(I)I(\alpha)m(I(\alpha))Kq;r_\mu r_p)\\ \times\mathfrak{R}(I(\alpha)m(I(\alpha))I'm(I')Kq';r_\mu r_{p'})|n'\kappa'J'm(J')\rangle.\end{aligned} \tag{47}$$

The sums over p and p' are over all protons, the sums over $I(\alpha)$ and $m(I(\alpha))$ are over the nuclear states belonging to the model space:

$$\mathfrak{R}(Im(I)I(\alpha)m(I(\alpha))Kq;r_\mu r_p)\equiv\langle\nu Im(I)K|\mathfrak{R}(q,r_\mu r_p)|\nu(\alpha)I(\alpha)m(I(\alpha))K\rangle$$

$$\mathfrak{R}(q,r_\mu r_p)=Y^q_2(\Omega_p)Y^{-q}_2(\Omega_\mu)T(r_\mu,r_p,2), \tag{47a}$$

$$\begin{aligned}&\mathfrak{M}(I(\alpha)m(I(\alpha))I'm(I')Kq';r_\mu r_{p'})\\ &\quad\equiv\langle\nu(\alpha)I(\alpha)m(I(\alpha))K|[E(Fm(F)I'J')-E(I(\alpha)K)-H_\mu]^{-1}\mathfrak{R}(q',r_\mu r_p)|\nu'I'm(I')K\rangle,\end{aligned} \tag{47b}$$

where

$$E(Fm(F)I'J')=E(\nu'I'K)+E(n'\kappa').$$

A manipulation, similar to that for the nuclear polarization correction, leads to the following result

$$\begin{aligned}\langle n|ES1|m\rangle=\sum_{\substack{\bar\kappa\\I(\alpha)}}(-1)^{J+\bar J+I(\alpha)+I'}\{[2I(\alpha)+1](2\bar J+1)(2J+1)(2I'+1)\}^{1/2}(J||2||\bar J)\left(\frac{\pi}{5}e^4Q_0{}^2\right)\\ \times(2I(\alpha)0K|IK)(2I'0K|I(\alpha)K)W(I(\alpha)\bar JI'J';F2)W(I(\alpha)\bar JIJ;F2)\\ \times\int_0^\infty dr_\mu r_\mu{}^2[g(n\kappa;r_\mu)\gamma'(\bar\kappa;r_\mu)+f(n\kappa;r_\mu)\phi'(\bar\kappa;r_\mu)]\frac{P(K;r_\mu)}{r_\mu{}^3}.\end{aligned} \tag{48}$$

For later use we introduce:

$$A(n\kappa\bar\kappa;r_\mu)=[g(n\kappa;r_\mu)\gamma'(\bar\kappa;r_\mu)+f(n\kappa;r_\mu)\phi'(\bar\kappa;r_\mu)]\frac{P(K;r_\mu)}{r_\mu{}^3},$$

where $g(n\kappa;r_\mu)$ and $f(n\kappa;r_\mu)$ are solutions of the homogeneous radial Dirac equations

$$\begin{aligned}&[E(n,\kappa)+mc^2-V(r_\mu)]f(n\kappa;r_\mu)-c\hbar\left[\frac{dg(n\kappa;r_\mu)}{dr_\mu}+\frac{(1+\kappa)}{r_\mu}g(n\kappa;r_\mu)\right]=0,\\ &[E(n,\kappa)-mc^2-V(r_\mu)]g(n\kappa;r_\mu)+c\hbar\left[\frac{df(n\kappa;r_\mu)}{dr_\mu}+\frac{(1-\kappa)}{r_\mu}f(n\kappa;r_\mu)\right]=0\end{aligned} \tag{48a}$$

and where $\gamma'(\bar\kappa,r_\mu)$ and $\phi'(\bar\kappa,r_\mu)$ satisfy the following inhomogeneous equations:

$$[E(F\,m(F)I'J') - E(I(\alpha)) + mc^2 - V(r_\mu)]\phi'(\bar{\kappa}; r_\mu) - c\hbar\left(\frac{d}{dr_\mu}\gamma'(\bar{\kappa}; r_\mu) + \frac{(1+\bar{\kappa})}{r_\mu}\gamma'(\bar{\kappa}; r_\mu)\right)$$
$$= (\bar{J}\,||\,2\,||\,J')\frac{P(K; r_\mu)}{(r_\mu)^3} f(n'\kappa'; r_\mu)\,,$$
$$[E(F\,m(F)I'J') - E(I(\alpha)) - mc^2 - V(r_\mu)]\gamma'(\bar{\kappa}; r_\mu) + c\hbar\left(\frac{d\phi'}{dr_\mu}(\bar{\kappa}; r_\mu) + \frac{(1-\bar{\kappa})}{r_\mu}\phi'(\bar{\kappa}; r_\mu)\right)$$
$$= (J\,||\,2\,||\,J')\frac{P(\kappa; r_\mu)}{(r_\mu)^3} g(n'\kappa'; r_\mu)\,. \tag{48b}$$

The sum over $\bar{\kappa}$ in Eq. (48) is over all muon states for which neither $(J\,||\,Y_2\,||\,\bar{J})$ nor the two Racah coefficients are zero. The sum over $I(\alpha)$ is over inner nuclear states:

$$Q_0 = 2\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\rho(r,\theta)P_2(\cos\theta)r^2 d^3r$$

and (48c)

$$P(K; r_\mu) = 1 + \frac{\int_0^{\pi}\int_0^{2\pi}\int_{r_\mu}^{\infty}\rho(r,\theta)P_2(\cos\theta)(r_\mu{}^2/r^3 - r^2/r_\mu{}^3)\,d^3r}{Q_0/2r_\mu{}^3}\,.$$

Now consider the second term in Eq. (46):

$$\langle n\,|ES\,2\,|m\rangle = -\sum_{\substack{m(J)\\ m(J')}}\sum_{\substack{m(I)\\ M(I')}}\sum_{\substack{m(I(\alpha))\\ I(\alpha)}}\sum_{\substack{\kappa(\beta),n(\beta)\\ m(J(\beta))}}\langle n\kappa J\,m(J)\,|R(I\,m(I)I(\alpha)\,m(I(\alpha))K; qr_\mu)$$
$$\times\,|n(\beta)\,\kappa(\beta)J(\beta)\,m(J(\beta))\rangle\,\langle n(\beta)\,\kappa(\beta)J(\beta)\,m(J(\beta))|$$
$$\times\,\tau(I(\alpha)\,m(I(\alpha))I'\,m(I')K; q'r_{\mu'})\,|n'\,\kappa'J'\,m(J')\rangle$$
$$\times(IJ\,m(I)\,m(J)|\,F\,m(F))\,(I'J'\,m(I')\,m(J')|\,F\,m(F))\,, \tag{49}$$

where

$$R(I\,m(I)I(\alpha)\,m(I(\alpha))K; qr_\mu) = \sum_p\sum_q\langle\nu I\,m(I)K\,|(-1)^q Y_2^{-q}(\Omega_\mu)Y_2^q(\Omega_p)\frac{4\pi}{5}(-e^2)T(r_\mu, r_p, 2)\,|\nu(\alpha)I(\alpha)\,m(I(\alpha))\,K\rangle \tag{49a}$$

and where

$$\tau(I(\alpha)\,m(I(\alpha))I'\,m(I')K; q'r_\mu) = \sum_{p'}\sum_{q'}\langle\nu(\alpha)I(\alpha)\,m(I(\alpha))\,K\,|[E(F\,m(F)I'J') - E(I(\alpha)) - H_{\mu'}]^{-1}$$
$$\times(-1)^{q'}Y_2^{-q'}(\Omega_{\mu'})Y_2^{q'}(\Omega_{p'})T(r_{\mu'}, r_{p'}, 2)\frac{4\pi}{5}(-e^2)\,|\nu'I'\,m(I')\,K\rangle\,. \tag{49b}$$

In the intermediate step we have used the orthonormal property of the Clebsch-Gordan coefficients

$$\sum_{\substack{F'\\ m(F')}}\left(I(\alpha)J(\beta)\,m(I(\alpha))\,m(J(\beta))|\,F'\,m(F')\right)\left(I(\alpha)J(\beta)\,m'(I(\alpha))\,m'(J(\beta))|\,F'\,m(F')\right)$$
$$= \delta\left(m(I(\alpha)), m'(I(\alpha))\right)\delta\left(m(J(\beta)), m'(J(\beta))\right)\,.$$

The sums over $I(\alpha)$, $m(I(\alpha))$ and $J(\beta)$,$m(J(\beta))$ are over nuclear states and muonic states within the model space.

The final result for this second part becomes

$$\langle n\,|ES2\,|m\rangle = -\sum_{\substack{n(\beta)\\ \bar{\kappa};I(\alpha)}}(-1)^{J+J(\beta)+I(\alpha)+I'}\{[2I(\alpha)+1](2I'+1)(2J+1)(2\bar{J}+1)\}^{1/2}(J\,||\,Y_2\,||\,\bar{J})$$
$$\times\,W(I(\alpha)\bar{J}IJ; F2)W(I(\alpha)\bar{J}\,I'J'; F2)(Q_0)^2e^4\frac{\pi}{5}(2I(\alpha)\,0K|\,IK)(2I'\,0K|\,I(\alpha)\,K)$$
$$\times\left(\int_0^{\infty}B(n\kappa\,n(\beta)\,\bar{\kappa}; r_\mu)\,r_\mu{}^2 dr_\mu\right)\left(\int_0^{\infty}C(n(\beta)\bar{\kappa}, r'_\mu)(r'_\mu)^2 dr'_\mu\right)\delta(\bar{\kappa}, \kappa(\beta)), \tag{50}$$

where

$$B(n\kappa n(\beta)\bar{\kappa};r_\mu)=g(n\kappa;r_\mu)[g(n(\beta)\kappa(\beta);r_\mu)+f(n\kappa;r_\mu)f(n(\beta)\kappa(\beta);r_\mu)]\frac{P(K;r_\mu)}{r_\mu^{\,3}} \tag{50a}$$

and

$$C(n(\beta)\bar{\kappa};r'_\mu)=[g(n(\beta)\bar{\kappa};r'_\mu)\gamma'(\bar{\kappa};r'_\mu)+f(n(\beta)\bar{\kappa};r'_\mu)\phi'(\bar{\kappa};r'_\mu)]\,. \tag{50b}$$

The total extended space correction is given by

$$\begin{aligned}\langle n|ES|m\rangle=\sum_{I(\alpha)\bar{\kappa}}(-1)^{J+I(\alpha)+I'+\bar{J}}Q_0{}^2e^4\frac{\pi}{5}\{[2I(\alpha)+1](2J+1)(2I'+1)(2\bar{J}+1)\}^{1/2}\\ \times(J||Y_2||\bar{J})(2I(\alpha)\,0K|IK)(2I'\,0K|I(\alpha)\,K)W(I(\alpha)\bar{J}IJ;F2)W(I(\alpha)\bar{J}I'J';F2)\\ \times\Big[\int_0^\infty A(n\kappa\bar{\kappa};r_\mu)r_\mu{}^2dr_\mu-\sum_{n(\beta)}\Big(\int_0^\infty B(n\kappa n(\beta)\bar{\kappa};r_\mu)r_\mu{}^2dr_\mu\int_0^\infty C(n(\beta)\bar{\kappa};r'_\mu)r'^2_\mu dr'_\mu\Big)\delta(\bar{\kappa},\kappa(\beta))\Big]\,.\end{aligned} \tag{51}$$

ANALYSIS OF THE EXPERIMENTAL DATA

The data on the x rays of muonic atoms of ^{181}Ta were taken at the synchrocylotron of Carnegie-Mellon University in a manner described in an earlier publication.[9] The spectrum of the double escape peaks of the K x rays is shown in Fig. 1. We present in Fig. 2 the spectrum of the L x rays. Spectra of the γ rays used for calibration were also obtained. The peak positions, corresponding to the energies of the γ rays, were obtained by fitting Gaussian curves to the peaks by a technique involving the minimization of χ^2.

If some of the peaks in the x-ray spectra appeared broader than expected, a calculation was done to obtain the multiplicity of the γ rays in the peak region. Reasonable values of c_0, t, and Q_0 were assumed and the energies of the transitions

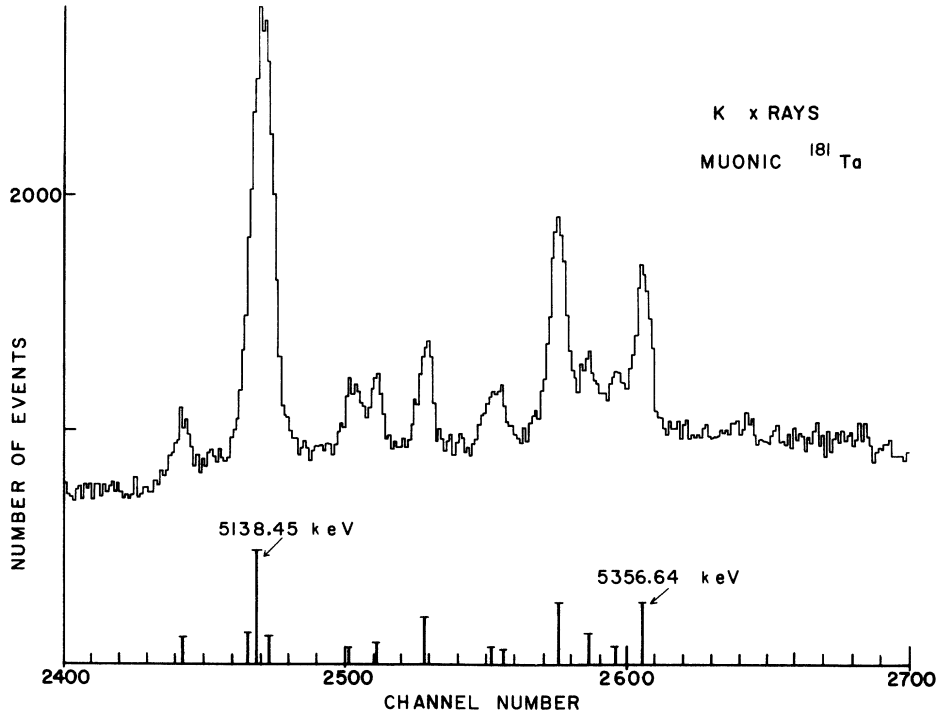

FIG. 1. Spectrum of double escape peaks of K transitions in muonic atoms of ^{181}Ta. The abscissa is the channel number and the ordinate represents the total number of counts in each channel. Vertical lines represent calculated transitions with heights proportional to intensities. Values of the parameters for the calculations are $c_0=1.1055$ fm, $t=2.389$ fm, and $Q_0=7.38$ b.

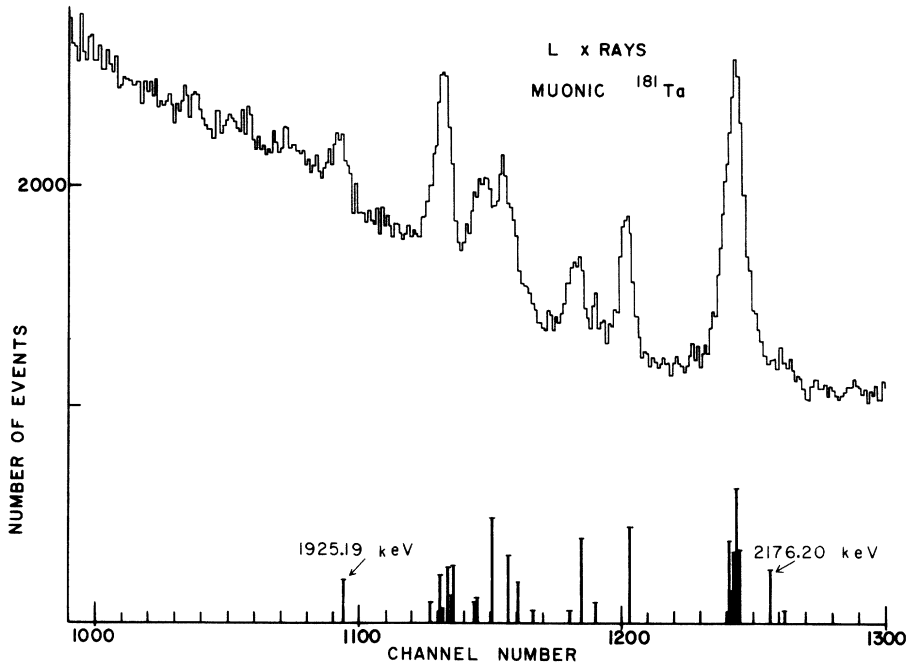

FIG. 2. Spectrum of L transitions in muonic atoms of ^{181}Ta. The legend is similar to that of Fig. 1.

were calculated. In addition, the intensities were calculated by assuming that the muonic atom cascaded from initial states of the $4f$ muonic atom by emitting E-1 radiation. A total angular momentum F was calculated for these initial states and the population of them was assumed to be proportional to $2F+1$.

This computation then suggested the number of components in the broad peak under study. The determination of the parameters of the constituents of the multipeak was then straightforward.

The values of the energies of the γ rays used for calibration are given in Table I. In Table II we present the numerical values of the physical constants which are required for the computation. The values as determined in this experiment of the energies of the K and L x-ray transition of muonic atoms of ^{181}Ta are presented in column 1 of Table III.

We were able to analyze the single escape peaks and the full energy peaks of 13 K x rays. This provided an extremely accurate determination of the slope of the linear response of the detecting system. We obtained a slope of 1.5974 ± 0.0014 keV/channel. The uncertainty in the slope is incorporated in the precision measures assigned to the experimental values of the energies of the transitions.

Only four peaks of the L x rays could be analyzed. The slope of the response of the system in the region of the L x rays was determined from energies and positions of calibration γ rays.

Theoretical values of the energies of the K and L transitions of the muonic atom of ^{181}Ta were calculated for many sets of values of the parameters c_0, t, and Q_0. A comparison of these energies with the experimentally obtained values was made by a calculation of χ^2. The region of fit for the values of the parameters was defined to give a value of χ^2 better than that for a confidence level of 90%.

TABLE I. γ rays used for calibration.

Values of energy of γ rays (keV)	Source	Reference
511.006 ± 0.000	^{22}Na	a
846.76 ± 0.05	^{56}Co	b
1238.34 ± 0.09	^{56}Co	b
1771.57 ± 0.01	^{56}Co	b
2035.03 ± 0.12	^{56}Co	b
2598.80 ± 0.12	^{56}Co	b
3253.82 ± 0.15	^{56}Co	b
5107.56 ± 0.24	^{16}N	c

[a] B. N. Taylor, W. H. Packer, and D. N. Langenberg, Rev. Mod. Phys. 41, 375 (1969).
[b] J. B. Marion, Nucl. Data. A4, 301 (1968).
[c] C. C. Hasman *et al.*, Phys. Rev. 159, 830 (1967).

TABLE II. Values of physical constants used in the calculation.

Constant	Value
Velocity of light	$2.997\,925\times10^{10}$ cm/sec
Mass of the electron	0.511 1006 MeV
Mass of the muon	105.659 MeV
Nuclear mass	931.478 (A +DA)
Mass defect (DA)	−0.052 020
Planck's constant	6.5820×10^{-22} MeV sec
Fine structure constant	1/137.0388
First excited state ^{181}Ta ($\frac{7}{2}^+$ band)	136.2 keV
Second excited state ($\frac{7}{2}^+$ band)	301.3 keV
Third excited state ($\frac{7}{2}^+$ band)	498.0 keV
Fourth excited state ($\frac{7}{2}^+$ band)	718.0 keV

This was accomplished by starting with fixed values of c_0 and t and obtaining the range of Q_0 for which the value of χ^2 was at the confidence level of 90% or better. The next step involved changing t and repeating the process for the new range of Q_0. Eventually we found values of t for which no range of Q_0 existed to give us a region of the required confidence.

We then change the value of c_0 and repeated the process. Ultimately we reached values of c_0, t, and Q_0 where we could not fit the experimental data with required confidence. In Table IV we present the values of the parameters which give the energies of the transitions with the confidence level of 90%.

We display in Figs. 1 and 2 the transitions calculated for an acceptable set of c_0, t, and Q_0 as vertical lines at the base of the figure. The heights

TABLE III. K and L transition energies calculated for:

$c_0 = 1.1055$ fm, $t = 2.3890$ fm, $Q_0 = 7.38$ b;

$\bar{E}_1 = -14.0$ MeV, $\bar{E}_2 = \bar{E}_3 = -15.0$ MeV.

Experimental value (keV)	Calculated excluding nuclear polarization correction and extended space correction (keV)	Calculated including nuclear polarization and extended space correction (keV)
	K transitions	
5098.3 ± 3.7	5089.01	5097.42
5133.5 ± 1.3	5126.99	5135.02
5139.5 ± 1.2	5130.92	5138.45
5145.5 ± 1.1	5236.54	5144.20
5192.2 ± 1.0	5181.90	5190.38
5205.3 ± 1.1	5196.98	5205.40
5234.2 ± 1.0	5225.01	5233.42
5267.6 ± 2.2	5262.94	5271.01
5275.7 ± 1.1	5269.24	5278.18
5308.9 ± 1.1	5300.12	5309.21
5325.7 ± 1.1	5317.90	5326.37
5342.1 ± 1.3	5333.29	5341.96
5357.8 ± 1.3	5347.02	5356.64
	L transitions	
1925.7 ± 2.8	1919.71	1925.19
2062.8 ± 0.6	2054.24	2060.61
2068.6 ± 0.5	2062.01	2068.35
2078.2 ± 0.9	2071.52	2077.35

TABLE IV. Range of values of parameters to obtain fits to the experimental data. Precision measures of Q_0 represent confidence limits of 90%.

c_0 (fm)	t (fm)	Q_0 (b)	χ^2
1.1010	2.4479	7.452 ± 0.008	6.4
	2.4413	7.503 ± 0.002	6.5
1.1055	2.3895	7.387 ± 0.005	6.4
	2.3767	7.480 ± 0.002	6.5
1.115	2.3049	7.316 ± 0.011	6.3
	2.2095	7.427 ± 0.007	6.4
1.1220	2.1416	7.251 ± 0.008	6.3
	2.1380	7.272 ± 0.008	6.3

of the lines are proportional to the intensity of the transitions which were calculated as indicated above. No attempt was made to fit the intensities for the experimental determination is not very precise. Qualitatively the intensities have the correct appearance.

In the calculation of the χ^2 the uncertainties of the experimentally determined values must be combined with the uncertainties of the calculation of the nuclear polarization correction. The latter uncertainties arise mainly from the use of wave functions for the spherically symmetric harmonic oscillator for the single particle functions. Chen[3,4] explored these uncertainties and estimated that the error in the calculation of the nuclear polarization correction was 30%. This error of 30% was combined in quadrature with the experimental errors of the energies of the K and L x-ray transitions.

In Fig. 3 we show plots of χ^2 vs Q_0 for a fixed $c_0 = 1.1240$ fm and for different values of t. Note that no curve reaches a value of χ^2 below 6.5, the value of which corresponds to a 90% confidence level. We assert, therefore, that for this value

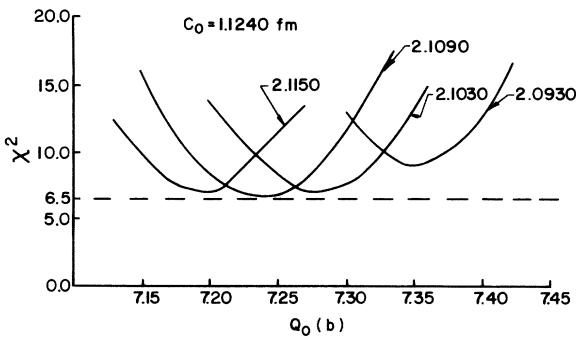

FIG. 3. Set of plots of $(\chi)^2$ versus Q_0, the intrinsic quadrupole moment. $c_0 = 1.1240$ fm. Each curve represents calculations for the specific value of t indicated on the figure.

of c_0 a fit is not possible. In Fig. 4 we show a similar curve for $c_0 = 1.055$ fm. In this figure one can see that a range of values of t will give acceptable fits. It was impossible to obtain an acceptable fit for any values of the parameters without the correction for nuclear polarization and extended space. In fact, the minimum χ^2 we obtained just trying to fit the energies of the K transitions was 45.8. The values of the parameters, corresponding to the unacceptable χ^2, were $c_0 = 1.150$ fm, $Q_0 = 7.20$ b, and $t = 1.548$ fm.

A set of values of c_0, t, and Q_0 was selected from well within the three-dimensional region of fit. A calculation of the energies of the K and L transitions, using this set of parameters, was performed. The results are presented in Table III. We also include in Table III the results of a calculation with these values of the parameters which neglect the nuclear polarization and extended space corrections.

It can be seen from Table IV that the bounds on the intrinsic quadrupole moment are 7.251 to 7.503 b. The surface thickness for the Fermi distribution lies between 2.1380 and 2.4479 fm. These ranges of values for the parameters obtain for the range of the half density radius, c_0, from 1.1010 to 1.1220 fm.

It might be noted here that there are several other methods for measuring the intrinsic quadrupole moment and we quote some of the results for comparison with ours. A typical spectroscopic measurement[10] gives a result of 8.36 ± 0.86 b. The giant dipole method[11-14] which measures the ratio of the energies of the two components of the (γ, n) reaction to obtain a value for Q_0 gives values ranging from 5.7 ± 0.3 to 7.1 ± 0.8 b with precision measures ranging from 0.3 to 1.3 b. A giant dipole experiment using a monochromatic beam of γ rays[15] yields 6.71 ± 0.74 b.

There are numerous measurements[16-19] using Coulomb excitation of the first excited state and second excited state of ^{181}Ta. The results for the intrinsic quadrupole moment vary from 5.2 ± 0.5 to 6.95 ± 0.27 b with precision measures ranging from 0.27 to 1.7 b.

Comparison with results from other muonic experiments is not very fruitful. A very early measurement[20] using detectors of NaI(Tl) obtained 8.4 ± 1.5 b. An experiment by the CERN Group[21] using a Ge(Li) detector produced a value of 7.5 ± 0.4 b for Q_0. This latter report consists of a three-parameter analysis of the experimental data with no corrections for nuclear polarization or extended space. As a result, the surface thickness reported by this group appears to be 1.50 ± 0.40 fm.

In the calculation of the nuclear polarization correction we assumed values for $\bar{E}_l$ as introduced in

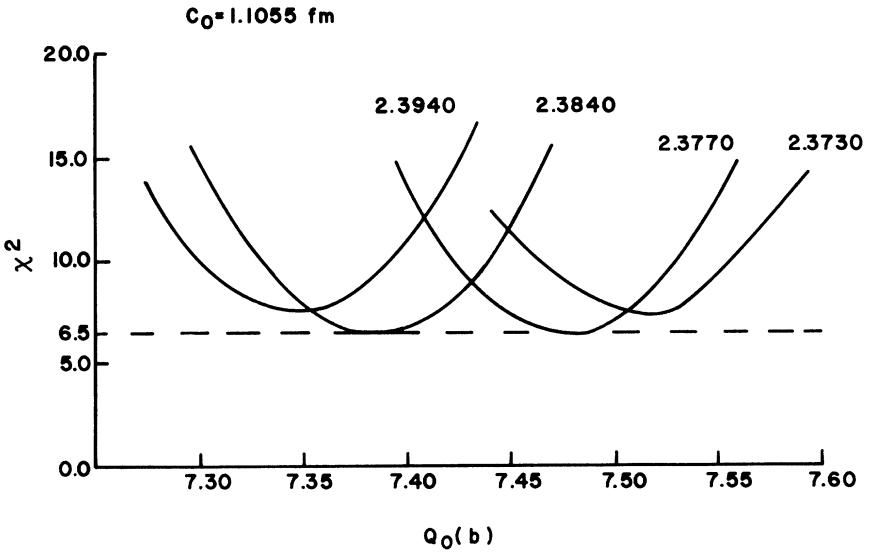

FIG. 4. Plots of $(\chi)^2$ versus Q_0 for $c_0 = 1.1055$ fm. Each curve represents calculations for the specific value of t indicated on the figure.

Eq. (26). The values assumed for the calculations in Table III are indicated. The results are not very sensitive to the values assumed for these mean energies as is shown in Tables V and VI. The sensitivity to values of $\bar{E}_3$ is not shown but is even less than the sensitivities given in Tables V and VI. It should be noted that in Eq. (26) we introduced the quantity $\bar{E}$. In our tables we talk of $\bar{E}_1$, $\bar{E}_2$, and $\bar{E}_3$. We did this at first to allow for different values with each term in the expansion of the electrostatic interaction. As is shown in the tables, one may assume the same value for all three $\bar{E}_I$.

TABLE V. Dependence of the nuclear polarization correction on $\bar{E}_1$: $c_0 = 1.1055$ fm, $t = 2.3890$ fm, $Q_0 = 7.38$ b; $\bar{E}_2 = \bar{E}_3 = -15.0$ MeV.

Line number	$\bar{E}_1 = -12.0$ MeV Nucl. pol. correction (keV)	$\bar{E}_1 = -14.0$ MeV Nucl. pol. correction (keV)	$\bar{E}_1 = -16.0$ MeV Nucl. pol. correction (MeV)
		K transitions	
1	6.56	6.54	6.38
2	6.52	6.52	6.36
3	6.46	6.46	6.32
4	6.56	6.46	6.31
5	6.54	6.55	6.40
6	6.56	6.51	6.38
7	6.52	6.54	6.38
8	6.63	6.52	6.36
9	6.65	6.59	6.44
10	6.57	6.60	6.45
11	6.54	6.55	6.36
12	6.52	6.51	6.47
13	6.67	6.64	6.38
		L transitions	
1	4.48	3.74	3.28
2	4.60	3.85	3.39
3	4.56	3.82	3.35
4	4.49	3.75	3.28

TABLE VI. Dependence of the nuclear polarization correction on $\bar{E}_2$:

$c_0 = 1.1055$ fm, $t = 2.3890$ fm, $Q_0 = 7.38$ b;
$E_1 = -14.0$ MeV, $E_3 = -15.0$ MeV.

Line number	$\bar{E}_2 = -10.0$ Nucl. pol. correction (keV)	$\bar{E}_2 = -15.0$ Nucl. pol. correction (keV)	$\bar{E}_2 = -25.0$ Nucl. pol. correction (keV)
	K transitions		
1	6.54	6.54	6.31
2	6.51	6.52	6.27
3	6.47	6.46	6.20
4	6.45	6.46	6.18
5	6.55	6.55	6.33
6	6.53	6.51	6.29
7	6.54	6.54	6.31
8	6.51	6.52	6.27
9	6.61	6.59	6.40
10	6.63	6.60	6.43
11	6.55	6.55	6.33
12	6.53	6.51	6.29
13	6.68	6.64	6.50
	L transitions		
1	4.17	3.74	3.35
2	4.29	3.85	3.50
3	4.25	3.82	3.47
4	4.18	3.75	3.38

In the nuclear polarization correction we calculated nonzero elements for $l = 1, 2, 3$ of the expansion of the electrostatic interaction. In the extended space correction we calculated contributions for $l = 2$ only.

The nuclear polarization corrections remain constant to within 0.2 keV for all transitions. The extended space correction and, consequently, the full corrections vary by as much as 2.1 keV. This indicates that adding a constant value to the calculated energies is not an adequate substitute for the calculations of the total correction.[22]

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APPENDIX

In Eq. (19) the expression for T is truncated by setting $T = T_0$. To see the validity of this approximation it is only necessary to consider the next term in the series for T; $T_0 V T_0$. This term will be seen to be small compared to T_0.

The matrix element of the operator $T_0 V T_0$, in the model space is:

$$\sum_{\alpha,\beta} \frac{\langle m|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|m'\rangle}{(E-E_\alpha)(E-E_\beta)},$$

where we remind the reader that the states specified by α, β lie outside the model space and those specified by m and m' lie in the model space. As is shown in Eq. (24), the sum over α is performed over two routes and similarly for the sum over β.

We get combinations of matrix elements which can be identified as being products of terms involving nuclear polarization only, products of nuclear polarization and extended space, and product of terms involving extended space only. When the products correspond to the extended space condition, the nuclear parts being in the model space, but the muonic parts being in outside states, the term which is truncated is small compared to the term which is kept. The energy difference which appears in the denominator $(E - E_\alpha)$ is at least of order 1000 keV. The presence of a second similar factor $(E - E_\beta)$ sharply reduces the size of the above matrix element.

When one or both of the factors correspond to the nuclear polarization condition, the terms in the above matrix element must be examined in greater detail.

When at least one of the factors comes from the nuclear polarization part, two situations occur. The first situation corresponds to the appearance of the product of two terms corresponding to the nuclear polarization. We next consider parts in which the sum over these outside states is made complete by adding and subtracting a term which contains only model space states.

The second situation corresponds to the extended space part in which the sum over the outside states is made complete only for the muonic states by adding and subtracting a term which contains both muonic and nuclear model space states. Each of these situations will be considered separately.

The first case to be considered is that in which the sums over α and β are carried out in part corresponding to the nuclear polarization treatment. Here four terms are generated in the matrix element. They are

$$\langle m|V|(E-\overline{E}-H_\mu)^{-1}|V|[E-\overline{E}-H_\mu]^{-1}|V|m'\rangle,$$

$$-\sum_{N^i}\langle m|V|N^i\rangle\langle N^i|[E-E_N(N^i)-H_\mu]^{-1}|V|(E-\overline{E}-H_\mu)^{-1}|V|m'\rangle,$$

$$-\sum_{N^i}\langle m|V|(E-\overline{E}-H_\mu)^{-1}|V|N^i\rangle\langle N^i|[E-E_N(N^i)-H_\mu]^{-1}|V|m'\rangle,$$

$$+\sum_{N^i,N^{i'}}\langle m|V|N^i\rangle\langle N^i|[E-E_N(N^i)-H_\mu]^{-1}|V|N^{i'}\rangle\langle N^{i'}|[E-E_N(N^{i'})-H_\mu]^{-1}V|m'\rangle\,.$$

The first and third terms are smaller by a factor of approximately $(\overline{E})^{-1}$ when compared to the terms which are kept in Eq. (25). Since $\overline{E}$ is of the order of 15 meV, these terms do not contribute significantly to the correction matrix. The second and fourth terms are identically zero, as is the subtracted term in Eq. (25), because of the nuclear integration of the interaction operator where a spherically symmetric approximation to the charge density is used.

The second case involves the part of the sum over α which corresponds to the nuclear polarization treatment coupled to that part of the sum over β which comes from the extended space correction. The forms are

$$\sum_{N^i}\langle m|V|N^i\rangle\langle N^i|[E-E_N(N^i)-H_\mu]^{-1}V(E-\overline{E}-H_\mu)^{-1}V|m'\rangle,$$

$$-\sum_{N^i,\mu^i}\langle m|V|N^i\rangle|\mu^i\rangle\langle\mu^i|\langle N^i|[E-E_N(N^i)-H_\mu]^{-1}V(E-\overline{E}-H_\mu)^{-1}V|m'\rangle,$$

$$-\sum_{N^{i'},N^i}\langle m|V|N^{i'}\rangle\langle N^{i'}|[E-E_N(N^{i'})-H_\mu]^{-1}V|N^i\rangle\langle N^i|[E-E_N(N^i)-H_\mu]^{-1}V|m'\rangle,$$

$$+\sum_{\mu^i,N^i,N^{i'}}\langle m|V|N^{i'}\rangle|\mu^i\rangle\langle\mu^i|N^{i'}|[E-E_N(N^{i'})-H_\mu]^{-1}V|N^i\rangle\langle N^i|[E-E_N(N^i)-H_\mu]^{-1}V|m'\rangle.$$

Each term vanishes for the same reason as the second and fourth terms in the previous case, i.e., the nuclear part of the matrix element $\langle m|V|N^i\rangle$ yields zero when a spherically symmetric approximation to the density is used.

In the third case, part of the sum over α corresponds to the extended space condition; the part indicated by β corresponds to the nuclear polarization condition. The four terms are

$$\sum_{N^i}\langle m|V|(E-\overline{E}-H_\mu)^{-1}V[E-E_N(N^i)-H_\mu]^{-1}N^i\rangle\langle N^i|V|m'\rangle,$$

$$-\sum_{\mu^i,N^i,N^{i'}}\langle m|V|N^{i'}\rangle|\mu^i\rangle\langle\mu^i|\langle N^{i'}|[E-E_N(N^{i'})-H_\mu]^{-1}V[E-E_N(N^i)-H_\mu]^{-1}N^i\rangle\langle N^i|V|m'\rangle,$$

$$-\sum_{\mu^i,N^i}\langle m|V|(E-\overline{E}-H_\mu)^{-1}V|\mu^i\rangle\langle\mu^i|[E-E_N(N^i)-H_\mu]^{-1}|N^i\rangle\langle N^i|V|m'\rangle,$$

$$+\sum_{\mu^i,N^i,N^{i'}}\langle m|V|N^{i'}\rangle\langle N^{i'}|[E-E_N(N^{i'})-H_\mu]^{-1}V|\mu^i\rangle\langle\mu^i|[E-E_N(N^i)-H_\mu]^{-1}N^i\rangle\langle N^i|V|m'\rangle.$$

Each term vanishes as in the previous case because of the presence of the nuclear parts of the matrix elements $\langle m'|V|N^{i'}\rangle$ or $\langle N^i|V|m'\rangle$.

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[1]L. Wilets, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 29, No. 3 (1954).

[2]B. A. Jacobsohn, Phys. Rev. 96, 1637 (1945).

[3]M. Y. Chen, Ph.D. thesis, Princeton University, 1968 (unpublished).

[4]M. Y. Chen, Phys. Rev. C 1, 1167 (1970).

[5]P.-O. Löwdin, Studies in Perturbation Theory, XII, Univ. of Florida preliminary research report No. 147 (unpublished), Quantum Theory Project.

[6]L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New

York, 1968).
[7]J. D. Rogers, Annu. Rev. Nucl. Sci. 15, 241 (1965).
[8]J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1967).
[9]R. E. Coté *et al.*, Phys. Rev. 179, 1134 (1969).
[10]T. Kamei, Phys. Rev. 99, 789 (1958).
[11]B. M. Spicer, H. H. Thies, J. G. Baglin, and F. R. Allum, Austral. J. Phys. 11, 298 (1958).
[12]R. W. Parsons and L. Katz, Can. J. Phys. 37, 809 (1959).
[13]O. V. Bogdankevich, B. I. Gonyachev, and V. A. Zapevalow, Z. Eksp. Teor. Fiz. 42, 1502 (1962) [Sov. Phys. -JETP 15, 1044 (1962)].
[14]E. G. Fuller and M. S. Weiss, Phys. Rev. 112, 560 (1968).
[15]R. L. Bramblett, J. T. Caldwell, G. F. Auchampaugh, and S. C. Fultz, Phys. Rev. 129, 2723 (1963).
[16]F. K. McGowan and P. H. Stelson, Phys. Rev. 99, 112 (1955); 109, 901 (1958).
[17]E. M. Bernstein and H. W. Lewis, Phys. Rev. 105, 1524 (1957).
[18]E. M. Bernstein and R. Graetzer, Phys. Rev. 119, 1321 (1960).
[19]C. McClelland, H. Mack, and C. Goodman, Phys. Rev. 97, 1191 (1955).
[20]S. Raboy *et al.*, Nucl. Phys. 73, 353 (1965).
[21]S. A. DeWit *et al.*, Nucl. Phys. 87, 657 (1967).
[22]D. McLoughlin, S. Raboy, and E. Deci, Phys. Rev. Lett. 35, 983 (1975).