Fluctuation cross section in the case of an isolated doorway resonance*

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In this article we discuss some aspects of the fluctuation part of the cross section at a doorway resonance. We show that in the strong absorption limit an inequality is derived for $\Gamma^\downarrow$, namely, $\Gamma^\downarrow \geq (2/N^{1/2}-1)\ \Gamma^\uparrow$ where N is the number of open channels. In the strong absorption limit no doorway resonance appears in the average cross section.

[NUCLEAR REACTIONS Intermediate structure, fluctuation cross section discussed.]

I. INTRODUCTION

In discussing the fluctuation part of the cross section at a doorway resonance the custom has been to assume that the absorption present in different channels is weak and thus is completely neglected in all channels. This then results in a simple expression for $\langle\sigma_{ab}^{fl}\rangle$, namely,

$$\langle\sigma_{ab}^{fl}\rangle_{\Delta E}=\frac{\Gamma_a\Gamma_b}{(E-E_R)^2+\frac{1}{4}\Gamma^2}\,\frac{\Gamma^\downarrow}{\Gamma^\uparrow}\,, \tag{1}$$

where $\Gamma_c<\Delta E<\Gamma$, Γ_c is a typical compound nucleus width (e.g., the $T_<$ states in the case of isobaric analog resonance). $\Gamma^\uparrow=\sum_a\Gamma_a$ and $\Gamma=\Gamma^\uparrow+\Gamma^\downarrow$. One knows, however, that in many situations, particularly in isobaric analog resonance phenomena, the absorption could be strong. The indiscriminate use of Eq. (1) to estimate $\langle\sigma_{ab}^{fl}\rangle$ for all cases certainly warrants a critical examination of the problem. In this work we shall look at the case when there is strong absorption in all channels. We shall also derive an expression for the fluctuation cross section in the case of intermediate absorption. And lastly we obtain an inequality for the spreading width $\Gamma^\downarrow$ of the doorway resonance. This inequality should be useful in cases where the number of channels N is small ($N<4$).

In what follows we shall omit all geometrical factors associated with the cross section.

II. *S*-MATRIX AND ITS AVERAGE

One can write the following expression for the S matrix describing the transition from state a to state b via the doorway state[1]:

$$\begin{aligned} S_{ab}&=S_{ab}^{(0)}-ie^{i(\delta_a+\delta_b)}\Gamma_a^{1/2}\Gamma_b^{1/2}f(E)\,,\\ S_{ab}^{(0)}&=\tau_a e^{2i\delta_a}\delta_{ab}\,,\\ f(E)&=\sum_\mu\frac{a_\mu^{\ 2}}{(E-E_\mu)+i\frac{1}{2}\Gamma_\mu}\,, \end{aligned} \tag{2}$$

where Γ_a is the partial width of the doorway resonance.

The sum $\sum_\mu$ is over all the complicated compound nucleus states that constitute the fine structure of the doorway resonance.

The doorway resonance becomes apparent when one considers the average S matrix averaged over an energy interval ΔE that satisfies $\Gamma_\mu<\Delta E<\Gamma$, where Γ is the total width of the doorway resonance.

Then

$$\langle S_{ab}\rangle_{\Delta E}=S_{ab}^{(0)}-ie^{i(\delta_a+\delta_b)}\frac{\Gamma_a^{1/2}\Gamma_b^{1/2}}{(E-E_R)+i\frac{1}{2}\Gamma}\,, \tag{3}$$

where $\langle S_{ab}^{(0)}\rangle_{\Delta E}=S_{ab}^{(0)}$ is used.

A very important fact appears conspicuously in Eqs. (1) and (2), namely, that even in the absence of direct reactions there is correlation among different channels due to the doorway state. This correlation manifests itself in the nondiagonal nature of S_{ab} and $\langle S_{ab}\rangle$. It is then interesting to investigate the effect of the correlation on the fluctuation cross section which in principle is determined from $\langle S_{ab}\rangle$ by diagonalization.[2] In the limiting cases of weak ($\tau_a\sim1$) and strong ($\tau_a\sim0$) absorption in all channels, however, a simple form for $\langle\sigma_{ab}^{fl}\rangle$ is obtained.

III. TRANSMISSION MATRIX

The transmission matrix is defined as usual by

$$P_{ab}=\delta_{ab}-\sum_c\langle S_{ac}\rangle\langle S_{bc}^*\rangle\,. \tag{4}$$

Using Eq. (2) one can find an expression for P_{ab} for any absorption τ_a,

$$\begin{aligned} P_{ab}=(1-\tau_a^{\ 2})\delta_{ab}+e^{i(\delta_a-\delta_b)}\Gamma_a^{\ 1/2}\Gamma_b^{\ 1/2}&\\ \times\Bigg\{i\left[\frac{\tau_b}{E-E_R+i\frac{1}{2}\Gamma}-\frac{\tau_a}{E-E_R-i\frac{1}{2}\Gamma}\right]&\\ -\frac{\Gamma^\uparrow}{(E-E_R)^2+\frac{1}{4}\Gamma^2}\Bigg\}\,,& \end{aligned} \tag{5}$$

where $\Gamma^{\dagger} = \sum_a \Gamma_a$.

P_{ab} is nondiagonal and it contains a nonresonant term and a resonant term (doorway resonance). The diagonalization of $\langle S_{ab} \rangle$ results in a diagonalization of P_{ab} itself by proper choice of the unitary operator that diagonalizes $\langle S_{ab} \rangle$.

The physical interpretation of P_{ab}, or rather, its diagonal part is that of a probability of passage from one channel to the other. Then the sum over all final states must be a positive quantity for all energies. Utilizing this fact we shall in the sequel derive an inequality.

Thus one demands $\mathrm{Tr} P \geq 0$ for all energies; therefore, from Eq. (5) one obtains

$$N - \sum_a \tau_a^2 + \frac{\sum_a \tau_a \Gamma_a \Gamma - (\Gamma^{\dagger})^2}{(E - E_R)^2 + \frac{1}{4}\Gamma^2} \geq 0 .$$

Evaluating the above at $E = E_R$ we get

$$N - \sum_a \tau_a^2 + \frac{4}{\Gamma^2}\left(\sum_a \tau_a \Gamma_a \Gamma - (\Gamma^{\dagger})^2 \right) \geq 0$$

or

$$N \geq \sum_a \tau_a^2 + \frac{4(\Gamma^{\dagger})^2}{\Gamma^2} - 4 \frac{\sum_a \Gamma_a \tau_a}{\Gamma} \tag{6a}$$

which is the desired inequality.

In the limit of strong absorption in all channels one obtains

$$\Gamma^{\dagger} \geq \left(\frac{2}{N^{1/2}} - 1 \right) \Gamma^{\dagger} . \tag{6b}$$

For large N the above inequality is a trivial one, but if $N < 4$ one gets an interesting *lower bound* on $\Gamma^{\dagger}$ which can be checked through a measurement of $\Gamma^{\dagger}$.

In general, however, one cannot realize the strong-absorption-limit-in-all-channel condition and thus Eq. (6a) is to be used instead. To be consistent with the condition of unitarity one would, in principle, vary τ_a in accordance with varying the resonance parameters. This is achieved through the evaluation of the following:

$$\left\langle \sum_c S_{ac} S_{bc}^* \right\rangle_{\Delta E} = \delta_{ab} \tag{7a}$$

which gives

$$\Gamma_a \Gamma^{\dagger} \langle |f(E)|^2 \rangle - \tau_a \Gamma_a \Gamma |\langle f(E) \rangle|^2 = 1 - \tau_a^2 . \tag{7b}$$

Equation (7b) is basically a relation between τ_a and the doorway resonance parameters Γ_a, Γ, $\Gamma^{\dagger}$, and E_R as well as the energy E.

Weak absorption in all channels implies an equation for $\langle |f|^2 \rangle$, namely,

$$\langle |f|^2 \rangle = |\langle f \rangle|^2 (\Gamma / \Gamma^{\dagger}) . \tag{8}$$

Strong absorption in all channels implies the following:

$$\Gamma_a \Gamma^{\dagger} \langle |f|^2 \rangle = 1 \tag{9}$$

which is meaningful only if there is only one channel or if Γ_a is the same in all channels which is approximately true when the number of open channels is large.

The best one can do in order to utilize Eq. (7b) is to evaluate $\langle |f|^2 \rangle$ by utilizing the statistics of a_μ's or through a dynamical model of the compound nucleus. But in the strong absorption case one can still make use of Eq. (9), since by summing over a one easily obtains $\langle |f|^2 \rangle = N/(\Gamma^{\dagger})^2$, i.e., no resonance appears in $\langle |f|^2 \rangle$. As we show in the next section this implies that the average cross section will show no doorway resonance in this limit.

IV. FLUCTUATION CROSS SECTION

In terms of $\langle |f|^2 \rangle$ and $|\langle f \rangle|^2$ the fluctuation cross section assumes the following form:

$$\langle \sigma_{ab}^{\mathrm{fl}} \rangle = \Gamma_a \Gamma_b (\langle |f|^2 \rangle - |\langle f \rangle|^2) . \tag{10}$$

Since $\sigma_{ab}(\langle S \rangle) = \Gamma_a \Gamma_b |\langle f \rangle|^2$, one immediately finds for the energy-averaged cross section

$$\langle \sigma_{ab} \rangle = \Gamma_a \Gamma_b \langle |f|^2 \rangle ; \tag{11}$$

the average $\langle f \rangle$ is known, namely,

$$\langle f \rangle = \frac{1}{E - E_R + i\frac{1}{2}\Gamma} . \tag{12}$$

The determination of $\langle |f|^2 \rangle$ thus completely determines both $\langle \sigma_{ab}^{\mathrm{fl}} \rangle$ as well as $\langle \sigma_{ab} \rangle$. One way of determining $\langle |f|^2 \rangle$ is to use Eq. (8) in the limit of weak absorption, $\tau_a \sim 1$; this gives Eq. (1). In the strong absorption limit we have

$$\langle \sigma_{ab} \rangle \to N \frac{\Gamma_a \Gamma_b}{(\Gamma^{\dagger})^2} . \tag{13}$$

Thus one sees that the strong absorption in all channels washes out the doorway resonance completely in the energy-averaged cross section.

The presence of N in Eq. (13) should not cause alarm in case the number of open channels N is large, because the quantity

$$\lim_{N \to \mathrm{large}} \frac{\Gamma^{\dagger}}{N}$$

can be considered as an average partial width that renormalizes Γ_a and Γ_b.

To obtain an expression for $\langle \sigma_{ab}^{\mathrm{fl}} \rangle$ valid in intermediate absorption cases where neither Eqs. (8) nor (9) are to be trusted requires, as was mentioned already, an explicit dynamical treatment of the resonances. However, one still hopes that if the number of open channels is large unitarity alone is sufficient to suggest a practical form for the fluctuation cross section. Utilizing Eq. (7b) again by summing over all channel indices, one obtains

$$\langle|f|^2\rangle(\Gamma^\dagger)^2-\sum_a\tau_a\Gamma_a\Gamma|\langle f\rangle|^2=N-\sum_a\tau_a^2\,. \quad (14)$$

In principle, the above equation should be considered as a unitarity constraint on the absorption coefficients τ_a, relating them to all possible variation in energy as well as the resonance parameters.

Denoting the average partial width by

$$\overline{\Gamma}\equiv\lim_{N\to\text{large}}\frac{\Gamma^\dagger}{N}\,;$$

the average absorption in all channels by

$$\overline{\tau}_a\equiv\lim_{N\to\text{large}}\frac{\sum_a\tau_a}{N}\,;$$

approximating

$$\lim_{N\to\text{large}}\frac{\sum_a\tau_a\Gamma_a}{N}\simeq\overline{\tau}\overline{\Gamma}$$

and

$$\lim_{N\to\text{large}}\frac{\sum_a\tau_a^2}{N}\simeq\overline{\tau}^2\,,$$

one thus obtains an approximate solution for $\langle|f|\rangle^2$, namely,

$$\langle|f|^2\rangle=\frac{1}{\Gamma^\dagger\overline{\Gamma}}\left[(1-\overline{\tau}^2)+\frac{\Gamma\overline{\tau}\overline{\Gamma}}{(E-E_R)^2+\frac{1}{4}\Gamma^2}\right]. \quad (15)$$

It should be realized that the above form for $\langle|f|^2\rangle$ is valid in the case where N, the number of open channels, is large, because then one may speak of average partial width by reason that the variation in Γ_a from one channel to the other would not be so significant. With the above form for $\langle|f|^2\rangle$ the energy-averaged fluctuation cross section becomes

$$\langle\sigma_{ab}^{\text{fl}}\rangle=\frac{\Gamma_a\Gamma_b}{\Gamma^\dagger\overline{\Gamma}}(1-\overline{\tau}^2)+\frac{\Gamma_a(\Gamma\overline{\tau}-\Gamma^\dagger)\Gamma_b}{(E-E_R)^2+\frac{1}{4}\Gamma^2}\frac{1}{\Gamma^\dagger} \quad (16a)$$

and the energy-averaged cross section,

$$\langle\sigma_{ab}\rangle=\frac{\Gamma_a\Gamma_b}{\Gamma^\dagger\overline{\Gamma}}(1-\overline{\tau}^2)+\frac{\Gamma_a\Gamma_b}{(E-E_R)^2+\frac{1}{4}\Gamma^2}\frac{\Gamma\overline{\tau}}{\Gamma^\dagger}\,. \quad (16b)$$

The above expression should be useful whenever one tries to approximate the absorption present in the reaction to be roughly the same in all channels, because then $\overline{\tau}$ would be the natural quantity to use to describe such an absorption rather than, say, individual τ_a's. *The larger the number of channels the better the above type of approximation is.* Of course, in the limit when this average absorption is weak ($\overline{\tau}\sim 1$) or strong ($\overline{\tau}\sim 0$), one recovers Eqs. (1) and (13), respectively.

Since a double check on Eqs. (16) is possible via $P_{aa}=\sum_b\langle\sigma_{ab}^{\text{fl}}\rangle$, we get, accordingly,

$$P_{aa}=\frac{\Gamma_a}{\overline{\Gamma}}(1-\overline{\tau}^2)+\frac{\Gamma_a(\Gamma\overline{\tau}-\Gamma^\dagger)}{(E-E_R)^2+\frac{1}{4}\Gamma^2}\,. \quad (17)$$

Naturally this is identified with Eq. (5) which indicates that, for Eq. (16) to be consistent with unitarity one must demand that $\Gamma_a=\overline{\Gamma}$ and $\tau_a=\overline{\tau}$. Since this connection between Γ_a and τ_a on the one hand and $\overline{\Gamma}$ and $\overline{\tau}$ on the other is only approximately true, one should thus keep this in mind when using Eqs. (16) for $\langle\sigma_{ab}^{\text{fl}}\rangle$.

From Eq. (16b) one sees that off resonance the average cross section is basically given by

$$\langle\sigma_{ab}\rangle_{\text{off}}\cong\frac{\Gamma_a\Gamma_b}{\Gamma^\dagger\overline{\Gamma}}(1-\overline{\tau}^2)\,.$$

At resonance one has

$$\langle\sigma_{ab}\rangle_{\text{R}}\cong\frac{\Gamma_a\Gamma_b}{(E-E_R)^2+\frac{1}{4}\Gamma^2}\frac{\Gamma\overline{\tau}}{\Gamma^\dagger}$$

$$\xrightarrow[E=E_R]{}\frac{4\Gamma_a\Gamma_b}{\Gamma}\frac{\overline{\tau}}{\Gamma^\dagger}\,;$$

the ratio between these two numbers is then independent of the channels involved, i.e.,

$$\frac{\langle\sigma_{ab}\rangle_{\text{R}}}{\langle\sigma_{ab}\rangle_{\text{off}}}\simeq\frac{4\overline{\tau}}{(1-\overline{\tau}^2)}\frac{\overline{\Gamma}}{\Gamma}\,. \quad (18)$$

Knowing Γ from other measurements one can, in principle, use the above equation to obtain the average absorption $\overline{\tau}$ which one needs to describe $\langle\sigma_{ab}\rangle$ for any other reaction $a\to b$.

V. DISCUSSIONS AND CONCLUSIONS

Several points are worth commenting upon in the light of the results we have obtained in this work.

A. Hauser-Feshbach theory

It is customary to express $\langle\sigma_{ab}^{\text{fl}}\rangle$ in terms of the diagonalized form of the transmission coefficient.[2] However, it seems to us that if the interest is just in the fluctuation cross section then the form (16) should be just as convenient to work with. As a matter of fact, the transmission matrix itself seems to be a more complicated object and one has to perform the Engelbrecht-Weidenmüller transformation in order to relate to the fluctuation cross section. Thus the need for a Hauser-

Feshbach calculation is certainly not so great in the cases we have discussed. It is the doorway nature of the intermediate structure that causes the cross section to have the simple form in Eq. (16). Of course, the above-mentioned Engelbrecht-Weidenmüller transformation becomes indispensable in the presence of direct (nonresonant) reactions. For this one needs the transmission matrix and thus a Hauser-Feshbach type of cross section.

B. Strong absorption

As we have seen in Sec. IV the presence of strong absorption in all channels results in smoothing out the intermediate structure in the energy-averaged cross section. This behavior can be traced to the condition that unitarity imposes on the absorption in its connection to the doorway resonance parameters and the energy. The averaged cross section one obtains in this limit, i.e., Eq. (13) should be considered to be mostly valid when the number of open channels N is large.

C. Intermediate absorption

In order to generalize the result obtained above to cases where the absorption is intermediate and equal in all channels we suggest a simple form, Eq. (16), valid only when the number of open channels is large. Unitarity imposes the further condition that the partial widths should be equal in all channels. Thus one would guess that our formula for $\langle\sigma_{ab}^{\text{fl}}\rangle$ in Eq. (16) is useful in elastic scattering. To discuss inelastic reactions via the doorway one has to consider Eq. (16) as an approximate one. Correction to $\langle\sigma_{ab}^{\text{fl}}\rangle$ as given in Eq. (16) can only be made if a detailed dynamical description of the resonances is made.

Lastly, we have exploited the positivity of the total reaction cross section to obtain an inequality which may be used in cases where N, the number of open channels, is small ($N< 4$) to give a lower limit to $\Gamma^{\downarrow}$, given an experimentally determined $\Gamma^{\uparrow}$. For large N the inequality becomes a trivial one $\Gamma^{\downarrow} \geq 0$.

It would be interesting to analyze the problem of the fluctuation cross section in the general case of many-doorway resonances and in the presence of direct, nonresonant reactions, using, e.g., the Englebrecht-Weidenmüller transformation or the Kawai-Kerman-McVoy approach.[3] We are presently exploring these extensions.

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