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Some effects of particle identity in many-body scattering using the method of channel coupling arrays

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Scattering events of the type $a + A \rightarrow A + a$, where A is composed of $(n-1)$ particles all identical to a, are studied within the framework of the channel coupling array approach to many-body scattering. The n coupled equations describing the scattering derived under the assumption that the n particles forming the scattering system are distinguishable, are shown to collapse to a single equation when particle identity is taken into account and a particular choice of a channel coupling array which includes the important special case of channel permuting arrays is made. For this latter choice, which guarantees connectedness, only the exchange Born potential explicitly enters the integral equation for the transition operator or the differential equation for the scattering function. The present derivation provides a justification of equations used previously in $e^- + H$ calculations.

[NUCLEAR REACTIONS Antisymmetrization effects in the Kouri-Levin-Tobocman coupled channel T operator nuclear reaction formalism are studied for the special case of a particle incident on a target containing $n-1$ particles identical to the projectile. The approach features the channel coupling arrays of Baer, Kouri, and Levin.]

We have recently developed a new method for describing many-body scattering, based on the use of the channel coupling array.[1,2] Coupled integral equations for the operators T_{jk} describing transitions between states of the arrangement channel Hamiltonians H_k and H_j have been derived and discussed under the assumption that the particles forming the scattering system are distinguishable.[3] The effects of particle identity are to be taken into account by forming proper linear combinations of the amplitudes for distinguishable (labeled) particles. For the case of two identical particles and a scattering center such as $e^- + H$, the pair of integral equations has been reduced to a single equation and calculations based on approximations to this single equation have been carried out[4]; we comment on the derivation of this equation below. It is well known by now that a similar reduction from three coupled equations to one also occurs[5] in the case of three identical particles. While a similar reduction to a single equation does not occur in the general case of n particles and N arrangement channels using an arbitrary channel coupling array, if does in fact occur for a particular choice, one sufficiently general to be of interest. Our purpose in this note is to examine this specific case and derive the resulting integral equation (and also the analogous differential equation).

The notation we use has been discussed elsewhere, and we only review it briefly here. The nonrelativistic scattering system, governed by a Hamiltonian H, is assumed to be composed of n particles, $1 \leq j \leq n$. In the most general case, the system can exist asymptotically in the states of N arrangement channels, corresponding to the asymptotic states of Ekstein.[6] For each arrangement channel k, $1 \leq k \leq N$, H can thus be partitioned into a channel Hamiltonian H_k and a channel perturbation V_k:

$$H = H_k + V_k, \quad 1 \leq k \leq N. \tag{1}$$

The asymptotic states $|\Phi_E(k)\rangle$ are eigenstates of H_k:

$$H_k |\Phi_E(k)\rangle = E |\Phi_E(k)\rangle \tag{2}$$

and are products of internal bound states for the clusters or subsystem of particles forming channel k and of plane wave relative motion states.[1-3] The V_k are assumed to vanish sufficiently rapidly when the relative separations of the clusters all become large so that the $|\Phi_E(k)\rangle$ can be defined.[6]

The transition operators $T_{jk}(E+i0)$ obey the coupled equations[1-3]

$$T_{jk}(+) = V_j W_{lk} + V_j \sum_i W_{li} G_i(+) T_{ik}(+), \tag{3}$$

where i, j, and l range over the arrangement channels of interest, the symbol + means $E+i0$, $G_i(+)$ is the ith channel Green's function defined by

$$G_i(+) = \lim_{\epsilon \to 0+} (E + i\epsilon - H_i)^{-1} \tag{4}$$

and W_{mn} is an element of the channel coupling array W. These elements obey

$$\sum_i W_{li} = 1, \tag{5}$$

with l free to be chosen as is convenient. The derivation of Eq. (3) has been given elsewhere[1-4] and need not be repeated here. We merely note that k must be a two-body channel. Methods for choosing l and the W_{mn} so as to ensure a connected iterate of the kernel $V_j W_{li} G_i^+(+)$ of Eq. (3) are discussed in Refs. 1 and 3.

The amplitude f_{jk} for making a transition from a state $|\Phi_E(k)\rangle$ of channel k to a state $|\Phi_E(j)\rangle$ of channel j, assuming all particles to be distinguishable, is

$$f_{jk} = \langle \Phi_E(j) | T_{jk}(+) | \Phi_E(k) \rangle, \tag{6}$$

where T_{jk} is to be determined from Eq. (3). If, however, some or all of the n particles of the scattering system are identical, then linear combinations of the f_{jk} must be formed. For an initial state k consisting of a single particle incident on a target of $(n-1)$ identical particles, the linear combination of the f_{jk} can be reduced to one "direct" term, and $(n-1)$ identical "exchange" terms in the case of elastic or inelastic scattering. The situation we shall consider here is restricted to final states corresponding to elastic or inelastic processes only, i.e., to two-body final channels consisting of a single particle and an $(n-1)$ particle bound state. Examples are neutron-nucleus scattering below the threshold for reactions to occur, or electron-neutral-atom scattering, below the ionization threshold.

Since only identical particles are involved, it is convenient to label the channels according to which labeled particle is chosen as the incident projectile. That is, in terms of the transition operators T_{jk}, there are n possible processes described by the operators T_{jk}, T_{lk}, etc., where k is the label of the particle chosen to initiate the reaction. We shall choose $k = 1$. Then n transition operators are T_{j1}, $1 \leq j \leq n$, with T_{11} describing the unique "direct" channel and the remaining T_{j1}, $j \neq 1$, describing the $(n-1)$ "exchange" channels.

We use the following notation. The initial state $|\Phi_E(1)\rangle$ is written as

$$|\Phi_E(1)\rangle = |\phi_0(2, 3, \ldots, n)\chi_{\vec{k}}(1)\rangle \tag{7}$$

and the final state in channel j, $|\Phi_E(j)\rangle$, is written

$$|\Phi_E(j)\rangle = |\phi_\alpha(1, 2, \ldots, j-1, j+1, \ldots, n)\chi_{\vec{k}'}(j)\rangle, \tag{8}$$

where $\{|\phi_\alpha(1, \ldots, j-1, j+1, \ldots, n)\rangle\}$ is a complete set of target states and $|\chi_{\vec{q}}(j)\rangle$ is a plane-wave state for particle j with momentum $\vec{q}$. We also assume

$$E = E_0 + \hbar^2 k^2/2\mu = E_\alpha + \hbar^2 k'^2/2\mu,$$

where E_α is the energy of the target state $|\phi_\alpha\rangle$ and μ is the reduced mass of the target-projectile system.

To simplify the labeling, we introduce a notation used previously[7]:

$$[j] \equiv (1, 2, \ldots, j-1, j+1, \ldots, n), \tag{9}$$

where serial ordering is maintained. Thus, we may rewrite Eqs. (7) and (8) as

$$|\Phi_E(1)\rangle = |\phi_0[1]\chi_{\vec{k}}(1)\rangle \tag{10}$$

and

$$|\Phi_E(j)\rangle = |\phi_\alpha[j]\chi_{\vec{k}'}(j)\rangle. \tag{11}$$

Finally, we shall work exclusively with systems of identical fermions, since in this case there occur phase factors depending on the particle labels, while for bosons all such phase factors are set equal to $+1$. Hence, we have that $|\phi_\alpha[j]\rangle$ is antisymmetric under interchange of any two labels in $[j]$:

$$P_{mn}|\phi_\alpha[j]\rangle = -|\phi_\alpha[j]\rangle, \quad m \neq j \neq n, \quad m \neq n,$$

where P_{mn} is the two-particle transposition operator. We also note that

$$|\phi_\alpha[j]\chi_{\vec{q}}(j)\rangle = (-1)^{j-l-1} P_{jl} |\phi_\alpha[l]\chi_{\vec{q}}(l)\rangle, \tag{12}$$

where $(-1)^{j-l-1}$ is the phase factor that occurs in restoring serial ordering to the state $|\phi_\alpha(1, \ldots, j-1, l, j+1, \ldots, l-1, l+1, \ldots n)\rangle$ resulting from application of the operator P_{jl} to $|\varphi_\alpha[l]\chi_{\vec{q}}(l)\rangle$.

The amplitude $A_{\alpha 0}$ for the process under consideration is[8]

$$A_{\alpha 0} = \langle \chi_{\vec{k}'}(1)\phi_\alpha[1] | T_{11}(+) | \phi_0[1]\chi_{\vec{k}}(1)\rangle - (n-1)\langle \chi_{\vec{k}'}(2)\phi_\alpha[2] | T_{21}(+) | \phi_0[1]\chi_{\vec{k}}(1)\rangle. \tag{13}$$

From Eq. (12), this may be written as

$$A_{\alpha 0} = \langle \chi_{\vec{k}'}(1)\phi_\alpha[1] | \tau(+) | \phi_0[1]\chi_{\vec{k}}(1)\rangle, \tag{14}$$

where

$$\tau(+) = T_{11}(+) - (n-1)P_{21}T_{21}(+). \tag{15}$$

Equation (13) is a statement that all the exchange amplitudes are identical. Our main purpose is to find an integral equation for the operator whose solution has matrix elements identical to Eq. (13). The general result we shall derive is

$$\tau = [V_1 W_{11} + (1 - W_{11})P_{21}V_2](1 + G_1\tau), \tag{16}$$

valid for any channel coupling array whose diagonal elements are all equal. This includes the important case of the *channel permuting array*[1] choice of

W. Such a W guarantees that the $(n-1)$st iteration of the kernel in Eq. (3) is connected. There are $(n-1)!$ choices of these particular W's, each of which has elements satisfying

$$W_{ii}=0, \quad \text{all } i\,; \tag{17}$$

properties of the other elements are given in the second of Ref. 1 and are noted below.

To derive Eq. (16), we use the most general form for $A_{\alpha 0}$ involving all the T_{j1} [we suppress the + symbol in $T_{j1}(+)$ and $G_j(+)$]:

$$A_{\alpha 0}=\left\langle \chi_{\vec{k}'}(1)\phi_\alpha[1]\right|\left(T_{11}-\sum_{l\neq 1}P_{l1}T_{l1}\right)\left|\phi_0[1]\chi_{\vec{k}}^{+}(1)\right\rangle, \tag{18}$$

where the equality of Eq. (18) with Eq. (13) is easy to demonstrate given the antisymmetry properties of the $\{|\phi_\alpha\rangle\}$ and the symmetry properties of the T_{l1}. For each T_{j1} we substitute Eq. (3) with k set equal to 1 and then make the choice $l=j$ in W_{1i}, this latter choice corresponding to the one initially used to specialize W to a channel permuting array. Then $A_{\alpha 0}$ may be written as

$$A_{\alpha 0}=\left\langle \chi_{\vec{k}'}(1)\phi_\alpha[1]\right|\left[\left(V_1W_{11}-\sum_{l\neq 1}P_{l1}V_lW_{l1}\right)(1+G_1T_{11})+\sum_{m\neq 1}\left(V_1W_{m1}-\sum_{l\neq 1}P_{l1}V_lW_{lm}\right)G_mT_{m1}\right]\left|\phi_0[1]\chi_{\vec{k}}^{+}(1)\right\rangle. \tag{19}$$

In this form, of course, τ depends on all the transition operators T_{l1}. However, just as in (13) where the symmetry is used to reduce $(n-1)$ different exchange terms to the same amplitude, the symmetry of the amplitudes can be used to transform this matrix element into one involving only the operators T_{11} and T_{21}. The key is the relabeling of coordinates in the terms of the sums over l, so that $l\to 2$ or $l\to 1$. To do this, we use the expansion

$$G_j(+)=\sum_\beta\int d^3q\,|\phi_\beta[j]\chi_{\vec{q}}^{+}(j)\rangle D_{\beta q}^{-1}\langle\phi_\beta[j]\chi_{\vec{q}}^{+}(j)|\,, \tag{20}$$

where $D_{\beta q}$ is an energy denominator independent of the label j. Notice, however, that the subscripts on the W_{ij} are not integration variables, and hence do not change under relabeling of coordinates, as was incorrectly assumed in the $e^-+\mathrm{H}$ calculations.[4]

There are three sums to be considered:

$$\sigma_1=-\sum_{l\neq 1}W_{l1}\langle\chi_{\vec{k}'}(1)\phi_\alpha[1]\,|\,P_{l1}V_l(1+G_1T_{11})\,|\,\phi_0[1]\chi_{\vec{k}}^{+}(1)\rangle=-\sum_{l\neq 1}W_{l1}M_l^{(1)}\,; \tag{21}$$

$$\sigma_2=\sum_{m\neq 1}W_{1m}\langle\chi_{\vec{k}'}(1)\phi_\alpha[1]|\,V_1G_mT_{m1}\,|\,\phi_0[1]\chi_{\vec{k}}^{+}(1)\rangle=\sum_{m\neq 1}W_{1m}M_m^{(2)}\,; \tag{22}$$

and

$$\sigma_3=-\sum_{m\neq 1}\sum_{l\neq 1}W_{lm}\langle\chi_{\vec{k}'}(1)\phi_\alpha[1]\,|\,P_{l1}V_lG_mT_{m1}\,|\,\phi_0[1]\chi_{\vec{k}}^{+}(1)\rangle=-\sum_{m\neq 1}\sum_{l\neq 1}W_{lm}M_{lm}^{(3)}\,. \tag{23}$$

The symmetry properties of G_1 and T_{11} mean that $|\phi_0[1]\chi_{\vec{k}}^{+}(1)\rangle$ and $G_1T_{11}|\phi_0[1]\chi_{\vec{k}}^{+}(1)\rangle$ have the same properties under particle interchange. Hence, by analyzing the matrix element $M_m^{(2)}$ in σ_2 first, we shall be able to deduce the behavior of $M_l^{(1)}$. Substituting Eq. (20) into the definition of $M_m^{(2)}$ in Eq. (22), we find

$$M_m^{(2)}=\sum_\beta\int d^3q M_m^{(2)}(\beta,\vec{q})\,, \tag{24}$$

where

$$M_m^{(2)}(\beta,\vec{q})=\langle\chi_{\vec{k}'}(1)\phi_\alpha[1]\,|\,V_1\,|\,\phi_\beta[m]\chi_{\vec{q}}^{+}(m)\rangle D_{\beta q}^{-1}\langle\chi_{\vec{q}}^{+}(m)\phi_\beta[m]|\,T_{m1}\,|\,\phi_0[1]\chi_{\vec{k}}^{+}(1)\rangle\,. \tag{25}$$

Since all coordinates are summation or integration variables, we may rewrite the first matrix element on the right in Eq. (25) as

$$\langle \chi_{\vec{k}'}(1)\phi_\alpha[1]|\, V_1 \,|\, \phi_\beta[m]\chi_{\vec{q}}(m)\rangle$$
$$= \langle \chi_{\vec{k}'}(1)\phi_\alpha(m,3,4,\ldots,m-1,2,m+1,\ldots,n)\,|\, V_1 \,|\, \phi_\beta(1,m,3,4,\ldots,m-1,m+1,\ldots,n)\chi_{\vec{q}}(2)\rangle$$
$$= (-1)^m\langle \chi_{\vec{k}'}(1)\phi_\alpha[1]|\, V_1 \,|\, \phi_\beta[2]\chi_{\vec{q}}(2)\rangle$$
$$= (-1)^m\langle \chi_{\vec{k}'}(2)\phi_\alpha[2]|\, V_2 \,|\, \phi_\beta[1]\chi_{\vec{q}}(1)\rangle\,, \tag{26}$$

where the first transformation involves relabeling so that $m \rightarrow 2$. Similarly, we find for the second matrix element

$$\langle \chi_{\vec{q}}(m)\phi_\beta[m]|\, T_{m1} \,|\, \phi_0[1]\chi_{\vec{k}}(1)\rangle = (-1)^m\langle \chi_{\vec{q}}(1)\phi_\beta[1]|\, P_{21}T_{21} \,|\, \phi_0[1]\chi_{\vec{k}}(1)\rangle\,. \tag{27}$$

Substituting (26) and (27) into (25) and this result back into (24) now gives

$$M_m^{(2)} = \langle \chi_{\vec{k}'}(1)\phi_\alpha[1]|\, P_{21}V_2G_1P_{21}T_{21} \,|\, \phi_0[1]\chi_{\vec{k}}(1)\rangle\,. \tag{28}$$

In similar fashion, $M_l^{(1)}$ of Eq. (21) becomes

$$M_l^{(1)} = \langle \chi_{\vec{k}'}(1)\phi_\alpha[1]|\, P_{21}V_2(1+G_1T_{11}) \,|\, \phi_0[1]\chi_{\vec{k}}(1)\rangle\,, \tag{29}$$

independent of l. The analysis for $M_{lm}^{(3)}$ of Eq. (23) is slightly more complicated because of the double sum. However, the procedure is made straightforward by isolating the term $l=m$, and we find

$$M_{mm}^{(3)} = \langle \chi_{\vec{k}'}(1)\phi_\alpha[1]|\, V_1G_1P_{21}T_{21} \,|\, \phi_0[1]\chi_{\vec{k}}(1)\rangle \tag{30}$$

and

$$M_{lm}^{(3)} = -\langle \chi_{\vec{k}'}(1)\phi_\alpha[1]|\, P_{21}V_2G_1P_{21}T_{21} \,|\, \phi_0[1]\chi_{\vec{k}}(1)\rangle, \quad l \neq m\,. \tag{31}$$

Putting the expressions for the matrix elements into the equations for the σ_i, substituting into Eq. (19), and then grouping terms gives for $A_{\alpha 0}$:

$$A_{\alpha 0} = \left\langle \chi_{\vec{k}'}(1)\phi_\alpha[1]\right| \Bigg\{ \left[V_1W_{11} - \left(\sum_{l\neq 1} W_{l1}\right)P_{21}V_2\right](1+G_1T_{11})$$
$$+ \left[\left(\sum_{\substack{m\neq 1\\ l\neq m}} W_{lm}\right)P_{21}V_2 - \left(\sum_{m\neq 1} W_{mm}\right)V_1\right]P_{21}T_{21}\Bigg\} \left|\, \phi_0[1]\chi_{\vec{k}}(1)\right\rangle. \tag{32}$$

Comparison of this result with Eqs. (14) and (15) shows that while T_{11} and T_{21} both occur in Eq. (32), the single linear combination τ does not enter the general expression above. Clearly, without specifying the array W further, no simple result occurs, since for example Eq. (5) by itself does not allow the sums of l and m in Eq. (32) to be evaluated. However, by restricting W slightly, these sums can be done. To do this, we first set $l=j$ in Eq. (3). Then the class of channel coupling arrays which enable one to obtain integral equations for single transition operator are those obeying the two conditions

$$W_{jj} = W_{11}, \quad j \neq 1$$

and

$$\sum_l W_{li} = 1\,. \tag{33}$$

The special case of channel permuting arrays[1] satisfies these conditions.

From Eq. (33) we easily find

$$\sum_{l\neq 1} W_{l1} = 1 - W_{11}\,,$$

$$\sum_{m\neq 1} W_{mm} = (n-1)W_{11}\,,$$

and

$$\sum_{\substack{m\neq 1\\ l\neq m}} W_{lm} = \sum_{m\neq 1}(1-W_{mm}) = (n-1)(1-W_{11})\,.$$

Substituting these results into Eq. (32) and simplifying yields:

$$A_{\alpha 0} = \langle \chi_{\vec{k}'}(1)\phi_\alpha[1]|\,[V_1W_{11} - (1-W_{11})P_{21}V_2]$$
$$\times(1+G_1\tau)\,|\,\phi_0[1]\chi_{\vec{k}}(1)\rangle\,. \tag{34}$$

Comparison of Eqs. (14) and (34) establishes Eq. (16) as the integral equation for the operator whose matrix elements evaluated as in Eq. (14) yield the transition amplitude $A_{\alpha 0}$. As yet we have found no other choice of the W_{mm} which yields an integral equation for τ. If only bound states are used in the spectral decomposition of G_1, there are no disconnected diagram divergences that arise in the solution of Eq. (16) and we are free to vary the W_{ji} away from the channel permuting array values ($W_{11}=0$), as in some of the e^-+H calculations.[4]

If we consider Eq. (16) for the channel permuting

array choice $W_{11}=0$, we notice that there is no "direct" Born term (V_1). Only the exchange term V_2 appears explicitly, although the direct channel Green's Function G_1 does occur. The absence of a direct Born term for $W_{11}=0$ at first glance would seem to ensure that our results cannot be correct. However, we have proved[3] that each T_{j1} in Eq. (3) has half-off-shell matrix elements identical to those of $T_{j1}=V_j+V_j(E+i0-H)^{-1}V_1$, the Low equation definition[5,6] of T_{j1}. Hence the T_{j1} of Eq. (3) lead to the correct results. Since Eq. (16) is derived from Eq. (3), it thus follows that on-shell matrix elements of τ will also be correct, i.e., will contain effects of a V_1 Born term. The explicit demonstration that the Born term is, in fact, present has been given elsewhere and need not be reproduced here. In addition, we note that in the case of the e^-+H calculations, use of Eq. (16) (with $W_{11}=0$) leads to results in reasonably good agreement with the variational calculations of Schwartz.[9] We would not expect such agreement unless the effects of V_1 were indeed being taken into account is some way. Hence, despite the lack of a manifest direct Born term, use of $W_{11}=0$ yields a correct result, and Eq. (16) is the correct operator to evaluate. Note of course that matrix elements of τ must be taken between states having the symmetry [1] and not the symmetry $[j]$, $j\neq 1$.

Let us now consider unitary approximations to Eq. (16) assuming that only bound states $|\phi_\alpha^{(b)}[1]\rangle$ are included. We do this in terms of a K, or reaction, operator.[10] Corresponding to Eq. (16) for τ is the reaction operator equation for K:

$$K=[V_1W_{11}-(1-W_{11})P_{21}V_2](1+G_1^pK)\,, \tag{35}$$

where $G_1^p=\mathrm{P}/(E-H_1)$ and P means principal value. Matrix elements of $K(E)$ taken between the states of H_1 will yield the elements of the K matrix, which for energies below the first excitation threshold, are proportional to $\tan\delta_l$, where δ_l is the lth order phase shift. The operators τ and K are related by the damping equations

$$\begin{aligned}\tau&=K-i\pi K\delta(E-H_1)\tau\\&=K-i\pi\tau\delta(E-H_1)K\,.\end{aligned} \tag{36}$$

These equations are consistent with the discontinuity equation[5] for τ, which is easily seen from Eq. (16) to be the unitarity relation

$$\mathrm{Im}\,\tau(+)=-\pi\tau(+)\delta(E-H_1)\tau(-)\,, \tag{37}$$

where $\tau(-)=\tau(E-i0)$ and $2i\,\mathrm{Im}\,\tau(+)\equiv\tau(+)-\tau(-)$. Since Eq. (36) also leads to Eq. (37), it readily follows that any approximation to K that has zero discontinuity will lead, via Eq. (36), to unitary matrix elements for τ.

It is also possible to obtain the differential equation whose solution (subject to outgoing-wave boundary conditions) yields the amplitude $A_{\alpha 0}=\langle\chi_{\vec{k}'}^-(1)\phi_\alpha[1]|\tau|\phi_b[1]\chi_{\vec{k}}^+(1)\rangle$. We omit the details of the derivation and state the result without proof; it is

$$[E-H_1-V_1W_{11}+(1-W_{11})P_{12}V_2]|\Psi_1\rangle=0\,. \tag{38}$$

If we expand $|\Psi_1\rangle$ in bound target states only, we obtain the equations

$$(E-E_\beta-K_1)|\Psi_\beta(1)\rangle=\sum_{\substack{\alpha\\ \text{bound}}}\langle\phi_\beta^{(b)}[1]|\,V_1W_{11}\,|\phi_\alpha^{(b)}[1]\Psi_\alpha(1)\rangle-\sum_{\substack{\alpha\\ \text{bound}}}\langle\phi_\beta^{(b)}[1]|(1-W_{11})V_1|\phi_\alpha^{(b)}[2]\Psi_\alpha(2)\rangle\,. \tag{39}$$

Here, K_1 is the relative kinetic-energy operator between particle 1 and the center of mass of the target. Equation (39) is reminiscent of the integro-differential equations found for the scattering of a particle by a system of particles identical to it when the full wave function is expanded in a term-wise antisymmetric sum of target states.[11] However, there is only a superficial similarity, as the detailed structure of the two sets is quite different. Indeed, $|\Psi_1\rangle$ is not the usual Schrödinger equation. Nevertheless, $|\Psi_\alpha(1)\rangle$ asymptotically yields the *correct* scattering amplitude $A_{\alpha 0}$ as is easily seen from Eq. (39), and thus Eq. (38) is a valid equation to solve. This is yet another example in which properly symmetrized amplitudes can be obtained from a state not having the proper symmetry under particle interchange.[12] Calculations based on Eq. (39) for the cases $n=2$ and $n=3$ are in progress and will be reported elsewhere.

We now return to Eq. (34) for the special case $n=2$. Here $\tau=T_{11}-P_{21}T_{21}$, and we note that

$$W_{11}=W_{22} \tag{40}$$

and also

$$W_{12}=W_{21}\,. \tag{41}$$

Use of Eqs. (40) and (41) leads to the equation

$$\tau=[V_1W_{11}-W_{21}P_{21}V_2](1+G_1\tau)\,. \tag{42}$$

Equation (42) is the one which was used in the e^-+H calculations.[4] Obviously, when we recall that $W_{21}=1-W_{11}$, Eq. (42) reduces to Eq. (16). Equation (42) was previously derived[4] assuming that in a term like $W_{jl}G_lT_{lm}$, interchange of particle labels in G_l would also interchange subscripts

on the W_{j1}, which is an invalid assumption. Nevertheless, the preceding analysis shows that Eq. (42) is correct, and hence that the e^-+H results are valid.[4] Because of its form, Eq. (42) can be used to study variations of W_{11} away from the value zero (and $W_{21} \neq 1$) when G_1 is approximated by expanding it in terms of bound states of particle 2 only, as in Ref. 4.

The analysis of this paper has been based on the assumption that the identical particles are fermions, so that under particle interchange, phase factors occur, as in Eqs. (12) and (26). All such phase factors become equal to unity in the case of identical bosons. To treat this latter case we merely need to change all factors $-P_{21}$ in the above equation to $+P_{21}$.

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