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Three-particle aspects in an N/D approach to nuclear reactions

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Several approximations in a generalized N/D approach to nuclear reactions are tested for the three-nucleon system. A detailed analysis of the analytic structure of the amplitudes for transitions between stable two-particle channels is presented. Three-particle unitarity is treated with a method which preserves the simple structure of the one variable N/D equations. Calculations for the three-nucleon system are compared with the exact solutions of the Amado equations. The importance of exchange contributions to three-particle unitarity is stressed.

[NUCLEAR REACTIONS Dispersion-relation method applied to the three-nucleon system. Analytic structure of scattering amplitudes. Three-particle unitarity.]

I. INTRODUCTION

In the last decade various methods have been developed to describe nuclear reactions. In the realm of few nucleon systems, the formulation of the rigorous three-body equations[1] has stimulated the development of several reaction theories which treat the three-particle aspects of the reaction along the lines of the Faddeev theory. If the number of nucleons exceeds three, the generalized Faddeev equations become very complicated so that one usually looks for methods to reduce the number of degrees of freedom. This should be done in such a way that the resulting amplitudes and the equations which they satisfy have a clear physical interpretation. Bound states, resonances, virtual states, and their interactions should find a natural embedding in such a theory.

An example of such a theory is the dispersion theoretic approach, which is based on the analytic properties of the amplitudes of various reaction processes. In this paper we will be concerned with an extension of the N/D theory of nuclear reactions formulated by Rinat and Stingl[2] (RS). Although the separation of the amplitudes into an N and D part serves a mathematical purpose, namely to linearize the unitarity equations, it has a physical relevance, too. It implies a separation of the dynamics of the reaction (i.e., the forces which act between the particles) and the fundamental constraint of matter conservation which is expressed by unitarity. In the N/D theory, both aspects can be treated to various degrees of sophistication, so that we can interpret the properties of the system in terms of these features.

In the present paper we will perform such an analysis for the three-nucleon system. The exact solution of the N/D equations for separable interactions should correspond with the solution of the Amado-Lovelace equations.[3] Since the latter solution is known,[4,5] we have a natural reference frame for performing our analysis.

In Sec. II we will review the many-channel N/D theory for two-body bound state channels. It has been shown by Rubin, Sugar, and Tiktopoulos[6] (RST) for the case of local Yukawa interactions, and by Stelbovics[7] in the case of the Amado-Lovelace model, that the amplitudes have the proper analytic properties for satisfying dispersion relations. The singularities in the energy E are situated along the real axis. Right-hand branch points correspond to physical thresholds; i.e., the energies at which reaction channels open. Left-hand branch points correspond to exchange processes involving nucleons, pions, or both. The cuts due to nucleon exchange are closest to the physical region. The corresponding forces therefore have a long range character. The range is roughly inversely proportional to the separation energy in one of the vertices. In our nonrelativistic approach, the pionic degrees of freedom are absorbed in the form factors of the vertices. The exchange forces of pionic nature have a range roughly inversely proportional to the pion mass and are therefore short ranged. The corresponding cuts lie far to the left and are thus well separated from the physical region. Other singularities in the amplitudes are bound state poles, whereas the N/D amplitudes may also contain ghost poles if the input in the N/D equations is inadequate. In the unphysical sheet there may be resonance poles off the real axis.

In Sec. III we will study the left-hand singularity structure of the first- and second-order partial wave amplitudes in detail. A similar study has been performed by RST in the case of local two-body Yukawa potentials; however, with a different purpose. They prove the analyticity of the unprojected on-shell amplitudes in a rigorous way, which is not the aim of the present paper. We concentrate on the case of two-body bound state channels and are interested in the actual positions of the singularities and their origin. With this knowledge we can construct the left-hand projection of the amplitude, which serves as input for our N/D equations. The amplitudes satisfying the Amado equations were also discussed by Stelbovics[7]; however, he only considers one two-body bound state channel, which is insufficient for our purposes.

In Sec. IV we will discuss three-particle unitarity. The exact treatment of three-particle channels is possible in dispersion theory[8]; however, this would complicate the equations to a level comparable to that of the Faddeev equations. Several approximations have been suggested to deal with this problem.[9,10,2] In this paper we reexamine this problem from the following point of view. Suppose that the Amado equations which manifestly preserve three-particle unitarity yield amplitudes $a(E)$ describing at all energies E the transition from one two-particle channel to another (or the same) two-fragment channel. One can now define new phase space factors $\rho(E)$ in terms of the amplitudes $a(E)$. Below the three-particle threshold they are identical with the original two-particle phase space factors. The solution of the N/D equations with this phase space matrix is exactly $a(E)$. Our aim, therefore, is to compute the thus modified phase space factors $\rho(E)$ as well as possible, without having to solve the full set of rigorous equations for $a(E)$. To accomplish this we use the isobar ansatz for the three-particle states. We will be particularly interested in exchange contributions to three-particle unitarity, which were hitherto neglected.

In Sec. V we apply the N/D approach in different orders of approximation to the three-nucleon system. We discuss both quartet and doublet scattering. In the latter case the singlet nucleon-nucleon interaction plays an important role. The corresponding t matrix has a pole close to the physical region (antibound state). In order to retain our formulation in terms of stable two-particle channels, we increase the strength of the singlet interaction somewhat so that it generates a bound state with small binding energy. The effect of the continuum singlet and triplet two-nucleon states can then be described in an approximate fashion by using modified phase space factors.

The introduction of the singlet state with a very small binding energy necessarily introduces an anomalous threshold. The present model of nucleon-deuteron scattering, therefore, is also a first application of the extended N/D equations introduced in Ref. 11. In that paper we showed that in the anomalous case N/D equations can be formulated which have the same structure as those for the normal case.

The results of the calculation are discussed in Sec. VI. Since our main purpose was to test various assumptions in the N/D approach, we did not try to introduce further refinements in our description of the three-nucleon system. We stress, however, that many refinements, such as the inclusion of noncentral and Coulomb forces, can be carried out in the N/D description, as has been shown in the recent study of the five-nucleon system.[12] We trust that the present study will clarify the conditions under which the N/D method can be successfully applied to nuclear reactions.

II. N/D FORMALISM

The N/D formalism has been discussed extensively in the literature,[13-15] so we only will mention the most important assumptions and equations. In dispersionlike treatments one assumes—and sometimes proves—that the scattering amplitudes which describe the different reactions are analytic in the energy plane. Since we restrict ourselves to central interactions, partial wave amplitudes can be defined according to

$$A_{ij}(E;\vec{k}_i,\vec{k}_j)=\sum_L \frac{(2L+1)}{4\pi}P_L(\vec{k}_i\cdot\vec{k}_j)\times a^L_{ij}(E;k_i,k_j)\,(k_ik_j)^L, \qquad (2.1)$$

where the center of mass (c.m.) energy is denoted by E, and the relative momenta in the in going and outgoing channels are $\vec{k}_i$ and $\vec{k}_j$. On-shell momenta satisfy

$$k_i^2=2M_i(E-E_i), \qquad (2.2)$$

where E_i is the threshold energy and M_i the reduced mass in channel i. The factor $(k_ik_j)^L$ was introduced to define partial wave amplitudes free of certain kinematical singularities. The on-shell partial wave amplitudes $a_{ij}(E)\equiv a^L_{ij}\,(E;k_i^{\rm on},k_j^{\rm on})$ satisfy unitarity equations in the physical region:

$$\underline{a}(E+i\epsilon)-\underline{a}(E-i\epsilon)=-2\pi i\underline{a}(E+i\epsilon)\times\underline{\rho}(E+i\epsilon)\,\underline{a}(E-i\epsilon), \qquad (2.3)$$

with phase space factors

$$\rho_{ij}(E+i\epsilon)=M_ik_i^{2L+1}\,\delta_{ij}\,\theta(E-E_i). \qquad (2.4)$$

Here we adopted a matrix notation for $\underline{a}$ and $\underline{\rho}$. Assuming the usual reality properties of scattering amplitudes,

$$\underline{a}^*(E) = \underline{a}(E^*), \tag{2.5}$$

one writes (2.3) above the relevant thresholds in the form

$$\text{disc}\,\underline{a}^{-1}(E+i\epsilon) = 2i\,\text{Im}\,\underline{a}^{-1}(E+i\epsilon) = 2\pi i \underline{\rho}(E). \tag{2.6}$$

In the N/D approach one writes the amplitude as a quotient of the numerator function n which contains the left-hand (dynamical) singularities and the denominator function D which carries the right-hand (unitarity) singularities,

$$\underline{a}(E) = \underline{n}(E)\,\underline{D}^{-1}(E). \tag{2.7}$$

Defining the input function $\underline{b}(E)$ as the left-hand projection of the amplitudes $\underline{a}(E)$, i.e., as a spectral integral over the left-hand cut L,

$$\underline{b}(E) = \frac{1}{\pi}\int_L dE'\,\text{Im}\,\underline{a}(E')/(E'-E), \tag{2.8}$$

one readily obtains the N/D equations (no subtractions)

$$\underline{n}(E) = \underline{b}(E) - \int_{E_i}^{\infty} dE' \frac{\underline{b}(E') - \underline{b}(E)}{E'-E}\,\underline{\rho}(E')\,\underline{n}(E'), \tag{2.9}$$

$$\underline{D}(E) = \underline{1} + \int_{E_i}^{\infty} dE'\,\underline{\rho}(E')\,\underline{n}(E')/(E'-E). \tag{2.10}$$

We note that in writing down the unitarity equations in the form (2.3) we neglected continuous channels, i.e., three-particle unitarity, and excluded anomalous thresholds since we assumed that the singularities of dynamical and kinematical nature are well separated. We come back to the first point in Sec. IV, whereas for the treatment of anomalous thresholds we refer to Ref. 11.

The input function $\underline{b}(E)$ is unknown, and the common strategy is to find good approximations. These may be based on perturbation expansions of the amplitudes $\underline{a}(E)$ if these are known, or on some diagrammatic method. The one cluster exchange (OCE) amplitudes and the repeated OCE amplitudes, as defined by Rinat and Stingl,[2] allow both a simple graphical interpretation and an explanation in terms of perturbation expansions. The nuclear models for the fragments enter via the dressed vertex functions. In nucleon-deuteron doublet scattering even third- and fourth-order input appear to be important, as was shown by Dodd and Stelbovics.[16] In the next section we will investigate the analytic structure of the amplitudes, in principle up to any order.

III. SINGULARITY STRUCTURE OF THE AMPLITUDES

The singularity structure of amplitudes in potential scattering has been studied extensively.[17,18] In particular, one has established the principle of nearest singularities which relates the proximity of the cut to the order of the amplitude concerned. For example, for the Yukawa potential the left-hand cut of the nth Born term starts at $-\frac{1}{4}n^2\mu^2$, where μ is the range parameter of the potential. The analytic properties of many-particle scattering amplitudes are far more complicated than those of the two-particle amplitudes, as was shown in the three-particle case by Rubin, Sugar, and Tiktopoulos[6] (RST). The analysis of the present section differs in several respects from this work. Starting points of RST were the Faddeev equations for local two-body interactions of the Yukawa type. Our basis is the many-channel reaction theory of Ref. 2, for which the driving mechanism is cluster exchange. Rather than with two-body potentials, we work with vertex functions which describe how composite particles are built up from constituent clusters. We choose the form factors to be of the Hulthên type. The analysis of RST concerns on-shell amplitudes before partial wave projection, whereas we will work throughout with partial wave amplitudes. Furthermore, our aim is to use the N/D method, which requires a distinction between left- and right-hand singularities. The latter in fact are an automatic consequence of the equations.

The basic mechanism in the theory of Rinat and Stingl[2] is cluster exchange. In first order there is OCE, in higher order there is repeated OCE. In the three-particle system the expressions for the amplitudes of successive order can be obtained from the Faddeev equations, or in the case of separable interactions from the Amado equations. The latter formulation is closer to that of Ref. 2. The OCE amplitude before partial wave projection is given by (cf. Fig. 1)

$$\langle \vec{k}|A_{ij}(E+i\epsilon)|\vec{k}'\rangle = C^A_{ij}C^{IS}_{ij}\frac{\Gamma^*(-\vec{k}'-\alpha\vec{k})\,\Gamma'(-\vec{k}-\alpha\vec{k}')}{E+i\epsilon-k^2/2m_b-k'^2/2m_a-(\vec{k}+\vec{k}')^2/2m_x}, \tag{3.1}$$

where C^{IS}_{ij} is a spin-isospin recoupling coefficient and C^A_{ij} is an antisymmetrization factor which—apart from its phase—is factorable in the vertices.[12] Furthermore, we introduced dimensionless mass ratios $\alpha = m_d/m_a$ and $\alpha' = m_b/m_c$. $\Gamma(\vec{q})$ is a vertex function which is chosen of the Hulthên form $q^l(q^2+\beta_l^2)^{-1/2}$, where β is a range parameter. This choice leads to an analytic expression for the partial wave amplitude. For central forces we have $l = l' = 0$ and obtain

$$a^{L(1)}_{ij}(E;k,k') = C^A_{ij}C^{IS}_{ij}\frac{m_x}{(kk')^{L+3}}\frac{N_{a,dx}N_{c,bx}}{4\alpha\alpha'} \times\left[\frac{Q_L(z)}{(z-x)(z-x')}+\frac{Q_L(x)}{(x-x')(x-z)}+\frac{Q_L(x')}{(x'-z)(x'-x)}\right], \tag{3.2}$$

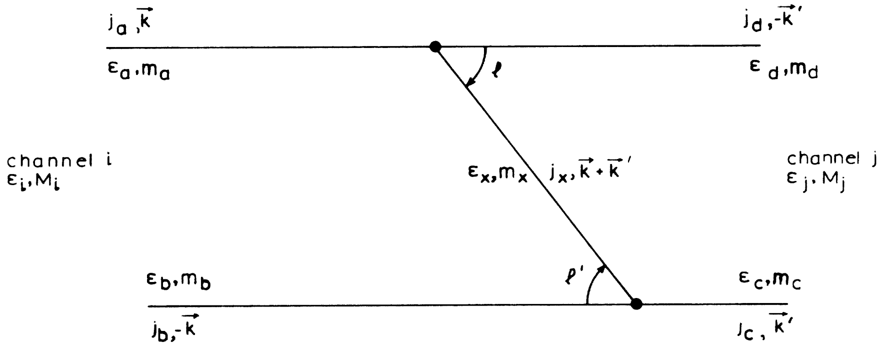

FIG. 1. One cluster exchange graph. The physical threshold for this process lies at $E_i=-\epsilon_a-\epsilon_b$ or $E_j=-\epsilon_c-\epsilon_d$.

with Q_L the well-known Legendre function of the second kind and

$$z=\frac{1}{kk'}\left[(E+\epsilon_x+\epsilon_b+\epsilon_d)\,m_x-\frac{k^2}{2\alpha}-\frac{k'^2}{2\alpha'}\right], \tag{3.3}$$

$$x=-\frac{k'^2+\alpha^2k^2+\beta^2}{2\alpha kk'}\,, \tag{3.4}$$

and

$$x'=-\frac{k^2+\alpha'^2k'^2+\beta'^2}{2\alpha' kk'}\,. \tag{3.5}$$

The following relation exists between coupling parameters and the separation energy $\kappa^2/2\mu_{dx}=\epsilon_a-\epsilon_d-\epsilon_x$:

$$N_{a,dx}{}^2=Z_{a,dx}\kappa\beta(\kappa+\beta)/\pi\mu_{dx}. \tag{3.6}$$

The spectroscopic factor $Z_{a,dx}$ represents the probability for finding fragments d and x in a. The definitions of the several binding energies ϵ, masses M, and reduced masses μ follow from Fig. 1. Remember that $M_i=m_am_b(m_a+m_b)^{-1}$. On-shell amplitudes are obtained from (3.2) by putting initial and final momenta on shell according to (2.2). The singularity structure of the physical on-shell amplitudes is determined by the square root branch points at the physical thresholds E_i and E_j and the logarithmic singularities of the Q_L functions for z, x and x' equal to ±1. For notational convenience we introduce some constants depending only on masses and binding energies involved:

$$\begin{aligned}\tau_{ij}&=\epsilon_c-\epsilon_b-\epsilon_x,\\ \Delta_{ij}&=\epsilon_c+\epsilon_d-\epsilon_a-\epsilon_b,\\ g_{ij}&=m_bm_d/Mm_x,\\ f_{ij}&=m_am_c/Mm_x=g_{ij}+1,\end{aligned} \tag{3.7}$$

and

$$\zeta_{ij}=-\tau_{ij}-g_{ij}\Delta_{ij},$$

where M is the total mass $m_a+m_b\approx m_c+m_d$. The logarithmic branch points of $a_{ij}^{(1)}(E)$ due to $Q_L(z)$ can be written as follows:

$$(l_{\lessgtr}^{z})_{ij}=-\epsilon_a-\epsilon_b-f_{ij}\tau_{ij}\left[1\mp\left(1+\frac{\zeta_{ij}}{f_{ij}\tau_{ij}}\right)^{1/2}\right]^2. \tag{3.8}$$

Similarly, the branch points due to $Q_L(x)$ are given by

$$(l_{\lessgtr}^{x})_{ij}=-\epsilon_a-\epsilon_b-[g_{ij}{}^{1/2}\hat\beta\mp f_{ij}{}^{1/2}(\Delta_{ij}+\hat\beta^2)^{1/2}]^2, \tag{3.9}$$

where $\hat\beta^2=f_{ij}\beta^2/2M_j$. Finally, the branch points due to $Q_L(x')$ are given by

$$(l_{\lessgtr}^{x'})_{ij}=(l_{\lessgtr}^{x})_{ji}. \tag{3.10}$$

In the following we will see that the left-hand singularities close to the physical threshold are determined mainly by the dynamical character of the reaction (z singularities), whereas the more distant singularities are determined by the finite range effects of the forces between the fragments d and x, and b and x (x and x' singularities). The contributions of the latter singularities turn out to be of particular interest in the higher order amplitudes, since many of the z contributions then vanish.

The cut structure of the $Q_L(z)$ term in $a_{ij}^{(1)}(E)$ is shown in Fig. 2. We have distinguished between three cases: $\zeta_{ij}<0$, $\zeta_{ij}=0$, and $\zeta_{ij}>0$. The latter case represents the so-called anomalous situation where right- and left-hand singularities of the on-shell amplitudes are no longer separated. In practice the $Q_L(x)$ and $Q_L(x')$ cuts never intertwine with the physical unitary cuts.

The second-order amplitudes can be given in terms of the half-shell amplitudes

$$\begin{aligned}&a_{ik}^{(1)}(E;k(E),k'(E'))\equiv a_{ik}^{(1)}(E;E,E'),\\ &a_{ij}^{(2)}(E)=\sum_k\int_{E_k}^{\infty}dE'a_{ik}^{(1)}(E;E,E')\rho_k(E')\\ &\qquad\times\tau_k(E-E')a_{kj}^{(1)}(E;E',E),\end{aligned} \tag{3.11}$$

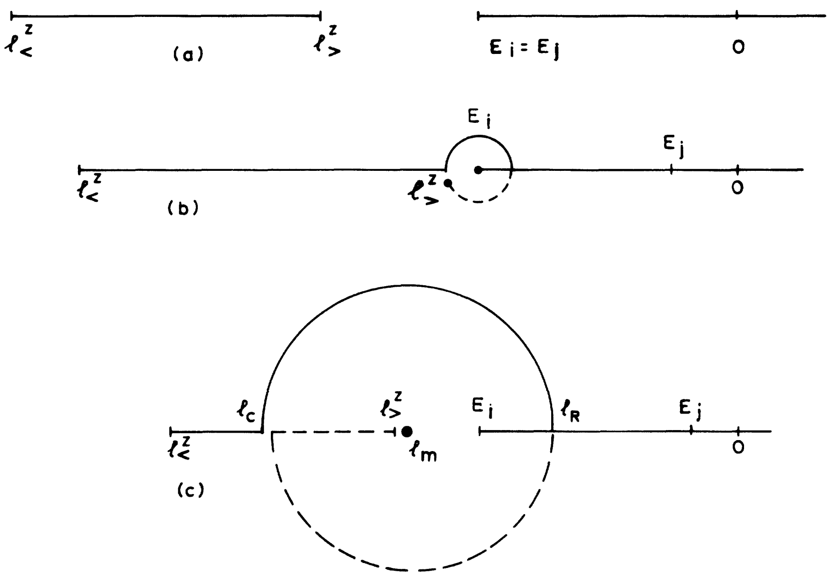

FIG. 2. Branch cuts of $a^{(1)}_{ij}(s)$ for some values of E_j: (a) $E_i = E_j$ and $\zeta < 0$; (b) $\zeta \approx 0$ and $|E_j| = |E^c_j| + \delta e^{-i\phi}$ ($\delta > 0$, $0 \le \phi \le \pi$); (c) $|E_j| < |E^c_j|$ and $\zeta > 0$. This cut is the image of the $[-1, 1]$ cut in the z plane.

where $k'^2(E') = 2M_k(E' - E_k)$. In Fig. 3 the dressed propagator of the intermediate state τ_k has been indicated by a blob. For separable potentials of the Hulthén form[3] one calculates

$$\tau_k(E - E') = \frac{(\kappa + \sqrt{-p^2})(\beta + \sqrt{-p^2})^2}{\kappa(\kappa+\beta)(\kappa + 2\beta + \sqrt{-p^2})(E - E')}, \tag{3.12}$$

where $p^2/2\mu_{dx} = E - E' - \kappa^2/2\mu_{dx}$ is the energy of the two-particle system (cf. Fig. 4) with $\kappa^2/2\mu_{dx}$ the separation energy defined before [Eq. (3.6)]. The propagator has a square root singularity at $p^2 = 0$ or $E - E' = \kappa^2/2\mu_{dx}$. The corresponding cut in $a^{(2)}_{ij}(E)$ starts at the three-particle threshold $E = -\epsilon_b - \epsilon_d - \epsilon_x$. Obviously, taking into account the contribution of the dressed propagator to the input functions (2.8) is only consistent if three-particle unitarity is also treated in a similar approximation. This will be worked out in the next section. The dressing of the propagator is important in actual calculations, however, in the determination of the left-hand cut it plays no role.

In order to determine the singularities of $a^{(2)}_{ij}(E)$ in the left half of the energy plane we proceed in the same way as in the study of anomalous singularities.[11] We start from (3.11), which is valid in the physical region below breakup. We trace the left-hand branch points of this amplitude by analytic continuation to lower values of the energy. To this end we need the branch points of the half-shell function $a^{(1)}_{ij}(E; E', E)$ in the E' plane (mind the ordering of the arguments):

$$[l^z_{\gtrless}(E)]_{ij} = E_i - [g_{ij}^{1/2}(-E + E_j)^{1/2} \mp \tau_{ij}^{1/2}]^2/f_{ij}, \tag{3.13}$$

and similar expressions for the x and x' singularities. As an example, we take nucleon-deuteron scattering ($\alpha = \alpha' = \frac{1}{2}$) where the rightmost branch

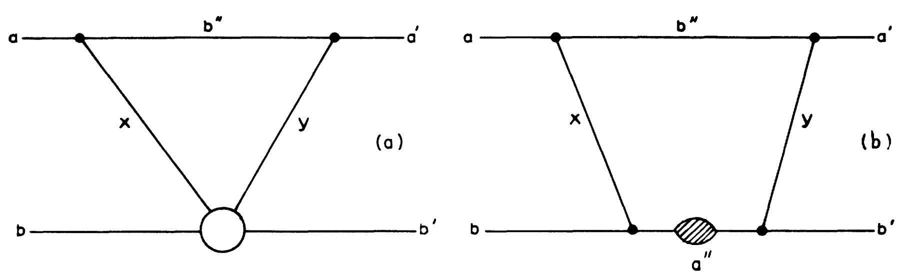

FIG. 3. (a) Triangular graph for the reaction $1 + 2 \rightarrow 1' + 2'$. (b) Repeated OCE graph as approximation of the triangular graph.

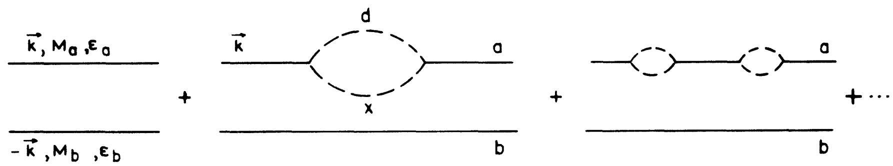

FIG. 4. Graphs contributing to the dressing of the propagator (3.12).

points correspond to the off-shell momenta k $\{=[2M_i(E'-E_i)]^{1/2}\}$:

$$k_{ij}^{z}(E)=i\{-[-\tfrac{1}{3}(E-E_j)]^{1/2}+(-E_j)^{1/2}\},$$
$$k_{ij}^{x'}(E)=2i\{-2[-\tfrac{1}{3}(E-E_j)]^{1/2}+\beta\}, \qquad (3.14)$$
$$k_{ij}^{x}(E)=i\{-[-\tfrac{1}{3}(E-E_j)]^{1/2}+\beta'\},$$

where $\beta(\beta')$ is the range parameter in the initial (final) vertex. The cut structure for the z case is shown in Fig. 5. As soon as the branch point of the half-shell functions interferes with the integration interval, the integration contour has to be deformed. The continuation of $a_{ii}^{(2)}(E)$ has been treated extensively in Ref. 11. It was shown that left-hand branch points in the energy plane can arise in two ways.

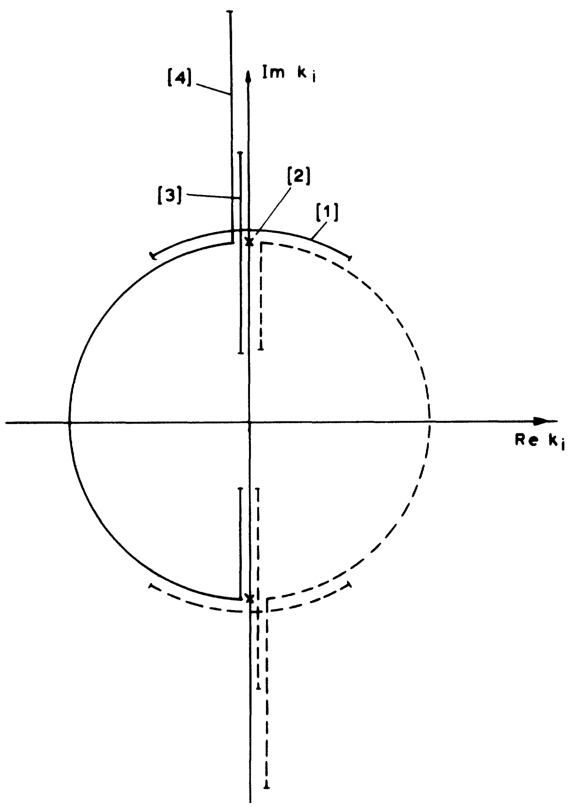

FIG. 5. Singularity curves of the $Q_L(z)$ part of $a_{ij}^{(1)}(E;E',E)$ in the k plane. The constants $g_{ij}=\frac{1}{3}$ and $\tau_{ij}=E_j$ are taken from nucleon-deuteron scattering. The curves denoted by (1), (2), (3), and (4) correspond to energies $E=0$, $-E_j$, $-2E_j$, and $-7E_j$. The solid (dashed) lines are the singularity curves which for $E>E_j$ lie in the (un)physical sheet.

First, it is possible that a branch point of one half-shell function $a_{ik}^{(1)}(E;E,E')$ encircles the endpoint of integration E_k, returns to the left, thereby deforming the integration path, and for a certain energy encounters a branch point of the other half-shell function $a_{kj}^{(1)}(E;E',E)$. The branch point in the E plane is then found as the solution of

$$k_{kj}^{\alpha}(E)=-k_{ki}^{\beta}(E)\,, \qquad (3.15)$$

where α and β may be z, x, or x'. The minus sign occurs because one of the branch points in the E' plane moved onto the adjacent Riemann sheet (see Fig. 5). Denoting the branch points of an ikj second-order term corresponding to the product of $Q_L(z)$ and $Q_L(x')$ as $(l_>^{zx'})_{ikj}$, one obtains the following condition for $E=(l_>^{zx'})_{iki}$:

$$k_{ki}^{z}(E)=-k_{ki}^{x'}(E), \qquad (3.16)$$

so that

$$(l_>^{zx'})_{iki}=E_i-\tfrac{3}{25}(\kappa+2\beta')^2, \qquad (3.17)$$

where we used the equality $\kappa=\sqrt{-E_i}$. Similarly, one finds

$$(l_>^{zx})_{iki}=E_i-\tfrac{3}{4}(\kappa+\beta)^2. \qquad (3.18)$$

Second, it is possible that the two singularities of the half-shell functions in (3.11) are the same. For the diagonal case the cut in the E plane starts as soon as the branch point of the half-shell function in the E' plane has reached the endpoint of integration E_k. Therefore, in this case the condition is simply

$$k_{ki}^{\alpha}(E)=0. \qquad (3.19)$$

Characteristic of the zz singularity is its independence of the intermediate channel

$$(l_>^{zz})_{iki}=4E_i. \qquad (3.20)$$

Observe that this is just the left-most branch point

$l_<^z$ of the zero range part of the OCE amplitude [cf. Eq. (3.8)].

The singularity structure of a general nth order partial wave amplitude can be analyzed similarly. Such an amplitude consists of many terms, corresponding to different contributions of intermediate channels $i_1, \ldots, i_{n-1}$, and the arguments z, x, and x' denoted by $\alpha_1, \ldots, \alpha_n$:

$$a(E)_{i i_1 \cdots i_{n-1} j}^{\alpha_1 \cdots \alpha_n} = \int_{E_{i_1}}^{\infty} dx_1 \cdots \int_{E_{i_{n-1}}}^{\infty} dx_{n-1}\, a_{i i_1}^{\alpha_1}(E;E,x_1)\, \rho_{i_1}(x_1)\, \tau_{i_1}(E-x_1) \cdots \tau_{i_{n-1}}(E-x_{n-1})\, a_{i_{n-1} j}^{\alpha_n}(E;x_{n-1},E). \tag{3.21}$$

In this expression the function a_{ij}^{α} represents the α component in the off-shell amplitude (3.2). If we continue (3.21) from the physical region to lower values of the energy we will reach the point where the first (or last) half-shell function becomes singular, so that the integration path must be deformed up to $\bar{x}_1 = [l_>^{\alpha_1}(E)]_{i i_1}$ [compare Eq. (3.13)]. Next one considers the singularities of the function $a_{i_1 i_2}^{\alpha_2}(E;\bar{x}_1(E),x_2)$ in the x_2 plane. For a certain value of E the rightmost singularity will reach the end point of integration E_{i_2} and the integration path must be deformed up to $\bar{x}_2(E)$. This process can be continued in a straightforward manner, both from the left and the right. In the case of odd n it may occur that finally all integration paths are deformed and that $a_{i_{m-1} i_m}^{\alpha_m}(E;\bar{x}_{m-1},\bar{x}_m)$ determines the branch point of (3.21) in the energy plane [$m = \frac{1}{2}(n+1)$]. For all other cases we are finally left with one undeformed integration path with an integrand containing, for example, $a_{i_{m-1} i_m}^{\alpha_m}(E;\bar{x}_{m-1},x_m)$ and $a_{i_m i_{m+1}}^{\alpha_{m+1}}(E;x_m,\bar{x}_{m+1})$. By construction, $\bar{x}_{m-1}$ depends on E and the labels $\alpha_1, \ldots, \alpha_{m-1}$ and $i, i_1, \ldots, i_{m-1}$. If the singularities of these remaining off-shell functions interfere with the integration path at the same time then the singularity of (3.21) follows from the requirement that $a(E;\bar{x}_{m-1}(E),E_{i_m})$ should be singular. If the off-shell functions are different then the singularity of (3.21) follows from the requirement that the branch points in the x_m plane coincide after approaching each other from opposite directions, so that either $a(E;\bar{x}_{m-1}(E),\bar{x}_m(E))$ or $a(E;\bar{x}_m(E),\bar{x}_{m+1}(E))$ determines the singularity.

Let us apply these techniques to the third-order with $\alpha_1 = \alpha_2 = \alpha_3 = z$, and $i = i_1 = i_2 = j$. In this case we should determine the singularities of the function $a^z(E;l_>^z(E),l_>^z(E))$. However, for $E < l_>^{zz}$—the point below which the two integration paths should be deformed—this function is free of singularities in the three-nucleon case. Therefore $a_{iii}^{zzz}(E)$ does not have left-hand singularities and does not contribute to the input function. The physical picture associated with this phenomenon has been given by RST, who characterize the zz and zzz singularities as rescattering singularities. RST determine the position of the singularities using the Landau rules. They demonstrate that the rescattering singularities of order n (all $\alpha = z$) should vanish if n equals (or exceeds) the maximum number of possible classical binary contact collisions in the three-particle system. This number obviously depends on the mass ratios of the particles involved, and reaches its minimum of 3 in the equal mass case.

We know that higher order input functions certainly cannot be neglected in some cases,[16] so we must conclude that the finite range effects of $Q_L(x)$ and $Q_L(x')$ are essential in giving a good representation of the input functions in these cases. The closest third-order left-hand branch point in the nucleon-deuteron case is found to be

$$l_>^{zxz} = E_i - 3(\kappa + \tfrac{1}{3}2\beta)^2, \tag{3.22}$$

where we omitted the subscript iii. Furthermore, one finds that the xzx, xzx', and $x'zx'$ terms have a vanishing left-hand projection.

In our formal analysis we showed that nth order branch points are obtained from a step by step process. In some cases this can be carried through completely. In particular we find

$$l_>^{x\cdots x\, x'\cdots x'} = E_i - 3\beta^2(1 - 2^{-m})^2, \quad m = n/2$$

and (3.23)

$$l_>^{x\cdots x\; x\; x'\cdots x'} = E_i - 3\beta^2(1 - \tfrac{2}{3}2^{-m})^2, \quad m = (n-1)/2\,.$$

Note that these branch points move monotonically to the left with increasing m; however, they have a finite accumulation point $E_i - 3\beta^2$, and thus have a behavior completely unknown in potential scattering. An example of a general set of branch points which have a more "regular" behavior is

$$l_>^{x'\cdots x'\; x\cdots x} = E_i - \tfrac{3}{4}\beta^2(3.2^{m-1} - 1)^2, \quad m = n/2\,. \tag{3.24}$$

Let us conclude this section by giving an explicit expression for the left-hand projection of the second order-amplitude (3.11). We give here an alternative to the spectral representation in terms of the left hand discontinuity [cf. Ref. 11, Eq. (5.15)] and use a dispersion relation for the dressed propagator $\tau_k(E')$ to write $b_{ij}^{(2)}(E) = \sum_k b_{ikj}^{(2)}(E)$ in the form[19]

$$b^{(2)}_{ikj}(E)=\int_{E_k}^{\infty}dE'\,\frac{a^{(1)}_{ik}(E;E,E')\,a^{(1)}_{kj}(E;E',E)-a^{(1)}_{ik}(E')\,a^{(1)}_{kj}(E')}{E-E'}\,\rho_k(E')$$
$$-\frac{1}{\kappa(\kappa+\beta)}\int_{E_k}^{\infty}dE'\,a^{(1)}_{ik}(E;E,E')\,\rho_k(E')\,a^{(1)}_{kj}(E;E',E)$$
$$+\frac{1}{\pi}\int_{E_k}^{\infty}dE'\,\rho_k(E')\int_{E'-E_k}^{\infty}\frac{dE''}{E'-E''}\,\Delta S_k(E''-E')$$
$$\times\frac{a^{(1)}_{ik}(E'';E'',E')\,a^{(1)}_{kj}(E'';E',E'')-a^{(1)}_{ik}(E;E,E')\,a^{(1)}_{kj}(E;E',E)}{E''-E}\,, \qquad (3.25)$$

where

$$\Delta S_k(E''-E')=-\frac{2|p|\beta(\kappa+\beta)}{\kappa[p^2+(\kappa+2\beta)^2]} \qquad (3.26)$$

and κ and β are the separation energy and range parameter of the dressed particle in the intermediate state k. The momentum p was defined in (3.12). The expression (3.26) holds in the normal case ($\zeta_{ij}<0$, $\zeta_{kj}<0$) and for large enough energy. The first integral which contains the contribution of the bound state pole in τ_k will be called the bare second-order input, the remainder is due to the dressing of the propagator. We also will need some of the input functions at energies for which the singularities of the half-shell function interfere with the integration contour. The extension of (3.25) to this case is obvious. Similarly, one can define the input function in the anomalous case, where the singularities of $a^{(1)}_{ik}(E')$ or $a^{(1)}_{kj}(E')$ may interfere with the integration contour. In this case the regularization implied in (3.25) persists although the integration paths for half- and on-shell functions may differ. One should, however, be aware that for the anomalous case the input functions are redefined in such a way[11] that integrals containing the square of the discontinuity of $a^{(1)}_{ik}(E')$ are treated separately.

IV. THREE-PARTICLE CHANNELS IN THE N/D APPROACH

In the N/D approach scattering amplitudes are determined by their singularities in the whole energy plane. For the general N-nucleon problem, therefore, the behavior of the amplitudes is also determined by three-, four-, up to N-particle unitarity singularities. Four- and higher-particle unitarity are usually neglected both for physical and practical reasons. Three-particle unitarity in this context has been considered by a number of authors.[9,10,2]

Contributions to three-particle unitarity arise from two sources.[20] First, the unitarity cuts of the two-particle scattering amplitudes contribute (the bound state poles in these scattering amplitudes are responsible for two-particle unitarity of the complete amplitudes). Second, the scattering amplitudes themselves contribute. The two contributions are shown in Fig. 6. Freedman *et al.*[20] have studied three-particle unitarity in the separable model. They proved in an elegant way that the discontinuity can also be obtained from the isobar ansatz. This isobar ansatz expresses the breakup amplitude in terms of bound state or resonance scattering amplitudes and two-particle propagators (see Fig. 7), and is exact in the separable model.

In order to give an explicit expression for this isobar ansatz we invoke the common three-particle notation where $\alpha=1$ means that pair (23) is coupled to particle 1, and indices a specify the states of pair α and particle α. Breakup channels are labeled by ν. The total mass is $M=m_1+m_2+m_3$, reduced masses are μ_α (internal in pair α) and M_α (of pair α and particle α).

The t matrix of pair α is assumed to be separable:

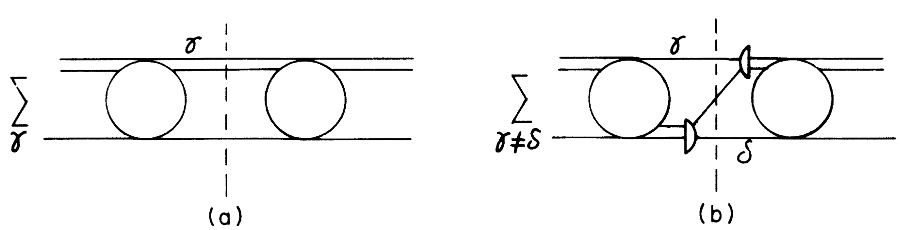

FIG. 6. Contributions of (a) the two-particle propagator and (b) of the OCE amplitude to three-particle unitarity.

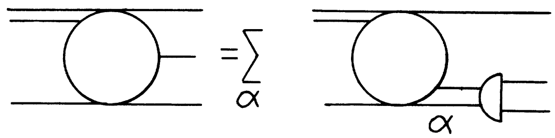

FIG. 7. Breakup amplitude in the isobar ansatz.

$$\langle \vec{p} \,|\, t_a(\omega) \,|\, \vec{p}\,' \rangle = \Gamma_a^*(\vec{p})\tau_a(\omega)\Gamma_a(\vec{p}\,') \,. \tag{4.1}$$

The propagator τ_a is a function of the two-particle energy $\omega = p^2/2\mu_\alpha$, normalized according to [cf. Eq. (3.12), $\kappa^2/2\mu_a = \epsilon_a$]

$$\lim_{\omega \to -\epsilon_a} (\omega + \epsilon_a)\tau_a(\omega) = 1 \,. \tag{4.2}$$

If the pair α cannot support a bound state, τ_a is still defined[3]; however, it no longer has the single pole at $\omega = \epsilon_a$ in the physical sheet. The isobar expression for the unprojected breakup amplitude $\bar{A}_{\alpha\nu}$ is formulated in terms of unsymmetrized two-body bound state amplitudes $\bar{A}_{\alpha\gamma}$ as follows:

$$\langle \alpha a \vec{k}_\alpha | \bar{A}_{\alpha\nu}(E+i\epsilon) | = \sum_{\gamma,c} \int d\vec{p}_\gamma \int d\vec{k}_\gamma \langle \alpha a \vec{k}_\alpha | \bar{A}_{\alpha\gamma}(E+i\epsilon) \,|\, \gamma c \vec{k}_\gamma \rangle \tau_c(E - k_\gamma^2/2M_\gamma - E_\nu + i\epsilon)\Gamma_c(\vec{p}_\gamma)\langle c\vec{p}_\gamma \vec{k}_\gamma | \,. \tag{4.3}$$

The bar over A indicates that we deal with unsymmetrized amplitudes. The labels γ and c run over all partitions and states consistent with the breakup channel ν. The three-body breakup threshold is characterized by E_ν.

The three-particle discontinuity is given by

$$\langle \alpha a \vec{k}_\alpha | \, \mathrm{disc}_\nu \bar{A}_{\alpha\beta}(E) | \beta b \vec{k}_\beta \rangle$$
$$= 2\pi i \langle \alpha a \vec{k}_\alpha | \bar{A}_{\alpha\nu}(E+i\epsilon)\Delta_\nu(E)\bar{A}_{\nu\beta}(E-i\epsilon) \,|\, \beta b \vec{k}_\beta \rangle \,. \tag{4.4}$$

The operator Δ_ν represents the discontinuity of the Green's function $(E-H_0)^{-1}$, where H_0 is the relevant free three-particle Hamiltonian. In momentum representation it is simply a δ function which dictates the on-shell condition $E = p_\gamma^2/2\mu_\gamma + k_\gamma^2/2M_\gamma + E_\nu$. If one inserts (4.3) into (4.4), thereby introducing two summations (γc) and (δd), two types of terms will appear. First the direct terms, for which $\gamma = \delta$. These terms come from the discontinuity of the two particle propagator [Fig. 6(a)]. An explicit expression of this contribution to three-particle unitarity has been given before by RS.

The contributions of the OCE amplitudes themselves, for which $\gamma \neq \delta$ can be considered as the effect of overlapping resonances or bound states. These exchange terms were neglected in the earlier calculations of $n-d$ scattering.[10] In calculating these terms we use the expression for the overlap integral

$$\langle c\vec{p}_\gamma \vec{k}_\gamma | \Delta_\nu(E) | \, d\vec{p}_\delta\,' \vec{k}_\delta\,' \rangle$$
$$= C_{cd}^{IS}\delta\left[\vec{p}_\delta\,' + \frac{m_\gamma}{m_\mu + m_\gamma}\vec{p}_\gamma - \frac{m_\mu M}{(m_\mu + m_\gamma)(m_\mu + m_\delta)}\vec{k}_\gamma\right]$$
$$\times \delta\left(\vec{k}_\delta\,' + \vec{p}_\gamma + \frac{m_\delta}{m_\mu + m_\delta}\vec{k}_\gamma\right)\delta\left(E - \frac{p_\gamma^2}{2\mu_\gamma} - \frac{k_\gamma^2}{2M_\gamma} - E_\nu\right), \tag{4.5}$$

where $(\gamma\delta\mu) = (123)$ or a cyclic permutation thereof. The overlap factor C_{cd}^{IS} was introduced in Eq. (3.1). We could stick to the notation of Sec. III by using $\alpha' = m_\gamma/(m_\mu + m_\gamma)$, $\alpha = m_\delta/(m_\mu + m_\delta)$, $f_{cd}^{-1} = m_\mu M/[(m_\mu + m_\delta)(m_\mu + m_\gamma)]$, and $m_\mu = m_x$; however the final results will be more easily expressible with the present notation.

The integrations over $\vec{p}_\delta\,'$ and $\vec{k}_\delta\,'$ can be trivially performed. Next we write the amplitudes in the partial wave series (2.1) and expand the whole integrand in terms of spherical harmonics of $\vec{p}_\gamma$, $\vec{k}_\gamma$, $\vec{k}_\alpha$, and $\vec{k}_\beta$. The angular integrations over $\vec{p}_\gamma$ and $\vec{k}_\gamma$ can now be performed. Symmetrization of the amplitudes simply leads to the factor C_{cd}^A which was introduced in Eq. (3.1). The resulting expression for the three-particle discontinuity of the symmetrized amplitude is

$$(-2\pi i)^{-1}\mathrm{disc}_\nu a_{ab}^L(E) = 2\pi \sum_{c,d} C_{cd}^A C_{cd}^{IS} \sum_{l=0}^{L} \binom{L}{l}(-2)^{L+2}\left(\frac{m_\gamma m_\delta}{M}\right)^{L-l/2+3/2} m_\mu^{3/2+l/2}(E-E_\nu)^{L+2}$$
$$\times \int_0^1 dx\, x^{2+l}(1-x^2)^{\frac{1}{2}+L-l/2}\, a_{ac}^L(E+i\epsilon;\, k_\alpha^{\mathrm{on}}, k_\gamma)\Gamma_c(p_\gamma)\tau_c\left(\frac{p_\gamma^2}{2\mu_\gamma}+i\epsilon\right)$$
$$\times \int_{-1}^1 d\cos\theta\, P_l(\cos\theta)\Gamma_d^*(p_\delta')\tau_d\left(\frac{p_\delta'^2}{2\mu_\delta}-i\epsilon\right)a_{db}^L(E-i\epsilon;\, k_\delta', k_\beta^{\mathrm{on}}) \,. \tag{4.6}$$

The momentum p_γ equals $x[2\mu_\gamma(E-E_\nu)]^{1/2}$, p_δ' and k_δ' are determined by the relation (4.5) and θ is the angle between $\vec{p}_\gamma$ and $\vec{k}_\gamma$. A simple way of understanding the occurrence of the spin and antisymmetrization factor is by realizing that (4.6) can also be obtained by directly cutting the OCE amplitude (3.1) according to Fig. 6(b).

We would like to express the result in terms of the phase space matrix $\underline{\rho}$ [Eq. (2.4)]. For this purpose we relate the off-shell partial wave amplitudes to the on-shell amplitudes. At this point we can follow Ref. 10, in which off-shell functions χ are defined according to

$$a_{ab}^L(E;\,k_\alpha^{\mathrm{on}},\;k_\beta)=a_{ab}^L(E)x_{ab}^L(E,\,k_\beta)\,, \tag{4.7}$$

or we can use a recently proposed[21] definition in the spirit of Kowalski and Noyes[22]:

$$a_{ab}^L(E;\,k_\alpha^{\mathrm{on}},\;k_\beta)=\sum_c a_{ac}^L(E)x_{cb}^L(E,\,k_\beta)\,. \tag{4.8}$$

We should realize that (4.7) or (4.8) are nothing else but definitions of the off-shell function χ in terms of unknown amplitudes a^L. Whether (4.7) or (4.8) is preferable will depend on whether they constitute a useful approximation scheme for the off-shell functions χ.

The first formulation leads to a phase space function ρ_{acdb} which also depends on the "outer" indices a and b, so that the unitarity relation cannot be inverted. Therefore (4.7) is only useful if one neglects the dependence on the "on-shell" indices αa:

$$\chi_{ab}^L(E,\,k_\beta)\approx\chi_b^L(E,\,k_\beta)\,. \tag{4.9}$$

If we restrict ourselves to the direct contributions of three-particle unitarity, then this approximation leads to modified phase space factors which are still diagonal, corresponding with the case considered in Ref. 10.

The second formulation (4.8) does not destroy the matrix character of the unitarity equations (3.1). It will, however, lead to a nondiagonal phase space matrix. In this case, solving the N/D equations is more complicated because of the nonreality of the n and D functions. In the present investigation we did not attempt to perform the corresponding extensions in our numerical work. Therefore we will use (4.7) in combination with (4.9). We will also include those exchange contributions of three-particle unitarity which contribute to the diagonal elements of $\underline{\rho}$.

We will apply the present theory in nucleon-deuteron scattering. In this case the antisymmetrization factor equals -2, independent of the channels involved. The minus sign arises because we have to relate the internal OCE coupling scheme with the coupling scheme used in performing the symmetrization. The latter coupling scheme is based on a logical enumeration of states in terms of permutation symmetry. Upon substituting (4.7) together with (4.9) into (4.6) one obtains an expression for the three-particle discontinuity, which has the same structure as (2.3). One can now include the three-particle contributions of the exchange type in the phase space matrix $\underline{\rho}$.

In order to assess the relative importance of two- and three-particle unitarity we introduce the dimensionless inelasticity parameter λ according to

$$\rho_{cd}^L(E)=(m_cm_d)^{1/2}(k_ck_d)^{L+1/2}\left[\theta(E-E_c)\,\delta_{cd}+\theta(E)\,\lambda_{cd}^L(E)\right], \tag{4.10}$$

where the three-nucleon breakup threshold is set equal to zero. For completeness we recall[10] the form of the direct contribution to three particle unitarity in the approximation (4.9):

$$\lambda_{cc}^L(E)_{\mathrm{direct}}=(E-E_c)^{-L-1/2}\,E^{L+2}(4\pi)\int_0^1 dx\,x^2(1-x^2)^{L+1/2}\left|\chi_c^L(E,k)\,\tau_c(p^2)\,\Gamma_c(p^2)\right|^2, \tag{4.11}$$

where $p=x\sqrt{E}$ and $k^2=\frac{4}{3}E(1-x^2)$.

For $E\to0$ all momenta in the integrand of (4.6) approach zero so that

$$\lambda_{cd}^L(E)\sim E^{L+2}\left[\delta_{cd}\,|\chi_c^L(E,0)\tau_c(0)|^2-2C_{cd}^{IS}(-\tfrac{1}{2})^L\chi_c^L(E,0)\;\tau_c(0)\Gamma_c(0)(\chi_d^L(E,0)\tau_d(0)\Gamma_d(0))^*\right]. \tag{4.12}$$

Obviously, the relative importance of the direct and exchange contributions in $\lambda_{cc}^L(E)$ is independent of the specific form of the off-shell functions. In nucleon-deuteron scattering the channel spin S can be either $\frac{3}{2}$ (quartet scattering) or $\frac{1}{2}$ (doublet scattering). The relevant recoupling coefficients are

$$\begin{aligned}&C_{dd}=\tfrac{1}{2},\quad S=\tfrac{3}{2},\quad I=\tfrac{1}{2},\\&C_{dd}=C_{ss}=-\tfrac{1}{4},\quad C_{ds}=C_{sd}=\tfrac{3}{4},\\&S=\tfrac{1}{2},\quad I=\tfrac{1}{2},\end{aligned} \tag{4.13}$$

where d represents a triplet two-particle state

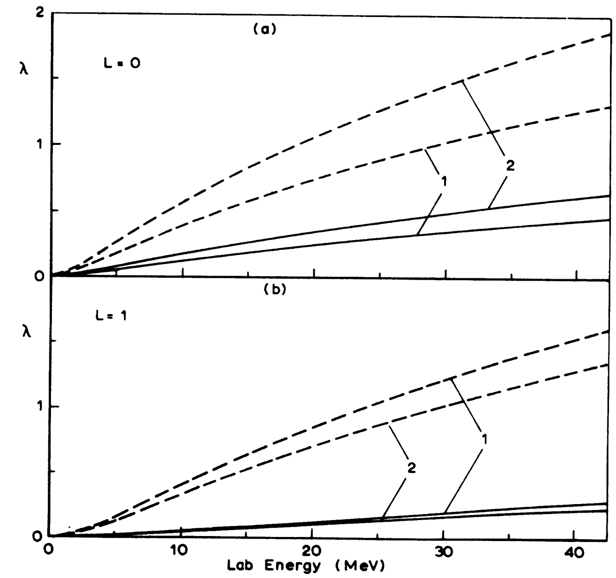

FIG. 8. Inelasticity parameter λ in the doublet case for (a) $L = 0$ and (b) $L = 1$. Notation: ——, $\chi_{cd} = 1$; ---, χ_z; 1 = λ_{direct}; 2 = $\lambda_{\text{direct}} + \lambda_{\text{exch}}$. Energy scale with breakup threshold in the origin.

(deuteron) and s a singlet two-particle state.

In S-wave quartet scattering the direct and exchange contribution cancel each other for $E \to 0$. This causes the slow decrease of the absorption coefficient $^4\eta_0$ for quartet scattering. In a previous calculation of nucleon-deuteron scattering Avishai, Ebenhöh, and Rinat[10] argued that the exchange contribution would be negligible. From (4.12) it is clear that this argument must be wrong. For doublet scattering one has

$$\lambda_{cc}^L(E) \sim E^{L+2}[1 + \tfrac{1}{2}(-\tfrac{1}{2})^L] \qquad (4.14)$$

both for $c = d$ (triplet channel) and $c = s$ (singlet channel). For S waves direct and exchange contributions add coherently, leading therefore to the rapid increase in $\lambda_{cc}^L(E)$ for $E \approx 0$. This behavior is well known from model calculations for the absorption coefficient $^2\eta_0$, which for small E is linearly related to $\lambda_{dd}^L(E)$.

In most of our calculations we used the simple model

$$\chi_{cd}^L(E; k_d) = 1; \qquad (4.15)$$

however, we also performed calculations with an off-shell function

$$\chi_z \equiv \chi_{cd}^L(E; k_d) = \left(\frac{5E + 8\kappa^2}{5E + 4\kappa^2 - p^2}\right)^{L+1}, \qquad (4.16)$$

where $p^2 = E - \frac{3}{4}k_d^2$ and κ^2 is the binding energy of the two-particle state in channel d. This model is based on the low energy behavior of the leading $Q_L(z)$ term in the first-order amplitude. In Figs. 8 and 9 we have shown the corresponding inelasticity for doublet and quartet case.

Finally, we note that one can avoid the introduction of the off-shell factors by employing the resonance character of the propagator τ. This could imply that we also treat amplitudes $a_{ac}^L(E; k_\alpha^{\text{on}}, [2M_c(E - E_\nu)]^{\frac{1}{2}})$ in our N/D approach. The implications of this approach have not yet been investigated.

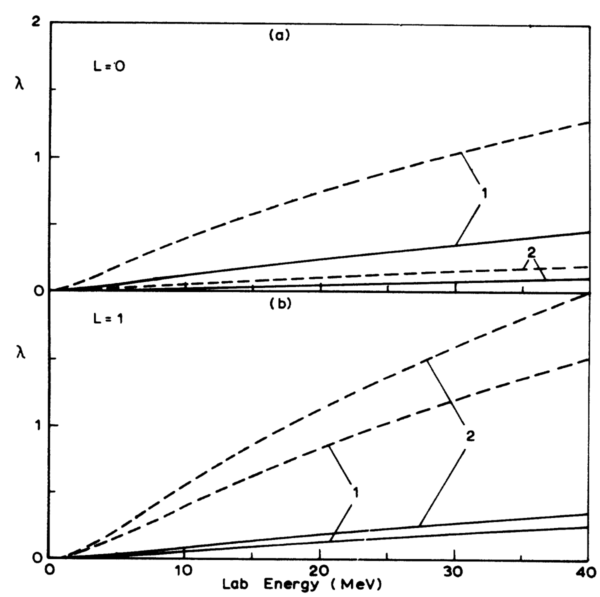

FIG. 9. Same as Fig. 8, but for the quartet case. The curves representing λ_{direct} are identical to the doublet case.

V. APPLICATION TO NUCLEON-DEUTERON SCATTERING

The nucleon-deuteron system has been studied before with dispersionlike methods by several authors from the elementary particle point of view,[23] from an on-shell point of view,[24] and from the off-shell point of view represented in Secs. II and III.[10] Some shortcomings of the application in Ref. 10 are: (1) Quartet scattering is discussed using only the direct part of the inelasticity λ. We just showed in the last section that this is very unsatisfactory, especially for low L. (2) Doublet scattering is described using experimental absorption coefficients which, moreover, turned out to be rather poorly determined.[4]

The approach sketched in the previous section leads to an improved description in these respects. As mentioned in the introduction, we consider a model system of three nucleons with central separable interactions. In addition to the deuteron state we assume the existence of a loosely bound

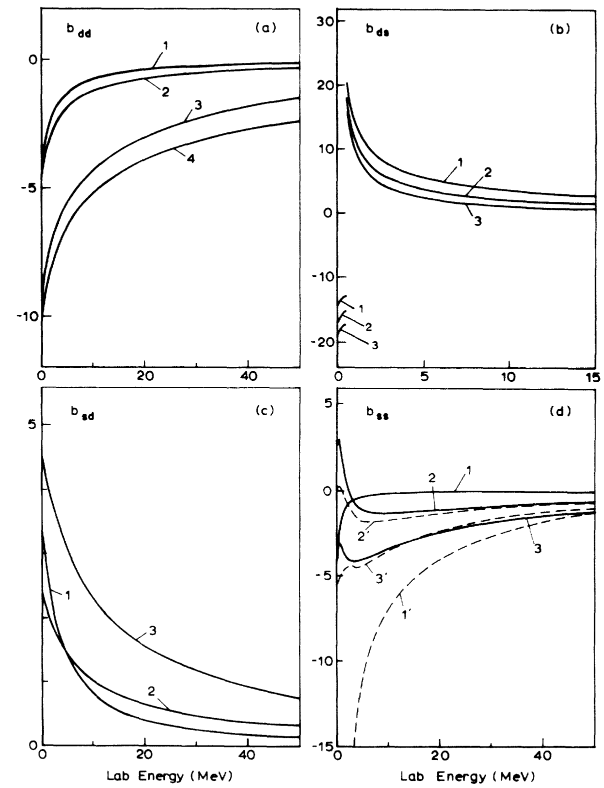

FIG. 10. Input functions for S-wave doublet scattering. The various curves correspond to: 1 = first Born term; 2 = 1 + second bare Born term; 3 = 1 + second dressed Born term; 4 = 3 + third-order pole; $n' = n$ + anomalous contributions. For (a) and (b) the origin coincides with the first physical threshold E_1 $(=-\epsilon_d)$; for (c) and (d), with the second physical threshold E_2 $(=-\epsilon_s)$.

singlet deuteron state. We therefore have one two-body channel for quartet scattering, and two two-body channels in nucleon-deuteron scattering. Because of the loosely bound singlet state an anomalous threshold occurs in the doublet case. This requires slight modifications in the N/D equations and input functions which were discussed in a previous paper.[11] Some of the input functions become discontinuous in a certain point l_R on the elastic cut, which is chosen for convenience at the energy where the argument of $Q_L(z)$ in (3.2) vanishes. This point can clearly be recognized in Fig. 10(b).

The bare second-order inputs shown in Fig. 10 are reduced in a sense discussed in Ref. 11. This reduction essentially means that singular parts arising from the anomalous threshold are treated analytically. These singular parts are necessary in order to define a unique continuation of the N/D equations from the normal to the anomalous case. We used a singlet deuteron binding energy ϵ_s of 0.1 MeV. The results did not depend sensitively on the value of ϵ_s. In S-wave doublet scattering we included a third-order pole contribution which was fitted to the third-order Born term given in Ref. 16. The pole position was −62 MeV.

In Fig. 11 part of the cut structure of b_{dd} is shown. In Fig. 10 we showed the input functions for the S-wave two-channel N/D equations. Obviously the dressing of second-order terms is very important. After inverting the N/D equations one obtains the scattering amplitudes and from these the nucleon-deuteron phase shifts and absorption coefficients. These phase shift parameters are depicted in Fig. 12 and Fig. 13, respectively. Also shown are the results of an Amado-Faddeev calculation for the corresponding case,[4] and one-channel calculations in which the singlet channel only enters as an intermediate state in the input functions. The range parameters used in these calculations are $\beta_d = 1.449\ \text{fm}^{-1}$ and $\beta_s = 1.161\ \text{fm}^{-1}$. Remember that only the diagonal contributions of Λ_{exch} are taken into account. Calculations for the less realistic input of first- + bare second-order input were also performed, but not shown in the figures. These results are included in Table I, in which the triton binding energies for different calculations are listed. As the first-order input is too weak to bind the triton we have not included this case in the table.

In quartet scattering the absence of channel coupling simplifies the problem greatly. Input functions for S- and P-wave scattering are shown in Fig. 14; the corresponding S-wave phase shift parameters are depicted in Fig. 15.

The various calculations presented in this section differ in their treatment of three-particle unitarity and the input. In the next section we will try to explain the results from these underlying approximations.

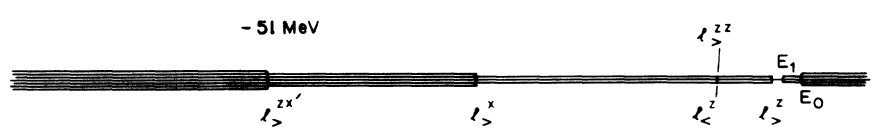

FIG. 11. Left-hand singularities of b_{dd}. Scale follows from the positions of the elastic $(E_1 = -2.225\ \text{MeV})$ and breakup threshold $(E_0 = 0)$. The second-order branch points are both branch points of the input terms with intermediate triplet and singlet channel. The first branch points not shown on the left (near −75 MeV) are of third order and arise from the intermediate singlet channel. The second-order branch point $l_>^{zx}$ lies even further to the left (−90 MeV).

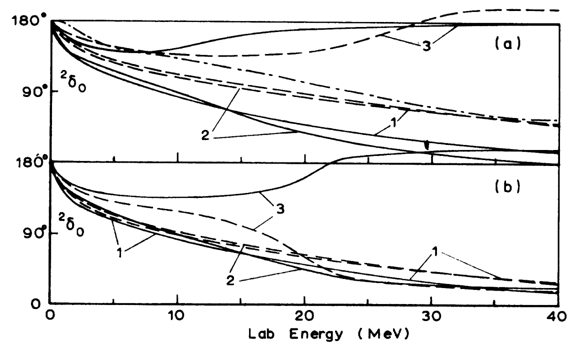

FIG. 12. Exact and calculated doublet phase shifts $^2\delta_0$. Notation: ——, two-channel calculations; ---, one-channel calculations; -.-.-, exact calculations; $1 = \lambda_{\text{direct}}$ with $\chi = 1$; $2 = \lambda_{\text{direct}} + \lambda_{\text{exch}}$ with $\chi = 1$; $3 = \lambda_{\text{direct}} + \lambda_{\text{exch}}$ with $\chi = \chi_s$; (a) shows calculations with the third-order pole included; (b) shows calculations with complete second-order term.

VI. DISCUSSIONS AND CONCLUSIONS

In this section we will analyze the results of the previous section and present a number of conclusions which we discuss pointwise.

(1) The dressing of the second-order input is very important in giving a reliable representation of the left-hand cut (Fig. 10, Fig. 14). Especially for the intermediate singlet channel this is the case since the contribution of the bound state singlet pole, i.e. the bare second-order input, vanishes for $\epsilon_s \to 0$. From Table I it is seen that the dressing has a huge effect on the position of the

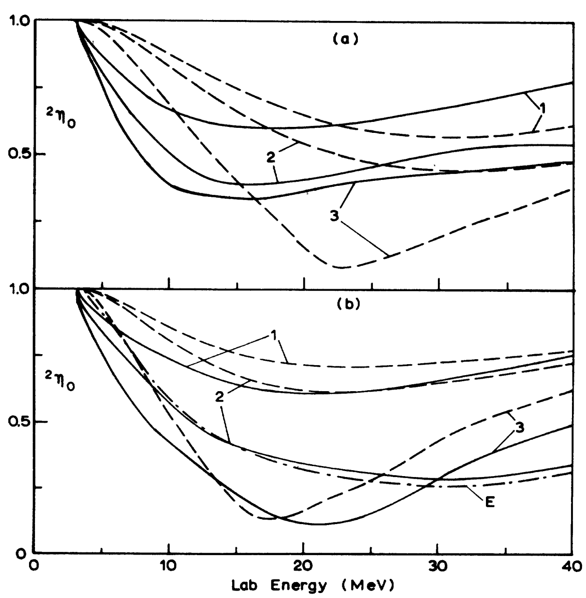

FIG. 13. Exact and calculated doublet absorption coefficients $^2\eta_0$. Notation as for Fig. 12.

TABLE I. Triton binding energy E_T for different λ models ($1 \equiv \lambda_{\text{direct}}$, $2 \equiv \lambda_{\text{direct}} + \lambda_{\text{exchange}}$ with $\chi = 1$, $3 \equiv \lambda_{\text{direct}} + \lambda_{\text{exchange}}$ with $\chi = \chi_s$) and inputs ($1 \equiv$ first Born input, $2 \equiv 1$ + bare second-order term, $3 \equiv 1$ + dressed second order term, $4 \equiv 3$ + third order pole). Both one-channel and (anomalous) two-channel computations are shown.

λ model	Input	E_T(MeV) (one channel)	E_T(MeV) (two channel)
1	2	2.269	2.225
	3	5.159	4.793
	4	7.340	5.092
2	2	2.273	2.231
	3	5.291	5.035
	4	7.561	5.573
3	2	2.304	2.274
	3	6.193	6.063
	4	8.733	7.568

triton pole. Similarly, the phase parameters are affected (these results are not shown).

(2) The contributions of the exchange term [Fig. 6(b)] to three-particle unitarity are quite important, especially in reproducing the exact absorption coefficient [Figs. 13, 15(b)]. In the quartet case we get very good agreement with the exact case[4] with the simple model $\chi = 1$ [Fig. 15(b)]. A similar improvement is obtained for P-wave quartet scattering.[19] In all cases the off-shell factor χ_s, which is much larger than 1, gives worse agreement. In the doublet case there are two effects which determine $^2\eta_0$, the channel coupling and the inelasticity parameter λ_{dd}. The coupling to the singlet channel in the N/D equations is accomplished by means of λ_{ss}, which for $\epsilon_s \to 0$ goes to infinity in such a way that the product $b_{ds}\,\lambda_{ss}\,b_{sd}$ remains finite. This stresses the sensitivity of the present model to the description of the inelasticity. As we neglect nondiagonal parts in the phase space factor $\underline{\rho}$, we include only part of the channel coupling. The neglect of nondiagonal phase space factors probably causes the fast decrease of $^2\delta_0$ for higher energies and its failure to go through 90° in case 3 (Fig. 12), although inadequacies in the off-shell factor can also lead to these phenomena. To understand the role of the inelasticity λ in determining the behavior of the phase parameters, we derived some bounds on the phase parameters. Under the assumption of diagonal λ (or ρ) one can derive the following bounds on η and δ[19]:

$$\frac{\lambda - 1}{\lambda - 1} \leq \eta \leq 1, \qquad (6.1)$$

$$4\eta \sin^2\delta \leq 2/\lambda, \text{ and } |\sin 2\delta| < \frac{\sqrt{1-a^2}}{\lambda},$$

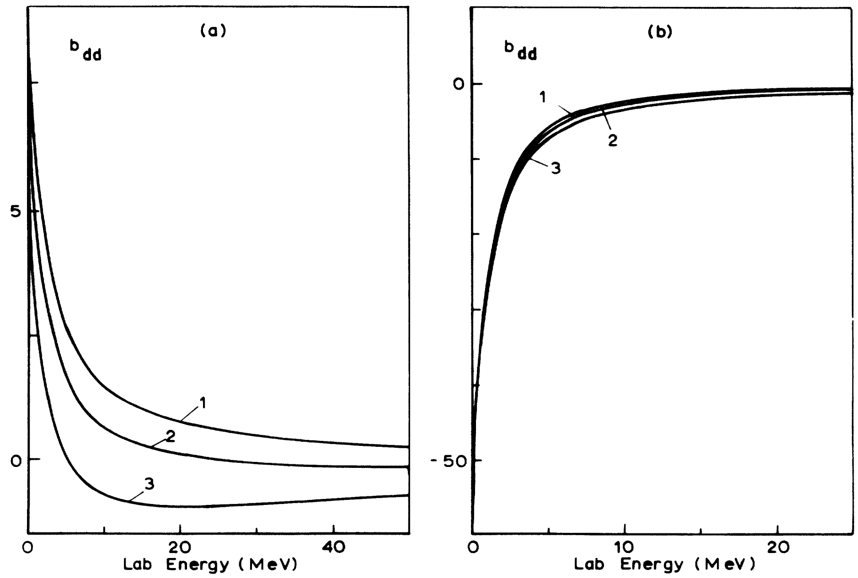

FIG. 14. Input for S- and P-wave quartet scattering. Notation as for Fig. 10.

where a^2 is a positive definite quantity determined by the channel coupling ($a=0$ in the one channel case). Since λ can become quite large in practice, especially for large χ (model χ_{ε}), these bounds can be very restrictive. For example, they force η to return to 1 for large λ, and thus force δ to be small for large λ, since

$$\sin^2\delta \leq \frac{1}{2\lambda}\frac{\lambda+1}{\lambda+1}. \tag{6.2}$$

Obviously, the strong bounds implied by unitarity when nondiagonal contributions are neglected explain the behavior of ${}^2\delta_0$ and ${}^2\eta_0$ at higher energies (≈ 40 MeV). These consequences of unitarity are reminiscent of certain anomalous phenomena which also can occur in the N/D method,[25] namely, anomalous resonances and anomalous bound states. These anomalies occur if the left-hand input is so poorly approximated that the scattering amplitude

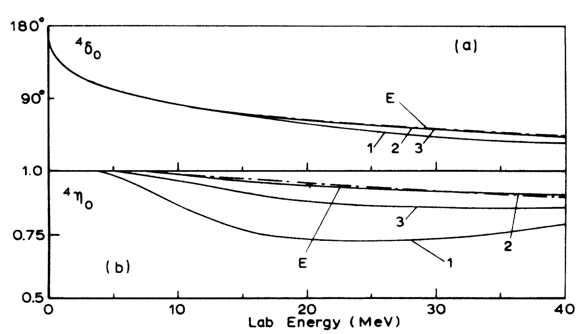

FIG. 15. Phase parameters for S-wave quartet scattering. Notation as for Fig. 12.

can only satisfy unitarity if at the same time anomalous pole singularities are introduced.

In the previous paragraph we stressed the importance of the nondiagonal inelasticity. We should, however, also remark that the results are very sensitive to the model for the off-shell function χ. For example, the coupling of triplet and singlet channel is roughly proportional to $|\chi_s|^2$. In the present model the phase of χ is not so important since it only plays a role in λ_{exch} which is partly neglected. In a full calculation including nondiagonal phase space factors it may, however, be quite important. Presently an investigation is carried out for the three-nucleon system to develop rigorous approximation schemes for the off-shell functions.[21] As the N/D method is designed for more general reactions, this investigation should also be extended to more general cases. Our third conclusion is related to the previous one.

(3) If one wants to improve on the one-channel calculations, in which the singlet state only occurs as an intermediate state in the input, by treating the singlet channel explicitly, then one cannot avoid the inclusion of the exchange part of the inelasticity, i.e., one has to do the N/D calculations with nondiagonal phase space matrices, and therefore complex n and D functions. As every OCE amplitude has its own exchange contribution to three-particle unitarity, this conclusion will also persist for other applications of the N/D method to nuclear reactions.

Another question is whether the present treatment of the singlet deuteron can be improved. For

$\epsilon_s = 0$ there may occur some simplifications such as the fact that the bare second-order input terms vanish if the intermediate state is a singlet deuteron. It is known that a resonance can be treated approximately by introducing a set of discrete channels[26]; however, whether a similar treatment is possible in the case of a virtual state is not clear.

(4) The low energy behavior of the S-wave phase shift is closely related to the position of the triton pole, which in the exact calculation lies at −11 MeV. Since our input is weaker than the exact input, this pole lies closer to threshold and the phase falls off too fast at low energies. Stelbovics and Dodd[16] showed that even fourth-order terms are important in reproducing the exact binding energy. From studies in potential scattering[25] we hope, however, that in general second-order inputs or at most third-order terms (which may be phenomenological poles) will be necessary. In addition, it may be possible to develop approximations for higher input functions on the basis of the left-hand spectral representation.

The present investigation has been a test of various assumptions in the N/D theory of nuclear reactions. Consequently we should now briefly comment on the usefulness of this theory in more quantitative applications. As mentioned in the introduction, various extensions can be and have already been performed in the N/D theory. Actually, noncentral and Coulomb forces have been used in the five-nucleon problem.[12] Other extensions which proved necessary in the present investigation are related to three-particle unitarity (which may also turn the N/D method into a useful tool for describing breakup reactions) and to the description of higher order inputs. The input may also be improved by using better form factors and by considering those pion-exchange graphs which are neglected in the Faddeev equations. Formal investigations may open the way to better treatments of resonances and virtual states. We realize that the present N/D calculations do not yet contain all these desirable ingredients, however, since most of the extensions which were mentioned are straightforward, we think that the N/D method may still develop into a quantitative tool for describing nuclear reactions and into a qualitative tool for understanding the underlying dynamics of the reaction.

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