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Background phase shift in R-matrix theory

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It has been shown recently by Cugnon in his generalized version of the R-matrix theory that the hard sphere phase shift used in the standard R-matrix theory is arbitrary and can be replaced, in general, by any other phase shift. In the present paper similar results have been derived in a straightforward manner, within the framework of the standard R-matrix theory, from the usual division of R matrix into a background matrix R_0 plus a matrix R' containing pole terms and further constraining R_0 to be a diagonal matrix. Thus Cugnon's theory and the standard R-matrix theory lead to algebraically equivalent results. An expression for the renormalization factor for the reduced width has also been given in terms of the new phase shift and the hard sphere phase shift.

[NUCLEAR REACTIONS R-matrix theory, arbitrary phase shift within standard formalism. Expression for reduced width.]

I. INTRODUCTION

R-matrix theory is most widely used by experimentalists to analyze resonance reactions. Other available theories are not always amenable to a simple parametrization of the experimental data. It has been found in many experimental studies that the R-matrix theory fails to reproduce the off resonance cross sections if the hard sphere phase shifts, as required by the theory, are used.[1] In an experimental study of the reaction $^{26}\mathrm{Mg}(\alpha,\alpha)$ in our laboratory,[2,3] it was found that the absolute value of the experimental cross section at backward angles differed by as much as a factor of 2 even though 19 levels were explicitly included in a two-channel multilevel expression of the R-matrix theory. When the R-matrix expression was modified, replacing the hard sphere phase shifts by the appropriate optical model phase shifts, the data could be fitted satisfactorily. On the basis of theoretical studies similar conclusions have been arrived at by Mahaux and Weidenmüller,[4] who found the one-level approximation of the R-matrix theory failing to fit a resonance produced by a mathematical model in which exact solutions could be arrived at by solving the Schrödinger equation. These authors also compared the formal expressions of the R-matrix theory with a theory based on the shell model. They concluded that expressions in the two theories are analogous, though the interpretation of the parameters differs in the two approaches. Mahaux and Weidenmüller suggested that the R-matrix expressions should be used, replacing the hard sphere phase shifts by the realistic optical model phase shifts. This was a practical suggestion but it did not have any sound basis. Recently the work of Cugnon[5] has clarified this situation. The arbitrariness involved in the use of the hard sphere phase shifts has been brought out clearly in his generalized version of the R-matrix theory. It has been further shown that any other background phase shifts can replace the hard sphere phase shifts. Essentially Cugnon[5] has investigated the continuity condition in the R-matrix theory using the projection operator formalism giving an alternative derivation of the R-matrix theory. He further shows the freedom of using two distinct boundary condition parameters in constructing the theory. The standard R-matrix theory is a special case when one of these two boundary condition parameters tends to infinity. With such an approach he generalizes the expression of the standard R-matrix theory.

In the present paper we show that the results of the standard R-matrix theory given by Lane and Thomas[6] easily yield the general expression of the scattering matrix derived by Cugnon. The mathematical structure of the two expressions for the scattering matrix obtained in the present approach using the standard R-matrix theory and in Cugnon's generalized version of the R-matrix theory is identical. In Sec. II we describe in brief the results of Cugnon.[5] In Sec. III Cugnon's results are derived from the standard R-matrix theory. In Sec. IV an expression has been derived for the factor renormalizing the reduced width in terms of the new phase shift and the hard sphere phase shift.

II. CUGNON'S R-MATRIX RESULTS

The new result obtained by Cugnon is essentially contained in Eq. (6.7) of his paper.[5] This equation as given in Ref. 5 has many printing errors in it. Correcting these errors, the expression for the scattering matrix U reads

$$U_{cc'} = \Omega_c^2(b_{2c})\delta_{cc'} - 2i\sum_{\lambda,\mu} \Omega_c q_c P_c^{1/2}\bar{A}_{\lambda\mu}^{-1} q_{c'} P_{c'}^{1/2}\gamma_{\lambda c}\gamma_{\mu c'}\Omega_{c'} . \quad (1)$$

In this expression c and c' stand for channels and λ and μ for levels in the compound system. $\Omega_c^2(b_{2c})$ is a function of the boundary condition parameter b_{2c} and is an element of new background phase shift matrix. P_c and $P_{c'}$ are penetration factors and Ω_c and $\Omega_{c'}$ are elements of the hard sphere phase shift matrix. $\bar{A}$ is a level matrix given by

$$\bar{A}_{\lambda\mu} = (E_\lambda - E)\delta_{\lambda\mu} + \sum_c q_c L_c^0 \gamma_{\lambda c}\gamma_{\mu c} . \quad (2)$$

$\bar{A}$ is a matrix of finite dimensions, as has been appropriately emphasized by Cugnon. Expression (1) therefore already assumes the existence of only a finite number of levels. L_c^0 is given by

$$L_c^0 = L_c - b_c = S_c + iP_c - b_c , \quad (3)$$

where L_c is an element of the diagonal logarithmic derivative matrix at the channel radius a_c, and b_c is an element of a real boundary condition parameter matrix used in the standard R-matrix theory. In Eq. (1), q_c is a new quantity appearing due to the general choice of the second boundary condition parameter matrix b_2. It has been shown in Ref. 5 that q_c is given by

$$q_c = \frac{b_c - b_{2c}}{L_c - b_{2c}} . \quad (4)$$

Another expression involving b_{2c} is given by

$$\Omega_c^2(b_{2c}) = \frac{L_c^* - b_{2c}}{L_c - b_{2c}}\Omega_c^2 . \quad (5)$$

Equations (1)–(5) describe the essential results of Ref. 5. Combining Eqs. (4) and (5), b_{2c} can be eliminated and yields q_c as

$$q_c = \frac{L_c^0\Omega_c^2(b_{2c}) - L_c^{0*}\Omega_c^2}{2iP_c\Omega_c^2} , \quad (6)$$

using the relationship $L_c - L_c^* = 2iP_c$.

III. DERIVATION OF CUGNON'S RESULTS FROM STANDARD R-MATRIX THEORY

It has been shown in Ref. 6 that if the R matrix is divided as

$$R = R^0 + R' , \quad (7)$$

where R' is given by

$$R'_{cc'} = \sum_\lambda \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E} , \quad (8)$$

the summation is over a finite number of levels, and R^0 is the remainder of the matrix R, then the scattering matrix is given by (Expression IX-1.4 of Ref. 6)

$$U_{cc'} = \Omega_c^2\delta_{cc'} + 2i\Omega_c P_c^{1/2}\sum_{c''}(1 - R^0L^0)_{cc''}^{-1}R^0_{c''c'}P_{c'}^{1/2}\Omega_{c'} + 2i\Omega_c P_c^{1/2}\sum_{\lambda,\mu}\alpha_{\lambda c}A_{\lambda\mu}\alpha_{\mu c'}P_{c'}^{1/2}\Omega_{c'} , \quad (9)$$

where $A_{\lambda\mu}$ is given by

$$(E_\lambda - E)A_{\lambda\mu} - \sum_{\nu,c,c'} L_c^0(1 - R^0L^0)_{cc'}^{-1}\gamma_{\lambda c'}\gamma_{\nu c}A_{\nu\mu} = \delta_{\lambda\mu} \quad (10)$$

and $\alpha_{\lambda c}$ is given by

$$\alpha_{\lambda c} = \sum_{c'}(1 - R^0L^0)_{cc'}^{-1}\gamma_{\lambda c'} . \quad (11)$$

Equation (9) goes to Eq. (1) when R^0 is constrained to be a diagonal matrix. Thus the standard R-matrix expression [Eq. (9)] is more general than the expression used by Cugnon [Eq. (1)]. This is shown in the following. With this constraint we set

$$U_{cc}^0 = \Omega_c^2 + 2i\Omega_c^2 P_c(1 - R_{cc}^0 L_{cc}^0)^{-1}R_{cc}^0 . \quad (12)$$

Thus $U_{cc}^0\delta_{cc'}$ replaces the first two terms of Eq. (9). Solving Eq. (12) for R_{cc}^0 we get

$$R_{cc}^0 = \frac{U_{cc}^0 - \Omega_c^2}{U_{cc}^0 L_c^0 - \Omega_c^2 L_c^{0*}} . \quad (13)$$

The result for R_{cc}^0 with the boundary condition $b_c = \delta_c$ has been given by Namjoshi.[3] In the present case, using Eq. (13), Eq. (11) becomes

$$\alpha_{\lambda c} = (1 - R_{cc}^0 L_c^0)^{-1}\gamma_{\lambda c} = \bar{q}_c\gamma_{\lambda c} , \quad (14)$$

where

$$\bar{q}_c = \frac{U_{cc}^0 L_c^0 - \Omega_c^2 L_c^{0*}}{2iP_c\Omega_c^2} . \quad (14a)$$

Using (14), Eq. (10) can be written as

$$(E_\lambda - E)A_{\lambda\mu} - \sum_{\nu,c} L_c^0\bar{q}_c\gamma_{\lambda c}\gamma_{\nu c}A_{\nu\mu} = \delta_{\lambda\mu} . \quad (15)$$

If we set $U_{cc}^0 = \Omega_c^2(b_{2c})$ then $\bar{q}_c = q_c$, as can be seen from Eqs. (6) and (14a). Also, by comparing equations (2) and (15) we see

$$A=-\bar{A}^{-1}, \tag{16}$$

and then (9) goes to

$$U_{cc'}=\Omega_c^2(b_{2c})\delta_{cc'} + 2i\Omega_c P_c^{1/2}\sum_{\lambda,\mu} q_c\gamma_{\lambda c}A_{\lambda\mu}q_{c'}\gamma_{\mu c'}P_{c'}^{1/2}\,\Omega_{c'}. \tag{17}$$

Equation (17) is identical to Eq. (1), keeping in mind Eq. (16). Thus Eq. (1) results from Eq. (9) with the constraint that R^0 is a diagonal matrix. It should also be mentioned that R^0 and b_2 are related and the relation between the two can be derived as

$$b_{2c}=b_c+R^{0\,-1}_{cc}. \tag{18}$$

In writing down both Eqs. (1) and (9), a common assumption, that the level matrix A is of a finite dimension, has been made. Hence expressions (1) and (17) are on equivalent footing and are identical to each other in mathematical structure. All the quantities appearing in the two cases have one to one correspondence. Therefore, it is a natural consequence of the present derivation that Cugnon's theory and the standard R-matrix theory are similar from an algebraic point of view. It should be mentioned that, though Cugnon's theory does start with a more general set of the continuity conditions while the standard R-matrix theory uses only a special set of the continuity conditions, surprisingly, the two theories led to equivalent expressions. This either implies that the generalization of the continuity conditions is redundant or the generalization is already built into the standard R-matrix theory. From the point of view of interpretation it can be argued that background arises in a different manner in the two approaches. In the standard R-matrix theory the background is due to the hard sphere scattering plus the scattering through distant levels, while in Cugnon's work the background arises from the boundary condition which allows the incident wave to be scattered differently from what it is by a hard sphere. This difference is not really so deep. It is well known[6] that in the standard R-matrix theory the scattering from distant levels without the random sign approximation for the partial widths gives rise to a direct process. In the present case of a diagonal R^0, this will correspond to a direct or potential elastic scattering. It can be further seen[6] that hard sphere scattering is more a label than a physically implied phenomena as far as the standard R-matrix theory is concerned. Remembering these points, the interpretation of the parameters in the two approaches is similar. Even from the point of view of an application the two theories have similar parametric forms, and therefore the parametrization of the experimental data cannot distinguish between the two theories.

IV. MODIFIED R-MATRIX EXPRESSION

Cugnon has transformed Eq. (1) in terms of observed widths but his definition is a departure from convention [see Eq. (6.10) of Ref. 5]. From Eq. (18) it can be shown that b_{2c} is real. With b_{2c} as real we get, combining Eqs. (4) and (5) (ref. 5),

$$\Omega_c^2(b_{2c})\,|q_c|^2=\Omega_c^2 q_c^2. \tag{19}$$

Using (19), we can transform expression (17) into

$$U_{cc'}=\Omega_c(b_{2c}) \times\left[\delta_{cc'}+2iP_c^{1/2}\sum_{\lambda,\mu}|q_c|\gamma_{\lambda c}A_{\lambda\mu}|q_{c'}|\gamma_{\mu c'}P_{c'}^{1/2}\right] \times\Omega_{c'}(b_{2c'}). \tag{20}$$

Defining partial width $\Gamma_{\lambda c}$ as

$$\Gamma_{\lambda c}=2P_c|q_c|^2\gamma_{\lambda c}^2, \tag{21}$$

(20) becomes

$$U_{cc'}=\Omega_c(b_{2c})\left[\delta_{cc'}+i\sum_{\lambda,\mu}\Gamma_{\lambda c}^{1/2}A_{\lambda\mu}\Gamma_{\mu c'}^{1/2}\right]\Omega_{c'}(b_{2c'}). \tag{22}$$

When b_{2c} tends to infinity, $\Omega_c(b_{2c})\to\Omega_c$, $q_c\to 1$, and $\Gamma_{\lambda c}\to 2P_c\gamma_{\lambda c}^2$. The latter is the conventional definition of the partial width. Therefore, $|q_c|^2$ is the factor modifying the extraction of reduced width from the observed partial width. As $\Omega_c=e^{i(\omega_c-\phi_c)}$, $\phi_c=\tan^{-1}F_c/G_c$ and ω_c is the relative Coulomb phase shift [see Eqs. III (4.5a,b) of Ref. 6], and if we set $\Omega_c(b_{2c})=e^{i(\omega_c+\delta_c)}$ we get, using Eq. (6),

$$|q_c|^2=\frac{[S_c^0\sin(\delta_c+\phi_c)+P_c\cos(\delta_c+\phi_c)]^2}{P_c^2}, \tag{23}$$

where $S_c^0=S_c-b_c$. With the boundary condition $b_c=S_c$ [expression (23)] simplifies to

$$|q_c|^2=\cos^2(\delta_c+\phi_c). \tag{23$'$}$$

It will be useful to mention that, using the method given in Ref. 6, expression (20) can also be written in the channel matrix form as

$$U=\Omega(b_2)[1+2iP^{1/2}(1-\bar{R}\bar{L}^0)^{-1}\bar{R}P^{1/2}]\,\Omega(b_2), \tag{24}$$

where $\bar{R}$ is the modified R matrix with matrix elements given by

$$\bar{R}_{cc'}=\sum_\lambda\frac{|q_c|\gamma_{\lambda c}|q_{c'}|\gamma_{\lambda c'}}{E_\lambda-E}, \tag{24a}$$

and $\bar{L}^0$ is

$$\bar{L}^0_{cc'} = L^0_c \frac{q_c}{|q_c|^2}\delta_{cc'} = \frac{L^0_c P_c e^{i(\delta_c+\phi_c)}\delta_{cc'}}{S^0_c \sin(\delta_c+\phi_c)+P_c\cos(\delta_c+\phi_c)}. \tag{24b}$$

In comparison to the R matrix, which contains a summation over an infinite number of levels, $\bar{R}$ contains a summation over only a finite number of levels and therefore will yield better results when approximated. This is a situation similar to the K-matrix theory based on the shell model, where the K matrix contains only a finite summation.[4] In practice $\Omega_c(b_{2c})$ may be considered to be similar to a phase shift matrix element produced by a single particle potential. In the above treatment $\Omega_c(b_{2c})$ has been assumed to be unimodular; phenomenologically nonunimodular $\Omega_c(b_{2c})$ produced by the complex potential well model (optical model) may be used. Both Eqs. (20) and (24) should be useful forms for analyzing the experimental data.

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