

Off-shell equivalence in three-body scattering*

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Explicit relationships are derived connecting the author's boundary condition formalism to the Faddeev theory of three-body scattering. In particular, it is shown that suitable input to the boundary condition formalism can always be chosen so as to exactly reproduce the Faddeev amplitudes. This is also true in the presence of explicit three-body forces. It is further shown that such forces cannot be distinguished from off-shell properties of the two-particle interaction on the basis of three-particle scattering observables. A previous analysis of the n - d breakup reaction is discussed in some detail.

[NUCLEAR REACTIONS Three-body scattering theory, three-body forces, N - d scattering.]

I. INTRODUCTION

Our present understanding of the nuclear force is largely empirical, and has developed in response to two-particle phenomena. The historical procedure has thus been to introduce just what is necessary in order to explain an enlarging class of experimental facts.¹ The constructs of such an empirical procedure are necessarily nonunique; these ambiguities are commonly referred to as the "off-shell" characteristics of the theory, and cannot (by definition) be resolved at the two-particle level. One can easily demonstrate at least a formal dependence of three-body systems on the off-shell properties, and this has stimulated a great deal of interest in studying the three-body problem.²

In principle, therefore, studies of the trinucleon system should provide a clear test of the off-shell characteristics of the nucleon-nucleon interaction, enabling one to choose from among the many phenomenological potentials which have been proposed. Early calculations of the triton ground state properties were very encouraging, in that considerable sensitivity was exhibited to the momentum dependence of the interaction, the type of short-range repulsion, tensor vs central forces, etc.³ However, after a great deal of labor involving increasingly more sophisticated models, it appears that the differences generated by competing "realistic" potentials are comparatively minor. Thus, by concentrating on models which produce identical (or closely similar) *two*-particle properties, much of the apparent off-shell sensitivity has been eliminated. In particular, theoretical values for the triton binding energy (E_T) differ by only a few tenths of an MeV for realistic potentials, although the 1.5 MeV missing as compared to experiment is a clear signal that the interaction has not been fully understood.⁴

What has emerged from this effort is the realization that the on-shell two-particle properties, and (more subtly) the constraints of three-body unitarity, to a large extent determine the three-particle observables. This understanding was obscured for a long time by the difficulties inherent in three-body calculations. The standard procedure has been to perform successive computations with different potentials, thus generating a selection of input and output for comparison. Inasmuch as the on-shell and off-shell properties are inextricably linked in the parameters characterizing the potential, this is a crude procedure at best. Nevertheless, if the numerical work were comparatively simple this would probably be adequate, but in practice it is slow, laborious, and expensive. Also, it is clearly quite important to require phase-equivalent input when making judgments about off-shell sensitivities, but this is not entirely practical within the conventional (Faddeev) framework.

Although N - d scattering calculations are considerably more difficult than the triton problem, and hence, in a comparatively early stage, the success of fairly trivial models in fitting both elastic and inelastic (breakup) data suggests that a similar picture will emerge once the Faddeev calculations have become sufficiently exhaustive.⁵ However, due to the inefficiencies of that approach, and the particular difficulties associated with the inclusion of local potentials, a definitive conclusion is likely to be some years away. This is particularly unfortunate from the standpoint of proposed experiments, since the outcome has clearly a great deal of bearing as to which will be most profitable.

The boundary condition formalism (BCF) proposed by this author was designed to shortcut this problem, and provide a practical, efficient framework for analyzing experimental sensitivities to

specific effects.⁶ As a test case, the technique was applied to the analysis of the n - d breakup reaction at 14.4 MeV.⁷ The results demonstrated that the differential cross sections are sensitive to only a single parameter, which may be fixed in terms of the n - d doublet scattering length (itself strongly correlated with E_T). This implies that with regard to these observables there is little to gain either from costly calculations with "realistic" potentials, or from the corresponding experiments. This conclusion is not dissimilar from that reached concerning the triton properties, but it was much easier to come by.

This result has stirred considerable controversy, particularly among those doing the "wrong" experiments. Various claims and counterclaims have arisen concerning the content, implications, and generality of the analysis.^{8,9} In particular, it has been suggested that the variations considered correspond to three-body forces rather than to two-particle off-shell properties, and that the results depend strongly on the particular N - N phase shifts employed as input.⁹ These claims are in direct conflict with previous statements by this author, and it is therefore clear that a certain amount of confusion and misunderstanding exists concerning the BCF technique in general, and the n - d analysis in particular. A major purpose of this article is therefore to make quite explicit the connection between the BCF and the Faddeev formalisms, with and without the inclusion of three-body forces. In particular, it will be shown that BCF input of the class considered in the n - d analysis can always be chosen so as to exactly reproduce the Faddeev results, irrespective of the phase shifts, two-body potentials, or possible three-body potentials. Thus a systematic variation of the BCF parameters must encompass any and all possibilities realizable in the Faddeev theory.

On a more general note, the remarks of Haftel and Petersen illustrate a persistent misunderstanding concerning the equivalence of off-shell properties and true three-body forces.⁹ Thus, there is a general impression that such effects can be distinguished experimentally by concentrating on specific regions of phase space.¹⁰ However, this is simply not the case, as will be shown via a simple extension of the off-shell equivalence proof. Specifically, there is no means by which one can distinguish a three-body force from off-shell properties even in principle, given a complete knowledge of three-particle *scattering* observables. This result is implicit in the interior-exterior separation proposed sometime ago by Noyes, but has not been widely appreciated.¹¹ Hopefully, the more explicit development pre-

sented below will serve to exorcise the recurrent confusion concerning this point. If studies of the three-body problem are to succeed in enhancing our knowledge of the nuclear force, it is essential that we clearly understand the limitations inherent in the problem. Thus, although the Faddeev equation provides a useful formalism in which to test a *specific* potential model, its complexity tends to obscure certain general features. In contrast, the BCF emphasizes three-body *observables*, and hence is a more suitable tool for experimental analysis.

The organization of this paper is as follows. In Sec. II we briefly derive the Faddeev equation for three spinless particles in the presence of an explicit three-body force. The equation is then cast into operator form for ease in subsequent manipulations. Section III is concerned with the reduction of the Faddeev equation to an effective one-variable form comparable to the BCF equation. The main content of the paper is presented in Sec. IV, which contains the explicit proof of off-shell equivalence. The essential ambiguity involved in distinguishing off-shell behavior from a true three-body force is also demonstrated in the context of this proof. Finally, Sec. V is devoted to a discussion of these results and the general problem of effectively utilizing three-body observables in investigating the nuclear force. In particular, assumptions underlying the 14.4 MeV analysis are discussed in considerably greater detail than was possible in previous letters. Relevant details concerning the BCF are provided in the Appendix.

II. FADDEEV EQUATIONS WITH THREE-BODY FORCES

Although the inclusion of three-body forces in the Faddeev formalism is straightforward, such forces have yet to be employed in scattering calculations and hence the formal development is largely unfamiliar.¹² We thus begin by briefly deriving the relevant equations.¹³ In order to avoid mathematical subtleties we shall assume that the potentials are sufficiently well behaved to guarantee the existence of various operator products and inverses employed below. For this purpose it is sufficient for the two-particle potentials to be bounded by a Yukawa potential, with a corresponding assumption regarding the three-body potential. In practice this includes all of the potential models (excluding Coulomb) actually employed in few-body calculations.¹⁴ For simplicity we shall ignore the spin, isospin degrees of freedom; this specialization clearly has little bearing on the questions to be addressed. Furthermore, it should be obvious that the operator for-

malism we employ is equally valid in the general case given a trivial expansion of the basis, and hence the proofs are actually quite general.

Consider the state vector $|\Psi\rangle$ describing the three-body system in its center of mass (c.m.), and let W denote the total energy in that frame. The Schrödinger equation may then be stated as

$$\left(H_0 + \sum_{\alpha} V_{\alpha} + V_3\right) |\Psi\rangle = W |\Psi\rangle. \quad (1)$$

Here H_0 is the free Hamiltonian (kinetic energy operator), V_{α} is the two-body potential for particles β and γ ($\alpha \neq \beta \neq \gamma$), and V_3 is the three-body potential. We consider a solution consisting of outgoing waves originating from an initial plane wave state $|\Phi\rangle$, and introduce the Green's functions

$$\begin{aligned} G_0 &= (H_0 - W - i\epsilon)^{-1}, \\ G_{\alpha} &= (H_0 + V_{\alpha} - W - i\epsilon)^{-1} \\ &\equiv G_0 - G_0 t_{\alpha} G_0. \end{aligned} \quad (2)$$

The latter equation defines for us t_{α} , the two-body scattering operator (t matrix), and implies that $G_{\alpha} V_{\alpha} = G_0 t_{\alpha}$.

As it stands, the solution $|\Psi\rangle$ of Eq. (1) must satisfy rather complicated boundary conditions related to the various types of asymptotic states, and hence we introduce the Faddeev channel decomposition, $|\Psi\rangle = \sum_{\alpha} |\psi_{\alpha}\rangle$. In the momentum-space representation this same decomposition solves the problem of disconnected graphs, as is well known. The $|\psi_{\alpha}\rangle$ then must satisfy

$$|\psi_{\alpha}\rangle = \frac{1}{3}(1 - G_0 t_{\alpha}) |\Phi\rangle - \frac{1}{3} G_{\alpha} V_3 |\Psi\rangle - G_0 t_{\alpha} \sum_{\beta \neq \alpha} |\psi_{\beta}\rangle. \quad (3)$$

If we define the three-particle t matrix T via the relation

$$|\Psi\rangle = (1 - G_0 T) |\Phi\rangle, \quad (4)$$

and expand $T = \sum_{\alpha} \tau_{\alpha}$, we have

$$|\psi_{\alpha}\rangle = \left(\frac{1}{3} - G_0 \tau_{\alpha}\right) |\Phi\rangle. \quad (5)$$

Substitution into Eq. (3) then yields the equation

$$\begin{aligned} \langle \alpha \vec{p}' \vec{q}' | I | \beta \vec{p} \vec{q} \rangle &= -\delta \left(\vec{p} + \frac{\mu_{\beta}}{m_{\gamma}} \vec{p}' + \frac{\mu_{\beta}}{M_{\alpha}} \vec{q}' \right) \delta \left(\vec{q} - \vec{p}' + \frac{\mu_{\alpha}}{m_{\gamma}} \vec{q}' \right) \text{ if } \alpha\beta\gamma \text{ are cyclic;} \\ &= -\delta \left(\vec{p} + \frac{\mu_{\beta}}{m_{\gamma}} \vec{p}' - \frac{\mu_{\beta}}{M_{\alpha}} \vec{q}' \right) \delta \left(\vec{q} + \vec{p}' + \frac{\mu_{\alpha}}{m_{\gamma}} \vec{q}' \right) \text{ if } \beta\alpha\gamma \text{ are cyclic;} \end{aligned} \quad (10)$$

here m_{α} is the mass of particle α . It follows that

$$\begin{aligned} I &= I^T, \\ I^{-1} &= \frac{1}{2}(1 + I), \\ (1 - I)^2 &= 3(1 - I). \end{aligned} \quad (11)$$

$$\tau_{\alpha} = t_{\alpha} + \frac{1}{3}(1 - t_{\alpha} G_0) V_3 (1 - G_0 T) - t_{\alpha} G_0 \sum_{\beta \neq \alpha} \tau_{\beta}, \quad (6)$$

since $|\Phi\rangle$ is arbitrary ($H_0 |\Phi\rangle = W |\Phi\rangle$). In the special case $V_3 = 0$, Eq. (6) is the usual expression of the Faddeev equations, and its solutions are known to be well defined for potentials V_{α} of the type considered. The mathematical properties of the equation are not altered for reasonable choices of $V_3 \neq 0$ (as in the present case), and hence τ_{α} is uniquely specified.

In order to perform the manipulations required below it is a great convenience to employ an operator notation which frees us from the explicit α indices. We shall thus describe our three-body state by the three sets of Jacobi variables $(\alpha \vec{p} \vec{q})$, where \vec{p} is the relative momentum of particles β and γ , and \vec{q} is the momentum of α relative to the $\beta\gamma$ c.m.¹⁵ The reason for employing three sets rather than one (only two vectors are linearly independent) is that t_{α} is much more simply described in terms of $(\alpha \vec{p} \vec{q})$ than $(\beta \vec{p} \vec{q})$. The \vec{p}, \vec{q} vectors correspond to the reduced masses μ_{α}, M_{α} , respectively, and a physical scattering state satisfies the *on-shell* condition

$$p^2/2\mu_{\alpha} + q^2/2M_{\alpha} = W. \quad (7)$$

Below we shall employ the momentum

$$\kappa_{\alpha} = [2\mu_{\alpha}(W - q^2/2M_{\alpha})]^{1/2}, \quad (8)$$

which is positive imaginary for $q > (2M_{\alpha}W)^{1/2} \equiv Q_{\alpha}$. Thus the physical states correspond to momenta $q \leq Q_{\alpha}$, with $p = \kappa_{\alpha}$.

We define a Hilbert space of states $|\alpha \vec{p} \vec{q}\rangle$ with the normalization

$$\begin{aligned} \langle \alpha \vec{p} \vec{q} | \beta \vec{p}' \vec{q}' \rangle &= \delta_{\alpha\beta} \delta(\vec{p} - \vec{p}') \delta(\vec{q} - \vec{q}'), \\ \sum_{\alpha} \int d\vec{p} d\vec{q} |\alpha \vec{p} \vec{q}\rangle \langle \alpha \vec{p} \vec{q}| &= 1. \end{aligned} \quad (9)$$

On this space we define a number of operators. The first is I , which "interconnects" the various Faddeev channels, and provides the transformation between the α and β representations. Specifically,

We also define operators t, G_0, V_3 such that

$$\begin{aligned} \langle \alpha \vec{p}' \vec{q}' | t | \beta \vec{p} \vec{q} \rangle &= \delta_{\alpha\beta} \delta(\vec{q} - \vec{q}') t_{\alpha}(\vec{p}', \vec{p}; W - q^2/2M_{\alpha}), \\ \langle \alpha \vec{p}' \vec{q}' | G_0 | \beta \vec{p} \vec{q} \rangle &= \frac{\delta_{\alpha\beta} \delta(\vec{q} - \vec{q}') \delta(\vec{p} - \vec{p}')}{p^2/2\mu_{\alpha} + q^2/2M_{\alpha} - W - i\epsilon}, \\ \langle \alpha \vec{p}' \vec{q}' | V_3 | \beta \vec{p} \vec{q} \rangle &= \delta_{\alpha\beta} V_3(\alpha \vec{p}' \vec{q}'; \alpha \vec{p} \vec{q}). \end{aligned} \quad (12)$$

Here $t_\alpha(\vec{p}', \vec{p}; s_\alpha)$ is the off-shell two-particle ($\beta\gamma$) t matrix. In addition we define

$$\bar{V} = \frac{1}{9}(1-I)V_3(1-I), \quad (13)$$

which implies that

$$\langle \alpha \vec{p}' \vec{q}' | \bar{V} | \alpha \vec{p} \vec{q} \rangle = \frac{1}{3} \langle \alpha \vec{p}' \vec{q}' | V_3 | \alpha \vec{p} \vec{q} \rangle. \quad (14)$$

We can now state Eq. (6) as an operator equation on the $|\alpha \vec{p} \vec{q}\rangle$ basis; the τ operator must satisfy

$$\tau = t(1-I) + (1-tG_0)\bar{V}(1-G_0\tau) + tIG_0\tau. \quad (15)$$

Finally, we observe that T may be expressed as $T = (1-I)\tau$.

III. REDUCTION TO ONE-DIMENSIONAL FORM

In order to establish the connection between the Faddeev theory and the boundary condition (BC) formalism, we first proceed to reduce Eq. (15) to an equivalent one-dimensional equation (only the q variable) analogous to the equation derived in BCA. We begin by observing that t may be quite generally decomposed as $t = t^s + t^r$, where, in a state of angular momentum l ,

$$t_{\alpha l}^s(p', p; s_\alpha) = f_{\alpha l}(p', \kappa_\alpha) t_{\alpha l}(s_\alpha) f_{\alpha l}(p, \kappa_\alpha), \quad (16)$$

$f_{\alpha l}(\kappa_\alpha, \kappa_\alpha) = 1$, and t^r vanishes half on shell. Here $t_{\alpha l}(s_\alpha)$ is the on-shell t matrix

$$t_{\alpha l}(s_\alpha) = -\frac{e^{i\delta_{\alpha l}} \sin \delta_{\alpha l}}{\pi \mu_\alpha \kappa_\alpha}. \quad (17)$$

This decomposition was discussed independently by Kowalski¹⁶ and Noyes,¹⁷ and is frequently referred to as the Kowalski-Noyes representation of the t matrix. Unfortunately, the representation possesses some undesirable properties in that t^s in general contains the left-hand cut structure of $t_{\alpha l}(s_\alpha)$; i.e., singularities for $s_\alpha \leq -\mathfrak{M}_\alpha^2/8\mu_\alpha$. These singularities are not proper to t_α , and are canceled by corresponding terms in t^r .

In order to avoid this problem we choose s_α^0 such that $-\mathfrak{M}_\alpha^2/8\mu_\alpha < s_\alpha^0 < s_\alpha^b$, where s_α^b is the energy of the $\beta\gamma$ ground state ($s_\alpha^b = 0$ if no bound states exist). We then define

$$f_{\alpha l}(p, \kappa_\alpha) = t_{\alpha l}(p, \kappa_\alpha; s_\alpha) / t_{\alpha l}(s_\alpha), \\ \bar{t}_{\alpha l}(s_\alpha) = t_{\alpha l}(s_\alpha), \quad \text{if } s_\alpha \geq s_\alpha^0$$

and

$$f_{\alpha l}(p, \kappa_\alpha) = N_{\alpha l}^{(0)}(p) / N_{\alpha l}^{(0)}(\kappa_\alpha), \\ \bar{t}_{\alpha l}(s_\alpha) = t_{\alpha l}^{BC}(s_\alpha), \quad \text{if } s_\alpha < s_\alpha^0. \quad (18)$$

Here $N_{\alpha l}^{(0)}(p)$ is the BC function defined in BCA, and $t_{\alpha l}^{BC}(s_\alpha) \equiv N_{\alpha l}(\kappa_\alpha) / D_{\alpha l}(\kappa_\alpha)$ is the BC representation of the (on-shell) t matrix. Precise definitions of these and associated quantities are given in the

Appendix. We now define t^s by Eq. (16) using the modified $f_{\alpha l}(p, \kappa_\alpha)$ function defined in Eq. (18), and also using $\bar{t}_{\alpha l}(s_\alpha)$ in place of $t_{\alpha l}(s_\alpha)$. Defining $t^r = t - t^s$, we deduce the following properties: (1) t^s contains all singularities of t , including the elastic cut for $s_\alpha > 0$ and the proper residues at bound state poles; (2) t^r is real, bounded, and analytic in the disjoint domains $s_\alpha < s_\alpha^0$ and $s_\alpha > s_\alpha^0$; (3) t^r vanishes half on shell for $s_\alpha > s_\alpha^0$. In particular, for *physical* three-particle scattering states ($s_\alpha > 0$) t^r vanishes; for such states G_0 possesses a right-hand cut with discontinuity ΔG_0 , and this fact can be stated in the form

$$\Delta G_0 t^r = t^r \Delta G_0 = 0. \quad (19)$$

Similarly, we define V^s such that

$$\langle \alpha \vec{p} \vec{q} | V^s | \beta \vec{p}' \vec{q}' \rangle = \sum_{lm} Y_{lm}(\hat{p}) f_{\alpha l}(p, \kappa_\alpha) \left(\frac{\kappa_\alpha}{|\kappa_\alpha|} \right)^\sigma \\ \times \langle \alpha | \kappa_\alpha | lm \vec{q} | \bar{V} | \beta \vec{p}' \vec{q}' \rangle, \quad (20)$$

where σ is 0 (1) if l is even (odd), and we have employed the partial-wave decomposition $\vec{p} \rightarrow (plm)$. Defining $V^r = \bar{V} - V^s$, it follows that (1) V^s and V^r are real valued and nonsingular; (2) $\Delta G_0 V^r = 0$. We may now introduce

$$K^r = -V^r G_0 + t^r G_0 \bar{V} G_0 + t^r I G_0, \\ K^s = -V^s + t^s G_0 \bar{V} + t^s I, \\ \Omega^r = t^r(1-I) + V^r - t^r G_0 \bar{V}, \\ \Omega^s = t^s(1-I) + V^s - t^s G_0 \bar{V}, \quad (21)$$

in terms of which Eq. (15) can be written as

$$\tau = \Omega^r + \Omega^s + (K^r + K^s G_0) \tau. \quad (22)$$

Given Eq. (19), one may easily verify that K^r is a real L_2 kernel, and hence there exists a unique inverse $Z^r = (1 - K^r)^{-1}$. Let

$$\tau = Z^r(\Omega^r + \tau_0); \quad (23)$$

then τ_0 satisfies

$$\tau_0 = \Omega_0 + K^s \bar{Z}^r G_0 \tau_0, \quad (24)$$

where $\bar{Z}^r = G_0 Z^r G_0^{-1}$, and

$$\Omega_0 = \Omega^s + K^s \bar{Z}^r G_0 \Omega^r \\ = t^s + K^s \bar{Z}^r (G_0 t^r - 1). \quad (25)$$

Due to properties of t^r and V^r , we observe that τ and τ_0 are identical on shell for physical states ($\Delta G_0 \tau = \Delta G_0 \tau_0$). The three-particle scattering amplitude (T) can thus be calculated from the knowledge of τ_0 alone. It is clear from the definitions of t^s , V^s that Eq. (24) may be reduced to a one-variable form. To exploit this it is con-

venient to introduce a partial-wave decomposition corresponding to the basis $|\alpha L M l \lambda p q\rangle$, where $\vec{l}(\vec{p})$ and $\vec{\lambda}(\vec{q})$ are coupled to a state of total angular momentum L ($L_z = M$). The normalization is

$$\langle \beta L' M' l' \lambda' p' q' | \alpha L M l \lambda p q \rangle = \delta_{\alpha\beta} \delta_{LL'} \delta_{MM'} \delta_{ll'} \delta_{\lambda\lambda'} \frac{\delta(p-p')}{p^2} \frac{\delta(q-q')}{q^2}. \quad (26)$$

Let F be a diagonal operator on this basis such that

$$F_{i\lambda}^{\alpha L}(p, q) = f_{\alpha i}(p, \kappa_\alpha), \quad (27)$$

and define operators $\hat{t}, \hat{V}, \hat{\Omega}, \hat{K}$ such that

$$\begin{aligned} t^s &= F \hat{t}, \\ V^s &= F \hat{V}, \\ \Omega_0 &= F \hat{\Omega}, \\ K^s &= F \hat{K}. \end{aligned} \quad (28)$$

Thus, for example,

$$\begin{aligned} \langle \beta L' M' l' \lambda' p' q' | \hat{t} | \alpha L M l \lambda p q \rangle \\ = \delta_{\alpha\beta} \delta_{LL'} \delta_{MM'} \delta_{ll'} \delta_{\lambda\lambda'} \frac{\delta(q-q')}{q^2} f_{\alpha i}(p, \kappa_\alpha) \tilde{t}_{\alpha i}(\kappa_\alpha); \end{aligned} \quad (29)$$

$$\langle \alpha l \lambda q | \Omega_f | \Phi \rangle = - \frac{N_{\alpha i}^{(0)}(\kappa_\alpha)}{\tilde{t}_{\alpha i}(\kappa_\alpha)} \langle \alpha L M l \lambda \kappa_\alpha q | t^s - K^s \tilde{Z}^r | \Phi \rangle, \quad (34)$$

$$\langle \alpha l \lambda q | K_f | \beta l' \lambda' q' \rangle = \frac{N_{\alpha i}^{(0)}(\kappa_\alpha)}{\tilde{t}_{\alpha i}(\kappa_\alpha)} \int_0^\infty \frac{dp' p'^2 2\mu_\beta}{p'^2 - \kappa_\beta^2} f_{\beta i'}(p', \kappa_\beta) \langle \alpha L M l \lambda \kappa_\alpha q | K^s \tilde{Z}^r | \beta L M l' \lambda' p' q' \rangle,$$

and we have used the fact that $t^r | \Phi \rangle$ vanishes since $| \Phi \rangle$ is a (physical) on-shell state. The above definitions imply that $d^{-1} \hat{K} \tilde{Z}^r$ is *real*, and hence that

$$\langle \alpha l \lambda q | \text{Im} K_f | \beta l' \lambda' q' \rangle = \frac{\pi \mu_\beta \kappa_\beta' N_{\alpha i}^{(0)}(\kappa_\alpha)}{\tilde{t}_{\alpha i}(\kappa_\alpha)} \langle \alpha L M l \lambda \kappa_\alpha q | K^s \tilde{Z}^r | \beta L M l' \lambda' \kappa_\beta' q' \rangle \quad (35)$$

for $q' \leq Q_\beta$ ($\text{Im} K_f$ vanishes otherwise). We observe that if the three-body potential vanishes identically, the on-shell matrix element of $K^s \tilde{Z}^r$ required in Ω_f and $\text{Im} K_f$ reduces to $t^s I$, and hence is determined entirely by on-shell information accessible in two-particle scattering.

Finally, we note that the relationship between the physical channel amplitude τ and X_f is given by

$$\langle \alpha L M l \lambda \kappa_\alpha q | \tau | \Phi \rangle = - \frac{\tilde{t}_{\alpha i}(\kappa_\alpha)}{N_{\alpha i}^{(0)}(\kappa_\alpha)} \langle \alpha l \lambda q | X_f | \Phi \rangle. \quad (36)$$

This is precisely the same relationship which exists between τ and the BC function X_b , defined as the solution of the equation

$$X_b = \Omega_b + K_b d X_b \quad (37)$$

discussed in BCA (and the Appendix).

i.e., the operators with the caret have no p' dependence. We further introduce the diagonal operator d such that

$$d_{i\lambda}^{\alpha L}(p, q) = \tilde{t}_{\alpha i}(\kappa_\alpha) / N_{\alpha i}^{(0)}(\kappa_\alpha). \quad (30)$$

Finally, setting $\tau_0 = -F d X_f$ in Eq. (24), we find that X_f satisfies the equation

$$X_f = -d^{-1} \hat{\Omega} + (d^{-1} \hat{K} \tilde{Z}^r G_0 F d) X_f. \quad (31)$$

Although Eq. (31) is expressed on the full $(\alpha \vec{p} \vec{q})$ basis, it is clear that the p label is superfluous. It is convenient to introduce a reduced basis $|\alpha l \lambda q\rangle$ such that

$$\langle \beta l' \lambda' q' | \alpha l \lambda q \rangle = \delta_{\alpha\beta} \delta_{ll'} \delta_{\lambda\lambda'} \frac{\delta(q-q')}{q^2}, \quad (32)$$

and to reexpress Eq. (31) as an operator equation on this basis. It is understood that the operators have an implicit dependence on the conserved quantities W , L , and M (only the driving term depends on the latter). We thus write

$$X_f = \Omega_f + K_f d X_f, \quad (33)$$

where

IV. OFF-SHELL EQUIVALENCE

In the preceding section we have demonstrated that the Faddeev equation for two- and three-body potentials can be reduced to a one-dimensional form whose solution (X_f) is simply related to the on-shell three-particle scattering amplitude. This sounds too good to be true, and of course one must recall that the kernel K_f defined in Eq. (34) can only be constructed via a knowledge of the operator \tilde{Z}^r . Except for separable two-particle potentials (and $V_3 \equiv 0$), it is necessary to solve a two-dimensional integral equation in order to determine \tilde{Z}^r . However, although computational techniques are irrelevant to our present purpose, it is worth noting that the K^r operator defined in Eq. (21) is *nonsingular*, and hence our reduction may prove useful in practical calculations.

We now consider the relationship between the

one-dimensional integral equations, Eqs. (33) and (37), which define X_f and X_b , respectively. Specifically, we recall that $\text{Im}K_b$ is uniquely determined by on-shell information, whereas $\text{Re}K_b$ is determined only up to an arbitrary *real-valued* operator A described in the Appendix. Below we demonstrate that such an A operator can always be chosen in such a way that the X_f and X_b amplitudes are identical on shell, and hence describe the same scattering observables. We also show that one cannot distinguish between off-shell two-body properties and true three-body forces on the basis of such observables.

We introduce the operator Ω^c such that

$$\begin{aligned} \langle \alpha l \lambda q | \Omega^c | \Phi \rangle &= \langle \alpha L M l \lambda \kappa_\alpha q | \hat{t}(I-1) | \Phi \rangle \\ &= \bar{t}_{\alpha l}(\kappa_\alpha) \int_0^\infty dp p^2 \langle \alpha L M l \lambda p q | I-1 | \Phi \rangle . \end{aligned} \quad (38)$$

We next demonstrate that a real operator Y_f may be defined such that

$$\Omega_f | \Phi \rangle = (1 - Y_f) d^{-1} \Omega^c | \Phi \rangle . \quad (39)$$

To do so we first consider any operator M on the full $|\alpha \vec{p} \vec{q}\rangle$ basis such that $M = \bar{M}(1 - I)$, or

$$M = \frac{1}{3} M(1 - I) \quad (40)$$

using Eq. (11). Examples of such operators include \bar{V} , \hat{V} , V^r , and $\hat{\Omega}$. Correspondingly, we define an operator M' such that

$$\begin{aligned} \langle \alpha L M l \lambda p q | M' | \beta L M' l' \lambda' p' q' \rangle \\ = \frac{\langle \alpha L M l \lambda p q | M | \beta L M' l' \lambda' \kappa'_\beta q' \rangle M_{\beta l'}(p')}{N_{\beta l'}^{(0)}(\kappa'_\beta)} , \end{aligned} \quad (41)$$

where $M_{\beta l'}(p')$ is an arbitrary function such that

$$\int_0^\infty dp p^2 M_{\beta l'}(p) = 1 . \quad (42)$$

One may then verify that

$$\begin{aligned} M | \Phi \rangle &= M' d^{-1} \hat{t} | \Phi \rangle \\ &= \frac{1}{3} M' d^{-1} \hat{t}(1 - I) | \Phi \rangle . \end{aligned} \quad (43)$$

In particular,

$$\begin{aligned} \hat{\Omega} | \Phi \rangle &= (\hat{t} - \hat{K} \bar{Z}^r) | \Phi \rangle \\ &= [1 - \frac{1}{3}(\hat{K} \bar{Z}^r - \hat{t} I)' d^{-1}] \hat{t}(1 - I) | \Phi \rangle . \end{aligned} \quad (44)$$

Recalling that $\Omega_f = -d^{-1} \hat{\Omega}$, it follows that

$$Y_f = \frac{1}{3} d^{-1} Y , \quad (45)$$

where

$$\begin{aligned} \langle \alpha l \lambda q | Y | \beta l' \lambda' q' \rangle \\ = \frac{\langle \alpha L M l \lambda \kappa_\alpha q | K^s \bar{Z}^r - t^s I | \beta L M' l' \lambda' \kappa'_\beta q' \rangle}{N_{\beta l'}^{(0)}(\kappa'_\beta)} . \end{aligned} \quad (46)$$

Since $d^{-1} K^s \bar{Z}^r$ and $d^{-1} t^s I$ are real, we have established Eq. (39). We observe that Y_f vanishes in the absence of three-body forces.

In similar fashion, one may verify that

$$\Omega_b | \Phi \rangle = (1 - Y_b) d^{-1} \Omega^c | \Phi \rangle , \quad (47)$$

where

$$\begin{aligned} Y_b &= \theta + \frac{1}{3}(1 - \theta) \bar{Y} , \\ \langle \alpha l \lambda q | \bar{Y} | \beta l' \lambda' q' \rangle \\ &= \frac{\langle \alpha L M l \lambda \kappa_\alpha q | \bar{Q} \hat{N}(I-1) | \beta L M' l' \lambda' \kappa'_\beta q' \rangle}{N_{\beta l'}^{(0)}(\kappa'_\beta)} \end{aligned} \quad (48)$$

in terms of operators θ, \bar{Q}, N defined in the Appendix. Physically, θ projects onto the interior region where the two-particle forces overlap. The development below is complicated by the fact that Y_b, Y_f are not compact, and hence we introduce the diagonal projection operator \mathcal{P} such that

$$\mathcal{P}_{i\lambda}^\alpha(q) = \theta[Q_\alpha - q] , \quad (49)$$

where $\theta[x]$ is the unit step function. Thus \mathcal{P} is unity acting on physical states such as $|\Phi\rangle$. It is sufficient for our purposes to work with

$$\begin{aligned} \bar{\Omega}_b &= (1 - Y_b \mathcal{P}) d^{-1} \Omega^c , \\ \bar{\Omega}_f &= (1 - Y_f \mathcal{P}) d^{-1} \Omega^c , \end{aligned} \quad (50)$$

which are equivalent to Ω_b, Ω_f when acting on $|\Phi\rangle$.

In order to establish the desired equivalence we now introduce an operator U such that

$$\begin{aligned} (1 - U)^{-1} (1 - Y_f \mathcal{P}) &= 1 - Y_b \mathcal{P} , \\ U &= 1 - (1 - Y_f \mathcal{P})(1 - Y_b \mathcal{P})^{-1} . \end{aligned} \quad (51)$$

The existence of U follows from the fact that $Y_b \mathcal{P}$ is a real compact operator on the $|\alpha l \lambda q\rangle$ basis. Defining V via the equation

$$K_f = U d^{-1} + V , \quad (52)$$

our two equations become

$$\begin{aligned} X_f &= \bar{\Omega}_b + (1 - U)^{-1} V d X_f , \\ X_b &= \bar{\Omega}_b + K_b d X_b , \end{aligned} \quad (53)$$

where it is understood that X_f, X_b are to act on $|\Phi\rangle$.

Inasmuch as $\text{Re}K_b$ is at our disposal, we can establish the desired relation

$$K_b = (1 - U)^{-1} V , \quad (54)$$

by proving that

$$\begin{aligned} (1 - U) \text{Im}K_b &= \text{Im}V \\ &= \text{Im}K_f + U \text{Im}d(dd^*)^{-1} \end{aligned} \quad (55)$$

since U is real. To do so we define G such that

$$\begin{aligned} \langle \alpha l \lambda q | G | \beta l' \lambda' q' \rangle \\ = \frac{\langle \alpha L M l \lambda \kappa_\alpha q | d^{-1} t^s I | \beta L M' l' \lambda' \kappa'_\beta q' \rangle}{N_{\beta l'}^{(0)}(\kappa'_\beta)} ; \end{aligned} \quad (56)$$

this definition implies that $G\mathcal{P} = \mathcal{P}G\mathcal{P}$. Comparing Eqs. (35) and (46), we find that

$$Y_f\mathcal{P} = -\frac{1}{3}[G + \text{Im}K_f d d^* (\text{Im}d)^{-1}] \mathcal{P}. \quad (57)$$

Furthermore, the formulas for K_b in the Appendix imply that

$$\text{Im}\theta K_b = -\theta \mathcal{P} \text{Im}d (d d^*)^{-1}, \quad (58)$$

$$\text{Im}(1 - \theta)K_b = -(1 - \theta)(G + 3Y_b)\mathcal{P} \text{Im}d (d d^*)^{-1}.$$

Solving Eq. (57) for $\text{Im}K_f$ and employing Eq. (58), Eq. (55) is equivalent to

$$[(1 - U)\theta + U](1 - G)\mathcal{P} = 3[Y_f - (1 - U)(Y_b - \theta)]\mathcal{P} \quad (59)$$

or

$$[Y_f - (1 - U)(Y_b - \theta)](2 + G)\mathcal{P} = 0, \quad (60)$$

using Eq. (51). However, the relations

$$\begin{aligned} Y_f G \mathcal{P} &= -2Y_f \mathcal{P}, \\ (Y_b - \theta)G \mathcal{P} &= -2(Y_b - \theta)\mathcal{P} \end{aligned} \quad (61)$$

follow from the above definitions in a fashion similar to Eq. (43). Note that

$$(1 - I)I = -2(1 - I) \quad (62)$$

as a consequence of Eq. (11). We have thus established Eq. (55), and can therefore guarantee Eq. (54) by requiring that

$$\begin{aligned} \text{Re}K_b &= (1 - U)^{-1} \text{Re}V \\ &= (1 - U)^{-1} [\text{Re}K_f - U \text{Re}d (d d^*)^{-1}]. \end{aligned} \quad (63)$$

Equivalently, in terms of the off-shell functions \hat{B}, \hat{C} ;

$$\begin{aligned} \langle \alpha \lambda q | \hat{B} | \beta \lambda' q' \rangle \\ = \frac{\langle \alpha \lambda q | (1 - \theta) [(1 - U)^{-1} \text{Re}V - \text{Re}K_b^{(0)}] | \beta \lambda' q' \rangle}{D_{\beta \lambda'}^{(0)}(\bar{\kappa}_\beta)}, \end{aligned} \quad (64)$$

$$\begin{aligned} \langle \alpha \lambda q | \hat{C} | \beta \lambda' q' \rangle \\ = \frac{\langle \alpha \lambda q | \theta [(1 - U)^{-1} \text{Re}V - \text{Re}K_b^{(0)}] | \beta \lambda' q' \rangle}{D_{\beta \lambda'}^{(0)}(\bar{\kappa}_\beta)} \end{aligned}$$

(for definitions see Appendix.) We note that $\bar{\kappa}_\alpha$ corresponds to Eq. (8) with W replaced by a negative energy parameter $\bar{W} < \min(s_\alpha^b)$. The purpose of this device is apparent in Eq. (64), in which $D_{\beta \lambda'}^{(0)}(\bar{\kappa}_\beta)$ compensates the exponential decline of V and $K_b^{(0)}$ in the limit $q' \rightarrow \infty$. The result is that \hat{B} and \hat{C} are L_2 functions decreasing at infinity according to negative powers of (q, q') . Furthermore, they are smooth functions for real (q, q') since they do not possess the κ'_β cut, and are analytic in a strip about the real axis characteristic

of the off-shell behavior. In particular, if μ is the mass of the lightest exchanged particle, \hat{B} and \hat{C} are analytic for $|\text{Im}q| < \min(\mu, |2M_\alpha \bar{W}|^{1/2})$. In practice we may choose \bar{W} such that the bound is given by μ ; note that \bar{W} is on the same footing as \hat{B}, \hat{C} in representing the off-shell structure arising from V_3 and the V_α . We also observe that in the special case $V_3 = 0$

$$(1 - \theta_b)(1 - U)^{-1} = 1 - \theta_b, \quad (65)$$

where θ_b describes a somewhat larger region of finite volume than does θ ($\theta_b \theta = \theta$). We then have

$$\begin{aligned} \langle \alpha \lambda q | (1 - \theta_b) \hat{B} | \beta \lambda' q' \rangle \\ = \frac{\langle \alpha \lambda q | (1 - \theta_b) [\text{Re}K_f - \text{Re}K_b^{(0)}] | \beta \lambda' q' \rangle}{D_{\beta \lambda'}^{(0)}(\bar{\kappa}'_\beta)}, \end{aligned} \quad (66)$$

so that the long-range part of the off-shell structure can be read off quite easily given K_f .

In the development above we have assumed that $W > 0$ and that the initial state consisted of three free particles. If $W > 0$ and the initial state consists of a two-particle bound state plus a spectator, it should be clear that the same operator U guarantees Eq. (54), and hence the equivalence of the kernels follows as before. However, for a $\beta\gamma$ bound state $q_{in} > Q_\alpha$, and hence $\mathcal{P}|\Phi\rangle = 0$. This would appear to indicate that the expressions for the driving terms must be modified, since now

$$\begin{aligned} \Omega_b |\Phi\rangle &= (1 - Y_b) d^{-1} \Omega^c |\Phi\rangle, \\ (1 - U)^{-1} \Omega_f |\Phi\rangle &= d^{-1} \Omega^c |\Phi\rangle. \end{aligned} \quad (67)$$

However, the prescription for calculating the physical amplitude (elastic bound state scattering or breakup) is to pick off the residue of the pole arising from $\tilde{t}_{\alpha i}(\kappa_\alpha)$ acting on $|\Phi\rangle$. Such terms arise upon iterating the X_f (or X_b) equation, since the factor d in the kernel can act on the $\delta(q - q_{in})/q^2$ term in Ω_f (or Ω_b). Thus, *effectively* one has

$$\begin{aligned} \langle \beta \lambda' q' | \Omega_b |\Phi\rangle &= \langle \beta \lambda' q' | (1 - U)^{-1} \Omega_f |\Phi\rangle \\ &= \langle \beta \lambda' q' | d^{-1} \Omega^c |\Phi\rangle \\ &= -\delta_{\alpha\beta} \delta_{\lambda\lambda'} \delta_{\lambda\lambda'} \frac{\delta(q' - q_{in})}{q'^2} N_{\alpha i}^{(0)}(\kappa_\alpha^{in}), \end{aligned} \quad (68)$$

assuming a $\beta\gamma$ bound state of angular momentum l , with a spectator of angular momentum λ ($\kappa_\alpha^{in} = i|2\mu_\alpha s_\alpha^b|^{1/2}$ for the ground state). This is also true for bound state scattering below the threshold for breakup [$\min(s_\alpha^b) < W < 0$], in which case $\mathcal{P}, U, \text{Im}K_b, \text{Im}K_f$ all vanish and Eq. (55) is trivially satisfied. Finally, these operators also vanish in calculations of the three-particle binding energy [$W < \min(s_\alpha^b)$]; in this case the equations are homogeneous ($\Omega_b = \Omega_f = 0$).

We have thus demonstrated that functions \hat{B}, \hat{C} [summarized by $A \equiv (1 - \theta)\hat{B} + \theta\hat{C}$] can always be constructed with the properties assumed in previous work providing one specifies two- and three-particle potentials of the usual class. We now consider whether it is possible to distinguish between off-shell two-particle properties and three-body potentials on the basis of three-particle scattering data. Suppose, for example, that it were possible to determine both Ω_f and K_f from the data using Eq. (33) to construct X_f . In the absence of three-body forces Ω_f and $\text{Im}K_f$ are given by on-shell matrix elements of $t^s(1 - I)$ and t^sI , respectively, and hence are uniquely determined by on-shell information. One could thus check whether the empirically determined values of Ω_f and $\text{Im}K_f$ agreed with two-particle data, and thereby prove the absence (or detect the presence) of a three-body force. Similarly, if one could reconstruct the Faddeev kernel [e.g., in Eq. (15)], one could distinguish the two terms in

$$K = -(1 - tG_0)\bar{V}G_0 + tIG_0 \quad (69)$$

from the fact that the $\alpha = \beta$ matrix elements of tIG_0 vanish (whereas those of \bar{V} do not).

However, the development above demonstrates that this is *not* in fact possible. Suppose that the data were adequate to completely specify \hat{B} and \hat{C} in the BC representation (experience with the three-nucleon system indicates that this assumption is incredibly optimistic). Knowing K_b , one could construct K_f from the relation

$$K_f = Ud^{-1} + (1 - U)K_b, \quad (70)$$

in which U is any real-valued L_2 operator. For a given U one could construct a corresponding Y_f via Eq. (51), and hence achieve an interpretation of U in terms of a particular combination of two- and three-particle forces [only certain matrix elements of the potentials are required in Eq. (46); the remaining degrees of freedom can be determined by Eq. (63)]. In particular, if one assumes *a priori* that $V_3 = 0$ and chooses U accordingly, an effective off-shell behavior can be deduced in accord with the data. We therefore conclude that a complete knowledge of three-particle scattering data is not sufficient, even in principle, to distinguish true three-body forces from the off-shell characteristics of the two-particle interactions.

V. DISCUSSION

Originally, the BCF was derived as a general solution of the three-body unitarity relation consistent with specified two-particle phase shifts. In the present work we have demonstrated the explicit connection between this representation of

the off-shell degrees of freedom (including three-body forces) and the conventional representation in terms of two- and three-particle potentials. Specifically, we have shown that for each set of such potentials, there exists a real-valued L_2 operator A such that the BCF generates precisely the same scattering observables as does the Faddeev equation. The same is true regarding three-particle binding energies, which appear as poles of the operator $(1 - K_b d)^{-1}$. Thus the BCF reproduces the primary singularity structure of the *on-shell* three-particle scattering matrix T (right-hand cut plus bound state poles), but not its left-hand cuts (except for the very important contribution arising from the exchange of a physical particle). Correspondingly, the BCF does not yield the correct half- or fully off-shell values of T , nor the bound state wave function. It is specifically designed for scattering calculations, which are most difficult in terms of the Faddeev formalism, and for which it possesses unique advantages.

Some of these advantages are apparent in the analysis of $n-d$ elastic scattering and breakup at 14.4 MeV.⁷ By taking advantage of the smooth behavior of A as a function of the integration variables (q', q) , and expanding A in a complete set [i.e., as an operator of finite rank; $A = \sum_k a_k |g_k\rangle\langle g_k|$ for a set $g_k^{\alpha\lambda}(q) = \langle \alpha | \lambda q | g_k \rangle$], it is trivial to reduce the BCF equation to a set of *algebraic* equations with coefficients determined by the minimal equation ($A \equiv 0$). This form is highly efficient for generating three-particle amplitudes corresponding to all possible values of A (arbitrary a_k); we have shown above that this must necessarily include all possible combinations of two- and three-particle potentials. In this way it was easy to demonstrate that the scattering observables at 14.4 MeV are sensitive only to a single off-shell parameter, the over-all scale of A , which could be normalized by fixing the value of the $n-d$ doublet scattering length a_2 . Thus, taking $A = \lambda_0 \bar{A}$, choosing an *arbitrary* operator \bar{A} , and varying λ_0 from zero to a value which gave $a_2 = 0.41$ fm (chosen to best represent low energy $n-d$ scattering), the differential cross sections were found to be independent of \bar{A} to the level of a few percent. The conclusion was thus that no off-shell information can be extracted from such data which is not already implicit in the value of a_2 .

Underlying this result are a number of specific assumptions concerning A . These are based partly on empirical experience with the trinucleon system, and partly on theoretical estimates linked to potential theory. In particular, one expects A (or the a_k) to be a slowly varying function of W ; this is clearly necessary if λ_0 is to be fixed at the $n-d$ threshold and employed at 14.4 MeV (or high-

er). Inasmuch as the dominant part of the (s -wave) nuclear force has a range ≤ 1 fm, one can infer quite generally that the appropriate scale is essentially $(M_n W)^{1/2} / [\frac{1}{2}(3m_n)]$, and hence that A is approximately constant for $T_L < 70$ MeV. This estimate is supported by numerical studies of specific models. In particular, these considerations rule out exotic energy dependence such as employed by Haftel and Petersen⁸; this is equivalent to specifying that T has conventional analyticity properties as a function of W .⁹

Secondly, since the a_2 parameter involves only the $L=0$ state (for s -wave N - N forces), our normalization procedure would not constrain the $L \geq 1$ contributions (which dominate the cross sections even at 14.4 MeV) if A were completely general. Here again one must appeal to potential theory, in which the various L states arise as the angular momentum projections of a function of the vectors \vec{p}, \vec{q} . For example, in the simple case of separable interactions the off-shell dependence is given in terms of form factors $g_\alpha(\vec{p}), g_\beta(\vec{p}')$, where \vec{p}, \vec{p}' are linear combinations of the integration variables \vec{q}, \vec{q}' . Thus, if A is to represent a plausible interaction, the operators $A^{(L)}$ are correlated and can be represented as angular momentum projections of a smooth function of q, q' and $\hat{q} \cdot \hat{q}'$. This was assumed in constructing the A set employed in the n - d analysis.

It should also be evident that for a given \bar{A} , the value of λ_0 which corresponds to a fixed a_2 is not unique. This is a simple consequence of the fact that if A is taken sufficiently large it will dominate K_L , and hence one can obtain almost arbitrary results (at least for $W < 0$). It was therefore assumed that off-shell corrections are a relatively small effect, and hence that the smallest value of λ_0 is the only plausible one. This is in accord with empirical experience regarding this particular three-body system, and may be inferred from the undramatic off-shell variations noted in the Faddeev calculations. Also, the next smallest value of λ_0 was typically an order of magnitude larger than the minimal value, and the sensitivity to this parameter was such that A was clearly dominating the calculation. Given these basic ground rules, A was allowed to vary widely, and even quite implausible shapes were ineffective in altering the calculated cross sections.

The assumptions noted may be summarized by a simple basic rule: *The search for off-shell sensitivity must be subject to reasonable theoretical guidelines.* The latter can be quite general within certain specified limits, but those limits must be applied if the results are to be meaningful. Thus, it is quite evident from the above discussion that the full generality explicated in the

BCF can be used to produce almost arbitrary "sensitivities" in predicted three-particle observables. However, these effects are irrelevant if they do not correspond to plausible interaction mechanisms. Thus, the three-body problem does not exist in isolation, and one must interpret three-particle data in terms of what is known about the N - N interaction, heavier nuclei, and the general postulates of nonrelativistic quantum mechanics. Otherwise, one tends to engage in mathematical games which have little bearing on the gaps in our basic understanding. It is certainly unwise (particularly in an era of limited resources) to justify present or proposed experiments on the basis of such "sensitivity."

An illustrative example is provided by the case of off-shell effects vs three-body forces. As noted above, one can define various combinations of two- and three-body potentials which generate identical scattering observables. Thus, although there are clearcut technical distinctions, one cannot distinguish them on the basis of scattering experiments. The only meaningful question one can pose is whether two-particle forces within a certain acceptable class can alone account for three-particle data. The definition of such an "acceptable class" must clearly be based on one's present theoretical understanding (and modified if found to be inadequate). As this author has noted previously, it is unnecessary to perform massive Faddeev calculations in order to answer this question.⁷ It will suffice to calculate E_T, a_2 (and perhaps a few additional parameters) to define an equivalent A operator; all consequences of the given model can then be quickly explored. In passing, it should be noted that the emphasis placed on *scattering* observables is related to the inability of the BCF to determine the actual wave function. As this author has previously pointed out, the electromagnetic properties of the triton may be used to deduce the presence of an effective three-nucleon force if mesonic corrections can be neglected (or estimated).¹⁸ Recent experimental results on the deuteron form factor appear to indicate that such corrections are far less significant than had been supposed,¹⁹ and an appropriate generalization of the previous technique could conceivably yield a definitive result.

Above we have demonstrated that for a *given* set of two-particle phase shifts, any and all off-shell variations can be realized within the context of the BCF. In conclusion we consider the possible model dependence of the n - d result as a consequence of the *particular* phase shifts employed. Thus, since the object of the analysis was to study off-shell dependence, it was argued that simple s -wave phases generated by a constant boundary

condition were adequate for the purpose. This choice was purely a matter of convenience given the computer program then available. However, Haftel and Petersen have argued that these phases uniquely determine an off-shell t matrix, and hence that the variations considered must be interpreted as due to three-body forces alone.⁹ This argument would be valid except for two major points. The first concerns the alleged uniqueness of the t matrix, which is apparently based on earlier work by this author.²⁰ However, that work uniquely linked properties of the *wave function* to the t matrix; the argument was not based on the phase shift. In fact, it is well known that one can always apply unitary transformations which leave the phase shift (but not the wave function) invariant.²¹ Secondly, it is clear from the presentation given in this paper that the results of a given calculation are limited only by the values of the on-shell t matrix in the domain $s_\alpha^b \leq s_\alpha \leq W$. The extension of t^{BC} to energies $s_\alpha < s_\alpha^b$ is quite arbitrary since it can be compensated by A ; and the use of a meromorphic logarithmic derivative (with a constant limit at infinity) is merely a computational device.²² For example, it is clear that T cannot depend on the parameter s_α^0 introduced in Sec. III, nor the particular values for $t_{\alpha l}$ employed for $s_\alpha < s_\alpha^0$. It is conceivable that the type of energy-dependent parametrization suggested previously for $t_{\alpha l}^{\text{BC}}$ might not yield $|t_{\alpha l} - t_{\alpha l}^{\text{BC}}| < \epsilon$ for arbitrary ϵ in the important region (s_α^b, W) , although this is unlikely to be important numerically. However, one could instead simply use $t_{\alpha l}$ itself in that region and match $t_{\alpha l}^{\text{BC}}$ smoothly at $s_\alpha = s_\alpha^b$. This is purely academic insofar as the 14.4 MeV analysis is concerned, since the domain (s_α^b, W) falls into the effective range region where the simple phases agree with any model.

It should therefore be clear that there are vast differences between the generalized BCF and the boundary condition model popularized by Feshbach and Lomon.²³ It is unfortunate that both the title and historical development of the approach have caused the two to be confused. Hopefully, the present article will serve to make clear the distinction. Although the trinucleon results are generally regarded as disappointing, there is nevertheless valuable information to be learned in that system. The very absence of off-shell sensitivity in the low energy region should make it possible to pin down hard to measure N - N properties such as the n - n effective range parameters, the 3P_1 phase, and the ϵ_1 mixing parameter.²⁴ Furthermore, this absence of sensitivity may be understood from the fact that momenta $q > 1 \text{ fm}^{-1}$ are needed to probe the region where A has struc-

ture, whereas the momenta which are numerically important correspond to κ_α physical, or $q \leq Q_\alpha$. This implies that for $T_L > 45 \text{ MeV}$ cross sections *could* begin to exhibit some sensitivity to the off-shell properties. This is not an easy regime to handle theoretically, since many angular momentum components of the N - N interaction will contribute. For this reason Faddeev-type calculations with other than trivial forces will not be feasible for this purpose in the near future. However, the simplified structure of the BCF makes it an ideal tool for probing this region.

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APPENDIX: BOUNDARY CONDITION FORMALISM

The on-shell properties of the $\beta\gamma$ subsystem are specified by a boundary condition at a relative displacement $x = a_\alpha$,

$$\lim_{\epsilon \rightarrow 0} \psi_{\alpha l}^{\text{ext}'}(a_\alpha + \epsilon) / \psi_{\alpha l}^{\text{ext}}(a_\alpha + \epsilon) = \lambda_{\alpha l}(\kappa_\alpha^2) = \lambda_{\alpha l}^{(0)} + \Delta_{\alpha l}(\kappa_\alpha^2), \quad (\text{A1})$$

applied to the asymptotic form of the two-particle (partial) wave function $\psi_{\alpha l}^{\text{ext}}$. This results in a specific representation for the t matrix;

$$\begin{aligned} t_{\alpha l}^{\text{BC}}(s_\alpha) &= N_{\alpha l}(\kappa_\alpha) / D_{\alpha l}(\kappa_\alpha), \\ N_{\alpha l}(\kappa_\alpha) &= (a_\alpha \lambda_{\alpha l} - l) j_l(a_\alpha \kappa_\alpha) + a_\alpha \kappa_\alpha j_{l+1}(a_\alpha \kappa_\alpha), \\ D_{\alpha l}(\kappa_\alpha) &= i\pi \mu_\alpha \kappa_\alpha [(a_\alpha \lambda_{\alpha l} - l) h_l(a_\alpha \kappa_\alpha) \\ &\quad + a_\alpha \kappa_\alpha h_{l+1}(a_\alpha \kappa_\alpha)]. \end{aligned} \quad (\text{A2})$$

In previous work it was assumed that a_α can be so chosen that $\lambda_{\alpha l}(\kappa_\alpha^2)$ is a meromorphic function of κ_α^2 approaching a constant $\lambda_{\alpha l}^{(0)}$ as $\kappa_\alpha^2 \rightarrow \infty$. This provides an essentially unique analytic continuation from the physical region ($\kappa_\alpha^2 > 0$) where $\lambda_{\alpha l}(\kappa_\alpha^2)$ is completely determined by the phase shift ($\delta_{\alpha l}$), to the domain $-\infty < \kappa_\alpha^2 < 0$ required by the Faddeev (or BCF) equations. This assumption is in accord with empirical experience concerning the N - N system²³ (and is apparently true for hadron-hadron scattering in general, insofar as the phases are known²⁵). However, we are concerned here with a purely mathematical statement regarding off-shell equivalence, and in general the phases will not be in accord with this assumption. Nevertheless, one can simply modify the previous prescription by taking $\lambda_{\alpha l}(\kappa_\alpha^2)$ directly from the model in the physical region (and such that the proper residues are generated at any bound state poles), and using the meromorphic form to the left of some matching energy $\bar{s}_\alpha < s_\alpha^b$. The value

of $t_{\alpha i}^{\text{BC}}$ will then be identical with that of the model in the accessible physical domain $s_{\alpha}^b \leq s_{\alpha} \leq W$, and hence $\tilde{t}_{\alpha i}(s_{\alpha}) = t_{\alpha i}^{\text{BC}}(s_{\alpha})$. Below and in the text we use $N_{\alpha i}^{(0)}$ and $D_{\alpha i}^{(0)}$ to denote $N_{\alpha i}$ and $D_{\alpha i}$ with $\lambda_{\alpha i}$ replaced by its asymptotic value $\lambda_{\alpha i}^{(0)}$.

The boundary conditions are applied to the three-particle wave function in the *exterior region*, defined by the requirement that each pair of particles $\beta\gamma$ is separated by a distance $x > a_{\alpha}$. The displacements \tilde{x}, \tilde{y} are taken (for a given α) to be conjugate to the momenta \tilde{p}, \tilde{q} , and hence the basis in the coordinate representation is $|\alpha\tilde{x}\tilde{y}\rangle$. A projection operator \mathcal{P}_{θ} on the exterior region is most simply defined in this representation

$$\langle \alpha\tilde{x}\tilde{y} | \mathcal{P}_{\theta} | \beta\tilde{x}'\tilde{y}' \rangle = \delta_{\alpha\beta} (\tilde{x} - \tilde{x}') \delta(\tilde{y} - \tilde{y}') \mathcal{P}_{\theta}^{(\alpha)}(\tilde{x}, \tilde{y}), \quad (\text{A3})$$

$$\mathcal{P}_{\theta}^{(\alpha)}(\tilde{x}, \tilde{y}) = \theta(x - a_{\alpha}) \theta(x_{\beta} - a_{\beta}) \theta(x_{\gamma} - a_{\gamma}),$$

where x_{β}, x_{γ} are appropriate linear combinations of \tilde{x} and \tilde{y} . Application of the boundary condition sets $x = a_{\alpha}$, bringing in $\mathcal{P}_{\theta}^{(\alpha)}(\tilde{x}, \tilde{y})$ as an explicit factor. For each channel α there exist displacements $y = y_{\alpha}^0$ and $y = b_{\alpha}$ such that

$$\begin{aligned} \mathcal{P}_{\theta}^{(\alpha)}(\tilde{x}, \tilde{y}) &= 1, & y > b_{\alpha} \\ &= 0, & y < y_{\alpha}^0. \end{aligned} \quad (\text{A4})$$

It is convenient to introduce operators θ and θ_b , which correspond to the step functions $\theta[y_{\alpha}^0 - y]$ and $\theta[b_{\alpha} - y]$, respectively, in the coordinate representation.

In practice, the short-range character of the interaction effectively restricts one to a finite set of l values in each channel ($l \leq l_{\alpha}^{\text{max}}$). For a given L , the equations are thus written on a truncated space corresponding to a finite range for l, λ . Defining an operator Q such that

$$\begin{aligned} \langle \alpha l \lambda q | K_b - K_b^{(0)} | \beta l' \lambda' q' \rangle &= \langle \alpha l \lambda q | (1 - \theta) \hat{B} + \theta \hat{C} | \beta l' \lambda' q' \rangle D_{\beta l'}^{(0)}(\bar{\kappa}'_b), \\ \langle \alpha l \lambda q | (1 - \theta) K_b^{(0)} | \beta l' \lambda' q' \rangle &= \int_0^{\infty} dp' p'^2 \langle \alpha L M l \lambda \kappa_{\alpha} q | (1 - \theta) \bar{Q} (N \mathcal{P}_{\theta} I - \hat{N}) G_0 g | \beta L M l' \lambda' p' q' \rangle, \\ \langle \alpha l \lambda q | \theta K_b^{(0)} | \beta l' \lambda' q' \rangle &= \langle \alpha l \lambda q | \theta | \beta l' \lambda' q' \rangle \left[N_{\beta l'}^{(0)}(\kappa'_b) \frac{D_{\beta l'}(\kappa'_b)}{N_{\beta l'}(\kappa'_b)} - D_{\beta l'}^{(0)}(\bar{\kappa}'_b) \right]. \end{aligned} \quad (\text{A10})$$

Here

$$\begin{aligned} \langle \alpha l \lambda q | \theta | \beta l' \lambda' q' \rangle &= \delta_{\alpha\beta} \delta_{l'l} \delta_{\lambda\lambda'} \theta_{\lambda}(q, q'; y_{\alpha}^0), \\ \theta_{\lambda}(q, q'; r) &= \frac{2r^2}{\pi} \left[\frac{q j_{\lambda+1}(rq) j_{\lambda}(rq') - q' j_{\lambda+1}(rq') j_{\lambda}(rq)}{q^2 - q'^2} \right], \end{aligned} \quad (\text{A11})$$

and $\bar{\kappa}'_b$ corresponds to Eq. (8) with W replaced by a negative energy parameter $\bar{W} < \min(s_{\alpha}^b)$. The purpose of this device is to compensate the exponential growth of d in the limit $q' \rightarrow \infty$ by the explicit factor $D_{\beta l'}^{(0)}(\bar{\kappa}'_b)$. With this convention the functions \hat{B}, \hat{C} are arbitrary real-valued L_2

$$\langle \alpha l \lambda y | Q | \beta l' \lambda' y' \rangle = \delta_{\alpha\beta} \frac{\delta(y - y')}{y^2} Q_{i\lambda i'\lambda'}^{\alpha\beta}(y), \quad (\text{A5})$$

$$Q_{i\lambda i'\lambda'}^{\alpha\beta}(y) = i^{l'+\lambda'-l-\lambda} P_{i\lambda i'\lambda'}^{\text{ext}}(a_{\alpha}, y),$$

where $P_{i\lambda i'\lambda'}^{\text{ext}}(a_{\alpha}, y)$ is the partial-wave projection of $\mathcal{P}_{\theta}^{(\alpha)}(\tilde{x}, \tilde{y})$, an inverse \bar{Q} may be defined on the *truncated space* such that

$$\begin{aligned} (1 - \theta) Q \bar{Q} &= (1 - \theta) \bar{Q} Q \\ &= 1 - \theta. \end{aligned} \quad (\text{A6})$$

As a consequence of Eq. (A4), it follows that

$$\begin{aligned} (1 - \theta_b) Q &= (1 - \theta_b) \bar{Q} = 1 - \theta_b, \\ \theta Q &= 0. \end{aligned} \quad (\text{A7})$$

The BCF equation can be stated in operator form by defining N, g such that

$$\begin{aligned} \langle \alpha \tilde{p} \tilde{q} | N | \beta \tilde{p}' \tilde{q}' \rangle &= \delta_{\alpha\beta} \delta(\tilde{q} - \tilde{q}') \\ &\times \sum_l \left(\frac{2l+1}{4\pi} \right) P_l(\hat{p} \cdot \hat{p}') N_{\alpha i}^{(0)}(p'), \\ \langle \alpha \tilde{p} \tilde{q} | g | \beta \tilde{p}' \tilde{q}' \rangle &= \delta_{\alpha\beta} \delta(\tilde{q} - \tilde{q}') \frac{\delta(p - p')}{p^2} \\ &\times \sum_l \left(\frac{2l+1}{4\pi} \right) P_l(\hat{p} \cdot \hat{p}') \frac{N_{\alpha i}^{(0)}(p)}{N_{\alpha i}^{(0)}(\kappa_{\alpha})}. \end{aligned} \quad (\text{A8})$$

The operator \hat{N} is then defined via the relation

$$N \mathcal{P}_{\theta} = (1 - \theta) (Q N + \hat{N}), \quad (\text{A9})$$

and corresponds to the function $\hat{N}^{\alpha\alpha; L}$ given in Eq. (41) of BCA. We note that $(1 - \theta)\hat{N}$ is nonzero only in the finite domain $y_{\alpha}^0 < y < b_{\alpha}$. With this notation the equations derived in BCA imply that

operators on the $|\alpha l \lambda q\rangle$ basis. For a given set of potentials they can be constructed via Eq. (64). The appearance of $\bar{\kappa}'_b$ in the expression for $\theta K_b^{(0)}$ is somewhat arbitrary; what has been done is to insure the correct expression for $\text{Im} \theta K_b^{(0)}$ [Eq. (58)] by choosing the simplest form which is

analytic and L_2 [note that the bracket in Eq. (A10) tends to zero faster than the exponential since $N_{\beta 1'} - N_{\beta 1'}^{(0)}$ and $D_{\beta 1'}(\kappa_{\beta}') - D_{\beta 1'}^{(0)}(\bar{\kappa}_{\beta}') as $q' \rightarrow \infty$]. The driving term is given by$

$$\langle \alpha l \lambda q | \Omega_b = \langle \alpha L M l \lambda \kappa_{\alpha} q | \bar{Q} N \Phi_e (I - 1) . \quad (A12)$$

In conclusion, we observe that although the equations stated above are designed to approach a specific limit (singular cores) as $\hat{B}, \hat{C} \rightarrow 0$, they possess a disadvantage for data analysis in that the operators become rather complicated in the

near-overlap region $y_{\alpha}^0 < y < b_{\alpha}$. Phenomenologically, there is no reason to insist on this picture, and one can greatly simplify the numerics by replacing θ by θ_b in Eq. (A10).²⁶ Noting that

$$(1 - \theta_b) \bar{Q} (N \Phi_e I - \hat{N}) = (1 - \theta_b) N I , \quad (A13)$$

the corresponding integral for $(1 - \theta_b) K_b^{(0)}$ can be done analytically, and is essentially given by Eq. (37) of BCA. One can easily verify that the equivalence theorem of Sec. IV goes through as before (e.g., Y_b becomes θ_b , etc.).

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¹This is well illustrated by the discussion of the nucleon-nucleon interaction given in M. J. Moravcsik and H. P. Noyes, *Annu. Rev. Nucl. Sci.* **11**, 95 (1961).

²See, for example, *Few-Particle Problems in the Nuclear Interaction*, edited by I. Slaus, S. Moszkowski, R. Haddock, and W. T. H. van Oers (North-Holland, Amsterdam, 1972), and earlier references given therein.

³L. M. Delves and A. C. Phillips, *Rev. Mod. Phys.* **41**, 497 (1969).

⁴An objective reading of the proceedings of the 1972 UCLA Conference (*Few-Particle Problems in the Nuclear Interaction*, Ref. 2) provides ample support for these assertions.

⁵R. Aaron and R. D. Amado, *Phys. Rev.* **150**, 857 (1966); R. T. Cahill and I. H. Sloan, *Nucl. Phys.* **A165**, 161 (1971).

⁶D. D. Brayshaw, *Phys. Rev. D* **8**, 952 (1973); hereafter we shall refer to this as BCA (boundary condition approach). An expanded discussion of this approach is contained in a recent article dealing with its relativistic generalization; see D. D. Brayshaw, *ibid.* **11**, 2583 (1975). Below we shall denote this by BCR (boundary condition relativistic).

⁷D. D. Brayshaw, *Phys. Rev. Lett.* **32**, 382 (1974). More detail is given in an invited talk presented at the Laval Conference; see *Few Body Problems in Nuclear and Particle Physics*, edited by R. J. Slobodrian *et al.* (Les Presses de l'Universite Laval, Quebec, 1975).

⁸M. I. Haftel and E. L. Petersen, *Phys. Rev. Lett.* **33**, 1229 (1974).

⁹M. I. Haftel and E. L. Petersen, *Phys. Rev. Lett.* **34**, 1480 (1975); this is a reply to D. D. Brayshaw *ibid.* **34**, 1478 (1975).

¹⁰W. M. Kloet and J. A. Tjon, *Nucl. Phys.* **A210**, 380 (1973).

¹¹H. P. Noyes, *Phys. Rev. Lett.* **23**, 1201 (1969).

¹²For an earlier theoretical treatment see K. L. Kowalski *Phys. Rev. D* **7**, 1806 (1973). However, this has apparently not been applied to actual calculations.

¹³The treatment below is somewhat different from that given in Ref. 12.

¹⁴Singular core models are an obvious exception, but these are clearly of the class considered in the BCF. See D. D. Brayshaw, *Phys. Rev. D* **7**, 1835 (1973).

¹⁵For more detail see BCR (Ref. 6).

¹⁶K. L. Kowalski, *Phys. Rev. Lett.* **15**, 798 (1965).

¹⁷H. P. Noyes, *Phys. Rev. Lett.* **15**, 538 (1965).

¹⁸D. D. Brayshaw, *Phys. Rev. C* **7**, 1731 (1973).

¹⁹R. G. Arnold *et al.*, *Phys. Rev. Lett.* **35**, 776 (1975).

²⁰D. D. Brayshaw, *Phys. Rev. C* **3**, 35 (1971).

²¹M. I. Haftel, *Phys. Rev. C* **7**, 80 (1973); M. I. Haftel and F. Tabakin, *ibid.* **3**, 921 (1971).

²²Other than simplicity, the virtue of this choice is that it preserves the analyticity properties of T (see discussion in Ref. 9); no singularities need be canceled by A .

²³H. Feshbach and E. Lomon, *Phys. Rev.* **102**, 891 (1956).

Such an approach was originally suggested by G. Breit and W. G. Bouricius, *Phys. Rev.* **75**, 1029 (1949).

²⁴See F. N. Rad *et al.*, *Phys. Rev. Lett.* **33**, 1227, 1579 (1974).

²⁵See, for example, W. W. S. Au and E. L. Lomon, *Phys. Lett.* **4**, 327 (1963); and other references cited therein.

²⁶Equations for this simplified procedure are derived in the Appendix of BCR (Ref. 6).