Relativistic effects on the wave function of a moving system*

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An alternative approach to the recent results of Krajcik and Foldy is presented which stresses a different point of view, but is completely equivalent to their results.

[NUCLEAR REACTIONS Relativistic corrections to wave functions.]

Recently, Krajcik and Foldy[1] (KF) published an extremely elegant and complete treatment of the center-of-mass (c.m.) variables for an interacting, relativistic, many-body system. The center-of-mass variables are so defined that the infinitesimal generators of the Poincaré group (the momentum, angular momentum, Hamiltonian, and boost operators), expressed initially in terms of the canonical variables of the individual particles, take the form they would have for a single particle when expressed in terms of the c.m. variables. Nonrelativistically this is accomplished in the usual way by defining the center-of-mass coordinate $\vec{R}$, total momentum $\vec{P}$, and *internal* coordinates and momenta for the ith particle in terms of the canonical coordinates and momenta $\vec{r}_i$ and $\vec{p}_i$, spin $\vec{s}_i$ and mass m_i:

$$\vec{r}_i' = \vec{r}_i - \vec{R}, \quad \vec{R} = \sum_{i=1}^{A} m_i \vec{r}_i / m_t, \quad m_t = \sum_{i=1}^{A} m_i \ , \tag{1a}$$

$$\vec{\Pi}_i = \vec{p}_i - \vec{P} m_i / m_t, \quad \vec{P} = \sum_{i=1}^{A} \vec{p}_i \ , \tag{1b}$$

where the total mass of the system of A particles is m_t and neglects the binding energy. The internal variables $\vec{r}_i'$ and $\vec{\Pi}_i$ are merely a convenient choice. In terms of the variables of Eq. (1), the nonrelativistic wave function of a moving many-body system takes a particularly simple form: it is the product of an internal wave function which describes the complexity of the system's structure and a plane wave which describes the over-all motion. An identical simplification results in the relativistic case when appropriate center-of-mass variables are used.[1,2]

The purpose of this note is to present an abridged alternative description of the results of KF which contains no new results, but a different point of view. We stress the latter point, since the results presented herein are precisely equivalent to the more elegant treatment by KF and certain features were modeled after their basic approach. Because the use of nonrelativistic c.m. variables is an ingrained feature of nuclear physics, the use of other c.m. variables is unfamiliar and it tends to obscure those features of Lorentz kinematics which are familiar. Classically we expect the Lorentz contraction to modify the shape of a moving object and the Thomas precession to affect the angular momentum. Clocks placed on individual nucleons may be synchronized in the rest frame, but will not in general be synchronized in any other frame. We expect these phenomena to affect the wave function of a moving quantum mechanical system as well, and our treatment will emphasize this aspect of the problem, which is bypassed in the more formal and elegant approach of KF. In addition, the alternative point of view we develop permits a direct comparison of the results of KF with the earlier treatment of relativistic corrections for the deuteron by Gross[3,4] using the Bethe-Salpeter equation. In this regard the approach is similar to treatment of Brodsky and Primack[5] and in particular, Grotch and Kashuba,[6] although their results were not complete to order (v/c).[2]

We restrict ourselves to relativistic correction terms of order $(v/c)^2$ beyond the nonrelativistic limit and because momentum divided by mass is dimensionally the same in our units as the ratio of velocity to the speed of light, we regard (v/c) as the equivalent of $(1/m)$, where m is any mass. This does not imply that the terms of order $(1/c)^n$ in KF necessarily correspond to $(1/m)^n$ in this work, but merely that their expansion is equivalent to ours. The nucleus is a weakly bound system and the kinetic and potential energies almost cancel, so we treat the potential as order $(1/m)$.

The three components of the total momentum operator P_i, the angular momentum operator J_i, the boost operator K_i, and the Hamiltonian H,

satisfy the Lie algebra of the Poincaré group:

$$[P_i, P_j] = [P_i, H] = [J_i, H] = 0 \ , \tag{2a}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad [J_i, P_j] = i\epsilon_{ijk} P_k \quad [J_i, K_j] = i\epsilon_{ijk} K_k \ , \tag{2b}$$

$$[K_i, P_j] = iH\delta_{ij} \quad [K_i, K_j] = -i\epsilon_{ijk} J_k \quad [K_i, H] = iP_i \ . \tag{2c}$$

As shown by KF, not all these relations are independent. In the nonrelativistic limit, the Lie algebra of the Galilean group is given by Eqs. (2a) and (2b) and the last of Eq. (2c), while the Hamiltonian in the first relation of Eq. (2c) should be replaced by m_t, and the right-hand side of the second relation should be replaced by zero. This limit is discussed in detail by KF and by Foldy.[7] For a noninteracting system, the wave function of the whole system is simply the direct product of the wave functions of the individual particles and the Poincaré group generators will be the sum of the generators of the individual particles. In the presence of an interaction between the constituents this is no longer true and we *choose* a representation where the total momentum and angular momentum are unchanged, while a potential U and interaction boost $\vec{V}$ (which vanishes if U vanishes) must be added to H and $\vec{K}$ in order to satisfy the commutation relations in Eq. (2). As shown by Foldy,[7] the interaction boost may be eliminated in the nonrelativistic limit by a choice of representation. Using the variables in Eq. (1) and the time t and including $(v/c)^2$ corrections we obtain from KF,

$$\vec{P} = \sum_i \vec{p}_i, \quad \vec{J} = \vec{S} + \vec{R} \times \vec{P}, \quad \vec{S} = \sum_i (\vec{r}_i' \times \vec{\Pi}_i) + \vec{s}_i \ , \tag{3a}$$

$$H = m_t + H_0 + \Delta H, \quad H_0 = \sum_i \vec{p}_i^{\,2}/2m_i + U_0, \tag{3b}$$

$$\Delta H = -\sum_i \vec{p}_i^{\,4}/8m_i^{\,3} + U_1 \ ,$$

$$\vec{K} = \vec{K}_0 + \Delta\vec{K}, \quad K_0 = m_t\vec{R} - t\vec{P}, \tag{3c}$$

$$\Delta\vec{K} = \sum_i (\{\tfrac{1}{2}\vec{p}_i^{\,2}, \vec{r}_i\} - \vec{s}_i \times \vec{p}_i)/2m_i + \vec{V} \ .$$

The relativistic correction to the nonrelativistic Hamiltonian H_0 (including potential U_0) is ΔH, which is of order $(1/m)^3$, and the relativistic correction to the nonrelativistic boost, $\vec{K}_0$, is $\Delta\vec{K}$. The usual canonical commutation relations of $\vec{r}_i$, $\vec{p}_i$, and $\vec{s}_i$ are sufficient to show that the quantities defined in Eq. (3) satisfy the commutation relations in Eq. (2) to the appropriate order in (v/c) in the absence of any interaction. These commutation relations and Eq. (2a) prove that the potential must be a translationally invariant scalar function and the last of Eq. (2b) demonstrates that $\vec{V}$ is a vector function under rotations. To order $(1/m)$ the first of Eq. (2c) yields

$$[V_i, P_j] = i\delta_{ij} U_0 \ , \tag{4a}$$

and anticipating the result $[\vec{R}, U_0] = 0$, Eq. (4a) allows us to write

$$\vec{V} = \vec{R}U_0 + \vec{w} \ , \tag{5a}$$

with $\vec{w}$ a translationally invariant function. To order $(1/m)^0$ the second of Eqs. (2c) yields the condition that the curl of $\vec{w}$ with respect to $\vec{P}$ vanishes:

$$\nabla_{\vec{P}} \times \vec{w} = 0 \ . \tag{5b}$$

The remaining commutation relation to order $(1/m)^0$ gives $[\vec{R}, U_0] = 0$ and to order $(1/m)^2$ gives a commutation relation which was originally solved by Foldy[7]:

$$U_1(\vec{P}) - U_1(0) = -\vec{P}^{\,2}U_0/2m_t - i[\chi_v, H_0] - i[\chi_0, U_0] \ , \tag{5c}$$

where[1]

$$\chi_v(\vec{P}) = -\frac{1}{2m_t}\int_0^{\vec{P}} \vec{w} \cdot d\vec{P}' + \text{H.c.} \ , \tag{6a}$$

$$\chi_0(\vec{P}) = -\tfrac{1}{2}\sum_i \vec{r}_i' \cdot \vec{P}\,\vec{\Pi}_i \cdot \vec{P}/2m_t^{\,2} - \tfrac{1}{2}\sum_i \vec{r}_i' \cdot \vec{P}\,\vec{\Pi}_i^{\,2}/2m_i m_t + \text{H.c.} + \sum_i \vec{s}_i \times \vec{\Pi}_i \cdot \vec{P}/2m_i m_t \ . \tag{6b}$$

Because of Eq. (5b) the line integral in Eq. (6a) is independent of the path. The commutation relations, Eq. (2), are therefore satisfied for any potentials U_0 and U_1 which are scalars independent of $\vec{R}$, and for U_0 independent of $\vec{P}$. In addition, the vector $\vec{w}$, defined in Eq. (5a), must be independent of $\vec{R}$ and satisfy Eq. (5b), while U_0, U_1, and $\vec{w}$ are related by Eq. (5c). As emphasized by KF, the interaction boost is quite arbitrary, as is the potential, within the restrictions stated above.

In order to determine the wave function of the system of particles in the rest frame we need to know the relativistic correction to the potential $U_1(0)$, as well as U_0, but only $U_1(\vec{P}) - U_1(0)$, or equivalently χ_v, is needed to determine the relativistic effect of *motion* on the wave function. That is, we are not interested in the wave function in the system rest frame, but in the relationship between the wave functions of the moving system and of the system at rest. We denote the time-independent wave function in the system rest frame by $|0\rangle$, and the time-dependent wave function of

the system moving with momentum $\vec{P}_0$ by $|\vec{P}_0\rangle$. The boost operator acting on the rest frame wave function produces the wave function of the moving object, and introducing the time dependence of the wave function we have from KF and Osborn[2]

$$|\vec{P}_0\rangle = (m_B/E_{P_0})^{1/2} \exp(i\theta\hat{v}\cdot\vec{K})|0\rangle \exp(-im_B t) \ , \tag{7a}$$

$$\tanh\theta = v, \quad m_B = m_t + \epsilon_b \ , \tag{7b}$$

$$E_{P_0} = (m_B{}^2 + \vec{P}_0{}^2)^{1/2} \cong m_B + \vec{P}_0{}^2/2m_B - \vec{P}_0{}^4/8m_t{}^3 \ , \tag{7c}$$

where m_B is the total mass including binding and E_{P_0} is the total energy. The factor in front of Eq. (7a) is necessary to maintain the conventional normalization of the plane wave part of the wave function of the moving system.

The time dependence can be removed from the exponential containing $\vec{K}$ by using the commutation relations of the Poincaré group and the following identity for the operators a, b, and c and the quantity $\alpha = \sqrt{\lambda\nu}$:

$$e^{a+b} = e^a \exp\left[b(\sinh\alpha/\alpha)\right] \exp\left(\frac{c}{\nu}(1-\cosh\alpha)\right) \ , \tag{8a}$$

which holds if

$$[a,b] = \lambda c, \quad [a,c] = \nu b, \quad [c,b] = 0 \ , \tag{8b}$$

Eq. (8b) is applicable since $a \sim \hat{v}\cdot\vec{K}(t=0)$, $b \sim \hat{v}\cdot\vec{P}t$, and $c \sim H$. The result is simply to replace m_B in the time exponential by E_{P_0} and to eliminate the time dependence in $\vec{K}$, which we will assume henceforth.

The remaining decomposition is more complex and we will perform it only approximately. Because θ is of order $(1/m)$, $\vec{K}_0$ is order (m), and $\Delta\vec{K}$ is order $(1/m)$, the factor in the exponential splits into two pieces, one of order $(1/m)^\circ$ and the other of order $(1/m)^2$. We are working only to order $(1/m)^2$ so it is sufficient to separate these two pieces and expand the result to order $(1/m)^2$. The $\vec{K}_0$ and $\Delta\vec{K}$ parts do not commute, but it is easy to see that repeated commutators of $\vec{K}_0$ with the potential independent part of $\Delta\vec{K}$ will eventually vanish. In fact, the double commutator is the last one we need consider. Repeated commutators of $\vec{K}_0$ with $\vec{w}$ need not terminate, but for simplicity we assume that two commutators suffice. If any powers of $\vec{P}$ in $\vec{w}$ come from a $\vec{P}/m$ expansion, there will be no more than one such power because $\vec{w}$ is order $(1/m)$. In the explicit examples considered in Close and Osborn[8] and KF, $\vec{w}$ was independent of $\vec{P}$. Neglecting terms of order $(1/m^4)$ and using the identity

$$e^{a+b} = e^b e^{c/2} e^a e^{\lambda/6} \ , \tag{9a}$$

which holds for operators a, b, and c satisfying

$$[a,b] = c, \quad [a,c] = \lambda, \quad [b,c] = 0 \ , \tag{9b}$$

the expression in Eq. (7a) can be simplified. Care must be taken to use the relativistic expression for the momentum, including the binding energy. The quantities a and b are the $\vec{K}_0$ and $\Delta\vec{K}$ pieces of $\vec{K}$, respectively, while b, c, and λ are of order $(1/m)^2$. Keeping only $(1/m^2)$ terms and performing considerable algebra we find

$$|\vec{P}_0\rangle = [1 - i\chi_0(\vec{P}) - i\chi_v(\vec{P})]\,|0\rangle e^{i(\vec{P}_0\cdot\vec{R} - E_{P_0}t)} \ . \tag{10}$$

Because the χ's are Hermitian, the usual normalization of the wave function obtains.

Without the factors of χ, we have the usual nonrelativistic result. These factors produce the kinematical effects one usually associates with relativistic phenomena. As argued by the present author,[9] the first term in Eq. (6b) produces the effect of Lorentz contraction on the internal wave function. The second term arises because the center of energy, rather than center of mass, moves uniformly in the absence of a force.[9] The third term is associated with the Thomas precession of the spins of the moving system.[2] The potential dependent term χ_v arises for a variety of reasons. In addition to the Lorentz contraction correction, Gross found a potential dependent correction[3,4,9] to the moving deuteron wave function in the form of Eq. (10). He attributed this to the lack of simultaneity of clocks on the two nucleons in different reference frames.[4]

Having determined the effect of relativity on the wave function, it is useful to recover explicitly the results of KF using the approach of Grotch and Kashuba.[6] Equation (10) has the form of a unitary transformation, and if we perform the inverse transformation on $|\vec{P}_0\rangle$ and *all* our operators, we have effectively changed variables to the center-of-mass variables of KF. Because χ_0 and χ_v do not depend on $\vec{R}$ and are scalars, this transformation does not affect $\vec{J}$ or $\vec{P}$. It does modify $\vec{K}$ and H and in the new representation denoted by primes we have

$$\vec{J}' = \vec{J}, \quad \vec{P}' = \vec{P}, \quad |\vec{P}_0'\rangle = |0\rangle \exp[i(\vec{P}_0\cdot\vec{R} - E_{P_0}t)] \ , \tag{11a}$$

$$\vec{K}' = \tfrac{1}{2}(H_0\vec{R} + \vec{R}H_0) - \vec{S}\times\vec{P}/2m_t - t\vec{P} \ , \tag{11b}$$

$$H' = \hat{M} + \vec{P}'^2/2\hat{M} - \vec{P}'^4/8m_t{}^3 \ , \tag{11c}$$

$$\hat{M} = m_t + \sum_i (\vec{\Pi}_i{}^2/2m_i - \vec{\Pi}_i{}^4/8m_i{}^3) + U_0 + U_1(0) \ , \tag{11d}$$

where $\hat{M}$ is the mass operator (including relativistic corrections) in the c.m. system of the nucleus.

The operators $\vec{J}'$, $\vec{K}'$, $\vec{P}'$ and H' are now in single particle form. In deriving Eq. (11c), Eq. (5b) has been used. The same transformation must be used for all other operators as well, such as the charge operator $\hat{\rho}(\vec{x})$,

$$\langle \vec{P}_f | \hat{\rho}(\vec{x}) | \vec{P}_i \rangle \cong \langle \vec{P}'_f | \hat{\rho}(\vec{x}) + i[\chi(\vec{P}), \hat{\rho}(\vec{x})] | \vec{P}'_i \rangle \ , \qquad (12)$$

where χ is the sum of the two individual χ's and the final matrix element in Eq. (12), explicitly exhibiting relativistic corrections to $\hat{\rho}$, uses center-of-mass wave functions which may be approximated by nonrelativistic wave functions if desired. This approach avoids the awkward expansion used in Ref. 8, which led to an error. The correct results of KF were obtained by evaluating the commutator in Eq. (12). It should be emphasized that the representation of KF, which uses the more complicated c.m. variables, leads to a simple wave function and a simple form for the group generators, while the simple nonrelativistic variables we have used result in a more complicated wave function. When calculating matrix elements of transition operators the representation of KF is more useful, as can be seen in Eq. (12).

In summary we have presented an abridged, though complete, treatment of lowest order relativistic corrections to the wave function of a moving nucleus which is equivalent to the more formal and elegant treatment of KF. This approach is useful in making comparisons with the works of others, such as Gross, and explicitly exhibits relativistic effects such as the Lorentz contraction. The reader is referred to Ref. 1 for the complete treatment of this problem, where many other aspects are considered.

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