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Response to dilatation and some soluble "squashing" models of interacting many-fermion systems

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The response of a many-fermion system to dilatation is studied through the introduction, in close analogy with the "cranking" and "pushing" models, of a "squashing" model characterized by the Hamiltonian, $H = H_0 - \dot{\epsilon} D_x$, where D_x is the generator of dilatations in the x direction. The evaluation of the "rigid" dilatational moment of such a noninteracting system is carried out for both standing wave and periodic boundary conditions in a cubic box of side L. As in the pushing model, the dilatation moment can be evaluated formally using the relation between D_x and the commutator $[x^2, H_0]$. For independent fermions moving in a harmonic oscillator one-body potential, this relation is shown to lead to further simplification after the introduction of perturbed one-body wave functions. In the case of periodic boundary conditions, it is shown that the rigid dilatational moment is not altered by an interparticle interaction to first order in the interaction strength; by comparison with the analogous cranking moment problem, this result extends to all orders in the interaction strength. The response to a symmetric dilatation, $D = D_x + D_y + D_z$, is studied in a soluble Hartree model of "nuclear" correlations wherein a separable monopole-monopole interaction acts between "nucleons." In this model the dilatation moment in the presence of pair correlations is found to be the rigid moment, with the kinetic energy of dilatation $\propto \dot{\epsilon}^2$, there being no higher order corrections.

[NUCLEAR STRUCTURE Approximate many-body methods applied to solvable models.]

I. INTRODUCTION

"Cranking"[1,2] and "pushing"[2,3] models of many-fermion systems were introduced a long time ago to discuss the inertial responses of such systems, these being characterized by the inertial moment in the one and the translational inertial mass in the other, as well as to investigate the effects[2] of particle-particle interaction on their "rigid" values, the respective inertial responses in the absence of particle-particle interaction. It is curious to find no mention in the literature of yet another such model, one in which we are invited to consider the response of this many-particle system to dilatation; we choose to term such a model colloquially the "squashing" model.

After briefly recapitulating the properties of the generator of dilatations (in one dimension) $D_x = \frac{1}{2}(xp_x + p_x x)$ in the context of the one-body quantum mechanics, we introduce the noninteracting squashing model of a many-fermion system, and carry out the evaluation of the rigid dilatational moment of such a system for both standing wave and periodic boundary conditions in a cubic box of side L. (The results are of course identical.) It comes as no surprise that the rigid value is found to receive its major contribution from states very near the Fermi surface.[2] We next prove in the case of periodic boundary conditions in a cubic box of side L that the rigid dilatational moment is not altered by interparticle interaction to first order in the interaction strength in the manner of the calculations of Brueckner and Amado[2] and of Rockmore[4] of long ago, and indicate the extension of this result to all orders in that coupling.[5]

While the squashing model is seen to resemble the cranking model on a number of counts, it is interesting to note that as in the pushing model the kinetic energy of dilation with the associated dilatational moment given by the familiar cranking formula[2]

$$\tfrac{1}{3}\mathfrak{D} = -2\sum_{n\neq 0}\frac{|\langle\psi_n|\,D_x\,|\,\psi_0\rangle|^2}{E_0 - E_n} \tag{1}$$

can be evaluated formally using the analogous relation between D_x and the commutator of x^2 with H_0.[2,6] In this connection it is illuminating to calculate the dilatational moment yet again in a soluble (Hartree) model of a many-fermion system with a (separable) monopole-monopole interaction.[7] The dilatation interaction (which we take to be symmetric with $D = D_x + D_y + D_z$) may be exactly transformed away in this case {although initially worked out in the familiar random phase or pair approximation [random phase approximation RPA]}[8] so that the dilatational shift in the ground state energy is *exactly* given by $-\frac{1}{2}\dot{\epsilon}^2\mathfrak{D}_{\text{rigid}}$, where $\dot{\epsilon}$ is the dilatational coupling. (See the discussion given below.) This result, though based on a soluble model of interacting fermions, reconfirms the recent observation made by Zamick[9] in the related calculation of the inertial parameter in the vibrating potential model[10] that the Inglis model works better for high frequencies than one would have expected.

II. SQUASHING IN THE NONINTERACTING SYSTEM. PARTICLES IN A BOX

Preliminary to considering the problem without interaction we recapitulate briefly the salient properties of D_x, the generator of dilatations in the x direction. Under an infinitesimal dilatation in the x direction

$$U_\epsilon = 1 - i(\epsilon/\hbar)D_x \; , \tag{2}$$

where ϵ is the infinitesimal extension, one has

$$U_\epsilon^\dagger f(x) U_\epsilon = f(x) + i(\epsilon/\hbar)[D_x, f(x)] = f(x) + \epsilon x \frac{df(x)}{dx} \; . \tag{3}$$

Similarly

$$U_\epsilon^\dagger g(p_x) U_\epsilon = g(p_x) - \epsilon p_x \frac{dg(p_x)}{dp_x} \; . \tag{4}$$

{In this connection, note that one has for the oscillator Hamiltonian (say, in one dimension) under a finite dilatation

$$U(\alpha)^\dagger h_0^{(x)} U(\alpha) = e^{i(\alpha/\hbar)D_x}[p_x^2/2M + \tfrac{1}{2}(Kx^2)]e^{-i(\alpha/\hbar)D_x} = p_x^2/(2Me^{2\alpha}) + \tfrac{1}{2}(Ke^{2\alpha}x^2) = h_0^{(x)\prime} \tag{5}$$

with

$$\omega' = [Ke^{2\alpha}/(Me^{2\alpha})]^{1/2} = \omega \; .\} \tag{6}$$

Proceeding in direct analogy with the cranking model,[2] where for rotation about the z axis with angular velocity ω one has

$$H = H_0 - \omega L_z \tag{7}$$

(L_z is the angular momentum operator about the z axis) and with the pushing model,[2] where for translation along the z axis with velocity V one has

$$H = H_0 - VP_z \tag{8}$$

(P_z is the linear momentum operator in the z direction) we construct the squashing model for noninteracting fermions:

$$H = H_0 - \dot{\epsilon} D_x \; , \tag{9}$$

where $\dot{\epsilon}$ is the angular frequency for vibrations in the x direction and D_x the appropriate dilatation operator. For orientation purposes we carry out an evaluation of the dilatation moment ($\frac{1}{3}\mathfrak{D}$) first in the case of standing-wave boundary conditions in a box of side L. The matrix element of D_x between single-particle states in the box with wave functions $\psi_{lmn}^{(0)}(xyz) = (8/L^3)^{1/2} \sin k_l x \sin k_m y \sin k_n z$ is

$$\langle l'm'n' | D_x | lmn \rangle = \delta_{mm'}\delta_{nn'}(-1)^{l-l'} \frac{\hbar}{i} \frac{k_{l'}^2 + k_l^2}{k_{l'}^2 - k_l^2} \; , \tag{10}$$

where

$$k_l = l\pi/L \quad (l = 1, 2, \ldots) \; . \tag{11}$$

The energy of a single-particle state is

$$E_{lmn} = \frac{\hbar^2\pi^2}{2ML^2}(l^2 + m^2 + n^2) \; . \tag{12}$$

Substituting into Eq. (1) we find

$$\tfrac{1}{3}\mathfrak{D} = 4M \sum_{l,l'} \frac{(k_{l'}^2 + k_l^2)^2}{(k_{l'}^2 - k_l^2)^3} \; . \tag{13}$$

Again as in the analogous cranking model calculation of Brueckner and Amado,[2] since the principal contribution to the sums comes from values of l and l' close to the maximum value

$$\Lambda = (F^2 - m^2 - n^2)^{1/2} \; , \tag{14}$$

we write

$$l'^2 - l^2 = 2\Lambda(l' - l) = 2\Lambda q \; . \tag{15}$$

Equation (13) now becomes

$$\tfrac{1}{3}\mathfrak{D} \cong \frac{2ML^2}{\pi^2} \sum_{m,n} \Lambda \sum_{q=1}^{\infty} \sum_{l=\Lambda-q}^{q} \frac{1}{q^3} \; . \tag{16}$$

The sum over q and l gives a ζ function

$$\sum_{q=1}^{\infty} \sum_{l=\Lambda-q}^{\Lambda} \frac{1}{q^3} = \sum_{q=1}^{\infty} \frac{1}{q^2} = \tfrac{1}{6}\pi^2 \tag{17}$$

and the sum over m and n gives the number of particles N. Thus the rigid dilatational moment

$$\mathfrak{D} = NML^2 \tag{18}$$

emerges, with the kinetic energy of dilatation $-\frac{1}{2}\dot{\epsilon}^2\mathfrak{D}_{\text{rigid}}$ characteristically coming from small values of q, i.e. from states very near the Fermi surface.[2]

As in the case of the pushing model, where the inertial mass

$$\mathfrak{M} = -2 \sum_{n \neq 0} \frac{|\langle \psi_n | P_z | \psi_0 \rangle|^2}{E_0 - E_n} \tag{19}$$

can be evaluated *formally* using the relation[2]

$$\frac{1}{i\hbar}[z, H_0] = \frac{P_z}{M} \; , \tag{20}$$

one may similarly evaluate the dilatational moment *formally* by using the relation

$$\frac{1}{i\hbar}[x^2, H_0] = \frac{2D_x}{M} \tag{21}$$

which leads to the alternative expression

$$\mathfrak{D} = 3M\langle\psi_0|\, x^2|\, \psi_0\rangle = M\langle\psi_0|\, \vec{r}^{\,2}|\, \psi_0\rangle \ . \tag{22}$$

However, note that we only encounter the analog of the perturbed single-particle wave function

$$\psi_{lmn}(xyz) = e^{iMVz/\hbar}\psi^{(0)}_{lmn}(xyz) \tag{23}$$

in terms of which the Hamiltonian [Eq. (8)] of the pushing model is diagonalized {through the introduction of the field operator Ψ in its expansion in *perturbed* single-particle wave functions [$\Psi = \sum_{lmn} c_{lmn}\psi_{lmn}(xyz)$] in place of its expansion in *unperturbed* single-particle wave functions [$\Psi = \sum_{lmn} c^{(0)}_{lmn}\psi^{(0)}_{lmn}(xyz)$] in H with

$$H = \int d\vec{r}\,\Psi^\dagger(\mathcal{H}_0 - Vp_z)\Psi$$

$$= \sum_{lmn} (E_{lmn} - \tfrac{1}{2}MV^2)c^\dagger_{lmn}c_{lmn} \ \} \tag{24}$$

in the case of *independent fermions moving in a harmonic oscillator one-body potential,* viz. for

$$H = \int d\vec{r}\,\Psi^\dagger(h_0 - \dot{\epsilon}D_x)\Psi \tag{25}$$

with

$$h_0 = h_0^{(x)} + h_0^{(y)} + h_0^{(z)} \ , \tag{26}$$

$$\Psi = \sum_{n_x n_y n_z} c_{n_x n_y n_z}\phi_{n_x}(x; K - \dot{\epsilon}^2 M)\phi^{(0)}_{n_y}(y;K)\phi^{(0)}_{n_z}(z;K) \ , \tag{27}$$

and

$$\phi_{n_x}(x; K - \dot{\epsilon}^2 M) = e^{i\dot{\epsilon}Mx^2/2\hbar}\phi^{(0)}_{n_x}(x; K - \dot{\epsilon}^2 M) \tag{28}$$

one has

$$H = \sum_{n_x n_y n_z} \{(n_x + \tfrac{1}{2})\hbar\omega[K - \dot{\epsilon}^2 M] + (n_y + n_z + 1)\hbar\omega[K]\}\, c^\dagger_{n_x n_y n_z} c_{n_x n_y n_z} \tag{29}$$

together with the expected result $O(\dot{\epsilon}^2)$,

$$H \cong \sum_{n_x n_y n_z} [E_{n_x n_y n_z} - \tfrac{1}{2}\dot{\epsilon}^2 M(x^2)_{n_x n_y n_z, n_x n_y n_z}] \times c^\dagger_{n_x n_y n_z} c_{n_x n_y n_z} \ . \tag{30}$$

It should be emphasized that these results are rather indifferent to the boundary conditions imposed. Thus, in the pushing model in the case of periodic boundary conditions (to simplify matters we use the formalism of second quantization) one has trivially by redefinition of k_z

$$H = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2M} c^\dagger_{\vec{k}} c_{\vec{k}} - V \sum_{\vec{k}} \hbar k_z c^\dagger_{\vec{k}} c_{\vec{k}}$$

$$= \sum_{\vec{k}} \left(\frac{\hbar^2 k^2}{2M} - \tfrac{1}{2}MV^2\right) c^\dagger_{\vec{k}} c_{\vec{k}} \ . \tag{31}$$

Analogously, for the squashing model in periodic boundary conditions {with single-particle wave functions $\psi^{(0)}_{lmn} = (1/L)^{3/2}\exp[i2\pi(lx + my + nz)/L] = (1/L)^{3/2}\exp(i\vec{k}\cdot\vec{r})$} with one-body matrix elements

$$(D_x)_{\vec{k}',\vec{k}} = \frac{i\hbar}{2}\left(\frac{k'_x + k_x}{k'_x - k_x}\right)\delta_{k'_y k_y}\delta_{k'_z k_z}, \quad (k'_x \neq k_x) \ , \tag{32}$$

$$(D_x)_{\vec{k},\vec{k}} = \tfrac{1}{2}\hbar k_x L \ , \tag{33}$$

one has (after again redefining k_x here)

$$H = \sum_{\vec{k}} \left(\frac{\hbar^2 k^2}{2M} - \tfrac{1}{8}\dot{\epsilon}^2 M L^2\right) c^\dagger_{\vec{k}} c_{\vec{k}} + H_1 \ , \tag{34}$$

$$H_1 = -\dot{\epsilon} \sum_{\substack{\vec{k}',\vec{k} \\ (\vec{k}' \neq \vec{k})}} (D_x)_{\vec{k}',\vec{k}}\, c^\dagger_{\vec{k}'} c_{\vec{k}} \tag{35}$$

with the nondiagonal contribution to $\mathfrak{D}$ given by

$$\tfrac{1}{3}\mathfrak{D}^{(\text{nondiag})} = -2\langle\psi_0|\, H_1 \frac{1}{E_0 - H_0} H_1\, |\psi_0\rangle$$

$$= M \sum_{\vec{k}\vec{k}'} \frac{(k'_x + k_x)^2}{(k'_x - k_x)^2(k'^2 - k^2)}$$

$$\simeq M \sum_{lmn} \sum_{q} \frac{L^2}{4\pi^2}\frac{2\Lambda}{q^3} \ , \tag{36}$$

where we have taken the limit of a large system[2] for which $\Lambda = (F^2 - m^2 - n^2)^{1/2}$ is large; thus we are able to write

$$|\, l^2 - l'^2| = 2\Lambda\, |\, l - l'|$$

and to introduce the summation variable $q = l' - l$. From the symmetry of contributions from positive and negative l and the relation[11] $\sum_{m,n} \Lambda = \frac{1}{2}N$, it follows that

$$\tfrac{1}{3}\mathfrak{D}^{(\text{nondiag})} = \tfrac{1}{12}NML^2 \ . \tag{37}$$

Thus, adding to the nondiagonal contribution given by Eq. (37) the diagonal contribution of Eq. (34) ($\frac{1}{3}\mathfrak{D}^{(\text{diag})} = \frac{1}{4}NML^2$), the earlier result for standing-wave boundary conditions $\mathfrak{D} = NML^2$ is recovered.

III. SQUASHING IN THE INTERACTING SYSTEM. PARTICLES IN A BOX

Here we sketch briefly the proof that for fermions in a box the rigid dilatational moment is unaltered by particle-particle interaction to first order in interaction strength and its subsequent extension to all orders. (It is not necessary to go into much detail here because of the very close resemblance of this problem and its resolution to that encountered a long time ago in the study of the cranking moment of the very same interacting many-particle system.[2,4]) In the case of periodic boundary conditions and under the characteristic assumption of a translation-invariant two-body interaction, one has now, as a consequence of the discussion given in Sec. II, to deal with the Hamiltonian

$$H=H_0+\mathcal{V}+H_1 \tag{38}$$

with

$$H_0+\mathcal{V}=\sum_{\vec{k}}\left(\frac{\hbar^2k^2}{2M}-\tfrac{1}{8}\dot{\epsilon}^2ML^2\right)c^\dagger_{\vec{k}}c_{\vec{k}}+\frac{1}{2}\sum_{\vec{k}_i}c^\dagger_{\vec{k}_1}c^\dagger_{\vec{k}_2}\langle \underline{1},\underline{2}|v|\underline{3},\underline{4}\rangle\times c_{\vec{k}_4}c_{\vec{k}_3}\delta_{\vec{k}_1+\vec{k}_2,\vec{k}_3+\vec{k}_4}\,; \tag{39}$$

thus it is sufficient to show that the effects of particle-particle interaction vanish in the calculation of the nondiagonal contribution to $\mathfrak{D}$:

$$\tfrac{1}{3}\mathfrak{D}^{(\text{nondiag})}=-2\left\langle\underline{\psi}_0\left|H_1\frac{1}{E_0-(H_0+\mathcal{V})}H_1\right|\underline{\psi}_0\right\rangle \tag{40}$$

which *to first order in V* is given by[2]

$$\tfrac{1}{3}\mathfrak{D}^{(\text{nondiag})}-\tfrac{1}{3}\mathfrak{D}^{(\text{nondiag})}_{\text{rigid}}=\Delta_1+\Delta_2+\Delta_3$$

$$=-2\sum_{n,m}\frac{\langle\psi_0|D_x|\psi_n\rangle(\langle\psi_n|v|\psi_m\rangle-\langle\psi_0|v|\psi_0\rangle\delta_{mn})\langle\psi_m|D_x|\psi_0\rangle}{(E_0-E_n)(E_0-E_m)}$$

$$-4\sum_{n,m}\frac{\langle\psi_0|D_x|\psi_n\rangle\langle\psi_n|D_x|\psi_m\rangle\langle\psi_m|v|\psi_0\rangle}{(E_0-E_n)(E_0-E_m)}\,. \tag{41}$$

The three interaction-dependent terms Δ_i $(i=1,2,3)$ given by

$$-\Delta_1=\frac{M^2L^4}{\pi^4\hbar^2}\sum_{l_1m_1n_1l_1'}\sum_{l_2m_2n_2}\left(\frac{l_1+l_1'}{l_1-l_1'}\right)^2\frac{1}{(l_1^2-l_1'^2)^2}\left(\langle\vec{p}_1'\vec{p}_2|v|\vec{p}_1'\vec{p}_2\rangle-\langle\vec{p}_1\vec{p}_2|v|\vec{p}_1\vec{p}_2\rangle\right), \tag{42}$$

$$-\Delta_2=\frac{M^2L^4}{\pi^4\hbar^2}\sum_{l_1m_1n_1l_1'}\sum_{l_2m_2n_2l_2'}\left(\frac{l_1+l_1'}{l_1-l_1'}\right)\left(\frac{l_2+l_2'}{l_2-l_2'}\right)\frac{\langle\vec{p}_1'\vec{p}_2|v|\vec{p}_1\vec{p}_2'\rangle}{(l_1^2-l_1'^2)(l_2^2-l_2'^2)}\,, \tag{43}$$

$$-\Delta_3=-\frac{M^2L^4}{\pi^4\hbar^2}\sum_{l_1m_1n_1l_1'}\sum_{l_2m_2m_2l_2'}\left(\frac{l_1+l_1'}{l_1-l_1'}\right)\left(\frac{l_2+l_2'}{l_2-l_2'}\right)\left(\frac{1}{l_1^2-l_1'^2}+\frac{1}{l_2^2-l_2'^2}\right)\frac{\langle\vec{p}_1'\vec{p}_2'|v|\vec{p}_1\vec{p}_2\rangle}{l_1^2+l_2^2-l_1'^2-l_2'^2} \tag{44}$$

are, following Ref. 2 successively transformed into

$$\Delta_1=-\frac{M^2L^2}{3}\frac{1}{p_F}\frac{dV}{dp_F}\sum_{m_1n_1}\Lambda_1=-\tfrac{2}{3}\mathfrak{D}^{(\text{nondiag})}\frac{1}{p_F}\frac{dV}{dp_F}\,, \tag{45}$$

$$\Delta_2+\Delta_3=-\frac{M^2L^4}{8\pi^4\hbar^2}\sum_{m_1n_1,m_2n_2}\left(\langle\vec{p}_1\vec{p}_2|v|\vec{p}_1\vec{p}_2\rangle-\langle\vec{p}_1,-\vec{p}_2|v|\vec{p}_1,-\vec{p}_2\rangle\right)_{p_1=p_2=p_F}\sum_{l_1=0}^{\Lambda_1}\sum_{q=1}^{\Lambda_1-l_1+1}\sum_{l_2=\Lambda_2-q+1}^{\Lambda_2}\frac{1}{q^4}$$

$$=-\frac{M^2L^4F^4}{96\pi^2\hbar^2}\int_{4\pi}d\Omega_1\int_{4\pi}d\Omega_2\sin\theta_1\sin\theta_2\sin\phi_1\sin\phi_2\langle\vec{p}_1\vec{p}_2|v|\vec{p}_1\vec{p}_2\rangle_{p_1=p_2=p_F}$$

$$=-\frac{M^2L^4F^4}{54\hbar^2}v_1(p_F,p_F)\,. \tag{46}$$

Since this last expression may cast into the form[2]

$$\Delta_2+\Delta_3=-\frac{M^2L^4F^4}{36\hbar^2}\times\int_{-1}^{1}d(\cos\theta_{12})\langle p_1p_2|v|p_1p_2\rangle_{p_1=p_2=p_F}\cos\theta_{12}=\tfrac{1}{6}NML^2\frac{M}{p_F}\frac{dV}{dp_F}\,, \tag{47}$$

we have

$$\Delta_1+\Delta_2+\Delta_3=0\ .$$

This first order cancellation of interaction effects is a consequence of the "generalized gradient theorem" (already familiar[5] from earlier discussions of the cranking moment) and the extension of this result to higher orders is straightforward,[5] i.e. we conclude that *interaction effects on the dilatation moment vanish to all orders in this model.*

IV. SQUASHING IN A SOLUBLE HARTREE MODEL OF FERMIONS WITH A SEPARABLE MONOPOLE-MONOPOLE INTERACTION

We consider finally the response to dilatation in a "soluble" model[7] of nuclear correlations wherein a separable monopole-monopole interaction[7] $v_{ij}=\lambda r_i^2r_j^2$ acts between nucleons (these we take to be spinless, adopting the definition that exchange terms vanish[7]). The result in this instructive and simple model may be indicative of the situation in the case of more realistic interactions. In the formalism of second quantization the Hamiltonian of this interacting system is given by

$$H=\int d\vec{r}\,\Psi^\dagger\frac{\vec{p}^{\,2}}{2M}\Psi+\tfrac{1}{2}\lambda:\left(\int d\vec{r}\,\Psi^\dagger\vec{r}^{\,2}\Psi\right)^2:. \tag{48}$$

The field operator $\Psi(\vec{r})$ is expanded in the eigenstates of a harmonic oscillator reference spectrum

$$\Psi(\vec{r})=\sum_{n_xn_yn_z}c_{n_xn_yn_z}\,\phi^{(o)}_{n_xn_yn_z}(\vec{r};K) \tag{49}$$

with the harmonic oscillator parameter $\hbar\omega=\hbar(K/M)^{1/2}$ variationally determined and with the associated annihilation (creation) operators $c_{n_xn_yn_z}$ ($c^\dagger_{n_xn_yn_z}$) satisfying the anticommutation relations

$$\{c_n,c^\dagger_{n'}\}=\delta_{n_x n_x'}\delta_{n_y n_y'}\delta_{n_z n_z'}\,. \tag{50}$$

(It will usually suffice to label operators and wave functions schematically, i.e.,

$$c_\alpha\to c_{n_xn_yn_z}\,,\quad \phi^{(o)}_\alpha\to\phi^{(o)}_{n_xn_yn_z}.)$$

Introducing operators for particles

$$c_\alpha=a_\alpha,\quad n(\alpha)>n_F \tag{51a}$$

[n_F refers to the Fermi surface, the last filled level $n_F(n_xn_yn_z)$ in the reference system.] and for holes

$$c_\alpha=b^\dagger_\alpha,\quad n(\alpha)\le n_F \tag{51b}$$

and normal ordering H, one has

$$H=\sum_\alpha^{n_F}T_{\alpha\alpha}+\tfrac{1}{2}\lambda\left(\sum_\alpha^{n_F}r_{\alpha\alpha}{}^2\right)^2+\left(\lambda\sum_\gamma r_{\gamma\gamma}{}^2-\tfrac{1}{2}M\omega^2\right)\sum_{\alpha\beta}r_{\alpha\beta}{}^2O_{\alpha\beta}+E_\alpha[\omega]O_{\alpha\alpha}+\tfrac{1}{2}\lambda\left(\sum_{\alpha\beta}r_{\alpha\beta}{}^2O_{\alpha\beta}\right)^2, \tag{52}$$

where

$$T_{\alpha\alpha}=\left(\phi^{(o)}_\alpha,\ \frac{\vec{p}^{\,2}}{2M}\phi^{(o)}_\alpha\right)=\tfrac{1}{2}(n+\tfrac{3}{2})\hbar\omega$$

and

$$\begin{aligned}O_{\alpha\alpha}&=N^{(a)}_\alpha-N^{(b)}_\alpha\,,\\ N^{(a)}_\alpha&=a^\dagger_\alpha a_\alpha\,,\\ N^{(b)}_\alpha&=b^\dagger_\alpha b_\alpha\,,\\ O_{\alpha\beta}&=c_{\alpha\beta}+c^\dagger_{\alpha\beta}+N^{(a)}_{\alpha\beta}-N^{(b)}_{\alpha\beta}\,,\\ c_{\alpha\beta}&=b_\alpha a_\beta\,,\\ N^{(a)}_{\alpha\beta}&=a^\dagger_\alpha a_\beta\,,\\ N^{(b)}_{\alpha\beta}&=b^\dagger_\alpha b_\beta\,.\end{aligned} \tag{53}$$

At the minimum of

$$\langle O|H[\omega]|O\rangle=\sum_\alpha^{n_F}T_{\alpha\alpha}+\tfrac{1}{2}\lambda\left(\sum_\alpha^{n_F}r_{\alpha\alpha}{}^2\right)^2,$$

determined by

$$\frac{\partial\langle O|H[\omega_0]|O\rangle}{\partial\omega_0}=\tfrac{1}{2}\hbar\sum-\frac{\lambda}{\omega_0}\left(\frac{\hbar}{M\omega_0}\right)^2\left(\sum\right)^2=0\,, \tag{54}$$

where $\sum$ denotes the sum over occupied states,

$$\sum\equiv\sum_{\alpha=0}^{n_F}(n_x+n_y+n_z+\tfrac{3}{2})=\tfrac{1}{8}(n_F+1)(n_F+2)^2(n_F+3)\,, \tag{55}$$

the pair excitations are decoupled from the uncorrelated ground state energy, viz.

$$H[\omega_0]=\langle O|H[\omega_0]|O\rangle+\sum_\alpha E_\alpha[\omega_0]O_{\alpha\alpha}+\tfrac{1}{2}\lambda\left(\sum_{\alpha\beta}r_{\alpha\beta}{}^2O_{\alpha\beta}\right)^2, \tag{56}$$

since there

$$\lambda \sum_{\gamma} r_{\gamma\gamma}{}^2[\omega_0] - \tfrac{1}{2} M\omega_0{}^2 = 0.$$

The statement of the problem is complete with the addition to H of the interaction with, say, a *symmetric* dilatation $D = D_x + D_y + D_z$:

$$H_1 = -\dot{\epsilon} \int d\vec{r}\, \Psi^\dagger D \Psi . \qquad (57)$$

The solution to this (so far) exact problem is obtained readily by comparison with its solution in the "boson" approximation (RPA).[8] The equivalent pair approximation to the exact problem is constructed by defining boson annihilation (creation) operators $\tilde{c}_{\alpha\beta}$ ($\tilde{c}^\dagger_{\alpha\beta}$) with commutation relations

$$[\tilde{c}_{\alpha\beta}, \tilde{c}^\dagger_{\alpha'\beta'}] = \delta_{\alpha\alpha'}\delta_{\beta\beta'} \qquad (58)$$

in terms of which

$$H + H_1 - \langle O | H[\omega_0] | O \rangle \rightarrow \mathcal{H}^{\mathrm{RPA}} = \mathcal{H}_0^{\mathrm{RPA}} + \mathcal{V}^{\mathrm{RPA}} + \mathcal{H}_1^{\mathrm{RPA}}$$

$$= \sum_{\alpha\beta} (E_\beta - E_\alpha) \tilde{c}^\dagger_{\alpha\beta} \tilde{c}_{\alpha\beta} + \tfrac{1}{2}\lambda \Big[\sum_{\alpha\beta} r_{\alpha\beta}{}^2 (\tilde{c}_{\alpha\beta} + \tilde{c}^\dagger_{\alpha\beta}) \Big]^2 - \dot{\epsilon} \sum_{\alpha\beta} d_{\alpha\beta} (\tilde{c}_{\alpha\beta} - \tilde{c}^\dagger_{\alpha\beta}), \qquad (59)$$

where

$$d_{\alpha\beta} = d^{(x)}_{\alpha\beta} + d^{(y)}_{\alpha\beta} + d^{(z)}_{\alpha\beta}$$
$$= (\phi^{(0)}_\alpha, D\phi^{(0)}_\beta) \qquad (60)$$

with

$$d^{(j)}_{\alpha\beta} = (\tfrac{1}{2} i\hbar) \Big(\{ [n_j(\beta)+1][n_j(\beta)+2] \}^{1/2} \delta_{n_j(\alpha), n_j(\beta)+2} - (\alpha \leftrightarrow \beta) \Big) \prod_{k \neq j} \delta_{n_k(\alpha), n_k(\beta)} . \qquad (61)$$

In the case of *no pair correlations* ($\lambda = 0$) the RPA Hamiltonian

$$\mathcal{H}_0^{\mathrm{RPA}} + \mathcal{H}_1^{\mathrm{RPA}} = \sum_{\alpha\beta} (E_\beta - E_\alpha) \tilde{c}^\dagger_{\alpha\beta} \tilde{c}_{\alpha\beta} - \dot{\epsilon} \sum_{\alpha\beta} d_{\alpha\beta} (\tilde{c}_{\alpha\beta} - \tilde{c}^\dagger_{\alpha\beta}) \qquad (62)$$

is diagonalized by the unitary transformation

$$S = \prod_{j=1}^{3} S^{(j)} ,$$
$$S^{(j)} = \exp[\dot{\epsilon} \sum_{\gamma\delta} g^{(j)}_{\gamma\delta} (\tilde{c}_{\gamma\delta} + \tilde{c}^\dagger_{\gamma\delta})] , \qquad (63)$$

where $g^{(j)}_{\gamma\delta}$ is pure imaginary. Thus,

$$S^\dagger (\mathcal{H}_0^{\mathrm{RPA}} + \mathcal{H}_1^{\mathrm{RPA}}) S = \sum_{\alpha\beta} (E_\beta - E_\alpha) \tilde{c}^\dagger_{\alpha\beta} \tilde{c}_{\alpha\beta} - 2\dot{\epsilon}^2 \sum_{j=1}^{3} \sum_{\alpha\beta} d^{(j)}_{\alpha\beta} g^{(j)}_{\alpha\beta} - \dot{\epsilon} \sum_{j=1}^{3} \sum_{\alpha\beta} (E_\beta - E_\alpha)(g^{(j)}_{\alpha\beta})^2$$
$$= \sum_{\alpha\beta} (E_\beta - E_\alpha) \tilde{c}^\dagger_{\alpha\beta} \tilde{c}_{\alpha\beta} - \dot{\epsilon}^2 \sum_{j=1}^{3} \sum_{\alpha\beta} \frac{|d^{(j)}_{\alpha\beta}|^2}{E_\beta - E_\alpha} \qquad (64)$$

provided

$$(E_\beta - E_\alpha) g^{(j)}_{\alpha\beta} = -d^{(j)}_{\alpha\beta} . \qquad (65)$$

The kinetic energy of dilatation is easily evaluated:

$$\dot{\epsilon}^2 \sum_{j=1}^{3} \sum_{\alpha\beta} \frac{|d^{(j)}_{\alpha\beta}|^2}{E_\beta - E_\alpha}$$
$$= \dot{\epsilon}^2 \frac{3\hbar}{8\omega_0} \sum_{n_x}^{n_F, n_F - 1} (n_x+1)(n_x+2) w_x(n_x) \qquad (66)$$

and, with some additional manipulation this last expression reduces to

$$\dot{\epsilon}^2 \frac{\hbar}{2\omega_0} \sum = \Big\langle 0 \Big| \frac{\dot{\epsilon}^2}{2\omega_0{}^2} \sum E_{n_x n_y n_z} c^\dagger_{n_x n_y n_z} c_{n_x n_y n_z} \Big| 0 \Big\rangle$$
$$= \Big\langle 0 \Big| \tfrac{1}{2} \dot{\epsilon}^2 M \sum_{n_x n_y n_z} (r^2)_{n_x n_y n_z, n_x n_y n_z} \times c^\dagger_{n_x n_y n_z} c_{n_x n_y n_z} \Big| 0 \Big\rangle . \qquad (67)$$

From the condition [Eq. (65)] with

$$(E_\beta - E_\alpha) g^{(j)}_{\alpha\beta} = (\phi^{(0)}_\alpha, [g^{(j)}, h_0[\omega_0]] \phi^{(0)}_\beta)$$
$$= -(\phi^{(0)}_\alpha, D_j \phi^{(0)}_\beta) \qquad (68)$$

we find

$$g^{(j)} = -(M/2i\hbar)r_j^{\,2}\,; \qquad (69)$$

thus

$$S = \exp\left[(i\dot{\epsilon}M/2\hbar)\sum_{\gamma\delta} r_{\gamma\delta}{}^2(\tilde{c}_{\gamma\delta} + \tilde{c}^\dagger_{\gamma\delta})\right] \qquad (70)$$

and S *commutes with* $\mathfrak{V}^{\text{RPA}}$ as well, i.e., *the dilatational moment in the presence of pair correlations is the rigid dilatational moment.* In the exact problem posed above at the outset of our discussion, the theorem is preserved intact if we simply make the correspondence[12]

$$S \to S_{(\text{exact})} = \exp\left[(i\dot{\epsilon}M/2\hbar)\sum_{\alpha\beta} r_{\gamma\delta}{}^2 O_{\gamma\delta}\right] \qquad (71)$$

again with

$$S^\dagger_{(\text{exact})}(H + H_1)S_{(\text{exact})} = H - \tfrac{1}{2}\dot{\epsilon}^2 \frac{\sum_{\text{occ}}\hbar\omega_0(n+\frac{3}{2})}{\omega_0{}^2}\,. \qquad (72)$$

This result is "quasiadiabatic" only in the sense that had we expanded Ψ in the *perturbed* oscillator states of Sec. II, we would have found that[13]

$$H[\omega] + H_1[\omega] = \sum_\alpha^{n_F} T_{\alpha\alpha}[\omega'] + \tfrac{1}{2}\lambda\left(\sum_\alpha^{n_F} r_{\alpha\alpha}{}^2[\omega']\right)^2 - \tfrac{1}{2}\dot{\epsilon}^2 M\sum_\alpha^{n_F} r_{\alpha\alpha}{}^2[\omega'] + \left(\lambda\sum_\gamma r_{\gamma\gamma}{}^2[\omega'] - \tfrac{1}{2}M\omega'^2\right)$$
$$\times\sum_{\alpha\beta} r_{\alpha\beta}{}^2[\omega']O_{\alpha\beta} + \sum_\alpha (E_\alpha[\omega'] - \tfrac{1}{2}\dot{\epsilon}^2 M r_{\alpha\alpha}{}^2[\omega'])O_{\alpha\alpha} + \tfrac{1}{2}\lambda\left(\sum_{\alpha\beta} r_{\alpha\beta}{}^2[\omega']O_{\alpha\beta}\right)^2. \qquad (73)$$

Neglecting the term $0(\dot{\epsilon}^2)$ in the minimization of $\sum_\alpha T_{\alpha\alpha}[\omega'] + \frac{1}{2}\lambda(\sum_\alpha r_{\alpha\alpha}{}^2[\omega'])^2$ produces our earlier result [Eq. (67)], since *after* minimization $\omega' = \omega_0$.

APPENDIX: PERTURBED SINGLE HARMONIC OSCILLATOR

It may be of some interest to solve the eigenvalue problem of the perturbed single harmonic oscillator of Sec. II by *algebraic* means. One introduces the operators

$$\eta_1 = (\tfrac{1}{4})\{(a^\dagger)^2 + a^2\},$$
$$\eta_2 = (\tfrac{1}{4}i)\{(a^\dagger)^2 - a^2\}, \qquad (A1)$$
$$\eta_3 = (\tfrac{1}{4})(aa^\dagger + a^\dagger a)$$

constructed of the annihilation (creation) operators a ($a^\dagger$) defined by

$$x = (\hbar/2M\omega)^{1/2}(a + a^\dagger),$$
$$p_x = (\tfrac{1}{2}\hbar M\omega)^{1/2}(a - a^\dagger)/i \qquad (A2)$$

with

$$[a, a^\dagger] = 1\,. \qquad (A3)$$

The η_i satisfy commutation relations analogous to those of angular momenta

$$[\eta_1, \eta_2] = -i\eta_3\,,$$
$$[\eta_2, \eta_3] = i\eta_1\,, \qquad (A4)$$
$$[\eta_3, \eta_1] = i\eta_2\,.$$

In terms of the η_i

$$h_0^{(x)} - \dot{\epsilon}D_x = 2\hbar\omega\eta_3 + 2\hbar\dot{\epsilon}\eta_2\,. \qquad (A5)$$

Under the unitary transformation $S = e^{i\phi\eta_1}$ (ϕ real) we have

$$\eta_2' = S^\dagger\eta_2 S = \eta_2\cosh\phi - \eta_3\sinh\phi,$$
$$\eta_3' = \eta_3\cosh\phi - \eta_2\sinh\phi, \qquad (A6)$$

so that for the particular choice, $\omega\sinh\phi = \dot{\epsilon}\cosh\phi$, one finds

$$S^\dagger(h_0^{(x)} - \dot{\epsilon}D_x)S = 2\hbar\omega(1 - \dot{\epsilon}^2/\omega^2)^{1/2}\eta_3\,. \qquad (A7)$$

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[11]Our fermions are taken to be spinless.

[12]This avoids the complications of working with the physical pairs of $\mathcal{H}_0^{\text{RPA}} + \mathfrak{V}^{\text{RPA}}$, whose readily calculable "breathing mode" eigenenergies $\hbar\Omega_{\alpha\beta} = (6\hbar\omega_0)^{1/2}$ are determined by the eigenvalue equation ($E_\delta - E_\eta = 2\hbar\omega_0$)

$$1 - \lambda\sum_{\eta\delta}(r_{\eta\delta}{}^2)^2\left(\frac{1}{\hbar\Omega_{\alpha\beta} - (E_\delta - E_\eta)} - \frac{1}{\hbar\Omega_{\alpha\beta} + (E_\delta - E_\eta)}\right) = 0$$

using the nontrivial relation $\sum_{\eta\delta}(r_{\eta\delta}{}^2)^2 = \hbar^2/(2M\omega_0)^2 4\sum$.

[13]$\omega' = (\omega^2 - \dot{\epsilon}^2)^{1/2}$.