# Exchange effects and $n$-body forces in nuclear matter

D. W. E. Blatt
*Department of Theoretical Physics, University of Sydney, Sydney, New South Wales, 2006, Australia and School of Computing Science, New South Wales Institute of Technology, Sydney, New South Wales, 2007, Australia**

Bruce H. J. McKellar
*School of Physics, University of Melbourne, Parkville, Victoria, 3052, Australia*

(Received 13 January 1975)

The $n$-pion-exchange $(n+1)$-body force is written explicitly in coordinate space and separated into its central and tensor parts. Exchanges of the spectator nucleons are shown to cause large cancellations. The ratio of $n$-body forces to $(n+1)$-body forces is estimated and compared with recent three- and four-body calculations.

[NUCLEAR STRUCTURE Nuclear many body forces; calculated effective two-body potential; includes correlations; estimated convergence ratio; calculated effect of spectator exchanges.]

## I. INTRODUCTION

Until recently, only three-body forces have been considered in any detail in computing the binding energy of nuclear matter. However, McKellar and Rajaraman[1] have pointed out that higher order many-body forces of the type represented by the diagram of Fig. 1 could produce some new physics especially at high nuclear densities, as in neutron stars. A detailed calculation by Blatt and McKellar[2] on the effect of a four-body force in nuclear matter at ordinary densities has shown good convergence with respect to two- and three-body effects. In that calculation both exchanges of intermediate nucleons in the four-body chain were taken into account, and correlations between all four nucleons. The work of McKellar and Rajaraman,[1] although taking some account of correlations (correlations between those nucleons which exchanged pions were included), took no account of exchanges between nucleons in the $n$-body chain.

In this paper, we calculate an exact coordinate space expression for the $n$-pion-exchange potential for $(n+1)$ nucleons, and then by assuming that correlation functions are spin and isospin independent, we split the direct term (i.e., the term considered in Ref. 1) into central and tensor parts, thereby generalizing the effective potential method of Loiseau, Nogami, and Ross.[3] Consideration of terms where the intermediate or spectator nucleons in the many-body force exchange among themselves, however, gives almost exact cancellation. Therefore, rather than summing over the $n$-body force direct terms only, as done in Ref. 1, we calculate an approximate relation between the $n$-body forces and the $(n+1)$-body forces, including the exchange effects, and show that this agrees with our earlier work on three-body forces[4] and four-body forces.[2]

## II. $n$PEP FOR $(n+1)$ NUCLEONS

In this section we find an expression for the general $n$-pion-exchange potential ($n$PEP) for $(n+1)$ nucleons; as in the three- and four-nucleon cases we exchange the outer (nonspectator) nucleons. We use the diagram rules given in Miyazawa.[5] To facilitate the calculations, we introduce the coordinates shown in Fig. 1. The $(n+1)$ nucleons are labeled $(\vec{r}_i\vec{\sigma}_i\vec{\tau}_i)$, $i=0,1,\ldots,n$, and the $n$ pions have momenta $\vec{q}_i$, $i=1,2,\ldots,n$.

We define relative coordinates

$$\vec{x}_i=\vec{r}_{i-1}-\vec{r}_i, \quad i=1,2,3,\ldots,n; \tag{2.1}$$

$$\vec{\nabla}_i=\frac{\partial}{\partial\vec{x}_i}, \quad i=1,2,3,\ldots,n. \tag{2.2}$$

This enables us to proceed with the analysis of $n$PEP in a manner analogous to the treatment of 2PEP and 3PEP.[2,4,6] For the direct term, the $(n-1)$ "spectator" nucleons satisfy the condition

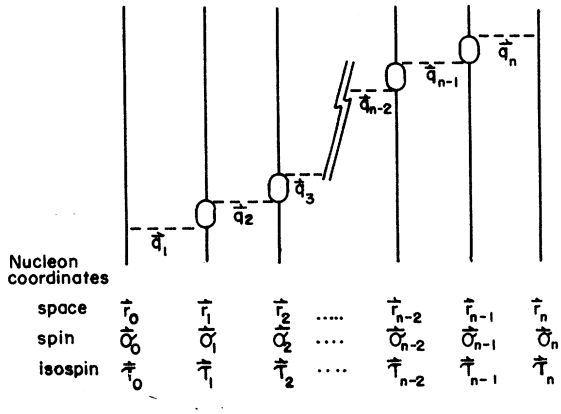



FIG. 1. The $n$-pion-exchange $(n+1)$-nucleon potential ($n$PEP).

of no isospin-flip. The effects of exchanging the spectator nucleons are examined in the next section. The general equation for the direct $n$-pion- exchange potential is then given by

$$W_{n\mathrm{PE}} = -\frac{4\pi f^2 \vec{\tau}_0\cdot\vec{\tau}_n}{(2\pi)^{3n}\mu^2}[2(A+B)]^{n-1}\iint\cdots\int d\vec{q}_1 d\vec{q}_2\cdots d\vec{q}_n e^{i\vec{q}_1\cdot\vec{x}_1}e^{i\vec{q}_2\cdot\vec{x}_2}\cdots e^{i\vec{q}_n\cdot\vec{x}_n}$$
$$\times\frac{K^2K'(q_1^2)}{q_1^2+\mu^2}\cdots\frac{K^2K'(q_n^2)}{q_n^2+\mu^2}\vec{\sigma}_0\cdot\vec{q}_1\vec{q}_1\cdot\vec{q}_2\cdot\cdots\vec{q}_{n-1}\cdot\vec{q}_n\vec{q}_n\cdot\vec{\sigma}_n, \tag{2.3}$$

where $K$ and $K'$ are the vertex and propagator form factors, $f^2=0.08$ is the $\pi N$-coupling constant, $\mu$ is the pion mass, and $A$ and $B$ are constants related to the $p$-wave $\pi N$-scattering cross section. These constants are all defined as in three-body calculations.[4,6,7] The form factors are assumed to have the form

$$\frac{K^2(q^2)K'(q^2)}{q^2+\mu^2}=\frac{1}{q^2+\mu^2}+\zeta\frac{1}{q^2+\eta^2}, \tag{2.4}$$

with the parameters $\zeta$ and $\eta$ specified in Table I. Form factor I is the case of a form factor which is identically 1 and, as in three-body calculations,[4,6,7] is included for comparison purposes only. Generalizing the three- and four-body expressions[2,4,6] we can write

$$W_{n\mathrm{PE}}(\vec{r}_0\vec{r}_1\cdots\vec{r}_n)=(-1)^{n+1}\frac{4\pi f^2\vec{\tau}_0\cdot\vec{\tau}_n}{(2\pi)^{3n}\mu^2}[2(A+B)]^{n-1}(2\mu\pi^2)^n\mu^{2n}g(\vec{x}_1\vec{x}_2\cdots\vec{x}_n), \tag{2.5}$$

where

$$g(x_1x_2\cdots x_n)=\frac{1}{\mu^{2n}}\vec{\sigma}_0\cdot\left(\underline{I}+\hat{x}_1\underline{\hat{x}}_1x_1\frac{\partial}{\partial x_1}\right)\cdot\left(\underline{I}+\hat{x}_2\underline{\hat{x}}_2x_2\frac{\partial}{\partial x_2}\right)\cdot\cdots\cdot\left(\underline{I}+\hat{x}_n\underline{\hat{x}}_nx_n\frac{\partial}{\partial x_n}\right)\cdot\vec{\sigma}_n\frac{1}{x_1}\frac{\partial}{\partial x_1}\frac{1}{x_2}\frac{\partial}{\partial x_2}$$
$$\times\cdots\frac{1}{x_n}\frac{\partial}{\partial x_n}\tilde{Y}_{x_1}\tilde{Y}_{x_2}\cdots\tilde{Y}_{x_n}. \tag{2.6}$$

Here $\underline{I}$ is the unit tensor of rank two and the functions $\tilde{Y}_x$ are given by

$$\tilde{Y}_x=Y_\mu(x)-\zeta\frac{\eta}{\mu}Y_\eta(x), \tag{2.7}$$

$$Y_\mu(x)=\frac{e^{-\mu x}}{\mu x}. \tag{2.8}$$

The differentiations in (2.6) can be done, giving

$$g(\vec{x}_1\vec{x}_2\cdots\vec{x}_n)=\vec{\sigma}_0\cdot\underline{G}^{(n)}\cdot\vec{\sigma}_n, \tag{2.9}$$

where

$$\underline{G}^{(n)}=\left[\tfrac{1}{3}(\bar{X}_1-\tilde{X}_1)\underline{I}+\tilde{X}_1\underline{\hat{x}}_1\underline{\hat{x}}_1\right]\cdot\left[\tfrac{1}{3}(\bar{X}_2-\tilde{X}_2)\underline{I}+\tilde{X}_2\underline{\hat{x}}_2\underline{\hat{x}}_2\right]\cdot\cdots\cdot\left[\tfrac{1}{3}(\bar{X}_n-\tilde{X}_n)I+\tilde{X}_n\underline{\hat{x}}_n\underline{\hat{x}}_n\right]Y_\mu(x_1)Y_\mu(x_2)\cdots Y_\mu(x_n) \tag{2.10}$$

and $\bar{X}_i$ and $\tilde{X}_i$ are functions of $X_i=\mu x_i$, $i=1,\ldots,n$, defined by

$$\bar{X}_i=1-\zeta\frac{\eta^2}{\mu^2}e^{-(\eta-\mu)x_i}, \tag{2.11}$$

$$\tilde{X}_i=\left(1+\frac{3}{\mu x_i}+\frac{3}{\mu^2x_i^2}\right)-\zeta\frac{\eta^2}{\mu^2}\left(1+\frac{3}{\eta x_i}+\frac{3}{\eta^2x_i^2}\right)e^{-(\eta-\mu)x_i}. \tag{2.12}$$

Equations (2.5) and (2.8)–(2.12) define the $n$PEP for $(n+1)$ nucleons. OPE (one-pion exchange), 2PE, and 3PE formulas are special cases of these, and are easily verified by putting $n=1$, 2, and 3 into these expressions. For example, $n=1$ gives the usual expression for OPEP:

$$W_{\mathrm{OPE}}(\vec{x}_1)=\tfrac{1}{3}f^2\mu\vec{\tau}_0\cdot\vec{\tau}_1[\vec{\sigma}_0\cdot\vec{\sigma}_1\bar{X}_1+S_{01}(\hat{x}_1)\tilde{X}_1]Y_\mu(x_1), \tag{2.13}$$

TABLE I. Form factor parameters.

| Form factor | $\zeta$ | $\left(\frac{\eta}{\mu}\right)^2$ |
|---|---|---|
| I | 0 | ⋯ |
| II | 0.72 | 5.73 |
| III | 1 | 10 |

where the tensor operator $S_{01}$ is defined by

$$S_{01}(\hat{x}_1) = 3\vec{\sigma}_0 \cdot \hat{x}_1 \vec{\sigma}_1 \cdot \hat{x}_1 - \vec{\sigma}_0 \cdot \vec{\sigma}_1 . \tag{2.14}$$

Notice that form factors are included in (2.13) for the reasons we outlined in Ref. 6. The case of unit form factor is given by $\bar{X}_1 \to 1$ and $\tilde{X}_1 \to 1 + 3/X_1 + 3/X_1^2$ (note that $X_i = \mu x_i$).

### III. EXCHANGE OF SPECTATOR NUCLEONS

In this section we estimate the effect of exchanges among the spectator nucleons in the $n$PEP for $(n+1)$ nucleons. We consider the general diagram in Fig. 1 with nucleons $0, 1, 2, \ldots, n-2, n-1, n$ in the initial state exchanged to $n, i_1, i_2, \ldots, i_{n-2}, i_{n-1}, 0$ in the final state:

$$P = \begin{pmatrix} 1 & 2 \cdots n-2 & n-1 \\ i_1 & i_2 \cdots i_{n-2} & i_{n-1} \end{pmatrix} \tag{3.1}$$

is an arbitrary permutation with parity $\epsilon_P$ of the spectator nucleons. There is a factor $\epsilon_P$ from antisymmetrization of the final state wave function.

To estimate the contribution from this process, we review the case of 3PEP.[2] Here there are only $2! = 2$ possible permutations of the spectator nucleons $\left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$. These are the direct term and the exchange term. As we demonstrated in Ref. 2, the exchange term is approximately a factor of 4 smaller than the direct term when it is assumed that the correlations are independent of spin and isospin. Notice also that

$$\langle 1 \rangle_{\substack{\text{spin-isospin} \\ \text{exch. av.}}} = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} . \tag{3.2}$$

The exchange average of 1 derives its only contributions from the initial states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ for spin, giving a factor of $\frac{1}{2}$, and a similar factor of $\frac{1}{2}$ is derived from the isospin average. That is, when the spin or isospin of the two spectator nucleons is not the same, the exchange gives a zero contribution, and the matrix element of 1 is zero.

Now consider the general case with its permutation $P$, Eq. (3.1). We write $P$ in cycle notation:

$$P = (j_1^{(1)} j_2^{(1)} \cdots j_{k_1}^{(1)})(j_1^{(2)} j_2^{(2)} \cdots j_{k_2}^{(2)}) \cdots (j_1^{(m)} \cdots j_{k_m}^{(m)}) . \tag{3.3}$$

Here $P$ is split into $m$ cycles of lengths $k_1, k_2, \ldots, k_m$; $k_1 + k_2 + \cdots + k_m = n-1$. In this notation, the identity permutation is written as

$$I_{n-1} = (1)(2)(3) \cdots (n-1) .$$

It has $(n-1)$ cycles each of length 1.

Now consider a particular cycle in (3.3). For a term in the spin-isospin average to contribute, all spins must be the same in the cycle and all isospins must be the same. Then the cycle will have no effect on the spin-isospin wave function and the overlap between the initial state and final state wave functions for nucleons in this cycle will be unity.

There are only four ways this condition can be satisfied (all spins up, all isospins up; all spins up, all isospins down; etc.). For a cycle of length $k$, there are $4^k$ possible spin-isospin wave functions which must be averaged over. The effect of this particular cycle on the spin-isospin average is therefore a factor of $4/4^k$. Equation (3.2) is a special case of this with $k = 2$.

All cycles in the permutation $P$ Eq. (3.3) can be summed over independently, giving the factor

$$\epsilon_P \frac{4}{4^{k_1}} \frac{4}{4^{k_2}} \frac{4}{4^{k_3}} \cdots \frac{4}{4^{k_m}} = \frac{(-4)^m}{(-4)^{n-1}} \tag{3.4}$$

for the spin-isospin average of $P$.

We must now sum over all permutations $P$. Suppose there are $N_m^{(n-1)}$ permutations of the $(n-1)$ nucleons with a cycle notation containing exactly $m$ cycles. The effect of exchanges is then to introduce a factor

$$\mathcal{E}_{n\text{PEP}} = \sum_{m=1}^{n-1} \frac{(-4)^m}{(-4)^{n-1}} \cdot N_m^{(n-1)} . \tag{3.5}$$

We can establish the recurrence relation for $N_m^{(n)}$:

$$\begin{aligned} N_m^{(n)} &= (n-1) N_m^{(n-1)}, \quad m = 1 \\ &= (n-1) N_m^{(n-1)} + N_{m-1}^{(n-1)}, \quad 1 < m < n \\ &= N_{m-1}^{(n-1)}, \quad m = n ; \end{aligned} \tag{3.6}$$

and then putting (3.6) into (3.5) gives the recurrence relation on $\mathcal{E}_{n\text{PEP}}$:

$$\mathcal{E}_{(n+1)\text{PEP}} = \frac{5-n}{4} \mathcal{E}_{n\text{PEP}} . \tag{3.7}$$

This has the interesting property that $\mathcal{E}_{n\text{PEP}}$ becomes identically equal to zero for $n > 5$. The general solution is

$$\begin{aligned} &\mathcal{E}_{2\text{PEP}} = 1 , \\ &\mathcal{E}_{3\text{PEP}} = \tfrac{3}{4} , \\ &\mathcal{E}_{4\text{PEP}} = \tfrac{3}{8} , \\ &\mathcal{E}_{5\text{PEP}} = \tfrac{3}{32} , \\ &\mathcal{E}_{n\text{PEP}} = 0 , \quad n > 5 . \end{aligned} \tag{3.8}$$

For the 3PEP we see that this type of exchange reduces the effective potential by a factor of $\frac{3}{4}$, in agreement with the results of Ref. 2.

We emphasize that not all exchange diagrams have been considered. Only exchanges among the $(n-1)$ spectator nucleons have been allowed. Furthermore, even the exchanges considered have only been treated approximately. While $\mathcal{E}_{n\text{PEP}} = 0$ for

$n>5$ only in this approximation, we would expect $\mathcal{S}_{n\mathrm{PEP}}$ to be small for large $n$.

A similar cancellation due to exchanges has been observed by Rajaraman and Trehan[8] in their work on many-body correlations in liquid helium and nuclear matter.

## IV. DECOMPOSING $n$PEP INTO CENTRAL AND TENSOR PARTS

As in the case of 2PEP and 3PEP, we define an effective two-body potential for $n$PEP by

$$V_{n\mathrm{PE}}(r)=\mathcal{S}_{n\mathrm{PEP}}\iint\cdots\int W_{n\mathrm{PE}}(\vec{\mathrm{r}}_0\cdots\vec{\mathrm{r}}_n)\times\rho(\vec{\mathrm{r}}_0\vec{\mathrm{r}}_1\cdots\vec{\mathrm{r}}_n)d\vec{\mathrm{r}}_1\cdots d\vec{\mathrm{r}}_{n-1},\quad n>1. \tag{4.1}$$

Here $\vec{\mathrm{r}}=\vec{\mathrm{r}}_0-\vec{\mathrm{r}}_n$ and $\rho$ is the square of a correlated $(n+1)$-body wave function. The factor $\mathcal{S}_{n\mathrm{PEP}}$ represents an approximate treatment of exchange diagrams where the "spectator" nucleons are permuted among themselves, and was obtained in the previous section.

The integrations leave $\vec{\mathrm{r}}$ as the only spatial-vector dependence remaining in the expression for the potential. It can therefore be reduced to a central and tensor part, provided only that $\rho$ has no spin dependence.

In principle, we can transform to a domain with $\phi$, the axial angle about $\hat{r}$, as our first coordinate in the integration. We can then treat the integrations in a similar manner to those of the 3PEP. The expansion of (2.9) will give terms such as $\vec{\sigma}_0\cdot\hat{x}_i\hat{x}_j\cdot\vec{\sigma}_n$ with coefficients which are dot products of the $\vec{\mathrm{x}}_i$'s. These coefficients are independent of $\phi$. They depend only on the shape of the $(n+1)$-sided skew polygon with sides $r, x_1, x_2, \cdots, x_n$. Hence we can write

$$\int\vec{\sigma}_0\cdot\hat{x}_i\hat{x}_j\cdot\vec{\sigma}_n d\phi=[\tfrac{1}{3}\vec{\sigma}_0\cdot\vec{\sigma}_n\hat{x}_i\cdot\hat{x}_j+\tfrac{1}{3}S_{0n}(\hat{r})\cdot(\tfrac{3}{2}\hat{x}_i\cdot\hat{r}\hat{x}_j\cdot\hat{r}-\tfrac{1}{2}\hat{x}_i\cdot\hat{x}_j)]\int d\phi. \tag{4.2}$$

[See Ref. 2, Eqs. (4.5)–(7), and discussion.] As the $n$PEP is constructed without any parity violating features, the parity violating term involving $\vec{\sigma}_0\times\vec{\sigma}_n\cdot\hat{r}$ must vanish in the general case as it did for 3PEP; so we have omitted it from Eq. (4.2).

Rewriting (4.2) we see that $\vec{\sigma}_0\cdot\hat{x}_i\hat{x}_j\cdot\vec{\sigma}_n$ is replaced by

$$[\tfrac{1}{3}\vec{\sigma}_0\cdot\vec{\sigma}_n-\tfrac{1}{6}S_{0n}(\hat{r})]\,\mathrm{Tr}(\hat{x}_i\hat{x}_j)+[\tfrac{1}{2}S_{0n}(\hat{r})]\hat{r}\cdot(\hat{x}_i\hat{x}_j)\cdot\hat{r}. \tag{4.3}$$

We have written $\hat{x}_i\cdot\hat{x}_j$ as the trace of the tensor $\hat{x}_i\hat{x}_j$ to emphasize the general form of (4.3). The integration of (2.9) with respect to $\phi$ can now be done by using (4.3) to replace the outer vectors $\vec{\sigma}_0$ and $\vec{\sigma}_n$ in the product, giving

$$\int\mathcal{G}(\vec{\mathrm{x}}_1\vec{\mathrm{x}}_2\cdots\vec{\mathrm{x}}_n)d\phi=\{[\tfrac{1}{3}\vec{\sigma}_0\cdot\vec{\sigma}_n-\tfrac{1}{6}S_{0n}(\hat{r})]\mathcal{G}_A+[\tfrac{1}{2}S_{0n}(\hat{r})]\mathcal{G}_B\}\int d\phi, \tag{4.4}$$

where $\mathcal{G}_A$ and $\mathcal{G}_B$ are spin-independent geometric functions of the $\frac{1}{2}(n+1)(n+2)$ interparticle distances, given by

$$\mathcal{G}_A=\mathrm{Tr}\underline{G}^{(n)},\qquad \mathcal{G}_B=\hat{r}\cdot\underline{G}^{(n)}\cdot\hat{r}. \tag{4.5}$$

Collecting together central and tensor parts of (4.4) then gives

$$\int\mathcal{G}(\vec{\mathrm{x}}_1\vec{\mathrm{x}}_2\cdots\vec{\mathrm{x}}_n)d\phi=[G_c^{(n)}\vec{\sigma}_0\cdot\vec{\sigma}_n+G_t^{(n)}S_{0n}(\hat{r})]\int d\phi, \tag{4.6}$$

where

$$G_c^{(n)}=\tfrac{1}{3}\,\mathrm{Tr}\underline{G}^{(n)},\qquad G_t^{(n)}=\tfrac{1}{2}\hat{r}\cdot\underline{G}^{(n)}\cdot\hat{r}-\tfrac{1}{6}\,\mathrm{Tr}\underline{G}^{(n)}. \tag{4.7}$$

This enables us to separate Eq. (4.1) for $V_{n\mathrm{PE}}$ into central and tensor parts, given by

$$V_{n\mathrm{PE}}(r)=\vec{\tau}_0\cdot\vec{\tau}_n[\vec{\sigma}_0\cdot\vec{\sigma}_n V_c^{n\mathrm{PE}}+S_{0n}(\hat{r})V_t^{n\mathrm{PE}}], \tag{4.8}$$

where

$$V_{c,t}^{n\mathrm{PE}}(r)=\mathcal{S}_{n\mathrm{PEP}}f^2\mu\left[\frac{-\mu^3(A+B)}{2\pi}\right]^{n-1}\times\iint\cdots\int G_{c,t}^{(n)}\cdot\rho(\vec{\mathrm{r}}_0\vec{\mathrm{r}}_1\cdots\vec{\mathrm{r}}_n)d\vec{\mathrm{r}}_1\cdots d\vec{\mathrm{r}}_{n-1}. \tag{4.9}$$

## V. CONVERGENCE OF THE $n$PEP

In infinite nuclear matter, all possible many-body forces are present and must therefore contribute to the binding energy. One of the main reasons for calculating the binding energy contribution due to 3PEP in Ref. 2 was to examine the size of many-body effects and check that four-body forces were indeed smaller than three-body forces.

Using the $n$-body potential we have developed in the previous section, it is possible to estimate the ratio of the energy contributions due to $n$PEP to those due to $(n-1)$PEP.

Consider the function

$$\underline{J}^{(n)}=\iint\cdots\int\underline{G}^{(n)}\rho(\vec{\mathrm{r}}_0\cdots\vec{\mathrm{r}}_n)d\vec{\mathrm{r}}_1\cdots d\vec{\mathrm{r}}_{n-1}. \tag{5.1}$$

$\underline{G}^{(n)}$ is a function of the internucleon distances

$\vec{x}_1, \ldots, \vec{x}_n$ [Eq. (2.10)]. It is reduced to the two scalar functions $G_c^{(n)}$ and $G_t^{(n)}$ in (4.7), which are then integrated in (4.9) to give $V_{c,t}^{n\mathrm{PE}}$. Hence, $V_{c,t}^{n\mathrm{PE}}$ can be written as appropriate scalar reductions of $\underline{J}^{(n)}$. As the only spatial direction not integrated over in (4.1) is $\hat{r}$, the tensor dependence in $\underline{J}^{(n)}$ must be on $\underline{I}$ and $\underline{\hat{r}\hat{r}}$. It is easy to verify that

$$\underline{J}^{(n)}(\vec{r}) = \frac{3}{\alpha_n}\left[\tfrac{1}{3}(V_c^{n\mathrm{PE}} - V_t^{n\mathrm{PE}})\underline{I} + V_t^{n\mathrm{PE}}\underline{\hat{r}\hat{r}}\right], \tag{5.2}$$

using (4.7) and (4.9). The constant $\alpha_n$ is given by

$$\alpha_n = \mathcal{S}_{n\mathrm{PEP}} f^2 \mu \left[\frac{-\mu^3(A+B)}{2\pi}\right]^{n-1}. \tag{5.3}$$

Now observe that $\underline{G}^{(n)}$ can be factorized:

$$\underline{G}^{(n)} = \underline{G}^{(n-1)} \cdot \left[\tfrac{1}{3}(\bar{X}_n - \tilde{X}_n)\underline{I} + \tilde{X}_n \underline{\hat{x}_n \hat{x}_n}\right] Y_\mu(x_n), \tag{5.4}$$

and in the approximation of step function correlation functions $\rho$ is given by

$$\rho(\vec{r}_0 \vec{r}_1 \cdots \vec{r}_n) = (\bar{\rho})^{n-1} \prod_{i<j=0}^{n} \theta(|\vec{r}_i - \vec{r}_j| - c)$$
$$= \rho(\vec{r}_0 \cdots \vec{r}_{n-1})\bar{\rho}\,\theta(x_n - c)\,\theta(r - c)\Theta_n, \tag{5.5}$$

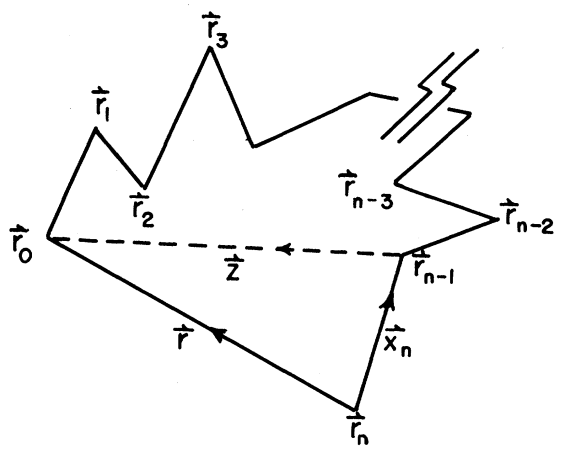



FIG. 2. Integration coordinates in the calculation of the effective potential for $n$PEP.

where

$$\Theta_n = \prod_{i=1}^{n-2} \theta(|\vec{r}_i - \vec{r}_n| - c). \tag{5.6}$$

($\bar{\rho}$ is the constant particle density of nuclear matter and $\vec{r} = \vec{r}_0 - \vec{r}_n$.) Substituting (5.4) and (5.5) into (5.1) gives

$$\underline{J}^{(n)} = \iint \cdots \int \underline{G}^{(n-1)} \cdot \left[\tfrac{1}{3}(\bar{X}_n - \tilde{X}_n)\underline{I} + \tilde{X}_n \underline{\hat{x}_n \hat{x}_n}\right] Y_\mu(x_n)\bar{\rho} \cdot \theta(r-c)\,\theta(x_n - c)\,\rho(\vec{r}_0 \cdots \vec{r}_{n-1})\Theta_n d\vec{r}_1 \cdots d\vec{r}_{n-2} d\vec{r}_{n-1}. \tag{5.7}$$

We now attempt the integrations over $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_{n-2}$. Since $\vec{x}_n$ is constant during these integrations, $\Theta_n$ is the only variable factor not present in $\underline{J}^{(n-1)}$. $\Theta_n$ prevents the points $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_{n-2}$ from being within a sphere of radius $c$ about the "external" point $\vec{r}_n$ (see Fig. 2).

As there are already $\frac{1}{2}n(n-1)$ $\theta$ functions in the integrations to produce $\underline{J}^{(n-1)}$, we neglect $\Theta_n$ containing $(n-2)$ $\theta$ functions, expecting it to be a second-order effect, and approximate (5.7) by

$$\underline{J}^{(n)}(\vec{r}) \approx \int \underline{J}^{(n-1)}(\vec{z})\left[\tfrac{1}{3}(\bar{X}_n - \tilde{X}_n)\underline{I} + \tilde{X}_n \underline{\hat{x}_n \hat{x}_n}\right] Y_\mu(x_n)\bar{\rho}\,\theta(x_n - c)\,\theta(r-c)\,d\vec{r}_{n-1}. \tag{5.8}$$

If we now substitute (5.2) for $\underline{J}^{(n-1)}$ we get

$$\underline{J}^{(n)}(\vec{r}) \approx \frac{3}{\alpha_{n-1}} \int \left\{\tfrac{1}{3}\left[V_c^{(n-1)\mathrm{PE}}(z) - V_t^{(n-1)\mathrm{PE}}(z)\right]\underline{I} + V_t^{(n-1)\mathrm{PE}}(z)\underline{\hat{z}\hat{z}}\right\}\left[\tfrac{1}{3}(\bar{X}_n - \tilde{X}_n)\underline{I} + \tilde{X}_n \underline{\hat{x}_n \hat{x}_n}\right] Y_\mu(x_n)\bar{\rho}\,\theta(x_n - c)\,\theta(r-c)\,d\vec{r}_{n-1}. \tag{5.9}$$

As $V_{c,t}^{(n-1)\mathrm{PE}}(z)$ contains a $\theta(z-c)$, the integration over $\vec{r}_{n-1}$ looks very much like the integration required[3,6] to get 2PEP:

$$\underline{J}^{(2)}(\vec{r}) = \int \left[\tfrac{1}{3}(\bar{Z} - \tilde{Z})\underline{I} + \tilde{Z}\underline{\hat{z}\hat{z}}\right] Y_\mu(z)\left[\tfrac{1}{3}(\bar{X}_n - \tilde{X}_n)\underline{I} + \tilde{X}_n \underline{\hat{x}_n \hat{x}_n}\right] Y_\mu(x_n)\bar{\rho}\,\theta(r-c)\,\theta(z-c)\,\theta(x_n - c)\,d\vec{r}_{n-1}. \tag{5.10}$$

After one further approximation, we can prove by induction that $V_{c,t}^{n\mathrm{PE}}$ are (approximately) a constant factor times $V_{c,t}^{(n-1)\mathrm{PE}}$. The necessary approximation is

$$V_c^{2\mathrm{PE}}(r) \approx C^{(2)} \bar{R} Y_\mu(r)\theta(r-c),$$
$$V_t^{2\mathrm{PE}}(r) \approx C^{(2)} \tilde{R} Y_\mu(r)\theta(r-c), \tag{5.11}$$

where $C^{(2)}$ is a constant. $\bar{R}$ and $\tilde{R}$ are defined similarly to $\bar{X}_t$ and $\tilde{X}_t$ in Eqs. (2.11) and (2.12). If $C^{(2)}$ were replaced by $C^{(1)} = \frac{1}{3} f^2 \mu$ (3.68 MeV), Eqs. (5.11) would be the equations for the central and tensor parts of OPEP with form factors included (compare with 2.13). As the 2PEP effective potentials calculated with cutoff correlations have the same shape as OPEP for form factors II and III, the approximation should be quite good for the central part (see Fig. 3). Looking at the OPEP and 2PEP potentials we estimate that

$$C^{(2)} \simeq 1.7 \text{ MeV (central)}$$
$$\simeq 0.7 \text{ MeV (tensor)}. \tag{5.12}$$

The approximation (5.11) is much better for the central part of the potential than the tensor part. However, the central and tensor components contribute approximately equally to the binding energy in both the 2PEP and 3PEP cases. The constant $C^{(2)}$ should therefore be chosen somewhere in the range 0.7 to 1.7 MeV to estimate ratios of binding energies.

We are now ready to proceed with a proof by induction. Our induction hypothesis is that

$$V_c^{(n-1)\mathrm{PE}}(r) = C^{(n-1)} \bar{R} Y_\mu(r)\theta(r-c),$$
$$V_t^{(n-1)\mathrm{PE}}(r) = C^{(n-1)} \tilde{R} Y_\mu(r)\theta(r-c), \tag{5.13}$$

where $C^{(n-1)}$ is some constant. Then (5.9) and (5.10) give

$$\underline{J}^{(n)}(r) = \frac{3}{\alpha_{n-1}} C^{(n-1)} \underline{J}^{(2)}(r), \tag{5.14}$$

since (5.9) with the hypothesis (5.13) is precisely the three-body integration with cutoffs required for the evaluation of 2PEP.

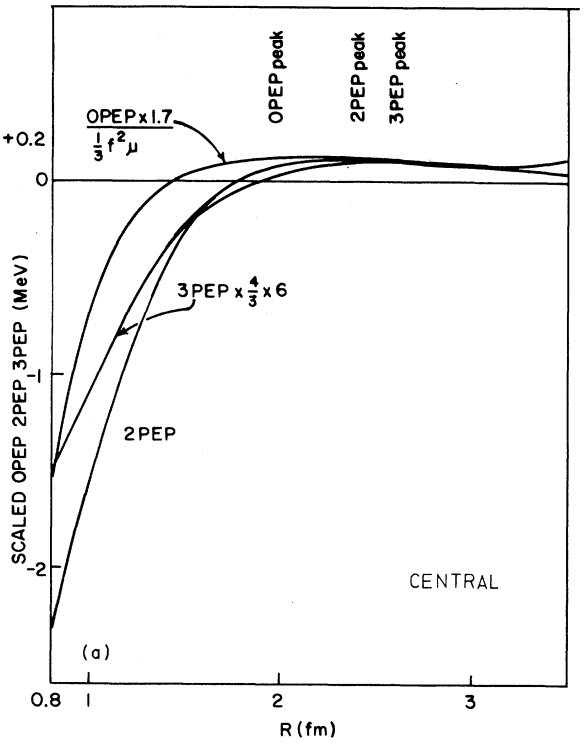
Using Eqs. (4.9) and (4.7) we then find that

$$V_{c,t}^{n\mathrm{PE}}(r) = 3 \frac{\alpha_n}{\alpha_{n-1}} C^{(n-1)} \frac{1}{\alpha_2} V_{c,t}^{2\mathrm{PE}}(r). \tag{5.15}$$

Hence, using (5.11) as a starting point, we have proved by induction on $n$ that there are proportionality constants $C^{(n)}$ to make (5.13) valid for all $n \geq 2$.

Finally, a comparison of (5.11) and (5.13) together with (5.3) allows us to write (5.15) as

$$\frac{V_{c,t}^{n\mathrm{PE}}(r)}{V_{c,t}^{(n-1)\mathrm{PE}}(r)} = \frac{\mathcal{G}_{n\mathrm{PEP}}}{\mathcal{G}_{(n-1)\mathrm{PEP}}} \frac{C^{(2)}}{\frac{1}{3} f^2 \mu}. \tag{5.16}$$



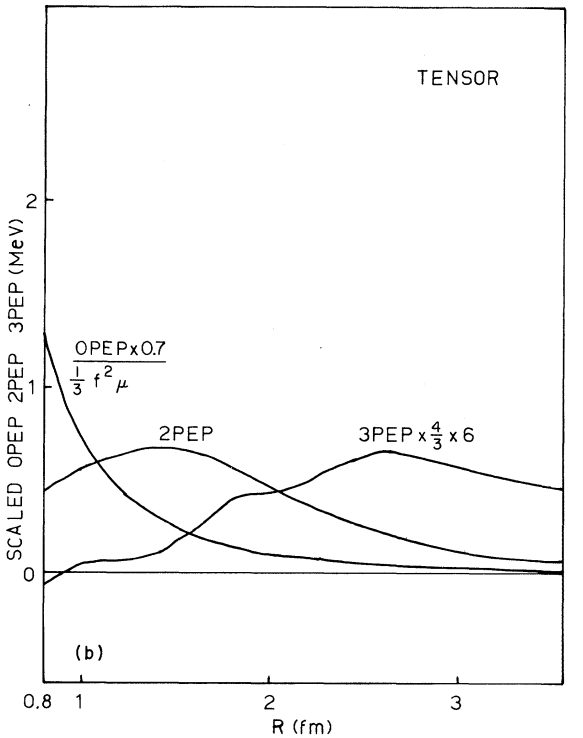




FIG. 3. Comparison of (a) the central part, and (b) the tensor part of the effective potential for OPEP, 2PEP, and 3PEP. Form factor II with a cutoff at 0.8 fm was used in the calculations.

The convergence ratio (5.16) can now be calculated using (5.12). For $C^{(2)}$ in the range 0.7 to 1.7 MeV, this ratio is between

$$0.2\mathcal{E}_{n\mathrm{PEP}}/\mathcal{E}_{(n-1)\mathrm{PEP}} \text{ and } 0.5\mathcal{E}_{n\mathrm{PEP}}/\mathcal{E}_{(n-1)\mathrm{PEP}} \quad (n>2). \tag{5.17}$$

The factor $\mathcal{E}_{n\mathrm{PEP}}/\mathcal{E}_{(n-1)\mathrm{PEP}}$ due to exchange diagrams was discussed in Sec. III; it was shown there that this factor causes the convergence ratio to drop to almost zero for $n \gtrsim 5$. For $n=3$, the factor is $\frac{3}{4}$.

Figure 3 shows the OPEP, 2PEP, and 3PEP central potentials with cutoff 0.8 fm and form factor II. The potentials are scaled so that the positive peaks in OPEP and 3PEP are close to the size of the 2PEP positive peak (0.12 MeV).

If the central and tensor parts of $n$PEP were really a constant factor times the corresponding parts of $(n-1)$PEP (i.e., independent of $r$), the ratio of the first-order energy contributions $\Delta E^{(1)}$ and the second-order cross terms $\Delta E_1^{(2)} = 2\left[V^{\mathrm{OPEP}}(Q/e)V^{n\mathrm{PEP}}\right]$ would also be given by this factor. The ratio for $\Delta E_2^{(2)} = \langle V^{n\mathrm{PEP}}(Q/e)V^{n\mathrm{PEP}}\rangle$ would be given by the square of this factor. A comparison of the actual results[4,6] for 3PEP and 2PEP gives the ratios

$$\begin{aligned} \Delta E^{(1)} &: 0.2\mathcal{E}_{3\mathrm{PEP}}/\mathcal{E}_{2\mathrm{PEP}}\,, \\ \Delta E_1^{(2)} &: 0.1\mathcal{E}_{3\mathrm{PEP}}/\mathcal{E}_{2\mathrm{PEP}}\,, \\ \Delta E_2^{(2)} &: 0.1\mathcal{E}_{3\mathrm{PEP}}/\mathcal{E}_{2\mathrm{PEP}}\,. \end{aligned} \tag{5.18}$$

## VI. CONCLUSION

In calculating the convergence ratios we have made two approximations. The first was to neglect $\Theta_n$ in the integrations over $\vec{r}_1\vec{r}_2\cdots\vec{r}_{n-2}$, which then gave $\underline{J}^{(n-1)}$. The inclusion of $\Theta_n$ in this integration would remove a sphere of radius $c$ about $\vec{r}_n$ from the domain in general. This would presumably reduce the result of (5.17) to a value similar to (5.18) for the 3PEP integration. The second approximation we have made was to say that 2PEP is proportional to OPEP with form factors included [Eqs. (5.11)]. This approximation is quite good for the central part of the potential, but is not nearly as good for the tensor part and, as we have seen, different proportionality constants are required in the two cases. This will introduce some error, and our final convergence ratio should not be expected to be more accurate than this difference in the constants $C^{(2)}$ of Eq. (5.12). However, the convergence ratios calculated are convincingly smaller than one. This is more spectacularly so when exchange diagrams are included. These exchanges cause strong cancellation so that for $n \gtrsim 5$ smaller order effects begin to dominate. This enhances the convergence considerably.

## ACKNOWLEDGMENTS

We wish to acknowledge helpful discussions with Dr. R. G. L. Hewitt. This work was supported in part by the Science Foundation for Physics within the University of Sydney and we are grateful to Professor H. Messel for his encouragement.

---

*Present address.

[1]B. H. J. McKellar and R. Rajaraman, Phys. Rev. Lett. 31, 1063 (1973); Phys. Rev. C 10, 871 (1974).
[2]D. W. E. Blatt and B. H. J. McKellar, Phys. Rev. C 11, 2040 (1975).
[3]B. A. Loiseau, Y. Nogami, and C. K. Ross, Nucl. Phys. A165, 601 (1971).
[4]D. W. E. Blatt and B. H. J. McKellar, Phys. Rev. C 11, 614 (1975).
[5]H. Miyazawa, Phys. Rev. 104, 1741 (1956).
[6]D. W. E. Blatt and B. H. J. McKellar, Phys. Lett. 52B, 10 (1974).
[7]R. K. Bhaduri, Y. Nogami, and C. K. Ross, Phys. Rev. C 2, 2082 (1970).
[8]R. Rajaraman and R. K. Trehan, Phys. Rev. A 5, 392 (1972).