Coulomb corrections for extracting spectroscopic factors using analyticity*

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The Coulomb scattering and Coulomb distortions are included in the optimal expansion technique in the $\cos\theta$ plane. From p-^{3}He elastic scattering from 4–20 MeV, an energy independent spectroscopic factor for $^3\mathrm{H}_e \to p+d$ is found, with an accuracy of a few percent. The method seems most promising for transfer reactions.

[NUCLEAR REACTIONS Dispersion relations, Coulomb corrections for $\cos\theta$ analyticity.]

I. INTRODUCTION

The amplitude of the asymptotic wave function in a breakup channel of a nucleus (say $X \to b+Y$) is an important nuclear structure property. The square of this amplitude, which we call a spectroscopic factor, appears in exchange Born terms in the amplitude for elastic scattering ($X+Y$ elastic scattering with b transfer) and gives characteristic peaks in the backward scattering. It also appears in the Born approximation for transfer cross sections ($X+A \to Y+B$ with b transferred) and gives characteristic transfer peaks. It was suggested long ago to use these features to extract spectroscopic factors directly from the data.[1] However, the experience since then has shown that the distorting effects are so large that one cannot do this directly; e.g., one cannot simply extrapolate the data to pick out the Born contributions.

From the point of view of the analytic properties of the scattering amplitudes these Born terms are poles with the residues proportional to the spectroscopic factors. In this language the problem of extracting spectroscopic factors is finding the residues of poles of scattering amplitudes in the background of other singularities—namely, cuts and other poles. Recently, it was suggested[2] that one could carry out such a procedure through the techniques of optimal expansions[3] which are being used in particle physics. The method was applied to n-d and n-^{4}He scattering with what seems to be consistent and reasonable results. It is the purpose of the present paper to study the inclusion of Coulomb effects so that one can treat the charged particle scattering and reactions with this method of conformal mapping.

The three-body problem offers an interesting test of the method. For p-^{3}He and n-^{3}H scattering the only transfer pole arises from deuteron transfer, and a successful application would be encouraging for application to many-nucleon transfer reactions. The theoretical value of the spectroscopic factor for the triton $^3\mathrm{H} \to d$-n has been obtained from accurate calculations using "realistic" two-body potentials, and it has been pointed out that this is an important and useful property of the three-body system.[4] The spectroscopic factor has also been determined from photodisintegration.[5]

In studies of the three-body spectroscopic factor the technique of Refs. 2 and 3 was applied to p-^{3}He elastic scattering[6] and n-^{3}He scattering.[7] Also, forward dispersion relations,[8] peripheral model calculations,[9,10] and partial wave dispersion relations,[11,12] as well as numerous conventional distorted wave Born approximation (DWBA) determinations have been made[13] for these three-body spectroscopic factors. The various determinations are not completely consistent.

The results of the conformal mapping calculation[6] were that the cut contributions cause the peripheral model[10] or forward dispersion relation calculation[8] to underestimate the spectroscopic factor in the three-body case. Although the results of Ref. 6 were consistent with the theoretical value of the spectroscopic factor,[4] the analysis of the p-^{3}He angular distributions gives an energy-dependent spectroscopic factor which was attributed to Coulomb effects. In the present paper we include the Coulomb effects to see if one can extract an energy-independent spectroscopic factor with the optimal expansion techniques. The method is seen to offer promise as a practical tool for analyzing data.

II. COULOMB CORRECTIONS

Coulomb corrections to pole diagrams have long been studied[14] and the complication of the Coulomb cut for nuclear applications of dispersion rela-

tions has been noted since the early attempts to use these methods[15,16]; of course the Coulomb effects are included in standard DWBA calculations. Coulomb distortions have been included in the peripheral model calculations for p-^{3}He scattering,[17] and recently for the study of the ^{2}H$(d,n)^3$He reaction, and also have been included in forward dispersion relation calculations.[18] We now show how we can include such effects for an analysis in the $\cos\theta$ plane as in Refs. 2 and 6.

There are many advantages in working in the $\cos\theta$ plane; the most important is that there is a clear physical signature for the pole term from the angular distribution. Also, one can obtain a spectroscopic factor for a variety of energies and check the consistency and accuracy of the results. However, since the conformal mapping technique makes use of the analytic structure of the amplitude near the physical region, it is essential to be able to treat the Coulomb cut accurately. Here we treat the elastic scattering, which is the more difficult case since there are cross terms between the Coulomb and strong amplitudes which are absent in transfer reactions (at least in the DWBA).

A. General considerations

Let us consider the elastic scattering $N+A \to N+A$, where N is an N-nucleon and A is an A-nucleon system. The general form of the amplitude for elastic scattering is

$$f_{\alpha'\alpha}(\theta) = f^c(\theta)\delta_{\alpha',\alpha} + f^{s(c)}_{\alpha'\alpha}(\theta)\,, \tag{1}$$

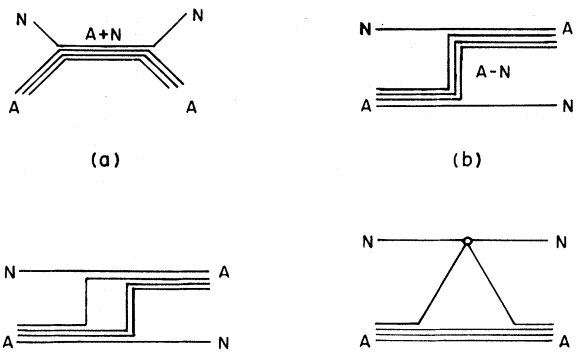

FIG. 1. Origin of nearest singularities in the scattering amplitude. The pole for $\cos\theta > 1$ arise from compound states (a) and transfer processes (b). The left-hand cut in $\cos\theta$ arises from continuum transfer (c) and the nearest right-hand cut from nucleon-nucleon scattering (d). Meson degrees of freedom are not explicitly shown.

where $f^c(\theta)$ is the Coulomb amplitude,

$$|f^c|^2 = \eta^2/4k^2(1-\cos\theta)^2$$

with $\eta = \mu Z_N Z_A e^2/\hbar^2 k$, μ the reduced mass, and k the center of mass momentum. In Eq. (1) $f^{s(c)}_{\alpha'\alpha}(\theta)$ is the nuclear amplitude modified by the Coulomb distortions with α and α' the spin, orbital, and projection quantum numbers. The charges of N and A are $Z_N e$ and $Z_A e$, respectively.

In the absence of Coulomb interactions the strong amplitude denoted as $f^s(\theta)$ has singularities nearest to the physical region corresponding to physical processes depicted in Fig. 1. Neglecting explicit meson degrees of freedom (mesons are included as form factors only) the poles correspond to s-channel (compound nucleus) bound states [Fig. 1(a)] and the transfer of a bound system [Fig. 1(b)]. The compound nuclear states of the $A+N$ system correspond to fixed energy poles with laboratory energy at the pole

$$E_{\text{pole}}\,(\text{compound nucleus}) = [M_{A+N}{}^2 - (M_N + M_A)^2]/2M_A\,. \tag{2}$$

These fixed energy poles are not of great interest for the $\cos\theta$ plane analysis, one of the characteristic features of the method. The transfer poles [Fig. 1(b)] give a pole in the $\cos\theta$ plane at fixed $u = M_{A-N}{}^2$, where u is the standard Mandelstam variable. In the $\cos\theta$ variable the pole is at

$$\cos\theta_p = [(M_A - M_N)^2 - 2M_A E_{\text{lab}} - (M_{A-1})^2]/2k^2 - 1\,, \tag{3}$$

where k is the center of mass momentum and E_{lab} is the laboratory energy. The nearest branch points correspond to diagrams of Figs. 1(c) and 1(d). Figure 1(c) depicts the transfer of a continuum state, characterized by a branch point with a minimum value of $u = u_c$, and a "left-hand" branch cut (i.e., $\cos\theta < -1$ along this cut). In the $\cos\theta$ plane the branch point is

$$\cos\theta_u = [(M_A - M_N)^2 - 2M_N E_{\text{lab}} - u_c]/2k^2 - 1. \tag{4}$$

The "right-hand" branch cut has a branch point given by the form factor [Fig. 1(d)] with a fixed t value t_c. The branch point is thus

$$\cos\theta_t = -1 + t_c/2k^2 \tag{5}$$

and $\cos\theta > 1$ along this cut. Details of the singularity structure will be given in another publication.[19] They are not needed to understand the present work.

The method of maximal convergence of Ref. 3 for the nuclear problem consists of mapping these two branch cuts onto an ellipse as described in

Ref. 2. As a result, the only singularities within the ellipse (for no Coulomb interaction) are the transfer poles, so for the case of a single pole of the type shown in Fig. 1(b)

$$f^s(\theta)=\frac{\Gamma^s}{\cos\theta-\cos\theta_p}+g^s(\theta)\,, \tag{6}$$

where $g^s(\theta)$ is regular except for the cuts. The residue of the pole Γ^s is given by the amplitude of the asymptotic part of the wave function which describes the relative $N-(A-N)$ system, i.e., defining r as the relative coordinate between the N and $(A-N)$ system, the asymptotic wave function of the system in the $A-N$, N channel as $r\to\infty$ is given by

$$\phi_A(r)\underset{r\to\infty}{\longrightarrow}\mathfrak{N}e^{-kr}/r[Y_lX]^J\,, \tag{7}$$

where l is the orbital angular momentum, J the total angular momentum, and X the spin wave function. The spectroscopic factor is

$$\text{Spectroscopic Factor}=\mathfrak{N}^2 \tag{8a}$$

and

$$\Gamma^s\propto\mathfrak{N}^2\,. \tag{8b}$$

The result (8b) can be understood from the expression (7) by recognizing that the vertex $A\to N+(A-N)$ is essentially the Fourier transform of the wave function $\phi_A(N,A-N)$, and that the Fourier transform of Yukawa e^{-kr}/r has a pole at $k^2+\kappa^2=0$. In other words,[20] for the case $l=0$

$$\sqrt{\Gamma^s}\propto\lim_{k\to\kappa}(k^2+\kappa^2)\int e^{-i\vec{k}'\cdot\vec{r}}\,\frac{e^{-\kappa r}}{r}\,e^{i\vec{k}\cdot\vec{r}}d^3r\,. \tag{9}$$

The form (9) is a useful one for obtaining the Coulomb correction to the residue of the pole. More precisely, in the presence of the Coulomb interaction a cut develops at that point. The residue of the pole is modified, so that even if one can successfully extract the residue, it is necessary to reinterpret the result. Just as in the DWBA, one can do this by using Coulomb wave functions, i.e., the plane waves $e^{i\vec{k}\cdot\vec{r}}$ and $e^{-i\vec{k}'\cdot\vec{r}}$ are replaced by appropriate Coulomb wave functions in the integral (9). This has been carried out explicitly in Ref. 14, and the explicit modification of the spectroscopic factor for the cross section $|f^{s(c)}|^2$ has been given in Ref. 16. One finds for elastic scattering, if the orbital angular momentum is zero,

$$\Gamma^{s(c)}=\rho(E)\Gamma^s \tag{10a}$$

with

$$\rho(E)=\frac{4\pi\eta}{e^{4\pi\eta}-1}\,e^{4\eta(\pi-\phi)}\,,\qquad \tan\phi=-2k/\kappa\,, \tag{10b}$$

where k, κ, and η are as previously defined. Note that $\kappa^2=-2\mu B/\hbar^2$, where B is the $N-(A-N)$ binding energy in the A nucleus and μ is the reduced mass. The expression for the integral over Coulomb wave functions is considerably more complicated for nonzero orbital angular momentum. In that case the integral (9) contains a Y_{lm}. Expressions for integrals of $(e^{-\kappa r}/r)Y_{lm}$ over Coulomb wave functions have been given,[21] with expressions becoming rapidly complicated as l becomes larger. Here we need only the $l=0$ result, but the extension to $l\neq0$ does not present an essential difficulty. From Eq. (10) one sees that the effect of the Coulomb distortion of the pole can be quite simply represented by the Coulomb distortion factor $\rho(E)$ in this approximation.

B. Case of p-^{3}He elastic scattering

The p-^{3}He system exists in a singlet and triplet state so that the unpolarized cross section is given as

$$\sigma(\theta)=|f^c|^2+f^c{*}f^{s(c)}+f^c f^{s(c)*}+\sigma^{s(c)}(\theta) \tag{11}$$

with

$$f^{s(c)}=\sum_{JII'SS'MM'}f^{s(c)J}_{l'S'M',lSM} \tag{11'}$$

in a representation of channel spin S, orbital angular momentum, and total spin J and projection M. Because of the cross terms $f^c{*}f^{s(c)}+f^cf^{s(c)*}$ the amplitudes $f^{s(c)J}_{l'S'M',lSM}$ are needed for the analysis. For other reactions, such as spin flip, elastic, pickup, and so forth these terms are not present since f^c cannot contribute and one deals solely with $\sigma^{s(c)}(\theta)$. For the present case of elastic scattering a phase shift analysis is needed for the cross terms. We can treat this case as a test of the method for handling Coulomb corrections and extracting spectroscopic factors by using the extensive phase shift analyses which have been carried out previously. Then using experimental cross sections for $\sigma(\theta)$, the expression

$$\sigma^{\exp}(\theta)-f^c{*}f^{s(c)}-f^cf^{s(c)*}-|f^c|^2\equiv S^{\exp}(\theta)=\sigma^{s(c)}(\theta) \tag{12}$$

is treated as in Ref. 6. One looks for the coefficients A_i in the expansion

$$k^4(\cos\theta-\cos\theta_p)^2S^{\exp}(\theta)=\sum_{i=0}^{P}A(i)P_i(\cos\theta)\,, \tag{13}$$

where $P_i(\cos\theta)$ are polynomials orthogonalized over the data. The latter are defined in the stan-

dard way by

$$\sum_n w_n P_\alpha(\cos\theta_n) P_\beta(\cos\theta_n) = \delta_{\alpha\beta} \tag{14}$$

in which the $\cos\theta_n$ are the experimental angles and the weights w_n are given by the errors. The mapping technique involves the replacement

$$\cos\theta \to Z(\theta)\,, \tag{15}$$

where $Z(\theta)$ is chosen[3] to map the cuts described in the last section onto an ellipse, with the branch points $\cos\theta_n$ and $\cos\theta_t$ the terminuses of the major axis. The parameters A_i are extracted from which one obtains[19] the spectroscopic factor:

$$k^4(\cos\theta - \cos\theta_p) S(\theta) \cos\theta \to \cos\theta_p = 3.8 \mathfrak{N}^4 \rho(E). \tag{16}$$

The units used in Eq. (16) are mb for $\sigma(\theta)$ and $S(\theta)$, GeV/c for k, and GeV/c for $\mathfrak{N}$. It is convenient to introduce the dimensionless spectroscopic factor $|C|^2$ of which Kim and Tubis[4] have discussed the various results. The relation between the parameters $\mathfrak{N}$ and C as defined in Eq. (7) and Ref. 4, respectively, is[22]

$$|\mathfrak{N}|^2 = \sqrt{2\mu B}\ |C|^2\,, \tag{17}$$

where B is the deuteron-nucleon binding energy in ^{3}He.

III. RESULTS AND CONCLUSIONS

In this section we give the results of the analysis of p-^{3}He elastic scattering.

Elastic p-^{3}He scattering data which is used here has been obtained for energies between 4.00 and 19.48 MeV,[23-25] and phase shifts have been determined for the energies 4.00, 5.51, 6.82, 10.77, and 19.48.[26,27] Using a Bessel interpolation from these known phase shifts to find phases for the energies 4.55, 6.52, 7.51, 8.51, 9.57, 10.38, and 11.48 MeV, we find that one can adequately fit the experimental cross sections. Although it is not possible to find accurate phase shifts for 13.6 and 16.232 MeV using this method, phases which most nearly reproduce the data for these energies can be used. This is adequate because the Coulomb subtractions become unimportant for energies greater than ~10 MeV.

The results of the analysis are shown in Figs. 2–5. At each energy the point shown is the value

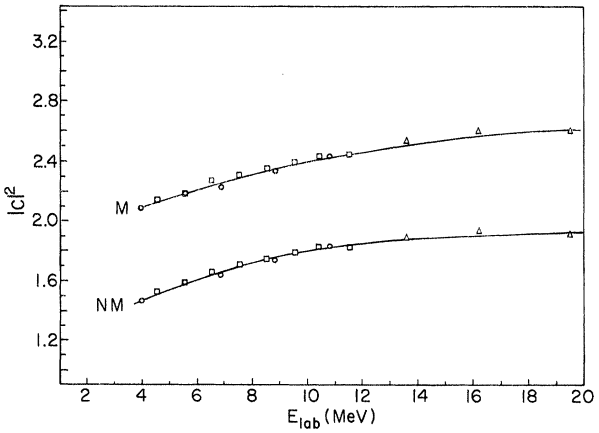

FIG. 2. The quantity $|C|^2$, defined in the text, extracted from the data from Wisconsin (Ref. 23) (○), Rice (Ref. 24) (□), and Los Alamos (Ref. 25) (△). Each point corresponds to an analysis at a fixed energy with the mapping procedure used for results along the upper curve (M) and without our mapping for results along the lower curve (NM). All Coulomb effects are neglected.

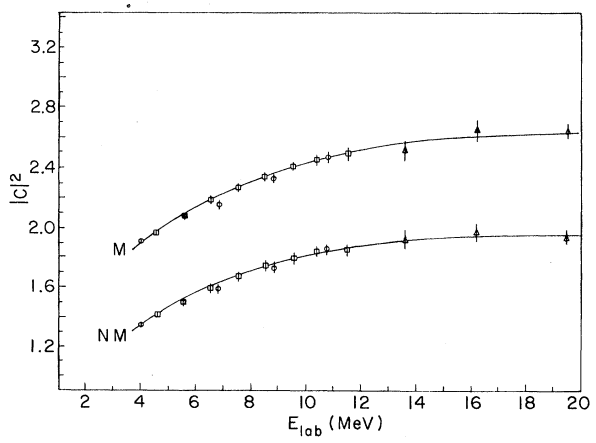

FIG. 3. The same as Fig. 2 but with a subtraction of the Coulomb scattering, including crows terms with the strong amplitude (see text).

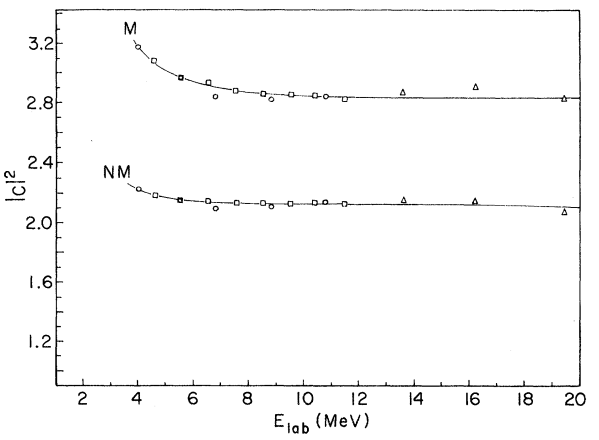

FIG. 4. The same as Fig. 2 but with the Coulomb distortion of the pole term included. Coulomb scattering is neglected (see text).

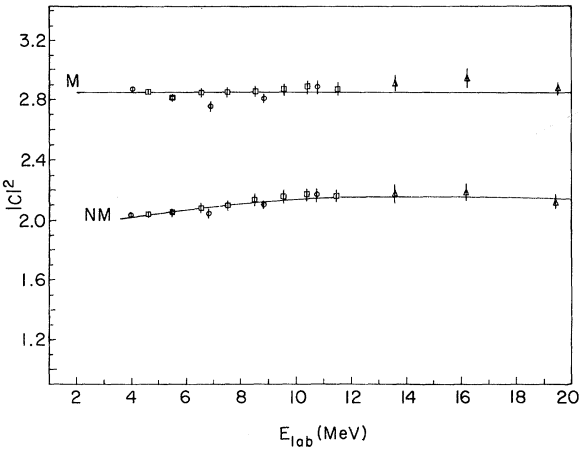

FIG. 5. The same as Fig. 2 but Coulomb scattering and distortion both included. The result labeled M, $|C|^2 = 2.85 \pm 0.07$, is the final result of the paper.

of the spectroscopic factor $|C|^2$ [defined in Eq. (17)] which is found by an analysis of the angular distribution as described in Sec. II. For comparison with the theoretical values[22,4] and the dicussion in Ref. 4, we use the experimental value of 5.49 MeV for the binding energy B, rather than the value 4.47 MeV [in Eq. (17)] which was used in Ref. 6 to compare with the Kim-Tubis result. In each figure the line drawn through the points is by eye and labeled M and NM corresponding to the results obtained with the mapping procedure and without, respectively.

Figure 2 shows the value of $|C|^2$ derived directly from the experimental cross sections with and without the mapping but without any Coulomb corrections. These results are just those of Ref. 6. The energy dependence of $|C|^2$ shows that the method is certainly not adequate below 10–15 MeV and casts doubt on the results even at 15–20 MeV.

The effect of the Coulomb scattering (but not distortion of the vertex) is shown in Fig. 3. The results in that figure are obtained by using the modified cross section $S(\theta)$ of Eq. (12), but neglecting the Coulomb modification of the pole term [taking $\rho(E) = 1$ in Eq. (16)]. The energy dependence of the $|C|^2$ remains, and in fact, is even a little stronger than before this correction. The effect of the Coulomb distortion at the pole term is shown in Fig. 4, which is obtained from the results shown in Fig. 2 simply by multiplying each value of $|C|^2$ by the Coulomb distortion factor $\rho(E)$ [Eq. (10)]. The resulting spectroscopic factors become approximately energy-independent above 8 MeV; however, there is still a considerable energy dependence below that energy.

The final results, including both the effects of Coulomb scattering and Coulomb distortion of the pole are shown in Fig. 5. Here one uses the modified cross sections (12) and includes the Coulomb distortion factor $\rho(E)$. Using a $\cos\theta$ analysis (no mapping) the spectroscopic factor still has an energy dependence and is considerably too small. With the mapping procedure, however, one obtains an essentially energy-independent result. This means that the cut contributions which are producing the difference between the M and NM results have an energy dependence.

The error bars which appear in Figs. 3 and 5 are derived from the estimated errors in the computed phase shifts[26,27] as a consequence of the errors in the cross sections. They do not reflect further errors which arise from the fitting procedure used to extract the spectroscopic factors. Thus these must be regarded as the minimum errors in the spectroscopic factors at each point. From Fig. 5 curve M we conclude that the ^{3}He $\rightarrow p + d$ spectroscopic factor $|C|^2$ is 2.85 ± 0.07. The numerical value of this result is in agreement with the value of 2.86 ± 0.03 obtained by Kim and Tubis.[4] However, there is some ambiguity in this comparison as these authors obtained a binding energy of 4.47 MeV in their calculation, which they used in finding $|C|^2$. Note also that in Ref. 6 the binding energy B of Eq. (17) was taken as 4.47 MeV to obtain $|C|^2$ from $|\mathfrak{N}|^2$ (which is unambiguously defined), so that our present result is actually about 10% larger than without Coulomb corrections (compare Figs. 2 and 5). Our results are consistent with Lim,[5] but smaller than Borbely *et al.*[9,17]

From the results in Figs. 2–5 it is seen that while only the complete analysis yields the energy-independent $|C|^2$ of Fig. 5, the use of the Coulomb distortion factor $\rho(E)$ and optimal expansion leads to an unambiguous value of $|C|^2$ which is correct for energies $\gtrsim 8$ MeV (Figs. 4 and 5). This suggests that for elastic scattering on light nuclei it is possible to treat Coulomb effects by the Coulomb distortion factor, including Coulomb scattering. This most certainly would not be true for heavier nuclei, where the Coulomb scattering is large, and one needs a representation for the amplitude in order to calculate the cross terms.

We conclude that one can satisfactorily include Coulomb effects and extract spectroscopic factors for charged particle reactions by using the analytic properties of the amplitudes in the $\cos\theta$ plane. The results are most promising for transfer reactions where one does not need cross terms with the Coulomb amplitude. The present work suggests that one can directly extract spectroscopic factors from the experimental data in such cases.

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