Comparison of realistic local and one-boson-exchange potentials in the three-nucleon system*

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(Received 1 April 1975)

The disagreement that currently exists regarding the calculated and experimental values of the triton binding energy (E_T) and the doublet scattering length (^2a) is analyzed in considerable detail. Using the unitary pole approximation, which has previously been tested with great success, we have calculated E_T and ^2a for a large class of realistic potentials. The results show that the one-boson-exchange potentials (OBEP) give in general ~ 1 MeV more binding for ^{3}H and correspondingly smaller ^2a than the local hard- and soft-core potentials. This difference is shown to be partly due to the fact that the OBEP have in general smaller deuteron D-state probability, and partly due to the fact that they fit the $^1S_0$ n-p rather than the p-p data. Finally, we demonstrate that the correlation between E_T and ^2a, predicted by Phillips on the basis of separable potential calculations, has validity for the more realistic interactions.

[NUCLEAR STRUCTURE ^{3}H; calculated binding energy, doublet scattering length. Faddeev approach, realistic N-N interactions.]

I. INTRODUCTION

The binding energy of ^{3}H, predicted by nucleon-nucleon (N-N) interactions, has been the subject of intensive study in recent years, and definitive results now exist for several realistic models.[1] These results show that the calculated triton binding energy (E_T) is some 1.5 MeV below the experimental value of $E_T = 8.48$ MeV, and the suggestion has been proposed that this discrepancy is indicative of a substantial contribution from three-body forces. However, such a large contribution from many-particle forces in a system as loosely bound as the triton seems very unlikely, but if substantiated, would have far reaching implications for calculations involving heavier nuclei. One question that has not been adequately studied is the possibility of explaining part of this difference on the basis of certain premises used in the construction of most nucleon-nucleon potentials, and it is this approach that we shall consider.

From basic symmetry considerations, the general form of a N-N interaction may be written in coordinate space to order p^2 as

$$V(r) = V_C(r) + V_T(r)S_{12} + V_{LS}\vec{L}\cdot\vec{S} + V_{LL}(r)L_{12} + V_{NL}\,, \tag{1}$$

where $\vec{L}\cdot\vec{S}$ and S_{12} have the usual meaning and where L_{12} involves combinations of $\vec{\sigma}_1\cdot\vec{\sigma}_2$, $\vec{L}^2$, and $(\vec{L}\cdot\vec{S})^2$. The momentum dependent, or nonlocal term V_{NL}, has the form

$$V_{NL} = \nabla^2 V_p(r) + V_p(r)\nabla^2\,. \tag{2}$$

Potentials of the Hamada-Johnston and Reid type are obtained from the general expression (1) by assuming a functional form for the $V_C(r)$, $V_T(r)$, $V_{LS}(r)$, and $V_{LL}(r)$ which involves unknown parameters. The numerical values of these parameters are then determined using the two-nucleon experimental data, in conjunction with the theoretical one-pion-exchange potential. The momentum dependent term V_{NL} is neglected in this procedure, making the resultant potential *local in each partial wave*. It has been clearly demonstrated that this phenomenological approach allows the existing N-N data to be quantitatively described with the only remaining uncertainty being the type of core used to parametrize the short-range interaction. Typically, we can have soft-core, super soft-core, or hard-core potentials corresponding to the detailed behavior at short distances. In this article we demonstrate that the assumed core is not the feature of the interaction that can account for the failure of these phenomenological potentials to reproduce the experimental triton binding energy and doublet n-d scattering length.

The remaining assumption to be tested is then clearly the neglect of the momentum dependence V_{NL}. It is quite feasible that while the neglect of this term allows one to fit the on-shell scattering data, the resulting potential may not provide an accurate method of extrapolating the fully off-shell scattering amplitude or T matrix. This would then account for the failure of phenomenological potentials in nuclear binding energy calculations which depend directly on the off-shell two nucleon T matrix.

The possibility of testing this conjecture is provided by the more fundamental theories which attempt to explain the N-N interaction using the mechanism of meson exchange. In particular, it

TABLE I. The potentials considered in this investigation of the trinucleon.

Potential	Type	Abbreviation	Reference
Reid hard core	hard core	RHC	3
Hamada-Johnston	hard core	HJ	4
Yale	hard core	Yale	5
Reid soft core	soft core	RSC	3
Alternate Reid soft core	soft core	RSCA	3
Tourreil-Sprung A	super soft core	TSA	6
Tourreil-Sprung B	super soft core	TSB	6
Tourreil-Sprung C	super soft core	TSC	6
Bryan-Scott	velocity dependent	BS	7
Bryan-Gersten	velocity dependent	BG	8
Stagat, Riewe, and Green	velocity dependent	SRG	9
Ueda-Green II	velocity dependent	UGII	10

has been shown that the exchange of pseudoscalar, vector, and scalar mesons accounts for most, if not all, of the important features of the internucleon force.[2] Moreover, the exchange of the latter two types of mesons leads to the term V_{NL}. While there are still difficulties in giving a precise formulation of the N-N interaction using this model, problems of unitarization (or cutoff) and the exact nature of the uncorrelated multimeson exchange contribution, several groups have calculated the nuclear force using the one-boson-exchange (OBEP) prescription and the success achieved is very satisfactory. The number of free parameters is substantially reduced in comparison with the phenomenological models, and good fits to the phase shifts are achieved. Finally, we note that the OBEP potentials usually fit the n-p rather than the p-p data.

Thus, to study the effect of both the momentum and charge dependence of the N-N interaction on the three-nucleon system, we have calculated the triton binding energy and n-d doublet scattering length for a large class of realistic potentials. These potentials are listed in Table I, together with abbreviations which we shall use throughout the subsequent discussion.

The triton binding energy (E_T) has been shown to depend almost exclusively on the nuclear interaction in the $^1S_0$ and $^3S_1$-$^3D_1$, partial waves; the contribution from higher partial waves amounting to less than 0.2 MeV.[1] The discrepancy between the calculated and experimental value of E_T is thus either due to uncertainties in the potentials used to represent the S-state interaction, or to a large (1.0–1.5 MeV) contribution from three-body forces and relativistic effects. In the present investigation we hope to study the degree of uncertainty due to the $^1S_0$ and $^3S_1$-$^3D_1$ potentials. If this uncertainty can be reduced, we may get a phenomenological value for the contribution of three-body forces and relativistic effects in the three-nucleon system. For the following discussion, we denote the triton binding energy for the $^1S_0$ and $^3S_1$-$^3D_1$ partial waves as E_T^*.

To calculate E_T^* and ^2a for a particular potential we will use the unitary pole approximation (UPA) to represent the off-shell two-nucleon T matrix, and then solve the appropriate form of the Faddeev equations. This procedure has been shown to be quite accurate for the Reid soft-core potential and in Sec. III we demonstrate its accuracy for other potentials.[11,12] The numerical techniques that allow the UPA to be efficiently utilized in such a comprehensive calculation are discussed in Sec. II.

The values obtained for E_T^* and ^2a using the interaction models of Table I are tabulated in Sec. IV. We find that the difference between the three-nucleon results for the OBEP and the other (local) potentials, is mainly due to the difference in the D-state probability of the deuteron and the $^1S_0$ effective range parameters (i.e., the difference between n-p and p-p data). We also find that the correlation between E_T and ^2a (i.e., the Phillips line[13]), shown on the basis of rank one separable potentials, is also present for the more realistic interactions.

Finally, in Sec. V we present some concluding remarks.

II. NUMERICAL METHODS TO FORM THE UNITARY POLE APPROXIMATION

To determine the unitary pole approximation (UPA) for a particular potential model, it is necessary to calculate the two-nucleon bound state wave functions in momentum space. In a previous application of the UPA to the Reid soft-core potential, this was accomplished by solving the homogeneous Lippmann-Schwinger equation.[12] However, some of the potentials listed in Table I contain a hard core and numerical problems prevent

us from adopting this procedure. We, instead, solve the bound state Schrödinger equation for the wave function and then perform the necessary transformation to momentum space. To achieve this without loss of accuracy, the deuteron wave function is expressed as a linear combination of N chosen functions with unknown expansion coefficients which are determined from the Schrödinger equation. By choosing basis functions with a known Bessel transform, the transition to momentum space does not incur any numerical error. This method is generally known as the method of moments and has been used successfully by Harms and Newton in obtaining the UPA for the $^1S_0$ state of the Reid soft-core potential.[14,15] Here we generalize the method to include hard-core potentials and apply it to the $^3S_1$-$^3D_1$ channel. Bhatt, Harms, and Levinger have previously used a similar type of approach.[16]

In the following discussion we shall derive all expressions including a hard-core radius r_c. The corresponding expressions for soft-core potentials are then obtained in the limit $r_c \to 0$. The partial wave Schrödinger equation appropriate to the $^3S_1$-$^3D_1$ channel may be written

$$\lambda\left(\frac{d^2}{dr^2}-\frac{6}{r^2}\delta_{l2}-k_d^2\right)u_l(r)=\sum_{L=0,2}V_{lL}(r)u_L(r), \quad (3)$$

where $l=0,2$ is the orbital angular momentum index, the deuteron wave functions $\psi_l(r)=u_l(r)/r$, and the functions $V_{lL}(r)$ represent the partial wave components of the potential. The expression (3) represents two coupled equations ($l=0$ and $l=2$) which are to be solved for the deuteron binding energy $E_d=\lambda k_d^2$ (where $\lambda=\hbar^2/M$, with M the nucleon mass) and for the components of the deuteron radial wave function $u_l(r)$. To solve Eq. (3), we express $u_l(r)$ as

$$\begin{aligned} &u_l(r)=0 \qquad \text{for } r<r_c, \\ &u_l(r)=-\sqrt{\tfrac{1}{2}\pi}\,\lambda\sum_{j=1}^{N}\alpha_l^j\phi_l^j(r) \qquad \text{for } r\geq r_c, \end{aligned} \quad (4)$$

and α_l^j are the expansion coefficients. The most convenient form for the functions $\phi_l^j(r)$ is the usual Yamaguchi deuteron wave functions whose momentum space form is well known and which have the correct asymptotic behavior for large r.[17] The effect of the hard core may be incorporated by the slight modification

$$\begin{aligned} \phi_0^j(r)&=e^{-k_d r}-\eta_0^j e^{-a_j r}, \quad (5)\\ \phi_2^j(r)&=2k_d^2A_{5/2}(k_d r) \\ &\quad -\eta_2^j[2a_j^2A_{5/2}(a_j r)-(k_d^2-a_j^2)a_j rA_{3/2}(a_j r)] \end{aligned}$$

with

$$A_{5/2}(\mu r)=\left(1+\frac{3}{\mu r}+\frac{3}{(\mu r)^2}\right)e^{-\mu r},$$

$$A_{3/2}(\mu r)=\left(1+\frac{1}{\mu r}\right)e^{-\mu r},$$

where the a_j, $j=1, N$ are predetermined ranges chosen so as to enable the sum in Eq. (4) to adequately represent the deuteron wave function. In practice, these were chosen between 0.7 and 20.0 fm. The lower cutoff is imposed so that the wave function has the correct asymptotic behavior and the upper cutoff because the solution was not sensitive to larger values. The coefficients η_l^j in Eq. (5) are determined from the equation $\phi_l^j(r_c)=0$.

Substituting for $u_l(r)$ in Eq. (3), the expression (4), and multiplying both sides by $\phi_l^i(r)$, then integrating from zero to infinity, yields the matrix equation

$$\begin{bmatrix} \underline{G}_{00} & \underline{0} \\ \underline{0} & \underline{G}_{22} \end{bmatrix}\begin{bmatrix} \underline{\alpha}_0 \\ \underline{\alpha}_2 \end{bmatrix}=\begin{bmatrix} \underline{V}_{00} & \underline{V}_{02} \\ \underline{V}_{20} & \underline{V}_{22} \end{bmatrix}\begin{bmatrix} \underline{\alpha}_0 \\ \underline{\alpha}_2 \end{bmatrix}, \quad (6)$$

where

$$G_{ll}^{ij}=\lambda\int_{r_c}^{\infty}\phi_l^i(r)\left(\frac{d^2}{dr^2}-\frac{6}{r^2}\delta_{l2}-k_d^2\right)\phi_l^j(r)\,dr, \quad (7)$$

$$V_{ll'}^{ij}=\int_{r_c}^{\infty}\phi_l^i(r)V_{ll'}(r)\phi_{l'}^j(r)\,dr. \quad (8)$$

Equation (6) represents a generalized eigenvalue problem of the form

$$\underline{G}\,\underline{\alpha}=\underline{V}\,\underline{\alpha}, \quad (9)$$

where the $\underline{\alpha}$ is a $2N$-dimensional column matrix of the expansion coefficients and the $(2N\times 2N)$ matrices $\underline{G}$ and $\underline{V}$ are functions of the deuteron binding energy E_d. This equation is solved numerically using the artifice of adding $\underline{I}\,\underline{\alpha}$ to each side

$$(\underline{G}-\underline{V}+\underline{I})\underline{\alpha}=\underline{\alpha} \quad (10)$$

which is now a standard eigenvalue problem with eigenvalue 1. The two-body binding energy is calculated by searching for the value of k_d satisfying

$$\det(\underline{G}-\underline{V})=0 \quad (11)$$

and then $\underline{\alpha}$ may be obtained by solution of Eq. (10) using one of the numerous available codes.

It was found that with $N=12$, the method yielded comparable accuracy in the binding energy and wave function as can be achieved by integrating Eq. (3) using the Numerov method.[18] Furthermore, the results were reasonably insensitive to the exact numerical values of the a_j and they were distributed in some convenient manner between 0.7

and 20.0 fm.

For the $^1S_0$ state where there is no bound state we formulate Eq. (6) as an eigenvalue problem of the form

$$\underline{G}\,\underline{\alpha} = \lambda \underline{V}\,\underline{\alpha} \tag{12}$$

with the energy fixed at zero. The eigenvalue λ of Eq. (12) is then calculated from the condition

$$\det(\underline{G} - \lambda \underline{V}) = 0 . \tag{13}$$

In this way we avoid the problem of analytically continuing the wave function onto the second energy sheet where the $^1S_0$ T matrix has a pole corresponding to the antibound state.[1,15] This procedure is justified by the proximity to zero energy of the antibound state ($E = -0.064$ MeV). Due to the absence of noncentral forces in the $^1S_0$ state, the matrices in Eq. (12) are of the rank N.

Once the deuteron wave function has been calculated, we may write the UPA potential in the $^3S_1$-$^3D_1$ channel as

$$V_{ll'}^{\mathrm{UPA}} = |\underline{\chi}\rangle \underline{\Delta}_l \underline{C}\, \underline{\Delta}_{l'} \langle \underline{\chi}| , \tag{14}$$

where $\underline{\Delta}_l$ is a projection operator onto the angular momentum state l and the elements of the strength matrix $\underline{C}$ are given by

$$C_{ll'} = (\rho_l \rho_{l'})^{-1} \langle \psi_l | V_{ll'} | \psi_{l'} \rangle$$

with

$$\rho = \sum_L \langle \psi_l | V_{lL} | \psi_L \rangle .$$

The matrix elements $\langle \psi_l | V_{ll'} | \psi_{l'} \rangle$ may then be written in terms of the expansion coefficients α_l^i of Eq. (4) as

$$\langle V_l | V_{ll'} | \psi_{l'} \rangle = \sum_{i,j=1}^{N} \alpha_l^i V_{ll'}^{ij} \alpha_{l'}^j , \tag{15}$$

where $V_{ll'}^{ij}$ are defined in Eq. (8). The form factor

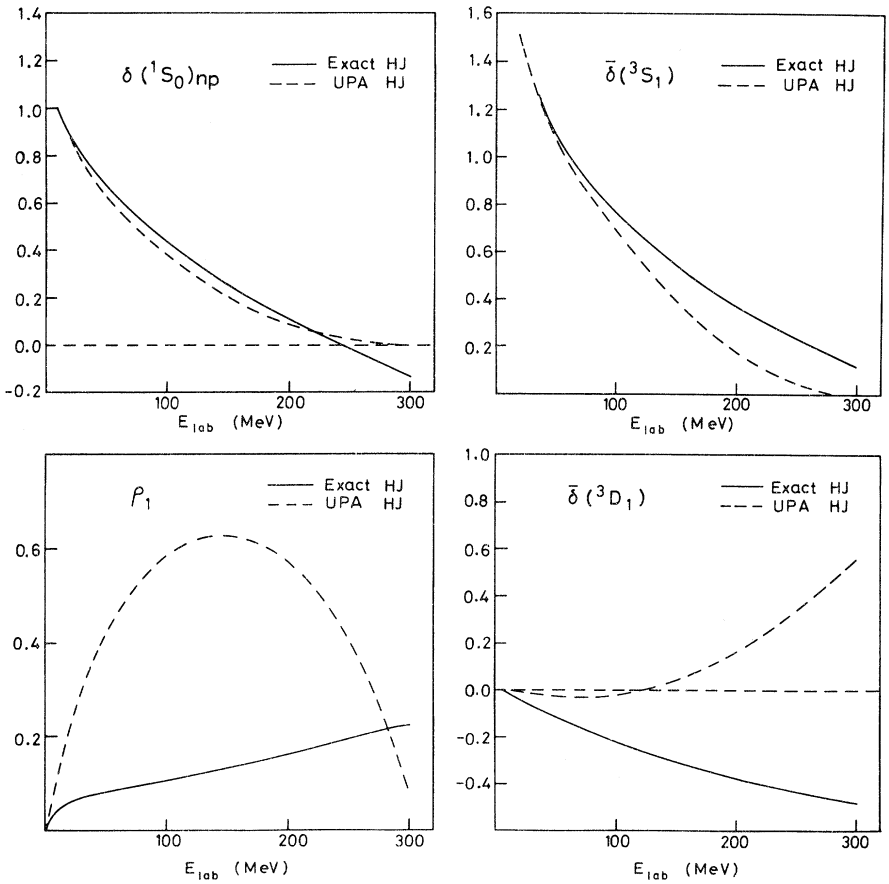

FIG. 1. The $^1S_0$ and $^3S_1$-$^3D_1$ nuclear bar phase parameters calculated from the Hamada-Johnston potential and its UPA counterpart.

$|\underline{\chi}\rangle$ is a two-dimensional row matrix

$$|\underline{\chi}\rangle = [|\chi_0\rangle |\chi_2\rangle] , \tag{16}$$

where the $|\chi_l\rangle$ are the form factors associated with the bound state wave functions

$$\begin{aligned}\chi_l(p) &= -\tau(p^2+k_d^2)\psi_l(p) \\ &= (p^2+k_d^2)\sum_{i=1}^{N}\alpha_l^i\phi_l^i(p)\end{aligned} \tag{17}$$

and the α_l^i are normalized so that $\sum_l\int_0^\infty \chi_l(p)p^2 [\tau(p^2+k_d^2)]^{-1}\chi_l(p)dp = 1$. The $\phi_l^i(p)$ are obtained from the expressions (5) by performing the integral appearing in

$$\phi_l^i(p) = \int_{r_c}^{\infty} r\,dr\, j_l(pr)\phi_l^i(r) \tag{18}$$

analytically.

The unitary pole approximation to the T matrix is obtained by solution of the Lippmann-Schwinger equation using the potential of Eq. (14) as

$$T_{ll'}(E) = |\underline{\chi}\rangle \underline{\Delta}_l M^{-1}(E)\underline{\Delta}_{l'}\langle\underline{\chi}| , \tag{19}$$

where the matrix $\underline{M}(E)$ is

$$\underline{M}(E) = \underline{C}^{-1} + \sum_L \underline{\Delta}_L\langle\underline{\chi}|G_0(E)|\underline{\chi}\rangle\underline{\Delta}_L \tag{20}$$

with $G_0(E) = (H_0 - E)^{-1}$ the free Green's function.

The explicit details required to utilize the above scheme are given elsewhere.[19] These include the numerical values of the expansion coefficients for each of the potentials in Table I, the explicit forms of the $\phi_l^i(p)$, and the analytic evaluation of the integrals of Eqs. (7) and (20). For $r_c = 0$ the $\chi_l(p)$ are the usual Yamaguchi form factors and all the relevant formula are well known.

III. ACCURACY OF THE UPA

The input to the bound state Faddeev equations is the fully off-shell two-nucleon T matrix at neg-

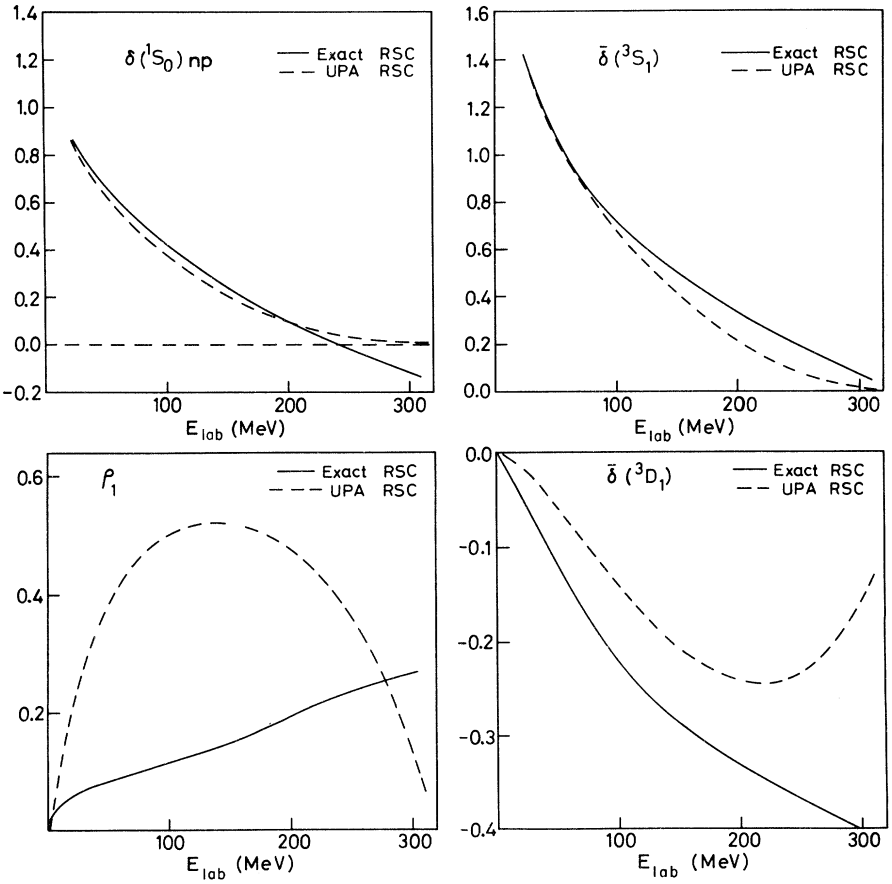

FIG. 2. The $^1S_0$ and $^3S_1$-$^3D_1$ nuclear bar phase parameters calculated from the Reid soft-core potential and its UPA counterpart.

ative energies. Thus, to test the accuracy of the UPA we need to compare the exact and UPA T-matrix elements at negative energy. This procedure has been carried out in some detail for the Reid soft-core potential and for this potential the UPA reproduces, with excellent accuracy, the T-matrix elements relevant to a calculation of E_T^*.[1,11] The comparison was also extended to positive energies and it was shown that the UPA accurately represents the $^1S_0$ and $^3S_1$ phase parameters. For our present purposes, we will adopt the latter procedure. This can be justified on the grounds that the triton is mainly sensitive to the $^1S_0$ and $^3S_1$ T-matrix elements and if the UPA gives a good representation of these functions at positive energy, then it will provide an even better approximation at negative energy. The reason for this is the fact that the two-body bound state pole dominates the behavior of the negative energy T matrix and the UPA reproduces exactly the position and residue of this pole.[11] Thus, to assess the ability of the UPA to represent the T matrices derived from the potentials in Table I, we shall compare the phase parameters in the important $^1S_0$ and $^3S_1$-$^3D_1$ channels.

Because many of the potentials in Table I are similar in their analytic form, we have chosen to present here a comparison of the UPA and exact phase parameters for one of the potentials of each type. The results for other potentials of similar construction exhibit identical behavior.[19] In Figs. 1 to 4 we compare the exact and UPA phase parameters for the $^1S_0$ and $^3S_1$-$^3D_1$ states obtained from the HJ, RSC, TSC, and BS potentials. The overall agreement of the $^1S_0$ and $^3S_1$ phase shifts, particularly at low energies, ensures that the UPA will give a good representation of the negative energy S-state T-matrix elements and thus of the trinucleon binding energy.[1] For the HJ potential, the UPA fails to reproduce the sign of the $^3D_1$ phase shifts, indicating that the UPA is more attractive in the D wave than the original potential. The sign of the $^3D_1$ phase shift is correctly reproduced for the TSC, RSC, and BS potentials, sug-

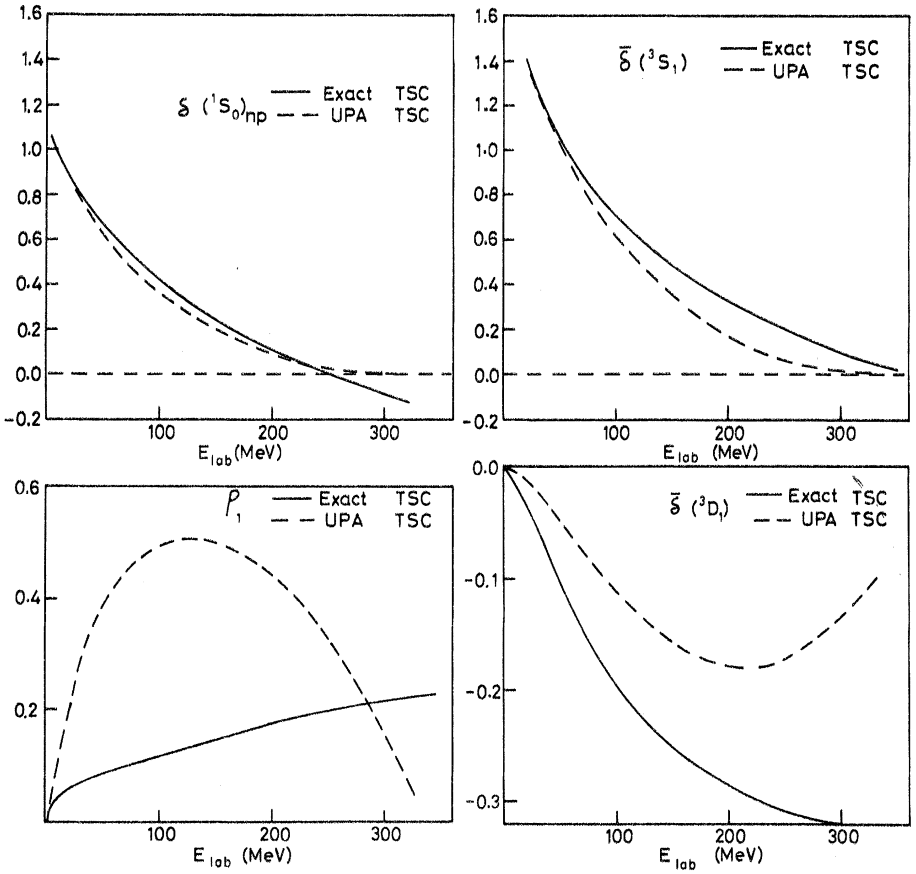

FIG. 3. The $^1S_0$ and $^3S_1$-$^3D_1$ nuclear bar phase parameters calculated from the Tourreil-Sprung potential (model C) and its UPA counterpart.

gesting that the UPA will be an accurate approximation for the D-wave negative energy T matrices derived from these models. To see the effect of too attractive a D wave in a triton calculation, we note that the approximation of Bhatt *et al.* when applied to the RSC potential predicts a large positive $^3D_1$ phase shift, whereas the UPA for this same potential correctly reproduces the sign of these phase shifts.[16,11,12] This is basically the main difference between these two approximations and the difference of 0.3 MeV in the corresponding values of E_T^*, when compared to the large differences in the $^3D_1$ wave phase shifts, demonstrates that E_T^* is not particularly sensitive to this quantity. However, a comparison of the UPA $^3D_1$ phase shifts with the exact values provides a method of assessing the relationship between the exact and approximate values of E_T^*. On this basis, the UPA values of E_T^* for the hard-core potentials will be some 0.1–0.2 MeV higher than the exact values. For the other potentials we would expect the UPA values to be very close to the actual values. The UPA for the OBE potentials in particular represents an extremely accurate approximation.

Finally, we notice that the UPA is not able to give an adequate representation of the $^3S_1$-$^3D_1$ coupling constant. This has been discussed elsewhere and the conclusion has been made that its failure is not important in calculating the properties of the triton.[11,12] The discrepancy in the coupling parameter for the UPA, and in fact for most separable potentials, can be understood using some results of Wong.[20] He showed that the low-energy coupling parameter ($E_{lab} < 38$ MeV) can be predicted from the $^3S_1$ phase shifts, by writing a dispersion relation including the deuteron pole and the one-pion-exchange branch cut. Since the UPA reproduces the $^3S_1$ phase shift and the position and residue of the deuteron pole, the disagreement concerning the coupling parameter shows that it is not able to simultaneously reproduce the one-meson branch cut in the mixing amplitude. Of course, this might have little bearing in a triton binding energy calculation where we only consider

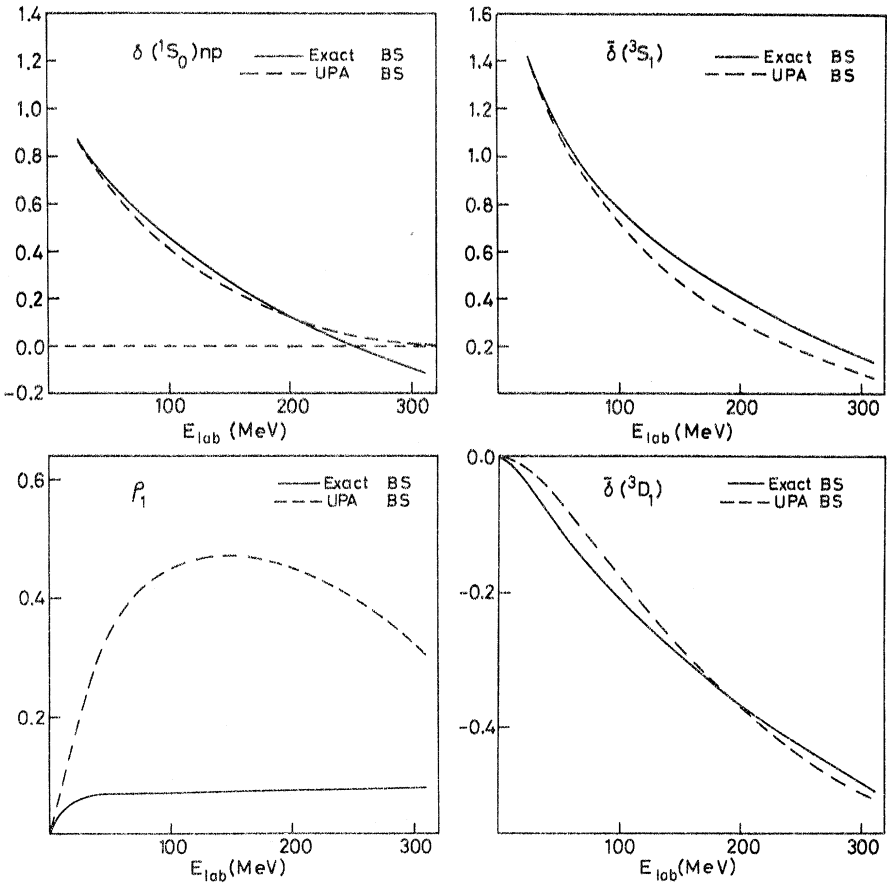

FIG. 4. The $^1S_0$ and $^3S_1$-$^3D_1$ nuclear bar phase parameters calculated from the Bryan-Scott potential and its UPA counterpart.

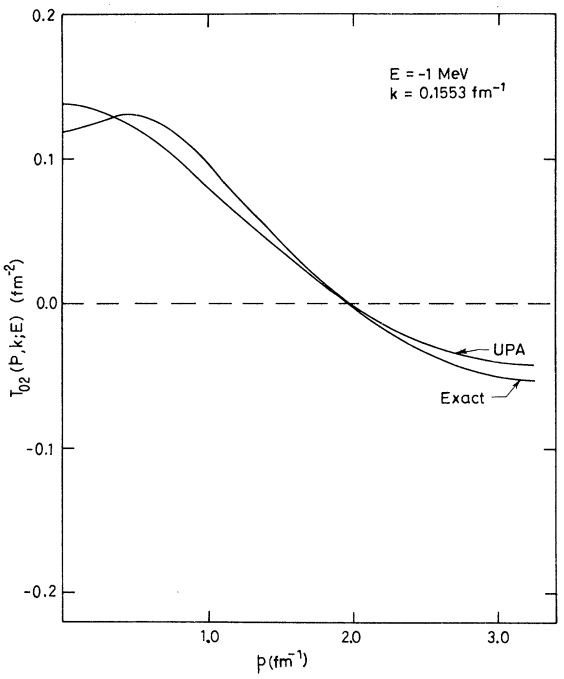

FIG. 5. A comparison of the exact and UPA values of the off-shell two-nucleon T-matrix elements $T_{02}(p, k; E)$ for the $^3S_1$-$^3D_1$ channel of the Reid soft-core potential. The center-of-mass energy $E = -1.0$ MeV and the momenta $k = 0.1553$ fm^{-1}.

the fully off-shell two-nucleon T matrix at negative energies, whose only analytic structure is the deuteron pole.[21] To show that the UPA represents a much better approximation of the off-shell two-nucleon T matrix in the S-D wave at negative energy, in Fig. 5 we plot the exact and UPA off-shell T-matrix elements for $E = -1$ MeV.

To have a direct verification that using the UPA in the Faddeev equations is an accurate calculational procedure for the three-nucleon bound state, in Table II we compare the UPA values of E_T with exact calculations of other authors. Some of these calculations have made use of the potential in all partial waves while others use the $^1S_0$ and $^3S_1$-$^3D_1$ channels only. The agreement between the UPA results and the more exact calculations are very good. For all potentials compared the difference is less than 0.2 MeV. Thus for the purpose of the present study, our calculation procedure should certainly be accurate enough to study the discrepancy of about 1.0 MeV in the triton binding energy.

IV. RESULTS

Over the past decade there has been an extensive study of the three-nucleon system using separable potentials. The results of these calculations have shown that the triton is mainly sensitive to the deuteron observables and to the S-state effective range parameters.[13,27,28] Although the one-term Yamaguchi potentials used to establish these results may be considered unrealistic for their lack of short-range repulsion, we would still expect these studies to have some relevance. Thus, in Table III we tabulate the S-state effective range parameters and deuteron observables for the potentials of interest.[29-31] A detailed examination of Table III shows clearly that there is considerable variation amongst the different interactions. Thus we find all the OBEP give a $^1S_0$ scat-

TABLE II. A comparison of the UPA values for the triton binding energy with results of other calculations.

Potential	E_T^* (UPA)	E_T	Method	Reference
Hamada-Johnston	6.96	6.5 ± 0.2 [a]	variational	22
		6.7 [a]	variational	23
Reid soft core	7.15	7.0 [b]	Faddeev r space	24
		7.02 [b]	T-matrix perturbation	12
		6.7 [b]	Faddeev q space	26
		7.75 ± 0.5 [a]	variational	25
Tourreil-Sprung A	7.52	7.64 [b]	Faddeev r space	24
Tourreil-Sprung B	7.62	7.71 [b]	Faddeev r space	24
Tourriel-Sprung C	7.42	7.46 [b]	Faddeev r space	24

[a] All partial waves included.
[b] Only the $^1S_0$ and $^3S_1$-$^3D_1$ partial waves included.

TABLE III. The low-energy parameters for the potentials of Table I.

Potential	E_d (MeV)	Q_d (fm^2)	P_d (%)	a_t (fm)	r_t (fm)	a_s (fm)	r_s (fm)
RHC	2.2246	0.277	6.50	5.40	1.72	−16.7	2.87
HJ	2.2689	0.284	7.02	5.38	1.75	−16.7	2.83
Yale	2.1888	0.276	6.95	5.45	1.75	−17.4	2.92
RSC	2.2246	0.280	6.47	5.39	1.72	−17.1	2.80
RSCA	2.2246	0.276	6.22	5.39	1.72	−17.1	2.80
TSA	2.2237	0.261	4.43	5.50	1.85	−17.3	2.84
TSB	2.2284	0.261	4.25	5.50	1.86	−17.3	2.84
TSC	2.2241	0.278	5.45	5.48	1.83	−17.3	2.84
BS	2.1977	0.258	5.47	5.39	1.70	−22.9	2.64
BG	2.3006	0.275	5.06	5.42	1.84	−23.7	2.75
SRG	2.3344	0.270	4.58	5.39	1.85	−32.3	2.66
UGII	2.7105	0.259	4.6	5.10	1.83	−25.5	2.76

TABLE IV. The triton binding energy and n-d doublet scattering length for the potentials of Table I.

Potential	E_T^* (MeV)	^2a (fm)
RHC	6.96	1.97
HJ	6.96	2.04
Yale	6.73	2.10
RSC	7.15	1.80
RSCA	7.32	1.68
TSA	7.52	1.46
TSB	7.62	1.38
TSC	7.42	1.52
BS	7.94	1.10
BG	7.81	1.43
SRG	8.31	1.14
UGII	8.83	1.59

tering length (a_s) of ~ -23 fm in agreement with the n-p scattering length, while the other potentials have a value of $a_s \sim -17$ fm, a reflection of the fact that they were fitted to the p-p scattering data. The sensitivity of E_T to the singlet effective range parameters has most recently been studied by Gibson and Stephenson,[32] and van Wageningen *et al.*[33] using separable potentials. They find that E_T increases very little when a_s goes from −17 to −23 fm. On the other hand E_T is very sensitive to the singlet effective range (r_s).

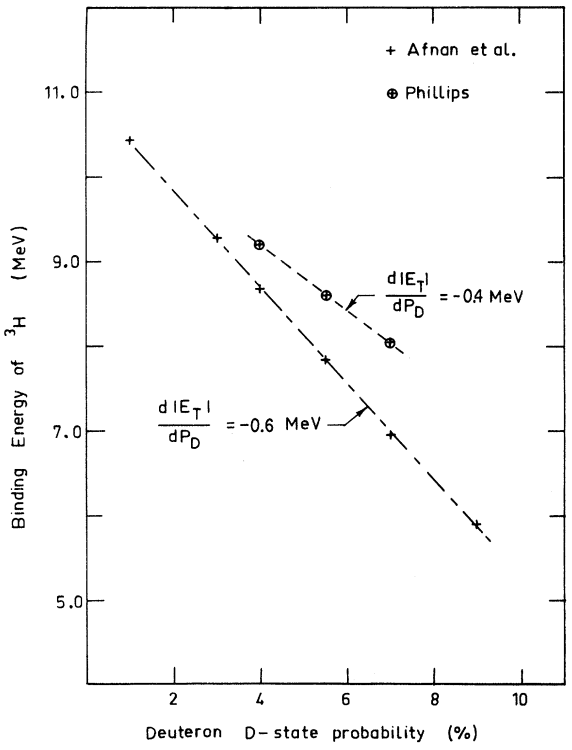

FIG. 6. The dependence of the triton binding energy on the percentage D state of the deuteron. The upper curve was constructed using the published results of Phillips (Ref. 13) and the lower curve from the results of our calculation using the one-term $^3S_1$-$^3D_1$ potentials of Afnan *et al.* (Ref. 34) with the $^1S_0$ potential of Phillips.

Another outstanding variation in Table III is the D-state probability of the deuteron which varies from 7% for the HJ potential to 4.25% for TSB potential. Phillips[13] has shown, using Yamaguchi potentials, that E_T and ^2a are sensitive functions of this parameter. We illustrate this variation in Fig. 6 with Phillips's original results and our calculations using the potentials of Afnan, Clement, and Serduke.[34] These latter potentials may be considered more realistic than the standard Yamaguchi form since they contain significant repulsion in the $^3D_1$ state. We observe that a decrease of 1% in P_d leads to an increase of 0.4–0.6 MeV in E_T, when all other parameters are fixed. We shall use these results for subsequent analysis of our three-nucleon results for the different potentials considered.

In Table IV we present the results for the binding energy of ^{3}H and the n-d doublet scattering length using the potentials in Table I. We observe that most of the hard-core potentials bind the triton at around 7 MeV and predict a scattering length of close to ~2 fm. The soft-core potentials, the RSC ($E_T^* = 7.15$ MeV), and RSCA ($E_T^* = 7.32$ MeV) give slightly more binding but not sufficient to make any major change to the estimated contribution of three-body forces and relativistic corrections. The difference in E_T^* for these very similar (RSC, RSCA) potentials is due to their different D-state probability.

To estimate the effect of the D state on E_T^*, we have calculated the trinucleon binding energy with $^1S_0$ RSC and $^3S_1$-$^3D_1$ RSCA potentials. In this way the only change in the input is the $^3S_1$-$^3D_1$ channel

TABLE V. The sensitivity of E_T to the D-state probability of the deuteron is illustrated by replacing the RSC by RSCA in $^3S_1$-$^3D_1$ channel.

$^3S_1$-$^3D_1$	$^1S_0$	P_d (%)	E_T^* (MeV)
RSC	RSC	6.47	7.15
RSCA	RSC	6.22	7.32
RSC	RSCA	6.47	7.15
RSCA	RSCA	6.22	7.32

keeping the $^1S_0$ unaltered. The resultant triton binding energy is 7.32 MeV as compared to 7.15 MeV for the case when the RSC potential is used in both $^1S_0$ and $^3S_1$-$^3D_1$ channels (see Table V). If we attempt to explain this difference as due to changes in the deuteron properties and $^3S_1$ effective range parameters we find that $dE_T^*/dP_d \sim 0.6$ MeV, since all the other parameters with the exception of Q_d remain the same. This result is consistent with those obtained using separable Yamaguchi potentials (Fig. 6). We note that although the UPA may introduce an error as large as 0.15 MeV in E_T^*, the error in ΔE_T^* (the change in binding energy by changing the $^3S_1$-$^3D_1$ potential) is much less. This reduction in error is due to (i) the $^1S_0$ potential being kept the same and thus not contributing to the error; (ii) the RSC and RSCA potentials being very similar in form. Thus one would expect the error due to the UPA to be approximately the same. This is illustrated in Table V where we replace the RSC $^1S_0$ by the RSCA potential. Finally we observe that the RSCA potential with a smaller deuteron D-state probability has also a smaller quadrupole moment (Table III).

The value of E_T^* for the Tourreil-Sprung models TSA ($E_T^* = 7.52$ MeV) and TSB ($E_T^* = 7.62$ MeV) are definitely in closer agreement with experiment. This increase in binding of about 0.5 MeV compared to the RSC potential can be due to their low D-state probability (4.43 and 4.25%), which would be consistent with the results of separable potentials. However, both of these potentials have a quadrupole moment for the deuteron $Q_d = 0.261$ fm^2, which is small compared to the experimental value of $Q_d = 0.287 \pm 0.002$ fm^2.[35] The Tourreil-Sprung model C (TSC) tries to remedy this small quadrupole moment by increasing its value to $Q_d = 0.278$ fm^2. In the process the D-state probability increases to 5.45% and the triton binding energy (E_T^*) decreases to 7.42 MeV.

From the above results for the Reid and Tourreil-Sprung potentials we conclude: First, that E_T^* increases by 0.4–0.6 MeV for each 1% decrease in the deuteron D-state probability. Second, for potentials of this form (i.e., local in each partial wave) a decrease in the deuteron D-state probability leads to a corresponding decrease in the quadrupole moment.[35] Thus to fit the latest value of $Q_d = 0.287$ fm^2 we expect a D-state probability of 7% or more for the Reid type of potentials and possibly 6.5% for the Tourreil-Sprung super soft core. Such high D-state probability will result in a triton binding energy of close to 7.0 MeV, leaving a discrepancy of 1.0–1.5 MeV compared to the experimental value. Finally we observe that for both the Reid and Tourreil-Sprung potentials, as the D-state probability decreases the doublet n-d scattering length decreases and at the same time approaches the experimental value of 0.65 ± 0.04 fm.[36]

We now turn to the three-nucleon results for the one-boson-exchange potentials in Table IV. We observe that for these potentials $E_T^* \simeq 8.0$ MeV, considerably higher than the value obtained for either the Reid or Tourreil-Sprung potentials. Furthermore the n-d doublet scattering lengths are closer to the experimental value. Thus for these potentials the contribution from three-body forces and relativistic effect need not be as large. To see if this is a real effect, we need to examine the effective range parameters and deuteron observables for these potentials (Table III).

From Table III we see that with the exception of the Ueda-Green (UGII) potential, which gives a deuteron binding energy of 2.71 MeV as compared to the experimental value of 2.2246 ± 0.0005 MeV, there are two distinct features these potentials have. First, their $^1S_0$ effective range parameters fit the n-p data ($a_{np} = -23.715 \pm 0.015$ fm, $r_{np} = 2.73 \pm 0.03$ fm)[37] rather than the n-n data ($a_{nn} = -16.4 \pm 0.9$ fm, $r_{nn} = 2.84 \pm 0.03$ fm),[37] which both the Reid and Tourreil-Sprung potentials are fitted to. Second, the OBEP seem to achieve higher values of the quadrupole moment with a low D-state probability. More important, if we compare the Bryan-Scott (BS) and Bryan-Gersten (BG) potentials, which are similar in their form, we find that BG has a D-state probability $P_d = 5.04\%$ and $Q_d = 0.275$ fm^2, while the BS potential gives $P_d = 5.47\%$ and $Q_d = 0.258$ fm^2. In other words, the BG potential has a larger quadrupole moment and a smaller D-state probability than the BS potential. This is a feature that was absent in both the Reid and Tourreil-Sprung potentials.[38]

In an attempt to understand the difference of ~1 MeV in E_T^* between the results for the RSC and OBE potentials, we have calculated the triton binding energy for different combinations of $^1S_0$ and $^3S_1$-$^3D_1$ potentials. These results are presented in Table VI, where we have also included the $^1S_0$ effective range parameters as predicted by the unitary pole approximation. On comparing the differ-

TABLE VI. To demonstrate the sensitivity of E_T^* to $^1S_0$ effective range parameter, we take different $^1S_0$ potentials keeping the $^3S_1$-$^3D_1$ fixed.

$^3S_1$-$^3D_1$	$^1S_0$	a_s	r_s	E_T (MeV)
RSC	RSC	−17.2	2.71	7.15
RSC	RSCA	−17.1	2.67	7.15
RSC	BS	−23.0	2.67	7.52
RSC	BG	−23.8	2.72	7.42
BS	BS	−23.0	2.67	7.94
BS	BG	−23.8	2.72	7.81
BG	BG	−23.8	2.72	7.81
BG	RSC	−17.2	2.71	7.52

ent cases in Table VI we find:

(a) The triton binding energy decreases with increasing $^1S_0$ effective range (r_s) as predicted previously on the basis of separable potentials.[32,33] However, if we assume for the moment that the UPA are just rank one separable potentials, then the change in E_T^* with r_s ($\Delta E_T^*/\Delta r_s$) varies anywhere from zero to -2.2 MeV fm^{-1}. To understand this variation in ($\Delta E_T^*/\Delta r_s$) we have examined the $^1S_0$ Kowalski-Noyes zero energy half-shell function $[f(p,k)=\langle p|\,t(k^2)|\,k\rangle/\langle k|\,t(k^2)|\,k\rangle$ for $k=0]$ for the potentials in Table VI. We find that the RSC and RSCA have the same half-shell function. On the other hand, the BS and BG half-shell functions differ particularly for momenta p less than 2 fm^{-1} by as much as 5–10%. If we compare two Yamaguchi potentials with different values of r_s we find differences comparable to those obtained in comparing the BS and BG half-shell functions. This difference in off-shell behavior and variation in ($\Delta E_T^*/\Delta r_s$) suggests that part of the sensitivity of E_T^* to r_s may be due to off-shell behavior.

(b) The binding energy of ^{3}H increases by ~0.3 MeV if one fits the n-p rather than the n-n (or Coulomb subtracted p-p) scattering length. This change in E_T^* is achieved by keeping the $^3S_1$-$^3D_1$ potential fixed, and taking either the RSC or BG potential in the $^1S_0$ channel. The off-shell effects are not as important in this case because both $^1S_0$ potentials have almost the same zero energy Kowalski-Noyes half-shell function for momenta less than ~2 fm^{-1}. If we compare the value of E_T obtained with RSCA and BS $^1S_0$ potential, the variation in E_T is slightly more. But then, these two potentials have slightly different off-shell behavior even for momenta less than 2.0 fm^{-1}.

(c) Finally, we see that changing the $^3S_1$-$^3D_1$ potential from the RSC to the BG we gain 0.3–0.4 MeV in binding. This is mainly due to the smaller D-state probability of the BG deuteron. Although this variation is smaller than one would expect based on the results of separable potentials, we should note that the RSC and BG $^3S_1$-$^3D_1$ potentials give slightly different deuteron binding energy and effective range parameters.

Finally in Table IV we have included the three-nucleon results for the potentials of Stagat, Riewe, and Green[9] (SRG), and Ueda-Green[10] (UGII). Here we find that the UGII potential, with a deuteron binding energy of 2.71 MeV, overbinds the triton with $E_T^*=8.83$ MeV. This clearly illustrates that any potential that overbinds the deuteron will give even more binding in a many-nucleon system. Similarly, the results for the SRG potential can be questioned due to the large singlet scattering length ($a_s=-32.3$ fm), and a slight overbinding of the deuteron ($E_d=2.33$ MeV).

On the basis of calculations using separable potentials of the Yamaguchi form, Phillips has shown that the n-d doublet scattering length ^2a and triton binding energy E_T^* are not independent quantities, but rather related by the so-called Phillips's line.[13] If this result holds in general for all nucleon-nucleon interactions and is not particularly affected by three-body forces and relativistic effects, then a correct determination of the triton binding energy should ensure a correct value of the doublet scattering length. Recently, Brayshaw has argued that all the information on the N-N T matrix to be gleaned from n-d scattering is contained in the value of ^2a.[39] If the results of Phillips and Brayshaw are combined, then it seems all the information one hopes to gain about the N-N interaction from the trinucleon will be present in the triton.

To illustrate the validity of the Phillips's line

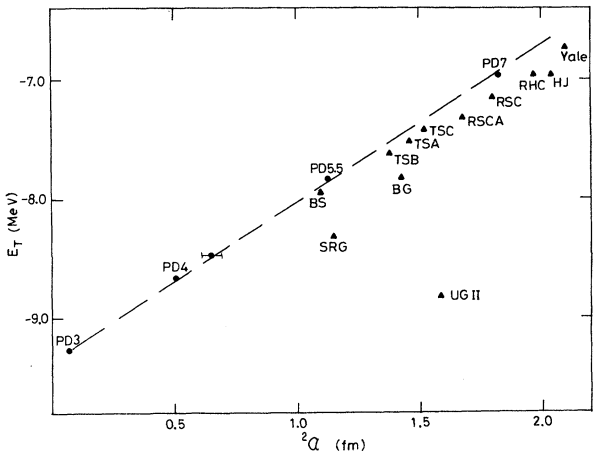

FIG. 7. The calculated values of the triton binding energy E_T^* plotted against the calculated values of the n-d doublet scattering length ^2a, for the potentials of Table I. The points designated PD3, PD4, PD5.5, and PD7 refer to the potentials of Ref. 34, and the experimental point (⊢●⊣) is included for comparison.

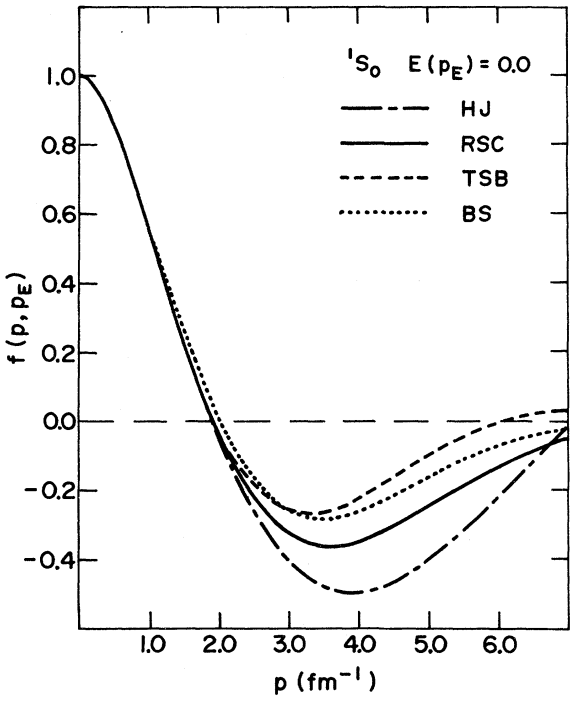

FIG. 8. The Kowalski-Noyes half-off-shell function for the HJ, RSC, TSB, and BS potentials.

for the potentials considered, in Fig. 7 we have plotted the binding energy of ^{3}H as a function of the n-d doublet scattering length ^2a. We find that the majority of potentials lie along a band running parallel to the Phillips's line obtained using the potentials of Afnan *et al.*[34] The main exception is the UGII potential which predicts erroneous values for several of the low-energy two-body parameters. Thus, our results suggest that the Phillips's line also has validity for realistic interactions provided they predict the correct deuteron properties and effective range parameters.

V. CONCLUSION

The main conclusions that may be inferred from the present investigation of the binding energy of ^{3}H and the n-d doublet scattering length using a variety of realsitic interactions are:

(i) The difference in the results for the different potentials can be explained on the basis of different deuteron observables and S-wave effective range parameters. For example, the difference of ~1 MeV in E_T^* between the results for the RSC and BS potentials can be attributed to differences in the deuteron D-state probability and $^1S_0$ effective range parameters. We also observe that the OBEP give a reasonable value of the quadrupole moment (0.275 fm^2) maintaining a low (5%) D-state probability. More important, if we compare Q_d and P_d for the BS and BG potentials (Table III), we find that within an OBEP model one may be able to increase the quadrupole moment to the present experimental value ($Q_d = 0.287 \pm 0.002$ fm^2),[33] yet maintain a low (~5%) D-state probability.[38] This in turn will give us more binding in ^{3}H than the present ~7 MeV for the RSC. If at the same time we distinguish between the n-p and n-n interaction we may be able to reduce the contribution of three-body forces and relativistic effects to ~0.5 MeV.

(ii) Comparing the results for the different potentials and taking into consideration the effective range parameters, we find the three-nucleon results are not particularly sensitive to the short-range behavior of the N-N interaction.

(iii) The correlation between the triton binding energy and n-d doublet scattering length (Fig. 7) is valid for most of the potentials considered. This result was in a way expected considering the fact that all the potentials have a one-pion-exchange tail, and thus the same long-range behavior.[40] In fact the requirement of one-pion-exchange tails determines the half-off-shell T matrix near the on-shell region which is most important for three-nucleon calculations. This is illustrated in Fig. 8 where we present the $^1S_0$ zero energy Kowalski-Noyes half-off-shell function for the different interactions. We find that all agree for momenta $p < 2$ fm^{-1}; the only exception is the BS potential. However, if one replaces the BS by the BG potential then the agreement for $p < 2$ fm^{-1} between the different potentials is much better. This difference between the BS and BG half-shell function might be due to the fact that the BS uses a different π-N coupling parameters ($g_\pi^2 = 12.5$ as compared to 14.0) in defining the one-pion-exchange potential.

ACKNOWLEDGMENT

The generous support afforded this work by the Australian Research Grants Committee (ARGC) is gratefully acknowledged. One of the authors (J. M. R.) was the recipient of an ARGC fellowship during the course of the present study and for this consideration he is sincerely indebted. We also wish to thank Professor I. E. McCarthy and Dr. A. W. Thomas for many enlightening discussions. One of us (I. R. A.) would like to thank Professor Erich Vogt for his hospitality during his stay at the University of British Columbia where the final manuscript was prepared.

*Research supported by the Australian Research Grants Committee and the Flinders University Research Budget.

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