# Coulomb effects in hadron-nucleus and nucleus-nucleus collisions and the hadron-neutron amplitude*

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Coulomb effects in high energy charged hadron-nucleus collisions are studied in a formalism which is exact within the framework of multiple diffraction theory and takes into account the extended charge effects of the incident hadron and the bound protons. Several more approximate but simpler formulas are also considered and their accuracy discussed. Results are extended to include heavy-ion collisions in the optical limit of the Glauber theory and in the Chou-Yang model. Applications are made to the proton-deuteron elastic and elastic plus quasielastic scattering measurements below 70 GeV to extract the ratio of real to imaginary part and the slope parameter of the proton-neutron elastic scattering amplitude. The results are compared with those from dispersion relation calculations and from earlier analyses. Approximate analytic formulas derived for $p$-$d$ scattering give results almost identical to the more exact expressions. For nucleus-nucleus collisions the Coulomb effects are found to be important over a rather wide range of momentum transfers.

NUCLEAR REACTIONS $d(p,p)$, $E=10$–$70$ GeV; calculated $\sigma(E,\theta)$ in Coulomb-nuclear interference region; deduced $p$-$n$ scattering amplitude parameters. $^{12}C$, $^{58}Ni$, $^{208}Pb(p,p)$, $E=1.04$ GeV, $^{12}C(^{12}C,{}^{12}C)$, $E=2.1$ GeV/nucleon; Coulomb effects, calculated $\sigma(\theta)$.

## I. INTRODUCTION

The study of Coulomb effects is of considerable importance for the accurate description of charged hadron-nucleus collisions and for the investigation of hadron-nucleon interactions by means of such collisions. At very small momentum transfers the contributions of the Coulomb interactions and of the strong interactions to the scattering amplitude are comparable, and hence the differential cross sections are quite sensitive to the interference between them. Information regarding strong interactions, such as the real part of the strong interaction amplitude, can be extracted from the analysis of the interference region. Since direct hadron-neutron scattering experiments, for the most part, are not feasible because of the unavailibility of neutron targets or the scarcity and short lives of most elementary particles, small angle hadron-deuteron scattering is of particular interest. Hadron-deuteron scattering experiments can be used together with hadron-proton measurements to extract information about the hadron-neutron amplitudes via a multiple scattering theory. Such analyses can be performed by means of the Glauber theory.[1,2] In the past, additional approximations and simplifications were made in the analyses[3,4] of proton-deuteron scattering data to obtain proton-neutron scattering parameters. It is desirable to investigate the accuracy of these analyses, since a detailed knowledge of the hadron-neutron amplitude is of fundamental importance in nuclear and particle physics studies.

Recently, extensive proton-deuteron elastic scattering experiments between 10 and 70 GeV were performed at Serpukhov[5] down to very small angles and additional small angle experiments are being carried out at Fermi National Accelerator Laboratory (FNAL) between 50 and 400 GeV. These measurements will be used to extract the relevant parameters in the proton-neutron scattering amplitudes. Since the high energy diffraction theory has generally proven to be quite accurate at small angles, it is appropriate to use it in undertaking a more careful study of Coulomb-nuclear interference and to analyze the $p$-$d$ measurements.

One can also hope to learn about hadron-nucleon interactions from the study of collisions of hadrons with other nuclei. For example, the minima which occur in hadron-nucleus elastic scattering intensities are quite sensitive to the ratio $\rho_n$ of the real to imaginary part of the hadron-neutron forward elastic scattering amplitude and hence the experimental data in that region can be used to determine $\rho_n$. As pointed out by Czyż, Lesniak, and Wolek,[6] Coulomb effects play a significant role near the minima; hence they must be carefully included. The Coulomb effects are expected to be even larger in heavy-ion collisions. Since such experiments are being planned at the Berkeley Bevalac, the detailed investigation of Coulomb effects in such collisions is of current interest.

In this paper, therefore, we also consider heavy-ion collisions and include the contributions due to the Coulomb interaction.

In Sec. II, we present theoretical expressions for elastic and elastic plus quasielastic differential cross sections for charged hadron-deuteron scattering. Our analysis is exact within the framework of the diffraction theory and we explicitly include the charge distribution effects of the incident hadron and the bound proton. We briefly describe some additional results which are slightly more approximate but significantly more simple in form and extremely easy to use in the analysis of data. In Sec. III we extend the formalism to include scattering by other nuclei and consider some approximate models due to Bethe[7] and to West and Yennie,[8] as well as an approximate phase formula. In each of these approximate models, Coulomb effects are assumed to originate from the nucleus as a whole rather than from each individual proton. In Sec. IV we include the extended charge Coulomb effects in nucleus-nucleus collisions. A comparison of the various approximate expressions with the more exact results is given in Sec. V and their accuracy is discussed. In Sec. VI we describe the deuteron form factor used in our analysis. We then apply our results to extract the ratio of real to imaginary parts of the proton-neutron elastic scattering amplitude from the proton-deuteron elastic scattering measurements in the energy region 10–70 GeV and from elastic plus quasielastic measurements at 19.3 GeV/$c$. We also estimate the slope parameter for the proton-neutron elastic scattering amplitude. We compare the results with those obtained from the approximate formulas discussed in Sec. II, with the results obtained by Beznogikh *et al.*,[4] and with some dispersion relation calculations. We also discuss briefly the possible effects of inelastic intermediate states on our results. We summarize our results in Sec. VII.

## II. HADRON-DEUTERON COLLISIONS

In the multiple diffraction theory of Glauber, the scattering amplitude operator for scattering by deuterons is given by[2]

$$F_d(\vec{q},\vec{s}) = \frac{ik}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}\,\Gamma_d(\vec{b},\vec{s})\,d^2b\ , \qquad (1)$$

where $\hbar\vec{k}$ is the incident momentum, $\hbar\vec{q}$ is the momentum transfer, $\vec{b}$ is the impact parameter, and $\vec{s}$ is the projection of the internal deuteron coordinate $\vec{r}$ on a plane perpendicular to $\vec{k}$. The profile function $\Gamma_d$ for the deuteron is given by

$$\Gamma_d(\vec{b},\vec{s}) = 1-\exp[i\chi_n(\vec{b}-\tfrac{1}{2}\vec{s}) + i\chi_p(\vec{b}+\tfrac{1}{2}\vec{s})]\ , \qquad (2)$$

where $\chi_n$ and $\chi_p$ are the phase shift functions for the scattering of the incident hadron by the neutron and proton, respectively. At high energies, we further assume[9] that

$$\chi_p(\vec{b}) = \chi_C(\vec{b}) + \chi_{ps}(\vec{b})\ , \qquad (3)$$

where $\chi_C$ is the phase shift function for the Coulomb scattering from the proton and $\chi_{ps}$ is that for scattering by the strong interaction alone. If we separate out the Coulomb phase shift function $\chi_C^{\rm pt}$ due to a point charge, we can write

$$\chi_C(\vec{b}) = \chi_C^{\rm pt}(\vec{b}) + \chi_C^{E}(\vec{b})\ , \qquad (4)$$

where $\chi_C^E$ denotes the correction to the Coulomb phase shift function due to extended charge effects. (In Appendix A we calculate $\chi_C^E$ explicitly for a specific charge distribution of the incident particle and the target proton.) Let us write the hadron-nucleon profile functions as

$$\Gamma_j(\vec{b}) = 1 - e^{i\chi_j(\vec{b})},\quad j=n, ps \qquad (5)$$

and also write

$$\Gamma_C^i(\vec{b}) = 1 - e^{i\chi_C^i(\vec{b})},\quad i={\rm pt},\ E\ . \qquad (6)$$

There are many ways of separating point charge and charge distribution effects. For example, one way that exhibits various multiple scattering effects explicitly is given by

$$\Gamma_d(\vec{b},\vec{s}) = \Gamma_C^{\rm pt} + \Gamma_C^E + \Gamma_{ps} + \Gamma_n - \Gamma_C^{\rm pt}(\Gamma_{ps}+\Gamma_n) - \Gamma_{ps}\Gamma_n - \Gamma_C^E(\Gamma_C^{\rm pt}+\Gamma_{ps}+\Gamma_n) + \Gamma_C^{\rm pt}\Gamma_{ps}\Gamma_n$$
$$+\Gamma_C^E(\Gamma_C^{\rm pt}\Gamma_{ps} + \Gamma_C^{\rm pt}\Gamma_n + \Gamma_{ps}\Gamma_n) - \Gamma_C^{\rm pt}\Gamma_C^E\Gamma_{ps}\Gamma_n\ , \qquad (7)$$

where the argument of $\Gamma_n$ is $\vec{b}-\frac{1}{2}\vec{s}$ and the argument of all the other $\Gamma$'s is $\vec{b}+\frac{1}{2}\vec{s}$.

The terms involving a single $\Gamma$ can be viewed as representing single scattering by the interactions denoted by the subscripts or superscripts, while a product of $\Gamma$'s corresponds to multiple scattering. The effects of charge distribution are embedded in the 2nd, 8th, 9th, 10th, and 12th through 15th terms. The 2nd term represents single scattering by the extended charge corrections to the Coulomb field. The 8th, 9th, and 10th are double scattering terms. For example, the 10th term represents double scattering by the extended charge corrections to the Coulomb field and by the neutron. The 12th, 13th, and 14th terms have similar triple scattering interpretations; the last term can be thought of as quadruple scattering by a point Coulomb field, the extended charge effects of the Coulomb field, the proton strong interaction, and the neutron. In the absence of an extended charge distribution, $\Gamma_C^E$ vanishes and Eq. (7) reduces to the results of Ref. 9.

Another way of writing Eq. (7) which is more convenient for numerical evaluation is[10]

$$\Gamma_d(\vec{b},\vec{s}) = \Gamma_C^{\rm pt}(\vec{b}+\tfrac{1}{2}\vec{s}) + e^{i\chi_C^{\rm pt}(\vec{b}+\frac{1}{2}\vec{s})}\Gamma_C^E(\vec{b}+\tfrac{1}{2}\vec{s}) + e^{i\chi_C(\vec{b}+\frac{1}{2}\vec{s})}[\Gamma_{ps}(\vec{b}+\tfrac{1}{2}\vec{s}) + \Gamma_n(\vec{b}-\tfrac{1}{2}\vec{s}) - \Gamma_{ps}(\vec{b}+\tfrac{1}{2}\vec{s})\Gamma_n(\vec{b}-\tfrac{1}{2}\vec{s})] \; . \quad (8)$$

For a screening radius $R$ and $q \gg 1/R$, we have[1]

$$\chi_C^{\rm pt}(\vec{b}) = 2n\ln(b/2R), \quad b<R$$
$$= 0, \quad b>R \; , \quad (9)$$

where $n = e^2/\hbar v$ and $v$ is the relative velocity between the projectile and the bound proton in the target.

For $R$ of atomic dimensions, this is valid for $\hbar^2 q^2 \gg 10^{-11}$ $(\mathrm{GeV}/c)^2$. Using Eqs. (6) and (9) we obtain

$$\frac{ik}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}\Gamma_C^{\rm pt}(\vec{b})d^2b = -\frac{2nk}{q^2}\exp\{-2i[n\ln(qR) - \arg\Gamma(1+in)]\}$$
$$= \exp[-2in\ln(2kR)]f_C^{\rm pt}(q) \; ,$$

where we have defined the Coulomb amplitude for a point charge as

$$f_C^{\rm pt}(q) = -\frac{2nk}{q^2}\exp\{-2i[n\ln(q/2k) - \arg\Gamma(1+in)]\} \quad (10)$$

with $\Gamma(z)$ being the gamma function

Hadron-nucleon profile functions are related to hadron-nucleon scattering amplitudes by[2]

$$\Gamma_j(\vec{b}) = \frac{1}{2\pi ik}\int e^{-i\vec{q}\cdot\vec{b}}f_j(\vec{q})d^2q, \quad j=n, ps \; .$$

Using this relation together with Eq. (1) we obtain

$$e^{2in\ln(2kR)}F_d(\vec{q},\vec{s}) = e^{-i\vec{q}\cdot\vec{s}/2}\Big\{f_C^{\rm pt}(q) + i\int_0^\infty J_0(qb)\Gamma_C^E(b)(kb)^{2in+1}db$$
$$+\frac{1}{4\pi^2}\int e^{i(\vec{q}-\vec{q}\,')\cdot\vec{b}}(kb)^{2in}e^{i\chi_C^E(b)}[f_{ps}(\vec{q}\,') + f_n(\vec{q}\,')e^{i\vec{q}\,'\cdot\vec{s}}]d^2q'd^2b$$
$$+\frac{i}{8\pi^3k}\int e^{i(\vec{q}-\vec{q}\,'-\vec{q}\,'')\cdot\vec{b}}(kb)^{2in}e^{i\vec{q}\,''\cdot\vec{s}}e^{i\chi_C^E(b)}f_{ps}(\vec{q}\,')f_n(\vec{q}\,'')d^2q'd^2q''d^2b\Big\} \; . \quad (11)$$

We will omit the phase $\exp[2in\ln(2kR)]$ from now on as it does not contribute to physically measured quantities.

We point out that the scattering amplitude for hadron-proton scattering $f_p(\vec{q})$ can be obtained as a special case by setting $\vec{s}$ and $f_n$ equal to zero in Eq. (11).

The differential cross section for elastic scattering by deuterium is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm el} = |F_{\rm el}(q)|^2 \; , \quad (12)$$

where the elastic scattering amplitude is the expectation value of $F_d(\vec{q},\vec{s})$ in the deuteron ground state and is given by

$$F_{\rm el}(\vec{q}) = \langle i|F_d(\vec{q},\vec{s})|i\rangle = S(\tfrac{1}{2}\vec{q})\Big[f_C^{\rm pt}(q) + i\int_0^\infty J_0(qb)(kb)^{2in+1}\Gamma_C^E(b)db + \frac{1}{4\pi^2}\int e^{i(\vec{q}-\vec{q}\,')\cdot\vec{b}}(kb)^{2in}e^{i\chi_C^E(b)}f_{ps}(\vec{q}\,')d^2q'd^2b\Big]$$
$$+\frac{1}{4\pi^2}\int e^{i(\vec{q}-\vec{q}\,')\cdot\vec{b}}(kb)^{2in}e^{i\chi_C^E(b)}S(\tfrac{1}{2}\vec{q}-\vec{q}\,')f_n(\vec{q}\,')d^2q'd^2b$$
$$+\frac{i}{8\pi^3k}\int e^{i(\vec{q}-\vec{q}\,'-\vec{q}\,'')\cdot\vec{b}}(kb)^{2in}e^{i\chi_C^E(b)}S(\tfrac{1}{2}\vec{q}-\vec{q}\,')f_{ps}(\vec{q}\,')f_n(\vec{q}\,'')d^2q'd^2q''d^2b \; . \quad (13)$$

Here $S(q)$ is the form factor of the ground state $|i\rangle$ of the deuteron.

If we parametrize the hadron-nucleon scattering amplitudes by

$$f_j(\vec{q}) = c_j\, e^{-a_j q^2/2}, \qquad j = n, ps \tag{14}$$

with

$$c_j = k\sigma_j(i+\rho_j)/4\pi\ ,$$

where $\sigma_j$ are incident hadron-nucleon total cross sections and $\rho_j$ are the ratios of real to imaginary parts of the forward hadron-nucleon elastic scattering amplitudes, and if we also define

$$T(b) = (kb)^{2in} e^{i\chi_C^E(b)}\, b\ , \tag{15}$$

then Eq. (11) simplifies to

$$F_d(\vec{q},\vec{s}) = e^{-i\vec{q}\cdot\vec{s}/2}\Big\{ f_C^{\rm pt}(q) + i\int_0^\infty J_0(qb)(kb)^{2in+1}\Gamma_C^E(b)db + (c_p/a_p)\int_0^\infty J_0(qb)T(b)e^{-b^2/2a_p}db$$
$$+ (c_n/a_n)e^{-s^2/a_n}\int_0^\infty J_0\Big(\Big|\vec{q}-\frac{i\vec{s}}{a_n}\Big| b\Big)T(b)e^{-b^2/2a_n}db$$
$$+\frac{ic_nc_p}{ka_na_p}e^{-s^2/a_n}\int_0^\infty J_0\Big(\Big|\vec{q}-\frac{i\vec{s}}{a_n}\Big| b\Big)T(b)\exp[-(a_n+a_p)b^2/2a_na_p]\Big\}\ , \tag{16}$$

where

$$\Big|\vec{q}-\frac{i\vec{s}}{a_n}\Big| = (q^2 - 2i\vec{q}\cdot\vec{s}/a_n - s^2/a_n{}^2)^{1/2}\ .$$

In this and future equations we suppress the subscript $s$ in $c_{ps}$ and $a_{ps}$. For hadron-nucleon amplitudes given by Eq. (14) and a deuteron form factor given by a sum of Gaussians

$$S(q) = \sum_j \alpha_j e^{-\beta_j q^2}\ , \tag{17}$$

the expression for the elastic scattering amplitude can be reduced to one dimensional integrals and is given by

$$F_{\rm el}(q) = \Big[ f_C^{\rm pt}(q) + i\int_0^\infty J_0(qb)(kb)^{2in+1}\Gamma_C^E(b)db + (c_p/a_p)\int_0^\infty J_0(qb)T(b)e^{-b^2/2a_p}db\Big]$$
$$\times\sum_j \alpha_j e^{-\beta_j q^2/4} + c_n\sum_j (\alpha_j/A_j)e^{-\beta_j q^2/4}e^{\beta_j{}^2q^2/2A_j}$$
$$\times\Big[\int_0^\infty J_0(G_jqb/A_j)T(b)e^{-b^2/2A_j}db + (ic_p/ka_p)\int_0^\infty J_0(G_jqb/A_j)T(b)e^{-H_jb^2/2A_ja_p}db\Big]\ , \tag{18}$$

where $G_j = a_n+\beta_j$, $A_j = G_j+\beta_j$, and $H_j = A_j + a_p$.

For the special case of point charges, $\chi_C^E$ vanishes and Eq. (18) can be evaluated analytically to yield

$$F_{\rm el}(q) = [f_C^{\rm pt}(q) + (2a_p)^{in}\Gamma(1+in)c_pe^{-a_pq^2/2}{}_1F_1(-in;1;a_pq^2/2)]\sum_j \alpha_je^{-\beta_jq^2/4}$$
$$+\sum_j \alpha_je^{-\beta_jq^2/4}(2A_j)^{in}c_n\Gamma(1+in)$$
$$\times\Big\{e^{-a_nq^2/2}{}_1F_1(-in;1;G_jq^2/2A_j) + \frac{i}{k}\frac{c_p}{H_j}\Big(\frac{a_p}{H_j}\Big)^{in}\exp[(\beta_j{}^2-a_pa_n)q^2/2H_j]\,{}_1F_1\Big(-in;1;\frac{a_pG_j{}^2q^2}{2A_jH_j}\Big)\Big\}\ , \tag{19}$$

which is a simple generalization of the results of Ref. 9 to the case of the deuteron form factor given by Eq. (17).

We can obtain an explicit expression for $\chi_C^E(b)$ by considering the incident hadron and the bound proton to have Gaussian charge distributions. If the respective charge form factors are given by $e^{-d^2q^2/4}$ and $e^{-c^2q^2/4}$, we obtain[10,11]

$$\chi_C^E(b) = nE_1[b^2/(c^2+d^2)]\ , \tag{20}$$

where $E_1(x) = -Ei(-x)$ is the exponential integral. This result is derived in Appendix A.

Since Eq. (18) requires numerical integration for its evaluation, it will be convenient to also consider an average phase approximation which leads to analytic results. For this approximation,[12] the scattering amplitude operator for hadron-deuteron collisions is given by

$$F_d^{\mathrm{av}}(\vec{q}, \vec{s}) = e^{-i\vec{q}\cdot\vec{s}/2}[f_C(\vec{q}) + e^{i\chi_{Cp}} f_{ps}(\vec{q})] + e^{i\vec{q}\cdot\vec{s}/2} e^{i\chi_{Cn}} f_n(\vec{q}) + \frac{i}{2\pi k} e^{i\chi_{Cpn}} \times \int e^{i\vec{q}'\cdot\vec{s}} f_n(\tfrac{1}{2}\vec{q} + \vec{q}') f_{ps}(\tfrac{1}{2}\vec{q} - \vec{q}') d^2q' , \tag{21}$$

where $\chi_{Cp}$, $\chi_{Cn}$, and $\chi_{Cpn}$ are average values of the Coulomb phase shift functions. Analytic expressions for them are derived in Appendix B for the case of Gaussian charge distributions for the incident hadron and the bound proton. The Coulomb amplitude may also be written in an analytic form as

$$f_C(q) = f_C^{\mathrm{pt}}(q) F_x(q) F_p(q) , \tag{22}$$

where $f_C^{\mathrm{pt}}(q)$ is given by Eq. (10) and $F_p(q)$ and $F_x(q)$ are the electromagnetic form factors of the proton and the incident hadron. The result that the Coulomb amplitude is proportional to the form factors of the colliding particles can be easily derived in the Born approximation. However, $f_C(q)$ given by Eq. (22) is more general as it has the correct phase in the point charge Coulomb amplitude. In Appendix B we show that this result can be derived from the more accurate Coulomb amplitude, given by the first two terms in Eq. (11), by dropping terms of $O(n^2)$.

For deuteron form factors given by Eq. (17) and hadron-nucleon amplitudes given by Eq. (14), the ground state expectation value of $F_d^{\mathrm{av}}$ reduces to

$$F_{\mathrm{el}}^{\mathrm{av}}(q) = [f_C(q) + e^{i\chi_{Cp}} c_p e^{-a_p q^2/2} + e^{i\chi_{Cn}} c_n e^{-a_n q^2/2}] \sum_j \alpha_j e^{-\beta_j q^2/4} + \frac{i c_n c_p}{k} e^{i\chi_{Cpn}} e^{-(a_p + a_n) q^2/8} \sum_j \frac{\alpha_j}{H_j} \exp[(a_p - a_n)^2 q^2 / 8H_j] . \tag{23}$$

Below 2 GeV, charge exchange effects are important and it is straightforward to extend this formula to include them. The final expression is given in Appendix C.

Measurements of the sum of $p$-$d$ elastic and quasielastic (i.e., deuteron breakup) scattering have also been used[3] to obtain values for $\rho_n$. The angular distribution for such processes is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{sc}} = \sum_f |\langle f | F_d(\vec{q}, \vec{s}) | i \rangle|^2 . \tag{24}$$

At high energies we can neglect the energy differences between the final states of the deuteron, and the completeness relation

$$\sum_f | f \rangle\langle f | = 1 \tag{25}$$

can be utilized to yield

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{sc}} = \langle i | \, | F_d(\vec{q}, \vec{s}) |^2 | i \rangle . \tag{26}$$

This expression can be evaluated using $F_d(\vec{q}, \vec{s})$ given by Eq. (11) but the result is a tediously long expression involving double integrals and an infinite sum. We shall write down the result in the average phase approximation where one can obtain an analytic result. Using $F_d(\vec{q}, \vec{s})$ in Eq. (26) we obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{sc}} = |f_C(q) + e^{i\chi_{Cp}} f_{ps}(q)|^2 + |f_n(q)|^2 + 2\,\mathrm{Re}[f_C(q) + e^{i\chi_{Cp}} f_{ps}(q)]^* \times \left[ e^{i\chi_{Cn}} f_n(q) \sum_j \alpha_j e^{-\beta_j q^2} + \frac{i c_p c_n}{k} e^{i\chi_{Cpn}} e^{-a_p q^2/2} \sum_{H_j} \frac{\alpha_j}{H_j} e^{a_p^2 q^2 / 2H_j} \right] + 2\,\mathrm{Re}\left\{ [e^{i\chi_{Cn}} f_n(q)]^* \frac{i c_n c_p}{k} e^{i\chi_{Cpn}} e^{-a_n q^2/2} \sum_j \frac{\alpha_j}{H_j} e^{a_n^2 q^2 / 2H_j} \right\} + \frac{|c_n c_p|^2}{k^2} \exp[-a_n a_p q^2/(a_n + a_p)] \sum_j \frac{\alpha_j}{(H_j^2 - 4\beta_j^2)} . \tag{27}$$

### III. HADRON-NUCLEUS COLLISIONS

The amplitude for elastic scattering between hadrons and nuclei with mass number $A$ is given by[1]

$$F_{ii}(q)=\frac{ik}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}\psi_i^*(\vec{r}_1,\ldots,\vec{r}_A)\delta^3\left(A^{-1}\sum_{j=1}^{A}\vec{r}_j\right)\left\{1-\prod_{j=1}^{A}[1-\Gamma_j(\vec{b}-\vec{s}_j)]\right\}\psi_i(\vec{r}_1,\ldots,\vec{r}_A)d\vec{r}_1\cdots d\vec{r}_A d^2b\ , \quad (28)$$

where $\psi_i$ is the ground state wave function of the nucleus and the other symbols are defined as in Sec. II. Since we will only be comparing theoretical expressions to exhibit the charge distribution effects and since accurate wave functions are needed only to reproduce the details of the structure in the differential cross section at large momentum transfers, we will assume that the nucleus can be described by a simple independent particle model where each nucleon is represented by a Gaussian, i.e.,

$$|\psi_i(\vec{r}_1,\ldots,\vec{r}_A)|^2=\prod_{j=1}^{A}|\phi_j(\vec{r}_j)|^2 =\prod_{j=1}^{A}\alpha^3\pi^{-3/2}e^{-\alpha^2r^2}\ , \quad (29)$$

where $\alpha$ is related to the rms radius of the nucleus.

Now, using the Gartenhaus-Schwartz transformation[13] to eliminate the $\delta$ function, we obtain

$$F_{ii}(q)=\frac{ik}{2\pi}K(q)\int e^{i\vec{q}\cdot\vec{b}}\left\{1-\prod_{p=1}^{Z}\langle\phi_p|[1-\Gamma_p(\vec{b}-\vec{s}_p)]|\phi_p\rangle\prod_{n=1}^{A-Z}\langle\phi_n|[1-\Gamma_n(\vec{b}-\vec{s}_n)]|\phi_n\rangle\right\}d^2b\ , \quad (30)$$

where $Z$ is the atomic number of the target nucleus and $K(q)=e^{q^2/4A\alpha^2}$. Again by writing

$$\Gamma_p(\vec{b}-\vec{s}_p)=\Gamma_C^{\rm pt}(\vec{b}-\vec{s}_p)+e^{i\chi_C^{\rm pt}(\vec{b}-\vec{s}_p)}\Gamma_C^E(\vec{b}-\vec{s}_p)+e^{i\chi_C(\vec{b}-\vec{s}_p)}\Gamma_{ps}(\vec{b}-\vec{s}_p) \quad (31)$$

we obtain

$$\langle\phi_p|[1-\Gamma_p(\vec{b}-\vec{s}_p)]|\phi_p\rangle=(2\alpha R)^{-2in}e^{-\alpha^2b^2}\left[\Gamma(1+in)\,{}_1F_1(in+1;1;\alpha^2b^2)-\int_0^\infty e^{-x}x^{in}\,\Gamma_C^E(\sqrt{x}/\alpha)I_0(2\alpha b\sqrt{x})dx - \frac{\sigma_p(1-i\rho_p)}{4\pi a_p}\int_0^\infty e^{-x}x^{in}e^{i\chi_C^E(\sqrt{x}/\alpha)}I_0(2\alpha b\sqrt{x})e^{-x/2a_p\alpha^2}dx\right]$$

$$=(2kR)^{-2in}A_p(b)\ , \quad (32)$$

$$\langle\phi_n|[1-\Gamma_n(\vec{b}-\vec{s}_n)]|\phi_n\rangle=1-\frac{\sigma_n(1-i\rho_n)}{2\pi}\left(\frac{\alpha^2}{2a_n\alpha^2+1}\right)\exp[-\alpha^2b^2/(2a_n\alpha^2+1)]$$

$$=A_n(b)\cdot \quad (33)$$

Equation (30), as it stands, converges very slowly and hence is not well suited for numerical evaluation. It can be made to converge rapidly by subtracting out the Coulomb amplitude due to a point nucleus of charge $Ze$. The Coulomb phase shift function $\chi_{CZ}^{\rm pt}(b)$ due to the point nucleus is given by[1]

$$\chi_{CZ}^{\rm pt}(b)=2nZ\ln(b/2R)\ . \quad (34)$$

By adding and subtracting $e^{i\chi_{CZ}^{\rm pt}}$ in the integrand of Eq. (30) we obtain

$$F_{ii}(q)=e^{-2inZ\ln(2kR)}K(q)\times\left\{f_{CZ}^{\rm pt}(q)-ik\int_0^\infty[A_p(b)^ZA_n(b)^{A-Z}-(kb)^{2inZ}]J_0(qb)b\,db\right\}\ , \quad (35)$$

where $f_{CZ}^{\rm pt}(q)$ is given by Eq. (10) with $n$ replaced by $Zn$. Now the phase factor can be dropped as it does not contribute to the cross section.

If we consider the special case where the incident hadron and the bound proton are point charges, $A_p(b)$ can be evaluated analytically and is given by

$$A_p(b) = (k/\alpha)^{2in} e^{-\alpha^2 b^2}\Gamma(1+in) \times \left[ {}_1F_1(in+1;1;\alpha^2 b^2) - \frac{\sigma_p(1-i\rho_p)}{4\pi a_p} \times \left(\frac{2a_p\alpha^2}{2a_p\alpha^2+1}\right)^{in+1} {}_1F_1\left(in+1;1;\frac{2a_p\alpha^4 b^2}{(2a_p\alpha^2+1)}\right)\right]. \tag{36}$$

This case has been considered in detail by Lesniak and Lesniak[14] for light nuclei.

Another way to include the Coulomb effects is by means of an approximation where one considers the Coulomb phase to originate from the nucleus as a whole rather than from each individual proton in the target. In that case the total phase shift function for the nucleus can be written as

$$\chi_{\rm tot}(\vec{b}) = \chi_{CZ}^{\rm pt}(\vec{b}) + \chi_{CZ}^{E}(\vec{b}) + \chi_s(\vec{b}) , \tag{37}$$

where $\chi_s(\vec{b})$ is the strong interaction phase shift function for the nucleus and $\chi_{CZ}^E(\vec{b})$ is the correction to the Coulomb phase shift due to the extended charge of the nucleus. One way to write the profile function for the nucleus is then

$$\Gamma_A(\vec{b}) = [1 - e^{i\chi_{CZ}^{\rm pt}(\vec{b})}] + e^{i\chi_{CZ}^{\rm pt}(\vec{b})}[1-e^{i\chi_{CZ}^{E}(\vec{b})}] + \exp[i\chi_{CZ}^{\rm pt}(\vec{b}) + i\chi_{CZ}^{E}(\vec{b})]\,\Gamma_s(\vec{b}) , \tag{38}$$

where

$$\Gamma_s(\vec{b}) = \langle \psi_i | \{1 - \exp[i\chi_s(\vec{b}, \vec{s}_1, \ldots, \vec{s}_A)]\} | \psi_i \rangle .$$

This can be rearranged for convenience in numerical evaluation and the elastic scattering amplitude is given by

$$F_{ii}(q) = K(q)\left(f_{CZ}^{\rm pt}(q) + i\int_0^\infty (kb)^{2inZ+1}\{1 - e^{i\chi_{CZ}^E(b)}[1-\Gamma_s(b)]\}J_0(qb)db\right) , \tag{39}$$

where we have again dropped an unimportant overall phase factor.

Another formula which has been frequently used is that due to Bethe[7] who showed that, under certain assumptions, the elastic scattering intensity for charged hadron-nucleus collisions may be written as

$$\frac{d\sigma}{d\Omega} = |f_s(q) + f_e(q)e^{inZ\phi}|^2 , \tag{40}$$

where $f_s(q)$ is the strong interaction amplitude and $f_e(q)$ is the Coulomb amplitude of the nucleus (apart from its phase). The amplitude $f_e(q)$ includes the electromagnetic form factor of the nucleus. The relative phase $\phi$ was given by

$$\phi \approx -2\ln(aq/1.06) ,$$

where $a$ is related to the strong interaction radius of the nucleus. Using relativistic methods, West and Yennie[8] obtained for the relative phase

$$\phi \approx -2\ln(a'q) - C \tag{41}$$

with

$$a' = \tfrac{1}{2}(a^2 + R_1{}^2 + R_2{}^2)^{1/2}, \quad C = 0.577\ldots ,$$

and where the charge form factors of the incident and target particles are given by $e^{-R_1^2q^2/4}$ and $e^{-R_2^2q^2/4}$. For the case of point charge particles incident on a point nucleus, this reduces to the phase given by Bethe.

## IV. NUCLEUS-NUCLEUS COLLISIONS

The Glauber theory has been extended to deuteron-deuteron collisions[15] and to heavy-ion collisions in general.[15,16] For heavy ions the full Glauber multiple scattering series reduces to a simple form only in the "optical limit," where the numbers of nucleons in the colliding nuclei are large. In this limit, for collisions of nuclei with respective mass numbers $A_1, A_2$ and atomic numbers $Z_1, Z_2$, the profile function is given by[16]

$$\Gamma_{A_1A_2}(b) = 1 - e^{-A_1A_2C(b)} \tag{42}$$

with

$$C(b) = \int d^2s\,d^2s'\,\rho_{A_1}(\vec{s})\Gamma(\vec{b}-\vec{s}+\vec{s}\,')\rho_{A_2}(\vec{s}\,') \tag{43}$$

and

$$\rho_{A_1,A_2}(\vec{s}) = \int_{-\infty}^{\infty}\rho_{A_1,A_2}(\vec{r})dz ,$$

where $\Gamma(b)$ is the nucleon-nucleon profile function. By taking Fourier transforms, $C(b)$ can be rewritten as

$$C(b) = \frac{1}{2\pi ik}\int d^2q\,e^{-i\vec{q}\cdot\vec{b}}S_{A_1}(\vec{q})f(\vec{q})S_{A_2}(-\vec{q}) , \tag{44}$$

where the $S_{A_i}(q)$ are the form factors of the two nuclei and $f(q)$ is the nucleon-nucleon scattering amplitude. If the form factors are given by $e^{-R_1^2q^2/4}$ and $e^{-R_2^2q^2/4}$, respectively, and $f(q)$ is given by the usual high energy parametrization of Eq. (14), we obtain

$$C(b)=\frac{\sigma(1-i\rho)}{2\pi R^2}e^{-b^2/R^2}\,, \qquad R^2=R_1^2+R_2^2+2a\,, \tag{45}$$

where $\sigma$, $a$, and $\rho$ are average nucleon-nucleon parameters defined as in Sec. II. An expression very similar to Eq. (44) is also obtained in the Chou-Yang model[17] (or the coherent droplet model). We can formally obtain the result of the Chou-Yang model by noting that if the sizes of the nuclei are large compared to the range of nucleon-nucleon interactions, the nuclear form factors will go to zero much more rapidly than the nucleon-nucleon amplitude $f(q)$ and Eq. (44) can then be approximated by

$$C(b)\simeq\frac{f(0)}{2\pi ik}\int d^2q e^{-i\vec{q}\cdot\vec{b}}S_{A_1}(\vec{q})S_{A_2}(-\vec{q})\,. \tag{46}$$

In this model one can take the view[18] that $x=A_1A_2\sigma(i+\rho)$ is a free parameter not related to the nucleon-nucleon amplitudes. Im$x$ may be estimated by nucleus-nucleus total cross sections, while $\rho$ or Re$x$ can be determined by fitting the scattering data. The most significant effect of $\rho$ is on the diffraction minima of the cross sections, and since the Coulomb effects are important near the minima they should be included in analyses which determine $\rho$ or Re$x$.

The extended charge Coulomb effects can be included by a formula similar to Eq. (39) and the full scattering amplitude is given by

$$F_{A_1A_2}(q)=K_{12}(q)\left(f^{\rm pt}_{CZ_1Z_2}(q)+i\int_0^\infty (kb)^{2inZ_1Z_2+1}\{1-e^{i\chi^E_{CZ_1Z_2}(b)}[1-\Gamma_{A_1A_2}(b)]\}J_0(qb)db\right)\,, \tag{47}$$

where

$$K_{12}(q)=\exp\{q^2[(R_1^2/4A_1)+(R_2^2/4A_2)]\}$$

is the center of mass correction and $f^{\rm pt}_{CZ_1Z_2}(q)$ is given by Eq. (10) with $n$ replaced by $Z_1Z_2n$. A similar formula was given by Czyż and Maximon[16] where the Coulomb phase shift function was given as a seven dimensional integral. However, we point out that for collisions between light nuclei where the charge form factors can be approximated by Gaussians (or sums of Gaussians), the Coulomb phase shift function incorporating the extended charge effects can be evaluated analytically. In Appendix A we show that $\chi^E_{CZ_1Z_2}(b)$, which is the correction to the Coulomb phase shift function due the the charge distribution of the colliding nuclei, is given by

$$\chi^E_{CZ_1Z_2}(b)=Z_1Z_2nE_1[b^2/(R_1^2+R_2^2)]\,. \tag{48}$$

With this result the numerical integration in Eq. (47) is straightforward. The problem of slow convergence of the integral (due to large impact parameter scattering by the Coulomb interactions) has been removed by subtracting out the point charge Coulomb scattering so that the integrand in Eq. (47) vanishes for large impact parameters.

## V. COMPARISON OF THEORETICAL EXPRESSIONS

In this section we compare the various approximations discussed in the text with the more exact results. Throughout our analysis the proton charge form factor is assumed to have the form $e^{-c^2q^2/4}$. The value of $c$ is taken to be 0.66 fm which corresponds to a rms radius of 0.81 fm. This value fits the experimentally observed form factor in the region $0.001\lesssim -t\lesssim 0.011$ and also $0.001\lesssim -t\lesssim 0.05$ $(\text{GeV}/c)^2$ which are the momentum transfer regions in which we perform our analysis of the $pd$ data.

We first consider the elastic scattering of protons by deuterium. The form factor used for the deuteron is a sum of Gaussians fitted to form factors obtained from "realistic" deuteron wave functions and will be described in detail in Sec. VI. In Fig. 1 we show the percent error in the $pd$ differential cross sections near the interference re-

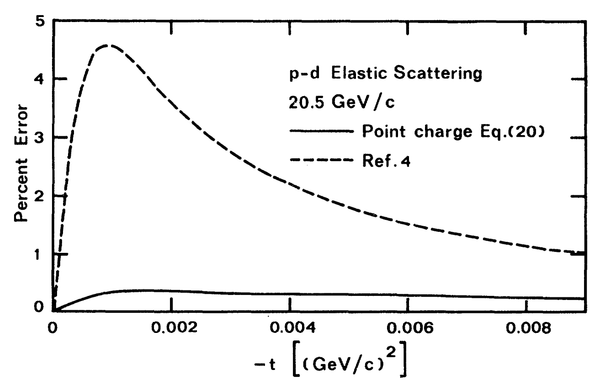


FIG. 1. Percent error in angular distributions for $p$–$d$ elastic scattering at 20.5 GeV/$c$ for the formula used in Ref. 4 and for the point charge solution of Eq. (19) compared to extended charge result given by Eq. (18).

gion, for the point charge solution given by Eq. (19) and for the formula used in Ref. 4 in an analysis of $p$-$d$ data. The "exact" values of the cross sections are obtained from Eq. (18). The error in the average phase approximation of Eq. (23) is exceedingly small in this region and hence is not shown. The maximum error in the formula of Ref. 4 in the interference region is ~4.6%. For $-t \lesssim 0.1$ $(\text{GeV}/c)^2$ the error in the point charge approximation is $\lesssim 0.5\%$, while the error in Eq. (23) is even smaller ($\lesssim 0.1\%$). However, near the minimum in the differential cross sections $[-t \sim 0.33$ $(\text{GeV}/c)^2]$, the error in Eq. (23) is as large as ~22%, while the error in Eq. (19) is only ~2%. For even larger angles, the error in the Coulomb amplitude due to the point charge approximation increases, but Coulomb contributions themselves become unimportant compared to nuclear contributions. The errors in Eqs. (19) and (23) depend negligibly upon the energy, while the error in the formula of Ref. 4 increases slightly with energy.

The reason for the increase in the error of the point charge solution with the momentum transfer is that the large angle scattering results from small impact parameter collisions where the point charge assumption starts to break down. On the other hand the error in the formula of Ref. 4 is in some measure due to the assumption that $\chi_{Cp}$, $\chi_{Cn}$, and $\chi_{Cpn}$ are all equal to 0.06 (at 70.2 GeV/$c$, for example, their values are, respectively, 0.086, 0.098, and 0.085), but is mostly due to the Coulomb amplitude in which Bethe's phase has been used incorrectly. We point out that Bethe's phase is the relative phase between the Coulomb amplitude (without its phase) and the strong interaction amplitude for the hadron-nucleus collisions, and it should not be present in the formula of Ref. 4 which has been obtained from a multiple scattering series where the strong interaction terms are already modified by a phase factor arising from the Coulomb interaction. The correct Coulomb amplitude that should be used is given by Eq. (22).

We next consider scattering by other nuclei. In Fig. 2 we show the percent error in the differential cross sections obtained from the various approximate formulas compared to the "exact" Glauber result given by Eq. (35). In Fig. 2(a) we show the results for $p$-$^{12}$C elastic scattering at 1.04 GeV. For $-t \lesssim 0.01$ $(\text{GeV}/c)^2$, the error in the approximate formula given by Eq. (39) is $\lesssim 0.8\%$, while the formula given by Eq. (40) [together with Eq. (41)] has errors of $\lesssim 1.5\%$. But at the larger momentum transfers the errors are as large as ~14 and 45% for the two expressions.

We find that the assumption of point charge for each proton in the target [i.e., Eq. (36)] is quite accurate except at the diffraction minima. At 1.04 GeV, the percent error in this approximation is $\lesssim 1\%$ near the minimum. However, at energies where the real part of the nucleon-nucleon amplitude is very small, almost the entire contribution to the cross sections near the minima is from the Coulomb scattering and the error in the point charge solution is larger. This error also de-

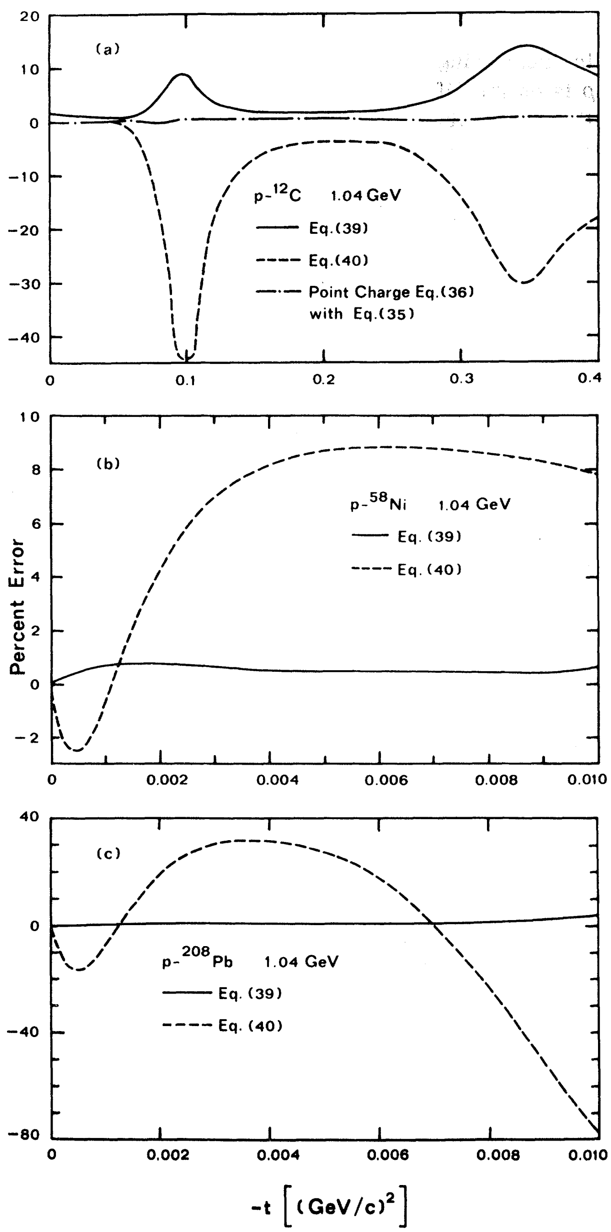


FIG. 2. Percent error in differential cross sections for $p$-nucleus elastic scattering at 1.04 GeV for the approximate formulas discussed in the text compared to the extended charge result given by Eq. (35). The figures correspond to (a) $p$-$^{12}$C, (b) $p$-$^{58}$Ni, and (c) $p$-$^{208}$Pb scattering.

pends on whether the Coulomb interaction is attractive or repulsive. For example at 180 MeV, the error at the first diffraction minimum in $\pi^-$-$^4$He scattering is ~3%, while for $\pi^+$-$^4$He scattering it is ~8%.

In Figs. 2(b) and 2(c) we show the results for $p$-$^{58}$Ni and $p$-$^{208}$Pb scattering at small momentum transfers. We note that Eq. (40) becomes worse than it was for light nuclei, with rather large errors even at small momentum transfers. For $-t \lesssim 0.01$ (GeV/$c$)$^2$ the errors are ~8% for $p$-Ni and ~30–80% for $p$-Pb. However, in this region, the approximate phase result of Eq. (39) does reasonably well with errors $\lesssim$1 and 4% for the two nuclei. Near the first diffraction minimum, the error in Eq. (39) is ~7% for both nuclei but increases at the other minima. For example, at $-t \sim 0.25$ (GeV/$c$)$^2$ the error in Eq. (39) for $p$-Ni scattering is ~23%.

It is interesting to note that treating the protons as point charges instead of extended charges in the exact Glauber series leads to a greater relative error in the scattering from deuterium than from other nuclei. The reason for this is that since in the heavier nuclei there are many protons close together, the point charge protons are smeared over the nuclear volume when the expectation value in the nuclear ground state is taken, and the result of smearing the extended charge protons is not much different. This is not true for the deuteron which has only one proton and is a loosely bound system. However, since the Coulomb effect itself is much smaller in deuterium, the point charge assumption does not lead to much error in the cross sections.

Among the other simpler formulas, the approximate amplitude of Eq. (39) is reasonable for small angles but leads to appreciable errors near the diffraction minima. At these larger angles one must consider the interference of Coulomb effects with strong interaction effects of each nucleon separately instead of considering the over-all Cou-

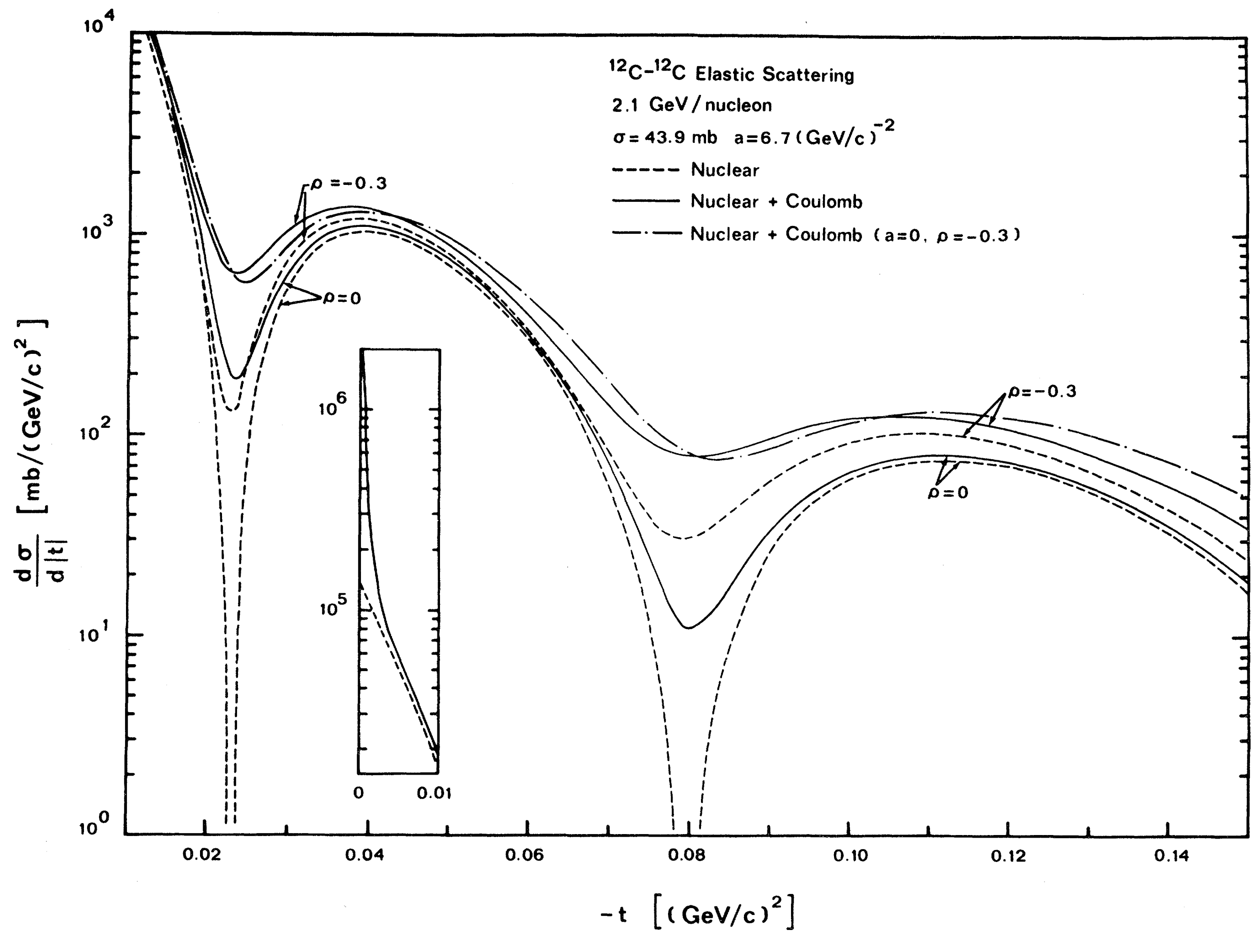


FIG. 3. The invariant differential cross section for $^{12}$C-$^{12}$C elastic scattering at 2.1 GeV/nucleon with and without Coulomb effects and for different values of nucleon-nucleon parameters.

lomb phase resulting from the nucleus as a whole. Equation (40) is good only for light nuclei at very small angles. Its derivation uses the average Coulomb phase shift of nuclear scattering evaluated at $q=0$ and neglects terms of $O(Z^2n^2)$. It can be improved[11] by introducing the $q$ dependence of the phase but the formula will still be inaccurate for collisions where $Zn \gtrsim 0.1$. We should mention, for completeness, that Coulomb effects can also be included by means of the optical limit. However, it was pointed out by Lesniak and Lesniak[14] that this approximation leads to significantly lower cross sections at large angles compared to those obtained from the full Glauber series with the assumption of each proton in the target being a point charge. Since we find that this point charge assumption is fairly accurate (except at the minima when the real parts of nucleon-nucleon amplitudes are small), our conclusions for the case of light nuclei are very similar to those of Ref. 14.

Since data may be forthcoming for $^{12}C$ ions at 2.1 GeV/nucleon, we have investigated the Coulomb effects for $^{12}C$-$^{12}C$ collisions at that energy using Eq. (47). We show in Fig. 3 that the Coulomb effects are important over a wide range of momentum transfers. They dominate at $-t \lesssim 0.003$ $(\text{GeV}/c)^2$ and at the diffraction minima. However, even at the subsidiary maxima, they increase the cross section by ~15–20%. If the data near the diffraction minima are used to extract $\rho$ or Re$x$ in the Chou-Yang model, the Coulomb effects will play a crucial role. For example, we note that a purely nuclear cross section with $\rho \sim -0.4$ corresponds to nuclear plus Coulomb cross section with $\rho \sim 0$ near the first minimum. We should also mention that if the full Glauber series were used instead of the optical limit for the nuclear cross sections, since the former are generally lower for light nuclei,[16] the Coulomb effects would be even more important. Also, Coulomb effects are larger at lower energies.

As pointed out in Sec. IV, the Chou-Yang model is essentially equivalent to the optical limit of Glauber theory. In fact Eq. (46) in the Chou-Yang model leads to the same result as Eq. (45) with $a=0$. To test the sensitivity of the cross sections to this assumption, we have plotted the cross sections with $a=0$. We find that the curves are very similar except that the positions of the minima are slightly shifted. (However, the curve of Fig. 3 with $a=0$ still has the center of mass correction in it. This correction factor is not present in the Chou-Yang model.) This indicates that the nucleus-nucleus amplitude is not very sensitive to the $q$ dependence of the nucleon-nucleon amplitudes. As discussed in Sec. IV, this is due to the fact that because of the large size of the colliding nuclei, the form factors are sharply peaked in the forward direction and pick up only the very small $q$ behavior of nucleon-nucleon amplitudes. Now this also implies that a weak $q$ dependence of $\rho$ should not have much effect on the nucleus-nucleus amplitude. Hence, assuming $\rho$ to be a constant, we show in Fig. 4 the ratio of real to imaginary part of the $^{12}C$-$^{12}C$ nuclear scattering amplitude at 2.1 GeV/nucleon for small momentum transfers. Again we notice that results are very similar if we set $a=0$.

## VI. ANALYSIS OF $p$-$d$ SCATTERING DATA

We first describe the deuteron form factor which will be used in our analyses. We note that the extended charge expressions derived in Sec. II are relatively easy to evaluate for deuteron form factors given by a Gaussian or sums of Gaussians. In order to both preserve this ease in numerical evaluations and have an accurate form factor, we have fitted sums of Gaussians to form factors given by realistic deuteron wave functions. The first wave function we choose is that obtained from the hard core potential of Reid[19] which is fitted accurately to the nucleon-nucleon ($NN$) phase shifts up to 350 MeV. Our second choice is the wave function obtained by Humberston[19] by slightly modifying the Hamada-Johnston[19] potential to give the observed binding energy of the deuteron. Since both these wave functions are obtained from hard core potentials, we also consider a wave function due to Bressel and Kerman[19] which is obtained from finite soft core potentials that reproduce the $NN$ phase shifts very well. All these wave functions have a small admixture of $D$ state. We obtained the form factors from these wave functions by numerical integration and we fitted sums of Gaussians to them. A sum of Gaussians which is

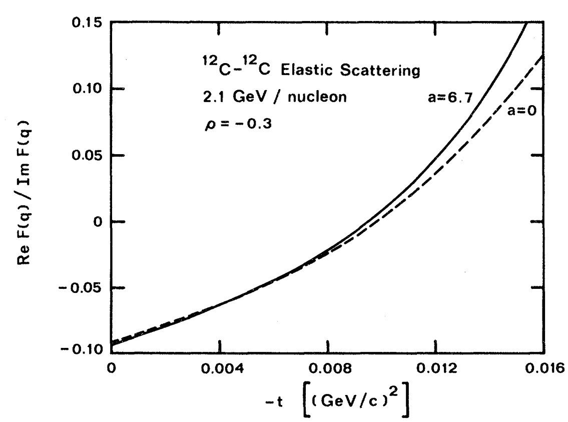


FIG. 4. The ratio of real to imaginary part of $^{12}C$-$^{12}C$ elastic scattering amplitude at 2.1 GeV/nucleon.

TABLE I. Results for $\rho_n$ and $a_n$ from the analyses of $pd$ data. The second and fourth columns give the values of $\rho_n$ obtained by $\chi^2$ fits to the $p$-$d$ elastic scattering data using either Eq. (18) (extended charge) or Eq. (23) (average phase) and point charge Eq. (19) with assumption $a_n = a_p$. The fifth column gives the results of Ref. 4 and the sixth column gives the result when the formula of Ref. 4 is used with the form factor given by Eq. (49). In the seventh column, we list the values of $a_n$ obtained by using the values of $\rho_n$ from the second column as input. The results at 19.3 GeV/$c$ are from the analysis of $p$-$d$ elastic plus quasielastic scattering data and are given in the third and eighth columns.

| $P$ (GeV/$c$) | $\rho_n$ Eqs. (18) and (23) | $\rho_n$ Eq. (27) | $\rho_n$ Eq. (19) | $\rho_n$ Ref. 4 | $\rho_n$ Ref. 4 with Eq. (49) | $a_n$ [(GeV/$c$)$^{-2}$] Eqs. (18) and (23) | $a_n$ [(GeV/$c$)$^{-2}$] Eq. (27) |
|---|---|---|---|---|---|---|---|
| 11.2 | −0.29 | | −0.28 | −0.21 | −0.26 | 7.3 | |
| 15.9 | −0.50 | | −0.49 | −0.38 | −0.46 | 7.0 | |
| 19.3 | | −0.32 | | | | | 10.25 |
| 20.5 | −0.48 | | −0.47 | −0.35 | −0.44 | 6.5 | |
| 26.5 | −0.45 | | −0.44 | −0.35 | −0.40 | 7.2 | |
| 34.8 | −0.38 | | −0.37 | −0.25 | −0.34 | 8.3 | |
| 48.9 | −0.33 | | −0.32 | −0.14 | −0.28 | 8.3 | |
| 57.2 | −0.36 | | −0.35 | −0.20 | −0.31 | 8.7 | |
| 60.8 | +0.06 | | +0.06 | | +0.14 | 11.7 | |
| 64.8 | −0.05 | | −0.04 | +0.13 | +0.03 | 9.3 | |
| 70.2 | −0.24 | | −0.23 | −0.04 | −0.18 | 8.4 | |

consistent with each of the three form factors is given by

$$S(q) = 0.34e^{-141.5q^2} + 0.58e^{-26.1q^2} + 0.08e^{-15.5q^2} \qquad (49)$$

with $q$ in GeV/$c$.

In the interference region $-t \lesssim 0.01$ (GeV/$c$)$^2$, the form factor $S(\frac{1}{2}q)$ given by Eq. (49) agrees with those given by realistic wave functions to better than 0.2%. For larger momentum transfers $0.01 \lesssim -t \lesssim 0.21$ (GeV/$c$)$^2$ it agrees to within 2.4%; but in this region the values of $S(\frac{1}{2}q)$ given by the Reid and by the Bressel-Kerman wave functions themselves differ by ~3%. The region where the form factor given by Eq. (49) differs significantly from that given by the above wave functions is at large momentum transfers. However, since the deuteron form factor is sharply peaked in the forward direction, the contribution from this region to the scattering amplitudes is negligible.

We can now utilize the proton-deuteron elastic scattering data obtained at Serpukhov in the energy range 10 to 70 GeV to extract proton-neutron scattering parameters such as $\rho_n$, the ratio of real to imaginary part of the forward proton-neutron elastic scattering amplitude, and $a_n$, the slope parameter. We do our calculations in two steps. For the purpose of calculating $\rho_n$ we restrict the analysis to small momentum transfers $-t \lesssim 0.011$ (GeV/$c$)$^2$ since beyond this the interference effects are negligible and also because we assume a constant phase for the hadron-nucleon strong interaction amplitudes (i.e., the same value as at $q=0$), an approximation which becomes less accurate away from the forward direction. We first calculate $\rho_n$ by assuming that $a_n = a_p$. Then using this value of $\rho_n$ we can use the larger momentum transfer region $-t \lesssim 0.05$ (GeV/$c$)$^2$ to calculate $a_n$. Using this value for $a_n$, we can then recalculate $\rho_n$ and perform this iteration until $\rho_n$ and $a_n$ do not change in value.

In our calculations the values of $a_p$ are taken from Ref. 20, $\rho_p$ from Ref. 21, and values of $\sigma_n$ and $\sigma_p$ are from Ref. 22. At any particular energy we have interpolated between the experimentally measured values if necessary and the actual values used in our calculations are given in Ref. 10. For the calculation of $\rho_n$, the angular distribution obtained from Eq. (18) was fitted to the proton-deuteron data by minimizing the $\chi^2$ with $\rho_n$ being the only free parameter. The values of $\chi^2$ per data point in these fits vary from 0.6 to 2.5 and the typical error bars for $\rho_n$ are ~±0.07 which also include the effects of varying the various experimental parameters that we used within their statistical errors. It also includes an uncertainty of ~±0.01 obtained by varying our Gaussian form factor of Eq. (49) to fit more closely the form factors given by the Reid hard core and Bressel-Kerman wave functions. The systematic errors in $\rho_n$ can be as large as ±0.13, the main source being the proton-deuteron elastic scattering data.[4]

In Table I, we list the values of $\rho_n$ obtained from various formulas by assuming $a_n = a_p$. The second column gives the results obtained from the extended charge expression Eq. (18) and also from Eq. (23). These two formulas yield the same results to two significant figures. The fifth column gives the results of Beznogikh *et al.*[4] who used a formula similar to Eq. (23) in their analyses with the deuteron form factor given by[23]

$$S^2(\tfrac{1}{2}q)=\exp(-25.9q^2+60q^4)\,. \qquad (50)$$

There is a significant difference between the two values of $\rho_n$ which is not only due to the different formula but also due to the difference in the form factors used. The deuteron form factor given by Eq. (50) is a fit to the numerical values of $S^2(\frac{1}{2}q)$ obtained by fitting the formula of Ref. 4 to the proton-deuteron cross section data for $0.002 \lesssim -t \lesssim 0.17$ $(\mathrm{GeV}/c)^2$ from 10 to 26 GeV with the assumptions that $\rho_n=\rho_p$ and $a_n=a_p$. In order to isolate the effects of the form factor, we have repeated the analyses of Ref. 4 with the deuteron form factor given by Eq. (49). The results are given in the sixth column and we find that the use of a more accurate form factor leads to a difference in $\rho_n$ of ~0.1 on the average. We should also point out that even though data exist at 60.8 GeV/$c$, they were not used in Ref. 4 for calculating $\rho_n$ and $a_n$. The fourth column in Table I shows the results for $\rho_n$ obtained from Eq. (19) (point charge). If charge exchange effects are included (using the formula given in Appendix C), we find that $\rho_n$ changes by $\lesssim 0.01$ at these energies.[24]

We can now use the value of $\rho_n$ obtained from Eq. (18) to calculate $a_n$ by analyzing the data for $0.002 \lesssim -t \lesssim 0.05$ $(\mathrm{GeV}/c)^2$. Again Eqs. (18) and (23) give identical results and the values of $a_n$ are listed in the last columns of Table I. One can now perform an iteration to recalculate $\rho_n$ and then $a_n$ but the values change very little because $\rho_n$ is relatively insensitive to variations in $a_n$. In our $\chi^2$ fit for $a_n$, the $\chi^2$ minimum varies from 1.0 to 2.3 per data point and typical results of our $\chi^2$ fit to the data are shown in Fig. 5. The error bars in $a_n$ in Table I are ~±1.5 and include the error due to the uncertainty in $\rho_n$. The smaller values of $a_n$ compared to $a_p$ are surprising since direct measurements[25] of $a_n$ (from $n$-$p$ scattering) and of $a_p$ at larger momentum transfers seem to agree with each other. However the value of $a_n$ is quite sensitive to the input value of $a_p$ in our analyses. If the values of $a_p$ are lowered, the values of $a_n$ increase by roughly the same amount. The input values of $a_p$ that we have used are from Ref. 20 and are slightly higher than the recent FNAL measurements in the overlapping energy region.

Since all the above results are based on the experiments of the same group, we have also used the elastic plus quasielastic scattering data at 19.3 GeV/$c$ of Bellettini *et al.*[3] to extract $\rho_n$. Using Eq. (27) with input parameters again from Refs. 20–22 (with values given in Fig. 6) we find $\rho_n=-0.32$ and $a_n \simeq a_p$. The fit to the data is shown in Fig. 6. If we use the same input parameters as used in Ref. 3 [$\sigma_n=\sigma_p=38.9$ mb, $a_n=a_p=10$ $(\mathrm{GeV}/c)^{-2}$, $\rho_p=-0.33$] we find $\rho_n=-0.25$. In Fig. 7 we show for comparison our calculated results for $\rho_n$ together with dispersion relation calculations by Barashenkov and Toneev[26] and also by Carter and Bugg.[27]

We should point out that our theoretical results for $p$-$d$ scattering do not include the effect of in-

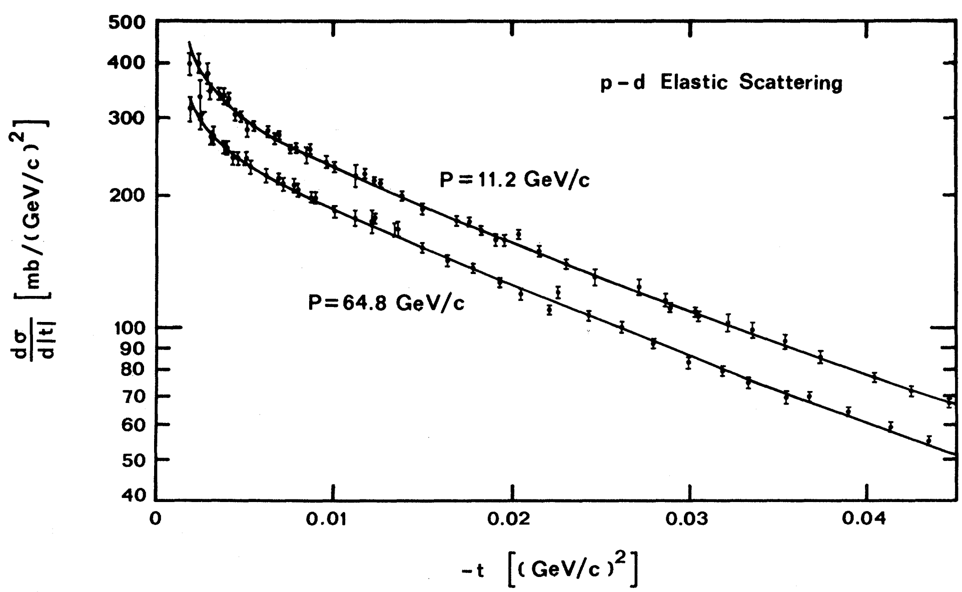


FIG. 5. Differential cross section for elastic $p$–$d$ scattering at 11.2 and 64.8 GeV/$c$. The data are from Ref. 5 and the curves correspond to Eq. (18).

elastic intermediate states[28] (for example, an $N^*$ can be created coherently in the nucleus and then decay back into a proton) on the elastic scattering. The Glauber approximation can be extended to include these effects.[29] But quantitatively reliable calculations of these effects are lacking because they depend critically upon the phases of the production amplitudes which are not well known. On the other hand, rough estimates of the effects that are not included in the Glauber approximation can be obtained from the $p$-$p$, $p$-$d$, and $n$-$p$ total cross section measurements. Let us define the experimental cross section defect in deuterium by

$$\delta\sigma_{\text{exp}} = \sigma_{np} + \sigma_{pp} - \sigma_{pd}$$

and the inelastic cross section defect by

$$\delta\sigma_{\text{inel}} = \delta\sigma_{\text{exp}} - \delta\sigma_{\text{tot}} ,$$

where $\delta\sigma_{\text{tot}}$ is calculated from the double scattering term of the elastic scattering amplitude in the Glauber approximation. By using experimental data wherever they exist at roughly the same energy,[22,30] we find that $|\delta\sigma_{\text{inel}}| \approx 0.1 \pm 0.9$ mb from 15 to 35 GeV. (For example, near 34 GeV, $\delta\sigma_{\text{exp}} = 3.16$ mb and using the deuteron wave function given by the Reid hard core potential we find that $\delta\sigma_{\text{tot}} = 3.05$ mb.) Since the effect of these inelastic processes on the invariant differential cross section $(d\sigma/dt)_{pd}$ in the forward direction is roughly proportional to $(\delta\sigma_{\text{inel}}/4\pi\sqrt{\hbar})^2$, these effects are quite small compared to the statistical errors in the $p$-$d$ data and hence should have very little effect on our results. However, for much higher energies the intermediate production processes may have significant effect.

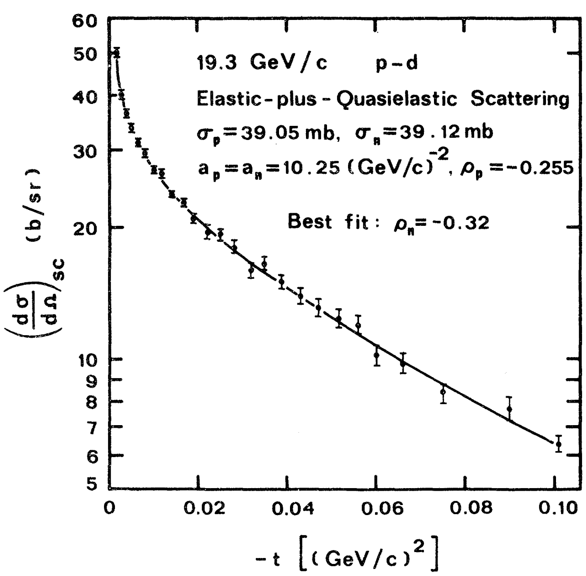


FIG. 6. Differential cross section for $p$-$d$ elastic plus quasielastic scattering at 19.3 GeV/$c$. The data are from Bellettini *et al.* (Ref. 3) and the curve corresponds to Eq. (27).

## VII. CONCLUSIONS

Since Coulomb-nuclear interference studies are important for obtaining $\rho_n$ and $a_n$ indirectly from $p$-$d$ scattering measurements, we have examined various approximate results together with the exact results in the diffraction theory. We find that the average phase approximation Eq. (23) gives results almost identical to the more accurate expressions given by Eq. (18) and together with Eq. (27) for $(d\sigma/d\Omega)_{sc}$, provides simple analytic formulas for the study of $p$-$d$ elastic and elastic plus quasielastic scattering. In this approximation one can easily include the effects of the quadrupole deformation of the deuteron, the momentum transfer dependence of $\rho$, and the spin-dependent effects which are important at medium energies. Using the experimental $p$-$d$ measurements from 10 to 70 GeV, we have calculated the parameters $\rho_n$ and $a_n$ for the $p$-$n$ forward scattering amplitude. Our results differ significantly from those obtained in the analysis of Ref. 4.

For hadron-nucleus scattering, we find that the formula given by Bethe and subsequently modified by West and Yennie gives reasonable results for light nuclei at very small angles. The approximate phase result of Eq. (39), where the Coulomb effects are considered to originate from the nucleus as a whole, works well for all nuclei at small angles but gives too large a cross section in the vicinity of the diffraction minima. However, if the Coulomb effects are incorporated in each proton in the Glauber series, then the assumption of point charges for the bound protons and the incident had-

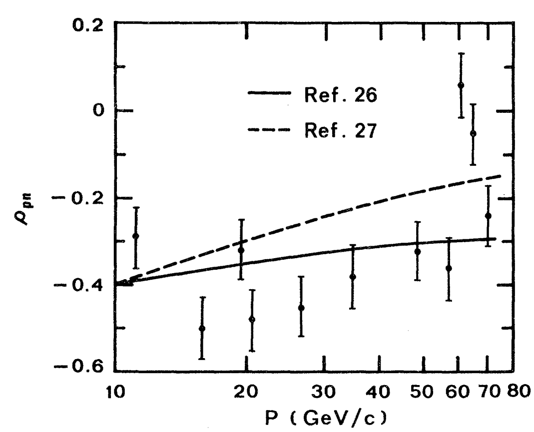


FIG. 7. Results for $\rho_n$ together with the dispersion relation calculations of Refs. 26 and 27.

ron is much more accurate than the previous approximations. Nevertheless, as mentioned in Sec. V, at the diffraction minimum this approximation can lead to errors ranging from ~0.5 to ~8% depending on the charge of the hadron and the real parts of the hadron-nucleon amplitudes. Therefore in any analysis of the experimental data on hadron-nucleus scattering, to obtain the real parts of the hadron-nucleon amplitudes the "exact" Eq. (35) should be used.

In the case of heavy-ion collisions, the Coulomb effects dominate at small angles and at the diffraction minima; furthermore they contribute significantly even at the maxima, increasing the cross sections by ~15–20% for $^{12}C$-$^{12}C$ collisions. We have shown that it is straightforward to include extended charge Coulomb effects at least for collision between light nuclei. We also find that the nucleus-nucleus amplitude is affected very little by the momentum transfer behavior of the nucleon-nucleon amplitudes so that the Chou-Yang model gives results very similar to those of the Glauber theory in the optical limit, except for a slight shift in the positions of the diffraction minima in the differential cross sections.

## APPENDIX A. COULOMB PHASE SHIFT FUNCTION FOR EXTENDED CHARGES

If both the incident and target particles are considered to be point charges, the Coulomb phase shift function is given by

$$\chi_C^{\rm pt}(b) = 2\eta \ln(b/2R)\,, \tag{A1}$$

where $\eta = Z_1 Z_2 e^2/\hbar v$.

If the target particle is considered to have a charge distribution given by $\rho_2(\vec{r})$, the Coulomb phase shift function becomes

$$\chi_1(b) = \frac{2Z_1 e}{\hbar v} \int \rho_2(\vec{r}) \ln\left(\frac{|\vec{b} - \vec{s}|}{2R}\right) d^3 r\,, \tag{A2}$$

where $\vec{s}$ is projection of $\vec{r}$ onto the impact parameter plane and $r^2 = s^2 + z^2$, if the $z$ axis is taken to be the direction of the incident beam. Assuming

$$\rho_2(\vec{r}) = Z_2 e \pi^{-3/2} c^{-3} e^{-r^2/c^2}\,, \tag{A3}$$

Eq. (A2) gives, upon straightforward integration,

$$\chi_1(b) = \chi_C^{\rm pt}(b) + \eta E_1(b^2/c^2)\,, \tag{A4}$$

where $E_1(x) = -Ei(-x)$ is the exponential integral.[31]

If we further assume that the incident particle also has a charge distribution which is given by $\rho_1(\vec{r})$, then the Coulomb phase shift function becomes

$$\chi_C(b) = \frac{2Z_2 e}{\hbar v} \int \rho_1(\vec{r}) \ln\left(\frac{|\vec{b} + \vec{s}|}{2R}\right) d^3 r + \frac{Z_2 e}{\hbar v} \int \rho_1(\vec{r}) E_1\left(\frac{|\vec{b} + \vec{s}|^2}{c^2}\right) d^3 r\,. \tag{A5}$$

If $\rho_1(\vec{r})$ also has a Gaussian form, given by

$$\rho_1(\vec{r}) = Z_1 e \pi^{-3/2} d^{-3} e^{-r^2/d^2}\,, \tag{A6}$$

equation (A5) becomes

$$\chi_C(b) = \chi_C^{\rm pt}(b) + \eta E_1(b^2/d^2) + I_3\,, \tag{A7}$$

where

$$I_3 = \eta \pi^{-3/2} d^{-3} \int e^{-r^2/d^2} E_1\left(\frac{|\vec{b} + \vec{s}|^2}{c^2}\right) d^3 r\,. \tag{A8}$$

With changes of variables and upon performing the angular integration, this leads to

$$I_3 = \frac{c^2}{d^2} \eta e^{-b^2/d^2} \int_0^\infty e^{-c^2 x/d^2} I_0\left(\frac{2bc}{d^2}\sqrt{x}\right) dx \int_x^\infty \frac{e^{-t}}{t} dt\,, \tag{A9}$$

where $I_0$ is the modified Bessel function.

Changing the order of the integrations, this can be cast in the form

$$I_3 = \eta \int_0^\infty \frac{e^{-t}}{t} dt \left[1 - J\left(\frac{c^2}{d^2}t, \frac{b^2}{d^2}\right)\right]\,, \tag{A10}$$

where

$$J(x,y) = 1 - e^{-y} \int_0^x e^{-t} I_0(2\sqrt{yt})\, dt\,. \tag{A11}$$

Equation (A10) can be rewritten as

$$I_3 = \eta \int_1^\infty dp \int_0^\infty e^{-pt} \left[1 - J\left(\frac{c^2}{d^2}t, \frac{b^2}{d^2}\right)\right] dt\,. \tag{A12}$$

Utilizing the result[32]

$$\int_0^\infty e^{-py}[1 - J(y,x)]\, dy = \frac{e^{-px/(p+1)}}{p(p+1)}\,, \tag{A13}$$

we obtain

$$I_3 = \eta \int_{d^2/c^2}^\infty e^{-b^2 u/d^2(u+1)} \frac{du}{u(u+1)}\,. \tag{A14}$$

If we let $b^2 u/d^2(u+1) = t$, $I_3$ may be written as

$$I_3 = \eta \int_{b^2/(c^2+d^2)}^{b^2/d^2} dt \frac{e^{-t}}{t}\,, \tag{A15}$$

which upon substitution into Eq. (A7) yields the final result

$$\chi_C(b) = \chi_C^{\rm pt}(b) + \eta E_1\left(\frac{b^2}{c^2 + d^2}\right)\,. \tag{A16}$$

## APPENDIX B. AVERAGE PHASE SHIFT FUNCTIONS

The scattering amplitude operator $F_d(\vec{q},\vec{s})$ for the deuteron, given by Eq. (11), is an exact expression within the framework of Glauber theory. However, it leads to Eq. (18) for elastic scattering amplitude which requires numerical integration for its evaluation. The result for elastic plus quasielastic scattering, involving double integrals and an infinite sum, is even more complicated. In this Appendix we obtain an approximation to Eq. (11) which leads to analytic results for both elastic and elastic plus quasielastic scattering by deuterium.

In terms of profile functions, Eq. (11) can be written as (apart from the unimportant phase factor)

$$F_d(\vec{q},\vec{s}) = e^{-i\vec{q}\cdot\vec{s}/2}\left[f_C^{\mathrm{pt}}(q) + i\int_0^\infty J_0(qb)\Gamma_C^E(b)(kb)^{2in+1}db\right] + \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}} e^{i\chi_C(\vec{b}+\frac{1}{2}\vec{s})}[\Gamma_{ps}(\vec{b}+\tfrac{1}{2}\vec{s}) + \Gamma_n(\vec{b}-\tfrac{1}{2}\vec{s}) - \Gamma_{ps}(\vec{b}+\tfrac{1}{2}\vec{s})\Gamma_n(\vec{b}-\tfrac{1}{2}\vec{s})]. \tag{B1}$$

Since for high-energy hadron-proton scattering $n$ is small (for example for $\pi^{\pm}$-$p$ or $p$-$p$ scattering $n < 10^{-2}$ at medium and high energies), we can write, to $O(n)$,

$$\Gamma_C^E(b) = 1 - \exp[inE_1(b^2/r^2)] \approx -inE_1(b^2/r^2)\,, \quad r^2 = c^2 + d^2\,, \tag{B2}$$

where we have assumed that the form factors $F_x(q)$ and $F_p(q)$ for the incident hadron and the bound proton are given by $e^{-d^2q^2/4}$ and $e^{-c^2q^2/4}$, respectively. The Coulomb amplitude in Eq. (B1) is then given by

$$f_C(q) = f_C^{\mathrm{pt}}(q) + i\int_0^\infty J_0(qb)(kb)^{2in+1}[-inE_1(b^2/r^2)]db\,. \tag{B3}$$

The integration can be performed by expanding the Bessel function in a power series and we obtain

$$f_C(q) = f_C^{\mathrm{pt}}(q) + \frac{n(kr)^{2in+1}r}{2} \times \sum_{m=0}^\infty \frac{(-1)^m(qr)^{2m}\Gamma(in+m+1)}{2^{2m}m!\,(in+m+1)\Gamma(m+1)}. \tag{B4}$$

For $f_C^{\mathrm{pt}}(q)$ given by Eq. (10), by dropping terms of $O(n^2)$ we obtain

$$f_C(q) = f_C^{\mathrm{pt}}(q)\left[1 + \sum_{m=0}^\infty (-1)^{m+1}\frac{(q^2r^2/4)^{m+1}}{(m+1)!}\right] = f_C^{\mathrm{pt}}(q)e^{-q^2r^2/4} = f_C^{\mathrm{pt}}(q)F_x(q)F_p(q)\,. \tag{B5}$$

The last three terms in Eq. (B1) can also be approximated. In these terms since the strong interaction profile functions become negligible for impact parameters which are larger than the sum of the deuteron size and the hadron-nucleon interaction radius, the long range part of the Coulomb interaction has little effect; and over the range of impact parameters where these terms are not negligible, the Coulomb phase shift varies slowly. We can therefore use Coulomb phase shift functions averaged over the appropriate profile functions and the deuteron ground state. They are defined as

$$\chi_{Cp} = \frac{\langle i|\int\chi_C(\vec{b}_p)\Gamma_{ps}(\vec{b}_p)d^2b|i\rangle}{\langle i|\int\Gamma_{ps}(\vec{b}_p)d^2b|i\rangle}, \tag{B6}$$

$$\chi_{Cn} = \frac{\langle i|\int\chi_C(\vec{b}_p)\Gamma_n(\vec{b}_n)d^2b|i\rangle}{\langle i|\int\Gamma_n(\vec{b}_n)d^2b|i\rangle}, \tag{B7}$$

$$\chi_{Cpn} = \frac{\langle i|\int\chi_C(\vec{b}_p)\Gamma_{ps}(\vec{b}_p)\Gamma_n(\vec{b}_n)d^2b|i\rangle}{\langle i|\int\Gamma_{ps}(\vec{b}_p)\Gamma_n(\vec{b}_n)d^2b|i\rangle}, \tag{B8}$$

where $\vec{b}_p = \vec{b} + \frac{1}{2}\vec{s}$, $\vec{b}_n = \vec{b} - \frac{1}{2}\vec{s}$, and $|i\rangle$ refers to the ground state of the deuteron. We will evaluate $\chi_{Cpn}$, which is the most complicated of the three, explicitly. For nucleon-nucleon amplitudes given by Eq. (14), we obtain

$$\Gamma_n(\vec{b}) = d_n e^{-b^2/2a_n}\,, \tag{B9}$$

$$\Gamma_{ps}(\vec{b}) = d_{ps}e^{-b^2/2a_p}\,, \tag{B10}$$

where $d_n$ and $d_{ps}$ are constants. By means of the relation

$$\int_{-\infty}^\infty |\phi(\vec{r})|^2dz = \frac{1}{4\pi^2}\int e^{-i\vec{q}\cdot\vec{s}}S(q)d^2q\,, \tag{B11}$$

where $\phi(\vec{r})$ is the wave function for the deuteron ground state, we may express $\chi_{Cpn}$ as

$$\chi_{Cpn} = \frac{\int_0^\infty \chi_C(b)e^{-b^2/2a_p}b\,db\int_0^\infty e^{-a_nq^2/2}J_0(qb)S(q)q\,dq}{a_p\int_0^\infty S(q)e^{-(a_n+a_p)q^2/2}q\,dq}. \tag{B12}$$

For the special case in which the form factor is a sum of Gaussians, as in Eq. (17), $\chi_{Cpn}$ reduces to

$$\chi_{Cpn}=\frac{\sum_i(\alpha_i/A_i)\int_0^\infty \chi_C(b)\exp[-b^2(a_p^{-1}+A_i^{-1})/2]b\,db}{a_p\sum_i(\alpha_i/H_i)}, \quad (B13)$$

where $A_i=a_n+2\beta_i$ and $H_i=a_n+a_p+2\beta_i$.

With $f_C^{pt}$ as defined in Eq. (10), the phase shift function $\chi_C(b)$ for Gaussian charge distributions Eqs. (A3) and (A6) is (see Appendix A)

$$\chi_C(b)=2n\ln(kb)+nE_1(b^2/r^2), \quad (B14)$$

where $r^2=c^2+d^2$ and $n=Z_1Z_2e^2/\hbar v$. To evaluate $\chi_{Cpn}$ with this expression for $\chi_C(b)$, we note that

$$\int_0^\infty e^{-\gamma b^2}[2n\ln(kb)+nE_1(b^2/r^2)]b\,db=\frac{n}{2\gamma}\{\ln[k^2(r^2+\gamma^{-1})]-C\}, \quad (B15)$$

where $C$ is Euler's constant, $C=0.577\ldots$. Using this result in Eq. (B13) we obtain

$$\chi_{Cpn}=\frac{n\sum_i\alpha_iH_i^{-1}\ln[k^2(r^2+2a_pA_iH_i^{-1}]}{\sum_i\alpha_iH_i^{-1}}-nC. \quad (B16)$$

In a similar fashion we find

$$\chi_{Cp}=n\ln[k^2(r^2+2a_p)]-nC, \quad (B17)$$

$$\chi_{Cn}=n\sum_i\alpha_i\ln[k^2(r^2+2A_i)]-nC. \quad (B18)$$

Using these approximations for $e^{i\chi_C(\vec{b}_p)}$ in Eq. (B1) together with Eq. (B5), we obtain as an approximation to Eq. (B1) the result

$$F_d^{av}(\vec{q},\vec{s})=e^{-i\vec{q}\cdot\vec{s}/2}f_C(q)+\frac{ik}{2\pi}\int d^2b\,e^{i\vec{q}\cdot\vec{b}}\times[e^{i\chi_{Cp}}\Gamma_{ps}(\vec{b}_p)+e^{i\chi_{Cn}}\Gamma_n(\vec{b}_n)-e^{i\chi_{Cpn}}\Gamma_p(\vec{b}_p)\Gamma_n(\vec{b}_n)] \quad (B19)$$

which leads to Eq. (21).

## APPENDIX C. CHARGE EXCHANGE EFFECTS

In this Appendix, we show that the effects of charge exchange in collisions of hadrons of isospin $\frac{1}{2}$ with deuterons can be easily incorporated in the average phase approximation. Since we are considering elastic scattering where no net transfer of charge occurs, we have to allow for a pair of successive collisions with two cancelling exchanges of charge. For elastic scattering, therefore, Eq. (21) can be rewritten[33] as

$$F_{el}^{av}(q)=S(-\tfrac{1}{2}\vec{q})[f_C(\vec{q})+e^{i\chi_{Cp}}f_{ps}(\vec{q})]+e^{i\chi_{Cn}}f_n(\vec{q})S(\tfrac{1}{2}\vec{q})+\frac{i}{2\pi k}e^{i\chi_{Cpn}}\int S(\vec{q}\,')\tfrac{1}{2}[f_{ps}(\tfrac{1}{2}\vec{q}+\vec{q}\,')f_n(\tfrac{1}{2}\vec{q}-\vec{q}\,')+f_n(\tfrac{1}{2}\vec{q}+\vec{q}\,')f_{ps}(\tfrac{1}{2}\vec{q}-\vec{q}\,')-f_{ce}(\tfrac{1}{2}\vec{q}+\vec{q}\,')f_{ce}(\tfrac{1}{2}\vec{q}-\vec{q}\,')]d^2q', \quad (C1)$$

where $f_{ce}(q)=f_{ps}(q)-f_n(q)$. Using the forms for $S(q)$, $f_{ps}$, and $f_n$ given in Sec. II, we obtain, upon integration, the result

$$F_{el}^{av}(q)=[f_C(q)+e^{i\chi_{Cp}}f_{ps}(q)+e^{i\chi_{Cn}}f_n(q)]\sum_j\alpha_je^{-\beta_jq^2/4}+\frac{i}{k}e^{i\chi_{Cpn}}\left[2c_nc_pe^{-(a_p+a_n)q^2/8}\sum_j\frac{\alpha_j}{H_j}e^{(a_p-a_n)q^2/8H_j}-\tfrac{1}{2}(c_p^2e^{-a_pq^2/4}+c_n^2e^{-a_nq^2/4})\sum_j\frac{\alpha_j}{H_j}\right]. \quad (C2)$$

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$$+B|c_n\Gamma(g)|^2\{a_nA^{-2}{}_2F_1(g,d;1;4\beta^2/A^2)+(2a_n/kAD)\times{}_2F_1(g,d;1;4\beta^2a_p/AD)\,\mathrm{Im}[c_p^*(D/a_pA)^{in}]+a_n|c_p|^2(kD)^{-2}{}_2F_1(g,d;1;4\beta^2a_p^2/D^2)\}.$$

Immediately after this equation, the definition $D = a_n A + a_p B$ should read $D = a_p A + a_n B$.

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