New methods for solving the Bethe-Goldstone equation

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Two methods of calculation of the Brueckner correlated wave function are suggested for the Bethe-Goldstone equation defined with the propagator $Q/(\omega + QTQ)$. The first method generalizes the Goldhammer-Pintar approach and is applicable when the energy denominators of the reference and exact propagators are different. The difference is treated as a perturbation. The second method consists in a direct inclusion of a harmonic Pauli projector. The only approximation is an arbitrary energy cutoff $\tilde{N}$ in the oscillator basis. Equations to be solved are written for partial waves. They are integral equations similar to the Schrödinger equation plus a set of linear algebraic equations. They are solved for a separable potential and transformed into differential ones in the case of a local potential. The number of equations is given and the error is *a priori* estimated using qualitative arguments.

[NUCLEAR STRUCTURE Two methods for solving Bethe-Goldstone equation suggested.]

I. INTRODUCTION

Goldhammer and Pintar[1] (GP) have recently found a new method for the calculation of the correlated wave function in the Brueckner theory. They used the reference spectrum method and energy denominator of the form $\omega + H_0$, i.e., the definition of the reaction matrix

$$G_0 = V - VA_0G_0, \quad A_0 = Q/(\omega + H_0), \tag{1}$$

where H_0 is an unperturbed (e.g., harmonic) two-particle Hamiltonian, Q is the Pauli projector on particle-particle states, and $\omega > 0$ is a parameter equal to the negative of the value of the so-called starting energy.

The idea of the method is to use the projector $P = 1 - Q$ instead of the original Q projector of Pauli. GP get a simple equation for the correlated function. The solution is expressed as a sum of reference functions. The coefficients in the sum are obtained by inverting the matrix of the overlap integrals.

In our work, the idea is applied in the case of the energy denominator of the form $\omega + QTQ$, which corresponds to the definition of the reaction matrix[2]

$$G = V - VA(\omega, Q)G, \quad A(\omega, Q) = Q/(\omega + QTQ). \tag{2}$$

The reasons for using such a type of the propagator modification in the definition (2) are given, e.g., by Baranger.[2] It seems that the recent calculations[3] of the third order diagrams' contribution to the energy of finite nuclei indicate the importance of self-consistent potential U insertions in higher orders of the perturbation series, thus providing arguments in favor of the definition (2), where such diagrams are partially summed up. The main trouble in using the definition (2) is of a technical nature and a new method is therefore desirable.

There are two methods suggested in our paper. The first uses the GP equation, where the necessary input functions are still fairly complicated. By means of the reference spectrum equation, those input functions are given in the form of a perturbation series. This method is described in Sec. II and may be adequate for comparisons of the definitions (1) and (2) or for approximate calculations.

The second method does not use the reference spectrum approach and is described in Sec. III. The resulting equations are very similar to the equations of GP. The correlated function is expressed as a sum of functions that are very similar to the reference functions. The expansion coefficients are obtained by solving a set of linear algebraic equations where the coefficients are again simple overlap integrals. Thus, the method can be considered as a counterpart to the GP method, adequate when dealing with an equation of type (2) instead of type (1).

II. REFERENCE SPECTRUM METHOD FOR CORRELATED FUNCTIONS

A. Goldhammer-Pintar approach

The correlated function $|\psi\rangle$ is defined in the Brueckner theory by the relation

$$G|\varphi\rangle = V|\psi\rangle, \tag{3}$$

where V is a potential, $|\varphi\rangle$ is the two-particle uncorrelated wave function, and the reaction matrix G satisfies the equation

$$G = V - VAG, \quad A = A^+ . \tag{4}$$

The propagator A is equal, in the case of GP, to A_0 from Eq. (1).

In the GP paper, the reference spectrum method is suggested for calculation of the function $|\psi\rangle$. The starting point is the reference equation

$$G_s = V - VA_sG_s, \quad A_s = A_s^+ , \tag{5}$$

where the propagator is chosen in the form $A_s = (\omega + H_0)^{-1}$ and the corresponding correlated reference function $|\psi_s\rangle$ is supposed to be known.

From the more general point of view, the propagators A and A_s can be quite arbitrary and only the relation

$$A_s - A = PA_s \tag{6}$$

must be fulfilled, where $P = 1 - Q = \sum_{j\in C_P} |\varphi_j\rangle\langle\varphi_j|$ is a projector defined in the two-particle uncorrelated function basis $|\varphi_i\rangle$ and C_P is a corresponding set of indices.

The usual reference spectrum equation

$$G = G_s - G(A - A_s)G_s \tag{7}$$

for the reaction matrix implies

$$AG = A_sG_s - (1 - AG)PA_sG_s ; \tag{8}$$

i.e.,

$$(1 - A_sG_s)|\varphi_i\rangle = (1 - AG)P(1 - A_sG_s)|\varphi_i\rangle \tag{9}$$

for $i \in C_P$. We can use the correlated functions $|\psi_i\rangle$ and rewrite Eq. (9) in the form

$$|\psi_{si}\rangle = \sum_{j\in C_P} |\psi_j\rangle\langle\varphi_j|\psi_{si}\rangle, \quad i \in C_P , \tag{10}$$

which is the form given by GP[1] [their Eq. (15)]. Thus, only the functions $|\psi_j\rangle$ with $j \in C_P$ are present in Eq. (10), not being coupled with $|\psi_l\rangle$, $l \notin C_P$. From (8) we can also derive the relation

$$|\psi_l\rangle = |\psi_{sl}\rangle - \sum_{j\in C_P} |\psi_j\rangle\langle\varphi_j|\psi_{sl}\rangle, \quad l \notin C_P , \tag{10'}$$

which defines the remaining functions $|\psi_l\rangle$, $l \notin C_P$ in terms of the functions $|\psi_i\rangle$, $i \in C_P$.

The infinite set of Eqs. (10) should be truncated. We define the finite subset of indices $C_P(\tilde{N}) \subset C_P$ such that $\lim C_P(\tilde{N}) = C_P$ for $\tilde{N} \to \infty$ and obtain the correlated functions $|\psi_j\rangle$ from Eq. (10) as a linear combination of the uncorrelated functions $|\psi_{si}\rangle$ by simply inverting the matrix $\langle\varphi_j|\psi_{si}\rangle$, $i, j \in C_P(\tilde{N})$. In principle, the approximated Pauli projector can be given with arbitrary accuracy, because

$$P = \lim_{\tilde{N}\to\infty} P_{\tilde{N}}, \quad P_{\tilde{N}} = \sum_{i\in C_P(\tilde{N})} |\varphi_i\rangle\langle\varphi_i| . \tag{11}$$

In the region of convergence, i.e., for sufficiently large $\tilde{N}$ and $i \in C_P(\tilde{N}) - C_P(\tilde{N} - 1)$, the influence of the projector

$$P_{\tilde{N}} - P_{\tilde{N}-1} = \sum_{i\in C_P(\tilde{N}) - C_P(\tilde{N}-1)} |\varphi_i\rangle\langle\varphi_i|$$

on the result should be negligible. This influence could be roughly measured by the ratio ζ of the number of functions $|\varphi_i\rangle$, $i \in C_P(\tilde{N}) - C_P(\tilde{N} - 1)$ to the number of all basis functions with the same quantum numbers. In the case of the magic nuclei and harmonic oscillator basis, this ratio is given in the Appendix. It is proportional to $\tilde{N}^{-2}$, therefore the expected convergence properties should be quite good. This qualitative argument could be affirmed by the results of some computations with an analogical approximation of the Pauli projector.[4]

B. Generalized GP approach

It can happen that Eq. (5) with the propagator A_s is still difficult to solve and that we want to start with the reference equation

$$G_r = V - VA_rG_r, \quad A_r = A_r^+ \neq A_s . \tag{12}$$

Let us define the operator D by the relation

$$A_r - A_s = A_rDA_s . \tag{13}$$

Then we can proceed in an analogous way and derive the necessary formulas to obtain the functions $|\psi_s\rangle$ from the previously known $|\psi_r\rangle$. First, let us introduce the operator Ω (wave operator) by the relation

$$\Omega|\varphi\rangle = |\psi\rangle . \tag{14}$$

The reference spectrum equation of the type (7),

$$G_s = G_r - G_r(A_s - A_r)G_s , \tag{15}$$

gives us the equation

$$(1 + \Omega_rA_rD)\Omega_s = \Omega_r(1 + A_rD) , \tag{16}$$

which can be represented as an infinite system of equations for the defect functions $|\chi\rangle = |\psi\rangle - |\varphi\rangle$,

$$(1 + \Omega_rA_rD)|\chi_{si}\rangle = |\chi_{ri}\rangle . \tag{17}$$

If we introduce the operator M by the equation

$$(1 + A_rV)M = A_r \tag{18}$$

then we can eliminate the operator Ω_r and write the set (17) in the form

$$(1 + MD)|\chi_{si}\rangle = |\chi_{ri}\rangle . \tag{19}$$

Taking the operator MD as a perturbation, we

finally obtain the series

$$|\psi_{si}\rangle = |\psi_{ri}\rangle + \sum_{k=1}^{\infty} (-MD)^k |\chi_{ri}\rangle \tag{20}$$

and insert it into Eq. (10). Thus, the nonzero value of the difference (13) between A_r and A_s implies the need to solve Eqs. (18) and (19) and then to use the GP equation (10).

Let us illustrate the method in the case of the definition (2) of the reaction matrix. We intend to find the solution of the Bethe-Goldstone equation

$$|\psi\rangle = |\varphi\rangle - A(\omega, Q)V|\psi\rangle \tag{21}$$

for the correlated function $|\psi\rangle$. We make the choice

$$A = A(\omega, Q), \quad A_s = (\omega + QTQ)^{-1},$$

which is consistent with the condition (5), so that we can use Eqs. (10) provided that we know the solution of the equations

$$|\psi_{si}\rangle = |\varphi_i\rangle - A_s V|\psi_{si}\rangle, \quad i \in C_P. \tag{22}$$

Writing (22) in the form of equations for the defect functions $|\chi_{si}\rangle$

$$(\omega + QTQ + V)|\chi_{si}\rangle = -V|\varphi_i\rangle, \quad i \in C_P, \tag{23}$$

we see that the problem has been only slightly simplified. On the other hand, the simple form of the reference equation

$$(\omega + T + V)|\chi_{ri}\rangle = V|\varphi_i\rangle \tag{24}$$

corresponds to the propagator $A_r = (\omega + T)^{-1}$ in Eq. (12), so that Eq. (19) can be used instead of (23). The properties of the usual approximations of Q[2,5] imply that the operator $D = QTQ - T$ can be treated as a perturbation and the expansion (20) of the solution of Eq. (23) can be safely truncated.

It remains to find the operator M. The special choice of the propagators in our case leads to the simplification of Eq. (18) into

$$(\omega + T + V)M = 1, \tag{25}$$

so that the problems of numerically solving Eqs. (24) and (25) are closely connected. If those problems are solved by some standard technique,[2] then the truncated set of Eqs. (10) with the insertion of the truncated sum (20) can be used instead of Eq. (21).

It is clear that the more accurate calculations with the expansion (20) will be difficult because of the presence of the Pauli projector in the definition of D. This shortcoming will be removed in the method of the next section.

III. DIRECT METHOD FOR CORRELATED FUNCTIONS

A. Equations of the direct method

The aim of the present section is to solve Eq. (21) in a more direct way than was done in the last section. We rewrite this equation for the defect function $|\chi\rangle = |\psi\rangle - |\varphi\rangle$,

$$[1 + A(\omega, Q)V]|\chi\rangle = -A(\omega, Q)V|\varphi\rangle \tag{26}$$

and multiplying it by the projector P from the left we get the simple relation

$$P|\chi\rangle = 0. \tag{27}$$

Multiplying Eq. (26) by the projector Q from the left, we get another equation,

$$[1 + A(\omega, Q)V]Q|\chi\rangle = -A(\omega, Q)V|\varphi\rangle, \tag{28}$$

which is equivalent, together with the condition (27), to Eq. (26). Multiplying Eq. (28) from the left by the nonsingular operator

$$\omega P + Q(\omega + T)Q, \tag{29}$$

we get an equivalent equation

$$Q(\omega + T + V)Q|\chi\rangle = -QV|\varphi\rangle, \tag{30}$$

where all the operators occur in the numerator only. In that equation we can replace all the Pauli projectors Q by the expression $1 - P$, which yields the desired formulation of the problem using the projector P.

We suggest that the set of Eqs. (27) and (30) be solved in the following way. We introduce the function $|\lambda\rangle$ by the relation

$$P|\lambda\rangle = P(\omega + T + V)|\chi\rangle + PV|\varphi\rangle \tag{31}$$

and combine this definition with Eq. (30) to form the equation

$$(\omega + T + V)|\chi\rangle = -V|\varphi\rangle + P|\lambda\rangle. \tag{32}$$

The purpose of such an approach lies in the simplicity of Eq. (32), where the sum $P|\lambda\rangle = \sum_{i \in C_P} |\varphi_i\rangle\langle\varphi_i|\lambda\rangle$ introduces into our solution of (32) parameters $c_i = \langle\varphi_i|\lambda\rangle$ to be determined from the conditions (27). The parameters c_i form an infinite set; the truncation introduced by GP can be used again. We approximate the projector P in the same way, $P = P_N^{\sim}$ (see Sec. II), so that the number of parameters becomes finite and equal to the number of equations in the set (27).

We now write

$$|\chi\rangle = |\kappa\rangle + |\rho\rangle, \tag{33}$$

where $|\rho\rangle = \sum_{i \in C_P(\tilde{N})} |\rho_i\rangle c_i$, $c_i = \langle\varphi_i|\lambda\rangle$, and the functions $|\kappa\rangle$ and $|\rho_i\rangle$ satisfy the equations

$$(\omega + T + V)|\kappa\rangle = -V|\varphi\rangle \tag{34}$$

and

$$(\omega + T + V)|\rho_i\rangle = |\varphi_i\rangle, \quad i \in C_P(\tilde{N}), \tag{35}$$

respectively.

Conditions (27) are now linear algebraic equations for the unknown coefficients c_i,

$$\langle\varphi_j|\kappa\rangle + \sum_{i \in C_P(\tilde{N})} \langle\varphi_j|\rho_i\rangle c_i = 0, \quad j \in C_P(\tilde{N}). \tag{36}$$

Thus Eqs. (34)–(36) are equivalent to the original Eq. (26) with the $P = P_{\tilde{N}}$ approximation only. The defect function is given as a sum (33) of $|\kappa\rangle$ and the sequence of Pauli corrections $|\rho_i\rangle c_i$. The set $|\rho_i\rangle$ is neither complete nor orthogonal (see Ref. 1). These basis functions are the solutions of Eqs. (35), which are linear equations very similar to the Schrödinger or reference equations. The zeroth order approximation $|\kappa\rangle$ is given by Eq. (34), which is equivalent to the usual reference Eq. (24). Only the coefficients c_i of the expansion depend on the level $\tilde{N}$ of approximation of the Pauli projector and are given by the simple matrix inversion in Eq. (36). The matrix elements in this last problem are overlap integrals of the functions $|\rho_i\rangle$ with the uncorrelated functions $|\varphi_j\rangle$, $j \in C_P(\tilde{N})$. The independence of the values c_i on the value $\tilde{N}$ must be tested numerically by solving Eqs. (36).

B. An example: equations with a separable potential

We introduce into Eqs. (34)–(36) the separable potential

$$V = \sum_{\alpha,\beta} |g_\alpha\rangle D_{\alpha\beta}\langle g_\beta|, \quad D_{\alpha\beta} = D^*_{\beta\alpha}. \tag{37}$$

Then we get from (34) the solution

$$|\kappa\rangle = -\sum_{\alpha,\beta} A_r |g_\alpha\rangle E_{\alpha\beta}\langle g_\beta|\varphi\rangle, \tag{38}$$

where $E_{\alpha\beta}$ is a solution of a matrix equation

$$\sum_\beta \left(\delta_{\alpha\beta} + \sum_\delta D_{\alpha\beta}\langle g_\delta|A_r|g_\beta\rangle\right) E_{\beta\gamma} = D_{\alpha\gamma} \tag{39}$$

and $A_r = (\omega + T)^{-1}$. From (35) we obtain

$$|\rho_i\rangle = \left[1 - \sum_{\alpha,\beta} A_r|g_\alpha\rangle E_{\alpha\beta}\langle g_\beta|\right] A_r|\varphi_i\rangle, \quad i \in C_P. \tag{40}$$

Equation (36) can then be given the form

$$\sum_{\alpha,\beta} \langle\varphi_j|A_r|g_\alpha\rangle E_{\alpha\beta}\langle g_\beta|\varphi\rangle = \sum_{i \in C_P} c_i \left(\langle\varphi_j|A_r|\rho_i\rangle - \sum_{\alpha,\beta} \langle\varphi_j|A_r|g_\alpha\rangle E_{\alpha\beta}\langle g_\beta|A_r|\varphi_i\rangle\right), \tag{41}$$

so that the only problem is the computation of the overlap integrals.

C. Partial wave representation of the equations

Equations (34)–(36) are written in an abstract operator form. In this section, we intend to rewrite them for partial waves. We use the relative and center-of-mass impulses $\vec{k} = (\vec{k}_1 - \vec{k}_2)/\sqrt{2}$, $\vec{K} = (\vec{k}_1 + \vec{k}_2)/\sqrt{2}$, where $\vec{k}_1$ and $\vec{k}_2$ are impulses of the nucleons. The operator $\omega + T + V$ in Eqs. (34) and (35) is then diagonal in the variable $\vec{K}$ and depends only on the $K = |\vec{K}|$ value, which can be treated as a parameter.

The first problem is to write explicitly the functions $\langle\vec{k}, \vec{K}|\varphi_i\rangle$. We shall investigate here only the case of the harmonic Pauli projector.[2] Then we can start with the definition

$$\langle\vec{k}_1, \vec{k}_2|\varphi_i\rangle = R_{n_{1i}l_{1i}}(k_1) R_{n_{2i}l_{2i}}(k_2) \langle\vec{k}_1/|\vec{k}_1|, \vec{k}_2/|\vec{k}_2|\,||\, l_{1i}, l_{2i}(\lambda_i) S_i J_i M_i T_i T_{zi}\rangle, \quad i \in C_P \tag{42}$$

of those functions in $\vec{k}_1$ and $\vec{k}_2$ variables. Here, the $R_{nl}(k)$ are functions of the linear harmonic oscillator and the coupling of the angular momenta is given by the schema $\vec{l}_1 + \vec{l}_2 = \vec{\lambda}$, $\vec{\lambda} + \vec{S} = \vec{J}$. S, T, and T_z denote the spin and isospin quantum numbers, respectively. The set $C_P(\tilde{N})$ of composite indices

$$i = [n_{1i}, n_{2i}, l_{1i}, l_{2i}, \lambda_i, S_i, J_i, M_i, T_i, T_{zi}] \tag{43}$$

is, in the simplest case of the ^{4}He nucleus, restricted by the condition[2] $(2n_{1i} + l_{1i})(2n_{2i} + l_{2i}) = 0$, while the parameter $\tilde{N}$ is chosen in the usual way as the maximum value of the energy quantum number $I_E = 2n_1 + l_1 + 2n_2 + l_2 \leq \tilde{N}$.

Moshinsky coefficients[6] $M(\ldots)$ and 6-j symbols[7] enable us to expand the function (42) in the basis

$$\langle \vec{k}, \vec{K} | nNl\beta_i \rangle = R_{nl}(k) R_{\mathfrak{N}\mathfrak{L}}(K) \langle \vec{k}/|\vec{k}|, \vec{K}/|\vec{K}| | l, \beta_i \rangle , \tag{44}$$

$$\beta_i = [S_i(j)\mathfrak{L} J_i M_i T_i T_{zi}] .$$

The expansion reads

$$\langle \vec{k}, \vec{K} | \varphi_i \rangle = \sum_{n\mathfrak{N}l\mathfrak{L}j} \tilde{M}_{n\mathfrak{N}}(l, \beta_i, i) R_{nl}(k) R_{\mathfrak{N}\mathfrak{L}}(K) \times \langle \vec{k}/|\vec{k}|, \vec{K}/|\vec{K}| | l, \beta_i \rangle , \tag{45}$$

where the coefficients

$$\tilde{M}_{n\mathfrak{N}}(l, \beta_i, i) = (-1)^{\mathfrak{L}+l+S+J} [(2\lambda+1)(2j+1)]^{1/2} \times \begin{Bmatrix} \mathfrak{L} & l & \lambda \\ S & J & j \end{Bmatrix} M_\lambda(nl\mathfrak{N}\mathfrak{L} | n_{1i} l_{1i} n_{2i} l_{2i}) \tag{46}$$

can be easily tabulated.

In simple cases, it is advantageous to derive the explicit expressions for $M_{n\mathfrak{N}}(l, \beta_i, i)$ using the Trlifaj formula[8] for Moshinsky coefficients; e.g., in the $^1S_0$ case we have

$$\tilde{M}_{n\mathfrak{N}}(0, [0(0)\mathfrak{L} J_i M_i T_i T_{zi}], i) = \left[\binom{n+\mathfrak{N}}{n} \frac{\Gamma(n+\mathfrak{N}+\mathfrak{L}+\frac{3}{2}) \Gamma(\frac{3}{2})}{2^{2\mathfrak{N}+2n+\mathfrak{L}} \Gamma(n+\frac{3}{2}) \Gamma(\mathfrak{N}+\mathfrak{L}+\frac{3}{2})} \right]^{1/2} . \tag{47}$$

Let us mention that the composite index $\beta_i = [S_i(j)\mathfrak{L} J_i M_i T_i T_{zi}]$ has the components with the subscript i fixed by the orthogonality of the angular part of the vectors (42) and (44). The range of the corresponding quantum numbers is restricted in ^{4}He by the triangular inequalities for the sets (γ, S, J), $(l, \mathfrak{L}, \gamma)$, $(J, \mathfrak{L}, j)$, and (l, S, j), where $\gamma = \max(l_1, l_2)$, and by the energy conservation $I_E = 2\mathfrak{N} + \mathfrak{L} + 2n + l = 2n_1 + l_1 + 2n_2 + l_2$ in the Moshinsky brackets. Symmetry properties of the Moshinsky brackets[6] imply that it is sufficient to investigate $\gamma = l_1$, because the $\gamma = l_2$ case differs only by the factor $(-1)^l$.

Now, Eqs. (34) and (35) can be written for partial waves in the form

$$(\omega + T)\kappa_l^\beta(k, K) + \sum_{l'} \int_0^\infty d\xi\, \xi^2 V_{ll'}^\beta(k, \xi)[\kappa_l^\beta(\xi, K) + \varphi_{l'}^\beta(\xi, K)] = 0 \tag{48}$$

and

$$(\omega + T)\rho_{i,l}^{\beta i}(k, K) + \sum_{l'} \int_0^\infty d\xi\, \xi^2 V_{ll'}^{\beta i}(k, \xi) \rho_{i,l'}^{\beta i}(\xi, K) = \varphi_{i,l}^{\beta i}(k, K) , \tag{49}$$

where

$$\varphi_{i,l}^{\beta i}(k, K) = \sum_{n\mathfrak{N}} R_{nl}(k) R_{\mathfrak{N}\mathfrak{L}}(K) \tilde{M}_{n\mathfrak{N}}(l, \beta_i, i) .$$

Equations (36) become

$$A^f + \sum_{i \in C_P(\tilde{N})} B^{fi} c_i = 0, \quad f \in C_P(\tilde{N}) , \tag{50}$$

where

$$A^f = \int_0^\infty dK\, K^2 \sum_{lj\mathfrak{L}} \int_0^\infty d\xi\, \xi^2 \varphi_{f,l}^{\beta f}(\xi, K) \kappa_l^{\beta f}(\xi, K)$$

and

$$B^{fi} = \int_0^\infty dK\, K^2 \sum_{lj\mathfrak{L}} \int_0^\infty d\xi\, \xi^2 \varphi_{f,l}^{\beta f}(\xi, K) \rho_{i,l}^{\beta i}(\xi, K) .$$

The quantum numbers M_i and T_{zi} do not influence the computed coefficients c_i and can therefore be omitted. The orthogonality of products $\langle \varphi_j | \varphi_i \rangle$ with respect to the quantum numbers T_i, J_i, and S_i implies the diagonal form of Eqs. (50) in those quantum numbers, which simplifies the problem considerably. Thus it is sufficient to substitute in ^{4}He a new composite index $i = [I, \gamma]$ for the previous one, Eq. (43). The numbers $0 \leq I \leq \tilde{N}$ and $0 \leq \gamma \leq I$ are defined as $I = 2n_1 + l_1$ and $\gamma = l_1$. The remaining components of the index (43) are fixed in the coupled subset of Eqs. (50) and can be varied independently. Thus, the number of algebraic equations (50) to be solved simultaneously is equal, for ^{4}He, to $\{[\tilde{N}/2]+1\}\{[(\tilde{N}+1)/2]+1\}$, where $[x]$ denotes the integer part of a real number x.

Let us consider the cutoff $\tilde{N} = 7$ which provides 20 functions $|\varphi_i\rangle$, $i \in C_P(\tilde{N})$ and is considered in GP[1] to be a reasonable approximation. The error estimate introduced in the Appendix is equal to $\zeta(1, 7) = 0.2$ in this case and seems to overestimate the actual error. Nevertheless, if we use this estimate together with the condition $\zeta(1, \tilde{N}) < 0.05$ then we must choose $\tilde{N} = 19$. Even this cutoff produces the 110×110 matrix B^{ji} in (50), which can be inverted quite easily on the computer.

D. An example: equations with a local potential

Because the Fourier transform of $R_{nl}(k)$ is proportional to $R_{nl}(r)$, the relative coordinate $\vec{r}$ can be used instead of the relative impulse $\vec{k}$ without

substantially changing the form (47) of the right-hand side functions $\varphi^{\beta}_{i,l}(k,K) \to \varphi^{\beta}_{i,l}(r,K)$. The presence of the δ function in the local potential operator enables us to transform the integral equations (34) and (35) into the differential ones, namely

$$(\omega+T)\kappa^{\beta}_{l}(r,K)+\sum_{l'} V^{\beta}_{ll'}(r)\kappa^{\beta}_{l'}(r,K) = -\sum_{l'} V^{\beta}_{ll'}(r)\varphi^{\beta}_{l'}(r,K) \quad (51)$$

and

$$(\omega+T)\rho^{\beta i}_{i,l}(r,K)+\sum_{l'} V^{\beta i}_{ll'}(r)\rho^{\beta i}_{i,l}(r,K)=\varphi^{\beta i}_{i,l}(r,K)\,. \quad (52)$$

The methods of solving these equations are standard.[2] The form of Eq. (50) does not change, because the integration variable ξ can be interpreted as a coordinate r.

IV. CONCLUSIONS

The proper treatment of the Pauli exclusion principle in the Brueckner theory of finite nuclei is accompanied by considerable numerical difficulties. The Goldhammer-Pintar approach[1] simplifies the correlated wave function problem in the case of the energy denominator $\omega+H_0$ which is diagonal in the single particle basis. The first method suggested in this paper generalizes the GP approach for the more general propagators and treats the difference of the energy denominators as a perturbation. Therefore, this method is adequate for the approximate calculations and complements the original GP method.

Second is a direct method of solving the Bethe-Goldstone equation for the correlated function with an exact (harmonic) Pauli projector and the energy denominator $\omega+QTQ$. In the practical calculations, the infinite harmonic oscillator basis must be truncated by means of the energy cutoff $\tilde{N}$. This is the only approximation used. Some *a priori* arguments can be given in favor of the second method even before it has been numerically tested.

(i) The formal analogy exists between GP and our method, so that the applicability range of both the methods will probably be the same.

(ii) The main advantage of the correlated function calculations lies in the decoupling of the equations. In the GP equations only hole-hole and hole-particle correlated wave functions are coupled [cf., e.g., Eq. (10) in Sec. IIB]. In our equations there is no coupling of correlated functions at all. This direct approach—without recourse to the reference spectrum method—gives about the same number of equations as GP, though we solve a problem with the much more complicated (nondiagonal) energy denominator $\omega+QTQ$.

(iii) The possibility of approximation of the Pauli projector with any degree of accuracy (at least in principle) makes the method useful for testing the usual Pauli projector approximations[2] which are motivated in a very intuitive way (e.g., diagonal approximation of the matrix Q in center-of-mass quantum numbers).

(iv) There is an exciting possibility of reaching, in numerical calculations, the region of stability of the results when changing the cutoff parameter $\tilde{N} \to \tilde{N}+1$, etc. This would mean that the results correspond to the exact Pauli projector (11).

APPENDIX

We can write

$$P_{\tilde{N}} - P_{\tilde{N}-1} = \sum_{i \in C_P(\tilde{N}) - C_P(\tilde{N}-1)} |\varphi_i\rangle\langle\varphi_i| = \sum_{i \in C_U(\tilde{N}) - C_U(\tilde{N}-1)} |\varphi_i\rangle q_i^{(\tilde{N})}\langle\varphi_i|\,, \quad (A1)$$

where $|\varphi_i\rangle$ are the functions (42), $C_U(\tilde{N})$ is the set of indices (43), and $C_P(\tilde{N})$ is its subset defined by the restriction[2]

$$2n_1 + l_1 < F \quad \text{or} \quad 2n_2 + l_2 < F\,. \quad (A2)$$

In the case of magic nuclei ^{4}He, ^{16}O, and ^{40}Ca, the number F is equal to 1, 2, and 3, respectively. The parameter $q_i^{(\tilde{N})}$ is equal to 1 or 0.

Let us denote by $n(\tilde{N})$ the number of all possible functions $R_{n_1 l_1}(k_1)R_{n_2 l_2}(k_2)$ for which $2n_1 + l_1 + 2n_2 + l_2 = \tilde{N}$ and by $n(F,\tilde{N})$ the number of the same functions with the restriction (A2). It is easy to derive the explicit form of these numbers. We have

$$n(1,\tilde{N}) = 2[\tfrac{1}{2}(\tilde{N}+2)], \quad n(2,\tilde{N}) = 2\tilde{N}+2\,, \quad n(3,\tilde{N}) = 4[\tfrac{1}{2}\tilde{N}] + 2\tilde{N} + 2 \quad (A3)$$

for $\tilde{N} > 2(F-1)$ and

$$n(2M) = \tfrac{1}{6}(M+1)(M+2)(2M+3)\,, \quad n(2M+1) = \tfrac{1}{3}(M+1)(M+2)(M+3)\,. \quad (A4)$$

According to (A1) and (42), the ratio $\zeta(F,\tilde{N}) = n(F,\tilde{N})/n(\tilde{N})$ is equal to the ratio of the number of nonzero values $q_i^{(\tilde{N})}$ to the number of all parameters $q_i^{(\tilde{N})}$ at given $\tilde{N}$. From (A3) and (A4) we get

$$\zeta(1,\tilde{N}) = \frac{12}{(M+2)(2M+3)}\,, \quad \zeta(2,\tilde{N}) = \frac{12(2M+1)}{(M+1)(M+2)(2M+3)}\,, \quad \zeta(3,\tilde{N}) = \frac{12(4M+1)}{(M+1)(M+2)(2M+3)} \quad (A5)$$

for $\tilde{N} = 2M > 2F-1$ and

$$\zeta(1,\tilde{N}) = \frac{6}{(M+2)(M+3)}\,,$$

$$\zeta(2,\tilde{N}) = \frac{12}{(M+2)(M+3)}\,, \qquad \text{(A6)}$$

$$\zeta(3,\tilde{N}) = \frac{12(2M+1)}{(M+1)(M+2)(M+3)}$$

for $\tilde{N} = 2M+1 > 2(F-1)$. For $\tilde{N} \leq 2F-1$ we have $\zeta(F,\tilde{N}) = 1$.

It is clear that the closer the value ζ approaches zero, the less influence the operator (A1) exhibits in the corresponding vector space. The value ζ can be used as an error estimate in the approximation $Q = 1 - P_{\tilde{N}}$ of the Pauli projector.

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