Simple relations for the excitation energies $E2$ and the transition probabilities $B(E2)$ of neighboring doubly even nuclides

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For even-even nuclei, the excitation energy $E2$ and the reduced transition probability $B(E2)$ between the ground state and the first excited 2^+ state have been considered. On the basis of different models, it is shown that for a nucleus $[N, Z]$ the relations

$$E2[N, Z] + E2[N+2, Z+2] - E2[N+2, Z] - E2[N, Z+2] \approx 0$$

and

$$B(E2)[N, Z] + B(E2)[N+2, Z+2] - B(E2)[N+2, Z] - B(E2)[N, Z+2] \approx 0$$

hold good, except in certain specified regions. The validity of these difference equations is tested with the available experimental data. The difference equation of Ross and Bhaduri is shown to follow from our approach. Some predictions of unmeasured $E2$ and $B(E2)$ values have been made.

[NUCLEAR STRUCTURE Simple relations for $E2$ transition probabilities and excitation energies of first excited 2^+ states in even-even nuclei.]

I. INTRODUCTION

In recent years, the mass sum rules given by Garvey and Kelson[1] have been found to be extremely successful in predicting the ground-state energies of nuclei. The success shows that the ground-state energies of a group of six neighboring nuclides are related through a difference equation. This may be attributed to the possible existence of a certain approximate symmetry in nuclear dynamics. It is natural to expect that such symmetry might also be manifested in properties other than the ground-state properties of nuclei. Ross and Bhaduri[2] (hereafter referred to as RB) have tried to find difference equations involving the properties of excited states. They have considered the excitation energies $E2$ of the first 2^+ excited states and the corresponding reduced transition probabilities $B(E2)$ of a group of four even-even neighboring nuclides, and found that for a nucleus $[N, Z]$ the quantity $F[N, Z] \equiv [E2\,B(E2)]^{-1}$ is related to the other three nuclides through the relation

$$F[N, Z] + F[N+2, Z+2] - F[N+2, Z] - F[N, Z+2] \simeq 0 . \tag{1}$$

They have shown by comparing with experimental data that the above relation is mostly true except in certain specified regions. In this paper, we have been able to establish even simpler difference equations involving either $E2$ or $B(E2)$ only for simpler sets of four even-even nuclides. The equations are

$$E2[N, Z] + E2[N+2, Z+2] - E2[N+2, Z] - E2[N, Z+2] \simeq 0 , \tag{2}$$

$$B(E2)[N, Z] + B(E2)[N+2, Z+2] - B(E2)[N+2, Z] - B(E2)[N, Z+2] \simeq 0 . \tag{3}$$

In Sec. II, the above difference equations have been derived in different models. In Sec. III, Eq. (1) due to Ross and Bhaduri has been shown to follow from our consideration. We have shown the consistency of Eqs. (2) and (3) with experimental data in Sec. IV. In Sec. V, we have discussed the limitations of these equations. Also, several unmeasured $B(E2)$ and $E2$ values have been predicted and compared with the available tentative experimental data. Our predictions of $B(E2)$ values have been compared with those predicted by RB.

II. DERIVATION OF THE DIFFERENCE EQUATIONS

The Garvey-Kelson mass sum rule,[1] which is remarkably successful, has in its background a single-particle picture of the nucleus. Its consistency with the microscopic calculation, in the frame work of the Hartree-Fock (HF) model of the nucleus, has been shown by Bassichis and Kel-

son[3] in the case of s-d shell nuclei. The success of this sum rule involving only ground-state energies is indicative of the possible existence of other sum rules relating properties of excited states which might be consistent with the HF model. In the following, we attempt to establish our difference equations (2) and (3) with this model.

The description of the deformed nuclei in the projected Hartree-Fock (PHF) theory[4,5] consists in calculating the intrinsic determinantal state ϕ_K through the minimization of the energy expectation value $\langle\phi_K|H|\phi_K\rangle$, and then generating the different members of the ground-state band from it by the use of the projection method of Peierls and Yoccoz.[6] In this prescription, the wave function for the state J belonging to the ground-state band K is given by

$$\Psi^J_{MK} = \frac{(2J+1)}{8\pi^2 N_{JK}^{1/2}} \int d\Omega\, D^{*J}_{MK}(\Omega) R(\Omega) |\phi_K\rangle , \tag{4}$$

where Ω stands for the Euler angles α, β, and γ, $R(\Omega)$ is the usual rotation operator $e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$, and $D^J_{MK}(\Omega)$ is the usual rotation matrix element. The normalization constant N_{JK} is given by

$$N_{JK} = \frac{(2J+1)}{8\pi^2} \int d\Omega\, D^{*J}_{KK} \langle\phi_K| R(\Omega)|\phi_K\rangle .$$

The matrix element of any tensor operator T^k_μ between two different angular momentum states J_i and J_f belonging to the same intrinsic wave function ϕ_K is

$$\langle \Psi^{J_f}_{M_f K} | T^k_\mu | \Psi^{J_i}_{M_i K} \rangle = \frac{32\pi^4}{(2J_f+1)(N_{J_f K} N_{J_i K})^{1/2}} \begin{bmatrix} J_i & k & J_f \\ M_i & \mu & M_f \end{bmatrix} \sum_\nu \begin{bmatrix} J_i & k & J_f \\ K-\nu & \nu & K \end{bmatrix} \int_0^\pi \sin\beta\, d\beta\, d^{J_i}_{K-\nu,K} \langle\phi_K| T^k_\nu e^{-i\beta J_y} |\phi_K\rangle . \tag{5}$$

The reduced $B(E2)$ transition probability between the initial state J_i and final state J_f can be obtained from Eq. (5) as

$$B(E2) = \frac{(2J_f+1)}{(2J_i+1)} |\langle J_f \| Q^2 \| J_i \rangle|^2 , \tag{6}$$

where

$$Q^2_\mu = (\tfrac{16}{5})^{1/2} \sum_{i=1}^{Z} r_i^2 Y^2_\mu(\theta_i \phi_i) \tag{7}$$

and

$$\langle J_f \| Q^2 \| J_i \rangle = \frac{32\pi^4}{(2J_f+1)(N_{J_i K} N_{J_f K})^{1/2}} \sum_\nu \begin{bmatrix} J_i & 2 & J_f \\ K-\nu & \nu & K \end{bmatrix} \int_0^\pi \sin\beta\, d\beta\, d^{J_i}_{K-\nu,K} \langle\phi_K| Q^2_\nu e^{-i\beta J_y} |\phi_K\rangle .$$

For the special case of even-even nuclei, for which $K=0$, $J_i=0$, and $J_f=2$, the above equation reduces to

$$\langle 2 \| Q^2 \| 0 \rangle = \frac{\int_0^\pi \sin\beta\, d\beta\, d^0_{00}(\beta) \langle\phi_0| Q_0^2 e^{-i\beta J_y} |\phi_0\rangle}{[\int_0^\pi \sin\beta\, d\beta\, d^0_{00}(\beta) \langle\phi_0| e^{-i\beta J_y} |\phi_0\rangle]^{1/2} [\int_0^\pi \sin\beta\, d\beta\, d^2_{00}(\beta) \langle\phi_0| e^{-i\beta J_y} |\phi_0\rangle]^{1/2}} .$$

For even-even nuclei, the overlap integral $\langle\phi_0| e^{-i\beta J_y} |\phi_0\rangle$ in the denominator is sharply peaked at $\beta=0$ and $\beta=\pi$ in cases where the intrinsic state ϕ_0 is sufficiently deformed. The overlap function

$$\eta(\beta) = \langle\phi_0| e^{-i\beta J_y} |\phi_0\rangle$$

for small β can be approximated as

$$\eta(\beta) = \exp(-\tfrac{1}{2}\beta^2 \langle\phi_0| J_y^2 |\phi_0\rangle) = \exp(-\tfrac{1}{4}\beta^2 \langle J^2 \rangle) .$$

This approximation[7] is exact up to and including β^3. In heavy nuclei, the expectation value of J^2 is expected to exceed 100 and, further, for even-even nuclei,

$$\eta(\beta) = \eta(\pi - \beta) .$$

Thus, $\eta(\beta)$ is extremely peaked at $\beta=0$ and $\beta=\pi$ for heavy nuclei; i.e., the region in which we are primarily concerned. The same behavior can be expected[8] from $\langle\phi_0| Q_0^2 e^{-i\beta J_y} |\phi_0\rangle$ in the numerator. Hence, assuming that the numerator and denominator contribute only at $\beta=0$ and $\beta=\pi$, we get

$$\langle 2 \| Q^2 \| 0 \rangle = \langle\phi_0| Q_0^2 |\phi_0\rangle = \sum_{i=1}^{Z} \langle i| Q_0^2 |i\rangle , \tag{8}$$

where $|i\rangle$ stands for the HF single-particle proton orbital. Using Eq. (8), we get the reduced transition probability from Eq. (6) as

$$B(E2) = 5 \left| \sum_{i=1}^{Z} \langle i| Q_0^2 |i\rangle \right|^2 . \tag{9}$$

One also arrives at Eq. (9), if one uses the corresponding wave function in the unified model of Bohr and Mottelson.

To establish our difference equation (3), we assume that the HF single-particle wave function varies very slowly in the domain of the groups of four nuclides considered here, and we take them to be identical in the same group. This approximation has also been made by Garvey and Kelson[1] to obtain their mass sum rule where six different nuclei are involved. It is also justified in view of the slow variation of deformation as shown in Table II. Hence, using Eq. (9), it is now trivial to derive difference equation (3);

$$B(E2)[N,Z]+B(E2)[N+2,Z+2]-B(E2)[N+2,Z]-B(E2)[N,Z+2]$$
$$=5\left[\left|\sum_{i=1}^{Z}\langle i|Q_0^2|i\rangle\right|^2+\left|\sum_{i=1}^{Z+2}\langle i|Q_0^2|i\rangle\right|^2-\left|\sum_{i=1}^{Z}\langle i|Q_0^2|i\rangle\right|^2-\left|\sum_{i=1}^{Z+2}\langle i|Q_0^2|i\rangle\right|^2\right]=0.$$

To establish the difference equation (2), we use Eq. (5) to calculate the energy of the state J of the ground-state band of even-even nuclei, by replacing T_μ^k with the nuclear Hamiltonian H and setting $J_i=J_f=J$ and $K=0$. Then Eq. (5) reduces to

$$E_0^J=\frac{\int_0^\pi \sin\beta\, d\beta\, d_{00}^J(\beta)\langle\phi_0|He^{-i\beta J_y}|\phi_0\rangle}{\int_0^\pi \sin\beta\, d\beta\, d_{00}^J(\beta)\langle\phi_0|e^{-i\beta J_y}|\phi_0\rangle}.$$

It has been shown approximately by Villars[9] and by Lamme and Boeker[7] that the above equation reduces to

$$E_0^J\simeq\langle\phi_0|H|\phi_0\rangle+\frac{1}{2\mathcal{I}}J(J+1), \tag{10}$$

where

$$\mathcal{I}=\frac{|\langle\phi_0|J_y^2|\phi_0\rangle|^2}{\langle\phi_0|HJ_y^2|\phi_0\rangle-\langle\phi_0|H|\phi_0\rangle\langle\phi_0|J_y^2|\phi_0\rangle} \tag{11}$$

is identified as the moment of inertia. So $E2$, the energy difference between 0^+ and 2^+ states, is

$$E2=3/\mathcal{I}. \tag{12}$$

Since it is difficult to derive the difference equation (2) using the expression of the moment of inertia given by Eq. (11), we use the simpler cranking formula for the moment of inertia:

$$\mathcal{I}=2\sum_{i,j}\frac{|\langle i|J_x|j\rangle|^2}{\epsilon_i-\epsilon_j}, \tag{13}$$

where i and j refer to occupied and unoccupied HF orbitals with energies ϵ_i and ϵ_j, respectively. This expression for the moment of inertia of the nucleus $[N,Z]$ can be written as the sum of a function $\mathcal{I}_p(Z)$ for protons and a function $\mathcal{I}_n(N)$ for neutrons:

$$\mathcal{I}[N,Z]=2\sum_{i,j}\frac{|\langle i|J_x|j\rangle|^2}{\epsilon_i-\epsilon_j}$$
$$=2\sum_{\substack{i=1,N\\ j=N+1,P}}\frac{|\langle i|J_x|j\rangle|^2}{\epsilon_i-\epsilon_j}+2\sum_{\substack{i=1,Z\\ j=Z+1,P}}\frac{|\langle i|J_x|j\rangle|^2}{\epsilon_i-\epsilon_j}$$
$$=\mathcal{I}_n(N)+\mathcal{I}_p(Z), \tag{14}$$

where P denotes the total number of neutron or proton single-particle HF orbitals in the model space. Since the HF orbitals for groups of four nuclides have been assumed to be the same, following Eq. (14) we can write the moments of inertia for a group as

$$\begin{aligned}\mathcal{I}[N+2,Z+2]&=\mathcal{I}_n(N+2)+\mathcal{I}_p(Z+2),\\ \mathcal{I}[N+2,Z]&=\mathcal{I}_n(N+2)+\mathcal{I}_p(Z),\\ \mathcal{I}[N,Z+2]&=\mathcal{I}_n(N)+\mathcal{I}_p(Z+2).\end{aligned} \tag{15}$$

The moment of inertia of $(N+2)$ neutrons can be expressed as

$$\mathcal{I}_n(N+2)=2\sum_{\substack{i=1,N+2\\ j=N+3,P}}\frac{|\langle i|J_x|j\rangle|^2}{\epsilon_i-\epsilon_j}$$
$$=2\sum_{\substack{i=1,N\\ j=N+1,P}}\frac{|\langle i|J_x|j\rangle|^2}{\epsilon_i-\epsilon_j}+2\sum_{\substack{i=(N+1),(N+2)\\ j=(N+3),P}}\frac{|\langle i|J_x|j\rangle|^2}{\epsilon_i-\epsilon_j}-2\sum_{\substack{i=1,N\\ j=(N+1),(N+2)}}\frac{|\langle i|J_x|j\rangle|^2}{\epsilon_i-\epsilon_j},$$

where the first term is the moment of inertia $\mathcal{I}_n(N)$ due to N neutrons, and the remaining two terms together represent the effective moment of inertia due to the last two neutrons added to the $(N+1)$th and $(N+2)$th orbitals. Obviously, these terms are very much smaller compared to the first term, and, since the numerators and denominators are positive, the sum $\delta_n(2)$ of these two terms will be still smaller. So we get

$$\mathcal{I}_n(N+2)=\mathcal{I}_n(N)+\delta_n(2). \tag{16}$$

Similarly, the moment of inertia due to $(Z+2)$ protons is

$$\mathcal{I}_p(Z+2)=\mathcal{I}_p(Z)+\delta_p(2). \tag{17}$$

Using Eqs. (12), (16), and (17) for a group of four nuclides, one can easily obtain

$$E2[N,Z]+E2[N+2,Z+2]-E2[N+2,Z]-E2[N,Z+2]=\frac{6\delta_p(2)\delta_n(2)}{\mathcal{I}[N+2,Z+2]\mathcal{I}[N+2,Z]\mathcal{I}[N,Z+2]}\simeq 0\,.$$

This proves our difference equation (2).

For spherical nuclei, it has not been possible to establish these difference equations in the framework of a microscopic theory like random-phase approximation (RPA). However, in the following we have been able to establish these difference equations for both spherical and deformed nuclei in the collective model. It is worthwhile to remark here that rotation has been recognized long since as a more pronounced feature of nuclear dynamics than vibration. Also, in recent years it has been found experimentally[10] that many soft nuclei do have well-defined ground-state bands.

For spherical nuclei, the collective vibrational model for excited states assumes that the nucleus executes small harmonic vibrations characterized by a restoring force C_2 and an inertial parameter B_2, where the subscript indicates that only quadrupole distortions are being considered. In this model, $E2$ and $B(E2)$ of a nucleus with mass number A and charge number Z are given by[11]

$$E2=\hbar(C_2/B_2)^{1/2}\,, \tag{18}$$

$$B(E2)=\frac{45\hbar Z^2e^2r_0{}^4A^{4/3}}{32\pi^2(B_2C_2)^{1/2}}\,, \tag{19}$$

where r_0 is related to the uniform nuclear radius R through $R=r_0A^{1/3}$. The values of B_2 and C_2 are given[12] in the hydrodynamical model with irrotational flow as

$$B_2=8\pi Mr_0{}^2A^{5/3}$$

and

$$C_2=4\pi r_0{}^2A^{2/3}S-\frac{3e^2Z^2}{10\pi r_0A^{1/3}}\,,$$

where S is the surface energy per unit area, which is approximately constant ($4\pi r_0{}^2S\cong 20$ MeV). On substitution of the values of B_2 and C_2 in Eqs. (18) and (19), the expressions for $E2$ and $B(E2)$ reduce to

$$E2=c'A^{-1/2}(1-b'Z^2/A)^{1/2} \tag{20}$$

and

$$B(E2)=a'Z^2A^{1/6}(1-b'Z^2/A)^{-1/2}\,, \tag{21}$$

where the constants a', b', and c' are, respectively, given by

$$a'=\frac{15}{16}\left(\frac{3}{2MS}\right)^{1/2}r_0{}^2e^2\hbar=0.000\,23\ e^2\mathrm{b}^2\,,$$

$$b'=3e^2/(40\pi r_0{}^3S)=0.0166\,,$$

$$c'=\hbar\left(\frac{32\pi S}{3M}\right)^{1/2}=36\,170\ \mathrm{keV}\,.$$

Thus, $E2$ and $B(E2)$ are smoothly varying functions of A and Z, and consequently of N and Z. Treating N and Z as continuous variables, $E2$ and $B(E2)$ for the nuclei $[N+2,Z+2]$, $[N+2,Z]$, and $[N,Z+2]$ can be expressed by performing Taylor's expansions in N and Z as

TABLE I. (a) Relative magnitudes of different derivatives of $E2[N,Z]$ and $B(E2)[N,Z]$ for some representative spherical nuclei. (b) Relative magnitudes of different derivatives of $E2[N,Z]$ and $B(E2)[N,Z]$ for some representative deformed nuclei.

Nucleus	$\frac{1}{B(E2)}\frac{\partial B(E2)}{\partial Z}$	$\frac{1}{B(E2)}\frac{\partial^2 B(E2)}{\partial Z^2}$	$\frac{1}{B(E2)}\frac{\partial B(E2)}{\partial N}$	$\frac{1}{B(E2)}\frac{\partial^2 B(E2)}{\partial N^2}$	$\frac{1}{E2}\frac{\partial E2}{\partial Z}$	$\frac{1}{E2}\frac{\partial^2 E2}{\partial Z^2}$	$\frac{1}{E2}\frac{\partial E2}{\partial N}$	$\frac{1}{E2}\frac{\partial^2 E2}{\partial N^2}$
			(a) For spherical nuclei					
$^{60}_{32}$Ni	0.0818	0.0043	0.0005	0.000 04	−0.0160	0.000 66	−0.0060	0.000 11
$^{114}_{48}$Cd	0.0514	0.0019	−0.0007	0.000 04	−0.0127	0.000 06	−0.0022	0.000 39
$^{124}_{52}$Te	0.0484	0.0018	−0.0009	0.000 04	−0.0126	0.000 05	−0.0017	−0.000 01
$^{206}_{82}$Pb	0.0367	0.0014	−0.0021	0.000 04	−0.0139	−0.000 07	−0.0004	−0.000 03
			(b) For deformed nuclei					
$^{154}_{64}$Gd	0.0399	0.001 02	0.008 65	0.000 187	−0.0108	0.000 56		
$^{164}_{66}$Dy	0.0384	0.001 00	0.008 00	0.000 010	−0.0101	0.000 49		
$^{176}_{70}$Yb	0.0361	0.000 95	0.007 60	0.000 014	−0.0094	0.000 43		
$^{244}_{96}$Cm	0.0263	0.000 24	0.005 40	0.000 007	−0.0068	0.000 22		

$$
\begin{aligned}
&E2[N+2,Z+2]=E2[N,Z]+2\frac{\partial E2}{\partial N}+2\frac{\partial E2}{\partial Z}+2\frac{\partial^2 E2}{\partial N^2}+2\frac{\partial^2 E2}{\partial Z^2}+2\frac{\partial^2 E2}{\partial N\partial Z}+\cdots,\\
&E2[N+2,Z]=E2[N,Z]+2\frac{\partial E2}{\partial N}+2\frac{\partial^2 E2}{\partial N^2}+\cdots,\\
&E2[N,Z+2]=E2[N,Z]+2\frac{\partial E2}{\partial Z}+2\frac{\partial^2 E2}{\partial Z^2}+\cdots,\\
&B(E2)[N+2,Z+2]=B(E2)[N,Z]+2\frac{\partial B(E2)}{\partial N}+2\frac{\partial B(E2)}{\partial Z}+2\frac{\partial^2 B(E2)}{\partial N^2}+2\frac{\partial^2 B(E2)}{\partial Z^2}+2\frac{\partial^2 B(E2)}{\partial N\partial Z}+\cdots,\\
&B(E2)[N+2,Z]=B(E2)[N,Z]+2\frac{\partial B(E2)}{\partial N}+2\frac{\partial^2 B(E2)}{\partial N^2}+\cdots,\\
&B(E2)[N,Z+2]=B(E2)[N,Z]+2\frac{\partial B(E2)}{\partial Z}+2\frac{\partial^2 B(E2)}{\partial Z^2}+\cdots.
\end{aligned}
\tag{22}
$$

The first terms in the first three and last three expansions stand for the $E2$ and $B(E2)$ for the nucleus $[N,Z]$, respectively. Retaining terms up to first order in Eqs. (22), one can easily obtain the difference equations (2) and (3). In order to show the validity of the Taylor's expansion and also justify retaining terms only up to first order, we have calculated [using Eqs. (20) and (21)] the quantities

$$
\frac{1}{B(E2)}\frac{\partial B(E2)}{\partial N},\quad \frac{1}{B(E2)}\frac{\partial B(E2)}{\partial Z},\quad \frac{1}{E2}\frac{\partial E2}{\partial N},\quad \frac{1}{E2}\frac{\partial E2}{\partial Z},\quad \frac{1}{B(E2)}\frac{\partial^2 B(E2)}{\partial N^2},\quad \frac{1}{B(E2)}\frac{\partial^2 B(E2)}{\partial Z^2},\quad \frac{1}{E2}\frac{\partial^2 E2}{\partial N^2},\quad \frac{1}{E2}\frac{\partial^2 E2}{\partial Z^2}
$$

for some representative nuclei and presented them in Table I(a). An analysis of these numerical values justifies our approximation.

We follow the same approach to establish Eqs. (2) and (3) for the rotational nuclei. In the collective rotational model, if the deformation is characterized by a single parameter β, then the quadrupole moment for a nucleus $[N,Z]$ is given by[13]

$$
Q_0=\frac{3}{5}\left(\frac{5}{\pi}\right)^{1/2}Zr_0^{\,2}A^{2/3}\beta, \tag{23}
$$

and the reduced transition probability for exciting the nucleus from ground state 0^+ to the 2^+ state is given by[13]

$$
B(E2)\uparrow=\frac{5}{16\pi}e^2Q_0^{\,2}. \tag{24}
$$

On substitution of the value of Q_0 from Eq. (23) in Eq. (24), one gets

$$
B(E2)\uparrow=\frac{9}{16\pi^2}e^2r_0^{\,4}Z^2A^{4/3}\beta^2. \tag{25}
$$

For axially symmetric nuclei, the excitation energy of the 2^+ state will be given by

$$
E2=3\hbar/\mathcal{I}_0, \tag{26}
$$

where $\mathcal{I}_0$ is the moment of inertia, which in hydrodynamical model is given by[14]

$$
\mathcal{I}_0=\frac{9}{8\pi}Mr_0^{\,2}A^{5/3}\beta^2. \tag{27}
$$

Here, $B(E2)$ and $E2$ are smoothly varying functions of N and Z, and also of β. We will assume that β is a very slowly varying function of N and Z, particularly in the domain of the groups of nuclides considered here. In Table II, we have presented the experimental values of β for representative groups of nuclides, the mean value of β for the groups, and the percentage of deviation from the mean values. It is found that the variation of β from the mean value of a group is very small for the nuclei shown in this table. Hence, we have treated β as a constant in performing the Taylor's

TABLE II. Experimental values of the deformation parameter β for groups of four nuclides. [These values have been extracted from the experimental $B(E2)$ values of Ref. 2.] In the third column, the mean value of β for the group is presented. The fourth column gives the percentage of deviation of each β value from the mean value.

Nucleus	Deformation β	Mean β	% deviation from mean β
$^{162}_{66}$Dy	0.2843	0.2867	−0.83
$^{164}_{66}$Dy	0.2961		+3.20
$^{164}_{68}$Er	0.2774		−3.20
$^{166}_{68}$Er	0.2892		+0.80
$^{236}_{92}$U	0.2404	0.2421	−0.70
$^{238}_{92}$U	0.2417		−0.16
$^{238}_{94}$Pu	0.2435		+0.60
$^{240}_{94}$Pu	0.2427		+0.24

expansion at $[N,Z]$ for $E2$ and $B(E2)$ of the rotational nuclei $[N+2,Z+2]$, $[N+2,Z]$, and $[N,Z+2]$. The numerical values of

$$\frac{1}{B(E2)}\frac{\partial B(E2)}{\partial N},\ \frac{1}{B(E2)}\frac{\partial B(E2)}{\partial Z},\ \frac{1}{E2}\frac{\partial E2}{\partial N},$$

$$\frac{1}{E2}\frac{\partial E2}{\partial Z},\ \frac{1}{B(E2)}\frac{\partial^2 B(E2)}{\partial N^2},\ \frac{1}{B(E2)}\frac{\partial^2 B(E2)}{\partial Z^2},$$

$$\frac{1}{E2}\frac{\partial^2 E2}{\partial N^2},\ \frac{1}{E2}\frac{\partial^2 E2}{\partial Z^2}$$

are presented in Table I(b) for some representative rotational nuclei. The relative values justify our approximation leading to the difference equations (2) and (3) for the rotational nuclei.

III. DERIVATION OF ROSS AND BHADURI DIFFERENCE EQUATION

In this section we show that the difference equation (1) due to Ross and Bhaduri follows from our approach, as shown below:

$$F[N,Z]+F[N+2,Z+2]-F[N+2,Z]-F[N,Z+2]=\frac{1}{E2[N,Z]B(E2)[N,Z]}+\frac{1}{E2[N+2,Z+2]B(E2)[N+2,Z+2]}$$
$$-\frac{1}{E2[N+2,Z]B(E2)[N+2,Z]}-\frac{1}{E2[N,Z+2]B(E2)[N,Z+2]}. \tag{28}$$

In the preceding section, it has been shown [Eq. (22)] that in the collective model, for both rotational and vibrational nuclei, it is a good approximation to expand $E2$ and $B(E2)$ for $[N+2,Z+2]$, $[N+2,Z]$, and $[N,Z+2]$ nuclei at N and Z in a Taylor's series and retain terms up to first order only. Making use of this fact, it is found that the right-hand side of Eq. (28) vanishes. Thus, in the collective model $F[N,Z]+F[N+2,Z+2]-F[N+2,Z]-F[N,Z+2]=0$ for both rotational and vibrational nuclei. In the PHF model also, this difference equation can be derived as follows. In this model it has been shown before that $E2$ and $B(E2)$ are given by Eqs. (12) and (9), respectively. When these expressions are utilized, the right-hand side of Eq. (28) reduces to

$$\frac{1}{15}\left[\frac{g[N,Z]}{\left|\sum_{i=1}^{Z}\langle i|Q_0^2|i\rangle\right|^2}+\frac{g[N+2,Z+2]}{\left|\sum_{i=1}^{Z+2}\langle i|Q_0^2|i\rangle\right|^2}\right.$$
$$\left.-\frac{g[N+2,Z]}{\left|\sum_{i=1}^{Z}\langle i|Q_0^2|i\rangle\right|^2}-\frac{g[N,Z+2]}{\left|\sum_{i=1}^{Z+2}\langle i|Q_0^2|i\rangle\right|^2}\right].$$

Following Eqs. (16) and (17), we put

$$g[N+2,Z+2]=g[N,Z+2]+\delta_n(2)$$

and

$$g[N+2,Z]=g[N,Z]+\delta_n(2)$$

in the above expression. Thus, Eq. (28) reduces to

$$F[N,Z]+F[N+2,Z+2]-F[N+2,Z]-F[N,Z+2]$$
$$=-\frac{\delta_n(2)}{15}\left[\frac{\left|\sum_{i=1}^{Z+2}\langle i|Q_0^2|i\rangle\right|^2-\left|\sum_{i=1}^{Z}\langle i|Q_0^2|i\rangle\right|^2}{\left|\sum_{i=1}^{Z}\langle i|Q_0^2|i\rangle\right|^2\left|\sum_{i=1}^{Z+2}\langle i|Q_0^2|i\rangle\right|^2}\right].$$

The quantity inside the square bracket is extremely small, since the numerator represents the contribution of the last two protons to the transition probability, while the denominator is the product of the transition probability due to Z and $(Z+2)$ protons. Further, this small fraction is multiplied by another small fraction, $\delta_n(2)/15$. Hence the above equation can be written as

$$F[N,Z]+F[N+2,Z+2]-F[N+2,Z]-F[N,Z+2]\simeq 0,$$

so the assumptions which have been made to derive our difference equations (2) and (3) in the collective model and PHF model are adequate enough to derive the difference equation (1) of Ross and Bhaduri. Hence it is reasonable to expect that Eqs. (2) and (3) will have the same degree of consistency with experimental data as Eq. (1) does.

IV. ANALYSIS OF EXPERIMENTAL DATA

We have exclusively taken the experimental $B(E2)$ and $E2$ values from the compilation of Ross and Bhaduri[2] and, in addition to this, some data on $E2$ have been included from Ref. 10. To test the consistency of our Eqs. (2) and (3), we have calculated the quantities

$$E=E2[N,Z]+E2[N+2,Z+2]-E2[N+2,Z]-E2[N,Z+2] \tag{29}$$

and

$$B=B(E2)[N,Z]+B(E2)[N+2,Z+2]-B(E2)[N+2,Z]-B(E2)[N,Z+2], \tag{30}$$

with these available experimental data, and presented them in the second and fourth columns of

TABLE III. Test of the difference Eqs. (2) and (3). The first column lists the first nucleus appearing in Eqs. (2) and (3). In the second and third columns are presented the values of E (in keV) and P (in %) of Eqs. (29) and (31), respectively, calculated with the experimental $E2$ values. The fourth and fifth columns list the values of B and δB of Eqs. (30) and (32), respectively, also calculated from experimental $B(E2)$ values.

Anchor nucleus	E (keV)	P (%)	B (e^2b^2)	δB (e^2b^2)	Anchor nucleus	E (keV)	P (%)	B (e^2b^2)	δB (e^2b^2)
$^{48}_{22}$Ti	54.0	4.5	−0.0378	±0.014 90	$^{140}_{58}$Ce	71.0	6.3	−0.05	±0.088 31
$^{52}_{24}$Cr	30.0	2.6	0.001	±0.018 74	$^{146}_{60}$Nd	−60.0	14.6	−0.188	±0.073 21
$^{56}_{26}$Fe	−79.0	7.1	−0.0169	±0.033 48	$^{148}_{60}$Nd	−45.15	20.3	−0.46	±0.143 20
$^{62}_{28}$Ni	−121.0	10.6	−0.024	±0.026 43	$^{150}_{62}$Sm	−6.75	2.9	0.08	±0.267 20
$^{68}_{30}$Zn	−11.0	1.1	0.008	±0.031 10	$^{152}_{62}$Sm	5.76	5.5	−0.03	±0.321 40
$^{72}_{32}$Ge	163.3	24.9	−0.107	±0.096 16	$^{152}_{64}$Gd	25.2	10.7		
$^{74}_{32}$Ge	87.7	15.0	−0.039	±0.059 43	$^{154}_{64}$Gd	−4.9	4.3	−0.12	±0.518 10
$^{74}_{34}$Se	106.7	20.6	0.093	±0.136 30	$^{156}_{64}$Gd	−3.8	4.27	−0.43	±0.555 00
$^{76}_{34}$Se	108.1	19.2	−0.11	±0.120 50	$^{158}_{64}$Gd	−1.9	2.3	−0.23	±0.487 30
$^{78}_{34}$Se	106.9	15.9	−0.073	±0.083 97	$^{154}_{66}$Dy	44.7	17.6		
$^{80}_{34}$Se	113.9	15.3	0.024	±0.059 37	$^{156}_{66}$Dy	−27.5	19.7	0.55	±0.500 90
$^{82}_{36}$Kr	179.9	20.4	−0.182	±0.112 90	$^{158}_{66}$Dy	−12.9	12.5	0.68	±0.555 00
$^{90}_{40}$Zr	579.0	42.3	0.125	±0.035 65	$^{160}_{66}$Dy	−3.4	3.8	0.000	±0.545 40
$^{92}_{40}$Zr	−78.5	8.9	0.048	±0.047 01	$^{162}_{66}$Dy	−2.7	3.3	0.2	±0.497 50
$^{94}_{42}$Mo	−86.5	11.0	0.16	±0.072 59	$^{156}_{68}$Er	36.8	13.0		
$^{96}_{42}$Mo	−122.7	17.78	0.199	±0.079 84	$^{158}_{68}$Er	−10.0	5.5		
$^{98}_{42}$Mo	186.5	31.9	−0.194	±0.193 00	$^{160}_{68}$Er	−18.8	14.6		
$^{102}_{44}$Ru	73.4	15.4	−0.046	±0.117 70	$^{162}_{68}$Er	−10.7	10.3		
$^{104}_{46}$Pd	44.2	7.5	−0.108	±0.074 71	$^{164}_{68}$Er	−4.4	4.8		
$^{106}_{46}$Pd	102.5	18.3	−0.077	±0.083 51	$^{166}_{68}$Er	−2.9	3.5	0.07	±0.461 90
$^{108}_{46}$Pd	19.6	3.7	−0.058	±0.097 16	$^{168}_{68}$Er	−5.1	6.3	0.70	±0.4602
$^{110}_{48}$Cd	83.5	8.7	−0.083	±0.063 45	$^{164}_{70}$Yb	−14.1	11.1		
$^{112}_{48}$Cd	51.9	5.6	−0.053	±0.064 46	$^{166}_{70}$Yb	−9.1	8.8		
$^{114}_{48}$Cd	−17.6	1.9	−0.062	±0.046 91	$^{168}_{70}$Yb	−2.7	3.0		
$^{118}_{50}$Sn	59.0	6.6	−0.127	±0.111 00	$^{170}_{70}$Yb	2.1	2.4	−0.29	±0.6048
$^{120}_{50}$Sn	74.0	8.4	−0.113	±0.019 05	$^{172}_{70}$Yb	−0.5	0.6	0.75	±0.5671
$^{122}_{50}$Sn	73.0	8.2	−0.039	±0.017 12	$^{174}_{70}$Yb	−0.8	0.9	−0.42	±0.5921
$^{124}_{52}$Te	−9.0	1.7	0.251	±0.206 60	$^{170}_{72}$Hf	−5.5	5.1		
$^{126}_{52}$Te	21.0	3.7	−0.176	±0.250 20	$^{172}_{72}$Hf	0.4	0.4		
$^{128}_{52}$Te	32.8	4.7	−0.178	±0.182 90	$^{174}_{72}$Hf	−2.1	2.1		
$^{122}_{54}$Xe	2.7	0.9			$^{176}_{72}$Hf	−6.9	7.3		
$^{124}_{54}$Xe	−11.4	3.5			$^{178}_{72}$Hf	−2.1	2.1	−0.03	±0.4754
$^{126}_{54}$Xe	25.0	7.8	−0.719	±0.436 60	$^{176}_{74}$W	5.3	4.4		
$^{128}_{54}$Xe	8.0	1.7	−0.1	±0.347 00	$^{178}_{74}$W	−3.3	2.8		
$^{130}_{54}$Xe	10.8	1.9	0.188	±0.254 90	$^{180}_{74}$W	−5.2	4.6		
$^{126}_{56}$Ba	23.9	9.6			$^{182}_{74}$W	6.3	5.4		
$^{128}_{56}$Ba	−8.7	2.8			$^{184}_{74}$W	6.4	4.8	−0.05	±0.3087
$^{130}_{56}$Ba	−21.2	5.4			$^{180}_{76}$Os	13.7	9.5		

TABLE III *(Continued)*.

Anchor nucleus	E (keV)	P (%)	B (e^2b^2)	δB (e^2b^2)	Anchor nucleus	E (keV)	P (%)	B (e^2b^2)	δB (e^2b^2)
$^{182}_{76}$Os	36.1	24.0			$^{224}_{88}$Ra	2.2	3.1	−0.97	±1.1270
$^{184}_{76}$Os	57.4	32.0			$^{226}_{88}$Ra	4.2	7.1	0.09	±1.1090
$^{186}_{76}$Os	12.6	5.7	0.17	±0.8772	$^{228}_{90}$Th	0.1	0.1	0.05	±1.8780
$^{188}_{76}$Os	−11.2	4.6	0.22	±0.7392	$^{230}_{90}$Th	−0.5	1.0	−1.25	±1.3350
$^{190}_{76}$Os	−7.0	2.7	−0.22	±0.2133	$^{232}_{90}$Th	3.8	8.0	2.95	±1.2110
$^{194}_{78}$Pt	−41.5	10.0	−0.12	±0.2527	$^{236}_{92}$U	−0.6	1.4	−0.23	±0.6905
$^{196}_{78}$Pt	−96.2	24.9	0.21	±0.2026	$^{240}_{94}$Pu	−0.92	2.1		
$^{202}_{80}$Hg	−86.0	13.3	0.28	±0.1197	$^{242}_{94}$Pu	−0.5	1.1	−0.73	±0.9956
$^{220}_{86}$Rn	28.7	18.4	−0.88	±0.4709					

Table III, respectively. (The "anchor nucleus," i.e., the first nucleus occurring in Eqs. (2) and (3), appears in the first column.) Thus E and B are the deviations from zero of the $E2$ sum and the $B(E2)$ sum in Eqs. (2) and (3), respectively. The validity of Eq. (2) cannot be fully judged from the smallness of the quantity E, since the relevant $E2$ values in the sum may be quite small. Hence, we define the quantity

$$P = 100|E|\{E2[N,Z]+E2[N+2,Z+2]+E2[N+2,Z] + E2[N,Z+2]\}^{-1}. \quad (31)$$

It is easy to see that $P \to 0$ for $E \to 0$; i.e, when Eq. (2) is exactly satisfied, and, $P \to 100$ when either the positive terms or the negative terms in Eq. (29) are comparatively smaller and can be neglected. This happens when Eq. (2) is severely violated. Thus, P represents a relevant quantity

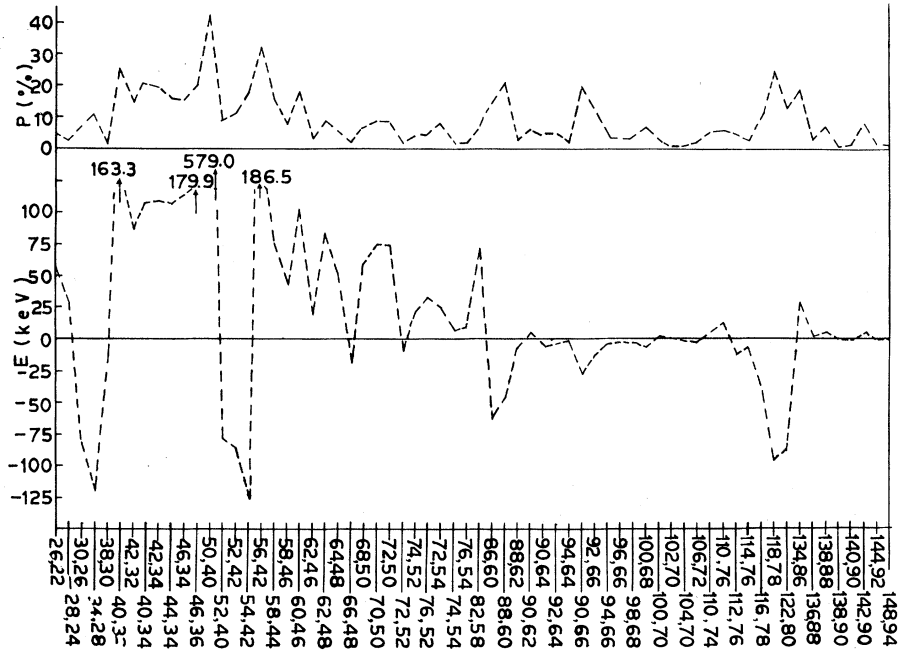

FIG. 1. Test of the difference equation for $E2$ [Eq. (2)] on the experimental data. The lower curve gives the deviation E as defined by Eq. (29). The upper curve gives the percentage deviation P as defined by Eq. (31). The case used to test the difference equation is labeled along the abscissa by the neutron number and the proton number of the anchor nucleus, i.e., the first nucleus occurring in Eq. (2).

which provides a true measure of the validity of Eq. (2). This is presented in the third column of Table III.

In Fig. 1, E and P have been plotted for 66 cases. In 45 cases P has values less than 10%, and only in seven cases does it have values more than 20%. The case with $^{90}_{40}Zr$ as anchor nucleus is extremely pathological. Here, the value of P is 42%. The failure of the difference equation may be attributed to the fact that three of the nuclei involved here are magic. Some other cases involving nuclei with $Z=40$ also show big descrepancies. Other cases where descrepancies occur mainly involve nuclei in the transitional regions around $Z=78$ and 88, and some nuclei with $N=88$. Except in these regions, it is reasonable to conclude that the difference equation (2) is quite successful. It is worthwhile to remark here that Fig. 1 clearly reveals the shell closure effect in the Periodic Table, through the values of P and E which are characteristic of the properties of excited states. Hence such plots are interesting.

The validity of the difference equation (3) can be judged from the smallness of the values of B calculated from Eq. (30). The percentage of deviation was not calculated here, since the experimental $B(E2)$ values have errors. On the other hand, we take the usual definition to compute the experimental uncertainty δB in the sum of Eq. (30) as

$$\delta B = \pm \left(\sum_{i=1}^{4} \{\delta B(E2)[N_i, Z_i]\}^2 \right)^{1/2}, \tag{32}$$

where $\delta B(E2)[N_i, Z_i]$ is the experimental error in $B(E2)[N_i, Z_i]$. In Fig. 2, δB is shown as an envelope about the horizontal axis along which are plotted N and Z of the anchor nucleus (although δB is defined only at a series of points, we have connected these points by straight lines to make the figure easier for interpretation). The values of B have been plotted in the same diagram and are joined by a broken line. The criterion for the validity of the difference equation (3) for a particular case is that $|B| \leq |\delta B|$, and from Fig. 2 this is generally found to be satisfied. However, there are regions where B is considerably outside the envelope. These regions are $Z=78$, 88, and $Z=40$, and Z or N with magic numbers, and are identical with those found from the analysis of Fig. 1. This is in conformity with the conclusions of RB.[2] Further, it is observed that the difference equation fails in the region of nuclei with $N=88$ and Z varying from 60 to 66. This is an additional feature emerging here in the analysis of both Figs. 1 and 2 that was not apparent in the analysis of RB. This feature is reminiscent of the fact that abrupt onset of deformation takes place at $N=88$, which has been theoretically shown by Nilsson *et al.*[15] and also empirically observed by others.[10,17] Hence,

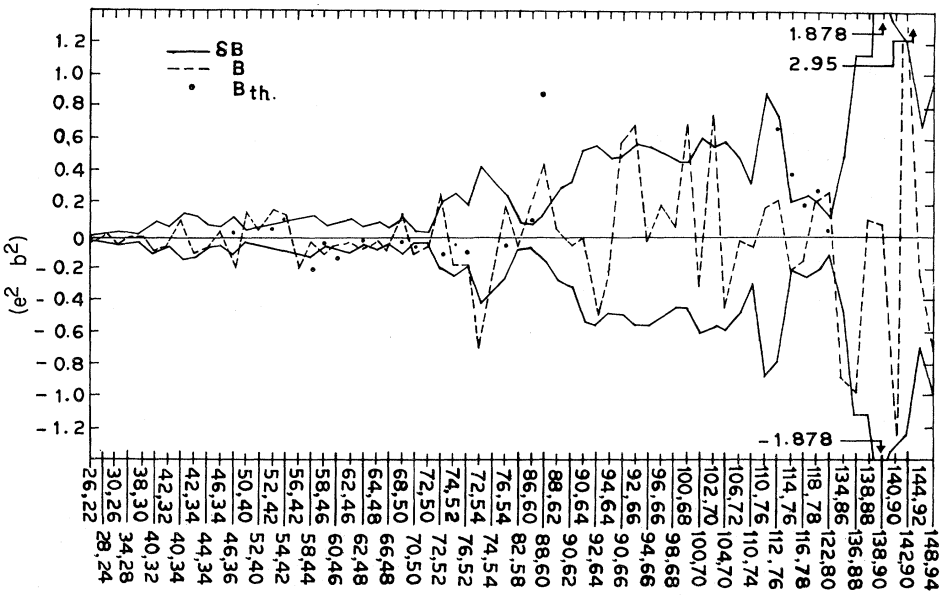

FIG. 2. Test of the difference equation for $B(E2)$ [Eq. (3)] on the experimental data. The deviation B as defined by Eq. (30) is given by the broken line. The experimental uncertainty δB, as defined by Eq. (32), is presented as an envelope about the horizontal axis. The theoretical values of B, calculated by using the $B(E2)$ values from Ref. 16, are presented as heavy dots. The case used to test the difference equation is labeled along the abscissa by the neutron number and the proton number of the anchor nucleus, i.e., the first nucleus appearing in Eq. (3).

TABLE IV. (a) Comparison of the predicted and experimental $B(E2)$ and $E2$ values. For a given nucleus, the second column gives our prediction and the third column gives the prediction of RB for the $B(E2)$ values. The tentatively assigned experimental values of $B(E2)$ are given in the fourth column. The fifth and sixth columns give our prediction and the experimental values of $E2$. If the prediction can be made in more than one way, the various results are shown along with their average value. (b) Predictions of unmeasured $B(E2)$ and $E2$ values. For a given nucleus the second and third columns give our prediction of $E2$ and $B(E2)$ values, respectively. The last column lists the $B(E2)$ value predicted by RB. The available experimental $E2$ values are presented in the second column inside parentheses.

(a) Comparison of predicted and experimental values

	$B(E2)$ (e^2b^2)			$E2$ (keV)	
Nucleus	Present prediction	RB Prediction	Experiment	Present prediction	Experiment
$^{108}_{44}$Ru	1.085± 0.14	1.25± 0.2	1.47± 0.29	219.8	242.3
$^{110}_{44}$Ru	1.124± 0.176	1.23± 0.25	1.45± 0.29	217.4	240.8
$^{112}_{44}$Ru	1.188± 0.214	1.28± 0.35	1.81± 0.36	224.8	236.8
$^{112}_{46}$Pd	0.899± 0.08	0.90± 0.1	>0.54	314.9	348.9
$^{114}_{46}$Pd	0.963± 0.092	0.97± 0.35	>0.68	303.5	332.9
$^{146}_{58}$Ce	1.05 ± 0.069	1.07± 0.1	0.79± 0.16	242.5	258.6
$^{182}_{76}$Os	3.75 ± 0.464	3.41± 0.46 3.5 ± 0.44 3.46± 0.54(av)	···	121.9	127.0
$^{184}_{76}$Os	3.60 ± 0.295 3.49 ± 0.346 3.545± 0.227(av)	3.62± 0.33 3.16± 0.38 3.41± 0.35(av)	3.20± 0.62	126.2	120.0

(b) Predictions of unmeasured values

	Present predictions		RB predictions of
Nucleus	$E2$ (keV)	$B(E2)$ (e^2b^2)	$B(E2)$ (e^2b^2)
$^{68}_{32}$Ge	100.1	0.198± 0.018	
$^{78}_{32}$Ge	615.2	0.157± 0.044	
$^{72}_{34}$Se	840.0	0.437± 0.084	
$^{106}_{44}$Ru	279.9	0.971± 0.107	
$^{118}_{52}$Te	623.0 550.3 586.6(av) (600.0)	0.548± 0.110 0.36 ± 0.185 0.454± 0.108(av)	
$^{124}_{54}$Xe	335.5 347.0 362.4 348.3(av) (355.0± 10)	1.225± 0.149 0.898± 0.052 1.171± 0.480 1.098± 0.168(av)	1.34± 0.43 0.93± 0.18 0.97± 0.07 0.99± 0.14(av)
$^{144}_{58}$Ce	410.0 (397.5)	0.64 ± 0.047	0.66± 0.11
$^{152}_{60}$Nd	91.14 (75.9)	3.64 ± 0.18	
$^{146}_{62}$Sm	795.0 791.0 793.0(av) (747.4)	0.33 ± 0.086 0.522± 0.046 0.426± 0.049(av)	0.58± 0.06

TABLE IV *(Continued)*

Nucleus	(b) Predictions of unmeasured values: Present predictions $E2$ (keV)	Present predictions $B(E2)$ (e^2b^2)	RB predictions of $B(E2)$ (e^2b^2)
$^{156}_{62}$Sm	72.5 (76.0)	4.83 ± 0.345	4.63 ± 0.12
$^{162}_{64}$Gd	68.1	6.19 ± 0.384	
$^{172}_{68}$Er	77.1	5.30 ± 0.420	
$^{194}_{76}$Os	232.9	1.84 ± 0.116	
$^{242}_{96}$Cm	41.3 (42.2 ± 0.1)		

except for these regions, the difference equation (3) can be considered to be reasonably valid. In Fig. 2, we have also displayed by heavy dots the values of B computed with the $B(E2)$ values calculated theoretically by Uher and Sorensen[16] for spherical nuclei in the RPA formalism. These values of $B(E2)$ were taken from Table V of Ref. 16, with an effective charge of 0.3 e for neutrons and 1.3 e for protons. It will be seen that, in most cases, the theoretical values of B lie within the error envelope. It is satisfying to note that, in the case of the anchor nucleus $N=88$, $Z=60$, both the theoretical and the experimental values of B fall outside the error envelope. Hence, it is natural to conclude that our difference equation (3) is reasonably valid, and has both theoretical and experimental support except in the specified regions.

V. PREDICTIONS AND CONCLUSION

In this section we would like to discuss the usefulness of these equations, make a few predictions of $E2$ and $B(E2)$ values, and compare with the tentative experimental values, wherever available. We will also compare our predictions with the $B(E2)$ values predicted by RB. Before we discuss our predictions, we summarize the regions where our difference equations are not likely to be applicable: (a) when one or more of the nuclei involved are singly or doubly magic, (b) when any of the nuclei has N or $Z=40$, (c) for some of the transition nuclei around $Z=78$ and $Z=88$, (d) when any of the nuclei has $N=88$ and Z between 60 and 66, and, (e) possibly, for low-mass nuclei ($A<50$).

In Table IV we have presented the predicted $E2$ and $B(E2)$ values of some representative nuclei. In some cases, where the predictions can be made in more than one way, all the calculated values along with the average value have been presented. Since the RB prescription can be used for prediction of $B(E2)$ values only when the $E2$ values for all the four nuclei and the $B(E2)$ values of the other three nuclei are known, it has not been possible to calculate the $B(E2)$ values in all cases. In Table IV(a), we present our predictions of $B(E2)$ and $E2$ values, RB predictions of $B(E2)$ values, and the corresponding experimental quantities. The agreement between our prediction and experimental values is satisfactory. It can also be seen that our predictions of $B(E2)$ values are quite close to those of RB in all the possible cases. This was expected, since our difference equations (2) and (3) are as strongly supported by theory and experiments as Eq. (1). Moreover, the RB equation has been shown to follow from the same considerations which have led to the establishment of Eqs. (2) and (3). It is also extremely satisfying to note that our predictions of $E2$ values compare [Table IV(a)] reasonably well with the experimental ones. In Table IV(b), we have presented a set of predictions for $B(E2)$ and $E2$ values for some representative nuclei.

In view of the theoretical and experimental evidences presented in this paper, the difference equations (2) and (3) can be reliably used to predict unmeasured $E2$ and $B(E2)$ values within an accuracy of ±20%, and often better. It is pleasing to note that the RB equation is compatible with our Eqs. (2) and (3), both theoretically and experimentally. Apart from the obvious simplicity of our equations compared to the RB equation, the former provide better guidelines for the understanding of nuclear dynamics than the latter. It further illustrates that the difference equation approach, which has been highly successful in the case of nuclear ground-state energies, can also be useful for correlating properties of excited states. We feel suitable difference equations similar to Eqs. (2) and (3), but involving other transition probabilities and excitation energies, may be found for odd-A nuclei.

[1]G. T. Garvey and I. Kelson, Phys. Rev. Lett. 16, 197 (1966); G. T. Garvey, W. J. Gerace, R. L. Jaffe, I. Talmi, and I. Kelson, Rev. Mod. Phys. 41, S1 (1969).
[2]C. K. Ross and R. K. Bhaduri, Nucl. Phys. A196, 369 (1972).
[3]W. H. Bassichis and I. Kelson, Phys. Rev. C 5, 1169 (1972).
[4]L. Satpathy and S. C. K. Nair, Phys. Lett. 26B, 257 (1968); L. Satpathy and S. C. K. Nair, Phys. Lett. 26B, 716 (1968); L. Satpathy, Phys. Rev. 174, 1324 (1968).
[5]G. Ripka, in *Advances in Nuclear Physics*, edited by M. Baranger and E. Vogt (Plenum, New York, 1968), Vol. I, p. 183.
[6]R. E. Peierls and J. Yoccoz, Proc. Phys. Soc. London, A70, 381 (1957).
[7]H. A. Lamme and E. Boeker, Nucl. Phys. A111, 492 (1968).
[8]B. J. Verhaar, Nucl. Phys. 54, 641 (1964).
[9]F. Villars, *Many-Body Description of Nuclear Structure and Reactions, in Proceedings of the International School of Physics, "Enrico Fermi," Course XXXVI*, edited by C. Bloch (Academic, New York and London, 1966), p. 1.
[10]M. A. J. Mariscotti, G. S. Goldhaber, and B. Buck, Phys. Rev. 178, 1864 (1969).
[11]D. J. Rowe, *Nuclear Collective Motion* (Methuen, London, 1970), p. 21.
[12]O. Nathan and S. G. Nilsson, in *Alpha-, Beta-, and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (North-Holland, Amsterdam, 1968), Vol. I, p. 601.
[13]S. G. Nilsson, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 29, No. 16 (1955).
[14]A. Bohr, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 26, No. 14 (1952).
[15]S. G. Nilsson, C. F. Tsang, A. Sobiczewski, Z. Szymanski, S. Wycech, C. Gustafson, I. Lamm, P. Möler, and B. Nilsson, Nucl. Phys. A131, 1 (1969).
[16]R. A. Uher and R. A. Sorensen, Nucl. Phys. 86, 1 (1966).
[17]J. B. Wilhemy, S. G. Thompson, R. C. Jared, and E. Cheifetz, Phys. Rev. Lett. 25, 1122 (1970).