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Calculation of the covariant pion-nucleus optical potential. I. Kinematical aspects*

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A discussion of several kinematical schemes used in the study of pion-nucleus scattering is presented. It is shown that various kinematical ambiguities of the noncovariant theories are resolved in a covariant analysis. In particular, the use of an *off-mass-shell* description for the projectile and the target particles provides a degree of freedom in the kinematical description which avoids various problems present in the noncovariant analyses. We present a comparison of the angle transformations, matrix elements, and differential cross sections obtained from the different kinematical specifications. In these comparisons we have attempted to maintain a clear separation of the dynamical and kinematical assumptions, using the same pion-nucleon interaction in all the cases considered. It is found that the large angle scattering is most sensitive to the details of the kinematical description.

[NUCLEAR REACTIONS Scattering theory, pion optical potential, comparison of kinematic transformations.]

I. INTRODUCTION

Recently, there has been a systematic effort in the use of pions to probe nuclear structure. The pion-nucleus interaction differs considerably from the nucleon-nucleus interaction. At a few hundred MeV, the kinetic energy of the pion is larger than its rest mass. Consequently, the pion must be treated as a relativistic particle. (The relativistic formalism is unnecessary at such energies in the case of the nucleon-nucleus interaction.) Furthermore, the presence of resonances in the medium energy domain makes the fundamental pion-nucleon (πN) interaction strongly energy dependent. This in turn requires a more careful treatment of binding effects and general off-shell effects.

Two different approaches have been used in the study of the scattering of pions from nuclei. A large number of the published works in this area rely on the nonrelativistic multiple-scattering theory of Watson, or of Kerman, McManus, and Thaler (KMT).[1] These treatments are very similar to those employed for nucleon-nucleus scattering except that πN scattering amplitude is used in place of the NN amplitude. Various *ad hoc* schemes have been adopted to introduce relativistic kinematics. We shall see that such an empirical conbination of nonrelativistic dynamics and relativistic kinematics is a source of confusion and ambiguity in dealing with the off-shell effects inherent in the analysis of pion-nucleus scattering.

Recently, a new covariant multiple-scattering theory has been developed.[2-4] The T matrix in this theory satisfies a three-dimensional covariant linear integral equation. (This equation can be extended to incorporate the desired crossing symmetry properties.[5] The resulting nonlinear equation for the T matrix is a generalization of the Low equation to the case of complex nuclear targets.) Here the use of relativistic kinematics is integral. Also, binding effects, off-shell effects, and the Fermi motion of the target nucleons may all be treated properly.

In this work, we compare a covariant pion-nucleus optical potential discussed previously[6] with the noncovariant optical potentials used in literature. For this purpose, we introduce into our covariant analysis the fixed scatterer approximation (FSA), an approximation which is not a prerequisite to the analysis but is convenient for the purpose of comparing our methods with various nonrelativistic schemes.

In Sec. II, we discuss the basic kinematical problems which appear in the construction of an optical potential for pion-nucleus scattering, and note the general defects of the noncovariant theories. The principal features of the covariant optical potential are then discussed in Sec. III. We shall see that, in addition to the usual Feynman diagrammatic analysis, the use of a special definition of a spinor for an off-mass-shell nucleon[7] is useful in eliminating the ambiguities in the kinematical transformations. One is able to construct a covariant optical potential which contains only true dynamical off-shell effects. Some numerical values for the calculated covariant optical potential are presented in Sec. IV, where pion-carbon elastic scattering cross sections are presented and compared with experimental data. Finally, we summarize some of our conclusions in Sec. V. (The reader who is not interested in the critique of the noncovariant kinematical schemes is advised to skip Sec. II.)

II. REVIEW OF NONCOVARIANT METHODS

It is well known that the basic pion-nucleon scattering amplitude appearing in any multiple scattering theory is "off-shell." The off-shell (πN) t matrix is often written in noncovariant theories as[8,9] (see Fig. 1):

$$\langle \vec{k}, \vec{p}_0 - \vec{q} | t(\omega_0) | \vec{k}, \vec{p}_0 \rangle = \gamma \langle \vec{k}_c' | t(\tilde{\omega}_0) | \vec{k}_c \rangle . \qquad (2.1)$$

Here, $\vec{k}'$ and $\vec{k}$ represent the pion momenta in the pion-*nucleus* c.m. frame, while $\vec{k}_c'$ and $\vec{k}_c$ refer to the relative momenta in the pion-*nucleon* (πN) c.m. frame. Also, $\vec{p}_0$ and $\vec{p}_0 - \vec{q}$ denote the nucleon momenta, and $\vec{q}$ is the momentum transfer. The coefficient γ is a factor related to the transformation of the πN amplitude from one frame to the other. A slightly different expression has also been used[10] to relate the πN scattering amplitude f (instead of the t matrix) in different frames:

$$f_{\rm lab}(\vec{k}', \vec{k}) = G(y) f_{{\rm c.m.}(\pi N)}(\vec{k}_c', \vec{k}_c), \qquad (2.2)$$

with

$$G^2(y) \equiv G^2(\cos\bar{\theta}_\pi) \equiv \frac{d(\cos\bar{\theta}_\pi)}{d(\cos\theta_{\rm lab})} \qquad (2.3)$$

representing the Jacobian of the angle transformation.

The necessity of introducing Eq. (2.1) or Eq. (2.2) arises from the fact that although the πN amplitude needed in constructing the π-nucleus optical potential is in the center-of-mass (c.m.) frame of the π-nucleus system, the phenomenological (either on-shell or off-shell) πN partial wave amplitudes are

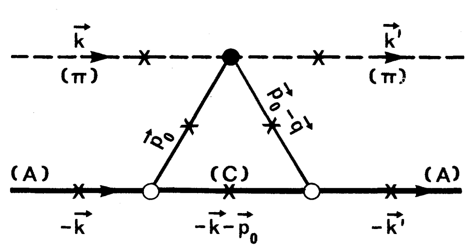

FIG. 1. The single-scattering diagram considered in noncovariant scattering theories. The dashed, the light, and the heavy lines represent, respectively, the pion, the nucleon, and various nuclei. The πN off-shell scattering amplitude $\langle \vec{k}', \vec{p}_0 - \vec{q} | t(\omega_0) | \vec{k}, \vec{p}_0 \rangle$ is denoted by the filled circle, while the nuclear vertex interactions are represented by the open circles. A cross denotes a particle which has been placed on its mass shell. Note that this figure is a schematic representation of the noncovariant calculation and is not a Feynman diagram.

always parametrized and computed in the c.m. frame of πN system. For instance, in a separable potential model we have,[11] with $\cos\bar{\theta}_\pi = \hat{k}_c' \cdot \hat{k}_c$,

$$\langle \vec{k}_c' | t(\tilde{\omega}_0) | \vec{k}_c \rangle \propto \sum_{l,T,j} \lambda^l_{2T,2j} \frac{v^l_{2T,2j}(|\vec{k}_c'|) v^l_{2T,2j}(|\vec{k}_c|)}{D^l_{2T,2j}(\tilde{\omega}_0)} P_l(\cos\bar{\theta}_\pi) \qquad (2.4)$$

as the specification in the πN center-of-mass frame.

We remark that in most applications the parameters, ω_0 and $\tilde{\omega}_0$ of Eq. (2.1) are somewhat ambiguous. However, in the case of *potential* models of the fundamental interaction, supplemented by nonrelativistic kinematics, it is possible to determine values for ω_0 and $\tilde{\omega}_0$ from a detailed many-body theory.[12] In that case, questions of transformation from one frame to another are readily resolved using standard *Galilean* transformations.

If we use the transformation of Eq. (2.1), it is unclear exactly which relativistic dynamical equation is implied in an *arbitrary* frame. Various authors have attempted to resolve these difficulties by using a potential theory which satisfies the requirements of Lorentz invariance[13]; however, this approach has not been applied to the study of pion-nucleus interactions.

In our work we have attempted to resolve the questions of the relativistic nature of the pionic motion by using a manifestly covariant diagrammatic technique. In this scheme we have the requirement of four-momentum conservation at each vertex as well as the necessity of considering the dynamics of *off-mass-shell* particles. Rather than being a disadvantage, the introduction of *off-mass-shell* considerations greatly simplifies the kinematical transformations. In the following we will attempt to point out some of the problems which arise if one insists on treating only the kinematics of on-mass-shell particles. (For the sake of clarity, we remark that in our covariant theory we will speak of off-*mass*-shell amplitudes; however, in the discussion of T matrices obtained from *three*-dimensional equations we will speak of on-*energy*-shell and off-*energy*-shell amplitudes, as is customary.)

As an example, we consider the case of a pion of momentum $\vec{k}_0$ incident on a nucleon of momentum $-\vec{k}_0/A$, as is appropriate to a "fixed scatterer" description of the target.[8] In keeping with the standard approach found in the literature, both the pion and the nucleon are placed on their mass shells. Then ω_0 [see Eq. (2.1)] is taken to be

$$\omega_0 = E_{\pi,\vec{k}_0} + E_{N,\vec{k}_0/A}. \qquad (2.5)$$

If p_π and p_N are the four-momenta of the pion and nucleon, one has

$$\tilde{\omega}_0^2 \equiv \tilde{s} = (p_\pi + p_N)^2 = (E_{\pi,\vec{\kappa}_0} + E_{N,\vec{\kappa}_0})^2 = (E_{\pi,\vec{k}_0} + E_{N,\vec{k}_0/A})^2 - |\vec{k}_0|^2(1-1/A)^2. \quad (2.6)$$

In Eq. (2.6) it has been assumed that $(p_\pi + p_N)^2$ is a Lorentz invariant and this quantity is then evaluated in the (πN) c.m. frame and in the π-nucleus c.m. frame.

Further, in order to use Eq. (2.1) it is necessary to specify γ. Now, for the on-energy-shell T matrices γ is given by

$$\gamma = E_{\pi,\vec{\kappa}_0} E_{N,\vec{\kappa}_0} / (E_{\pi,\vec{k}_0} E_{N,\vec{k}_0/A}). \quad (2.7)$$

Equation (2.7) is then generalized in the off-energy-shell case to[8]

$$\gamma = \left(\frac{E_{\pi,\vec{k}_c} E_{\pi,\vec{k}'_c} E_{N,\vec{k}_c} E_{N,\vec{k}'_c}}{E_{\pi,\vec{k}} E_{\pi,\vec{k}'} E_{N,\vec{k}/A} E_{N,\vec{k}'/A}} \right)^{1/2}. \quad (2.8)$$

In this scheme the magnitude of the relative momentum $\vec{k}_c$ is determined by the relation

$$(E_{\pi,\vec{k}_c} + E_{N,\vec{k}_c})^2 = (E_{\pi,\vec{k}} + E_{N,\vec{k}/A})^2 - |\vec{k}|^2(1-1/A)^2, \quad (2.9)$$

which is a general version of Eq. (2.6). The relation between the cosines of pion scattering angles, $\cos\theta_\pi \equiv \hat{k}' \cdot \hat{k}$ and $\cos\bar{\theta}_\pi \equiv \hat{k}'_c \cdot \hat{k}_c$, is obtained in this model from the assumed invariance of $t = (p_\pi - p'_\pi)^2$ [or of $(p_\pi \cdot p'_\pi)$]:

$$\cos\bar{\theta}_\pi = E_{\pi,\vec{k}_c} E_{\pi,\vec{k}'_c} - E_{\pi,\vec{k}} E_{\pi,\vec{k}'} + kk' \cos\theta_\pi / k'_c k_c. \quad (2.10)$$

We may criticize the calculational scheme described above on several grounds:

(1) The energy parameter $\tilde{\omega}_0$ is determined by Eq. (2.6) once and for all. It is then used as the argument of the D function in the separable model of Eq. (2.4). This approximation neglects the fact that the energy available should be calculated separately for each elementary collision. This fact is of particular significance for the πN system where the denominator D is a strong function of energy.

(2) In the original application[8] of the FSA described here, the T matrix which appears in the theory does not conserve three-momentum or energy (except in the case of forward scattering). For example, if the momentum transfer is $\vec{q}$, this symmetric FSA analysis implies the use of kinematical transformations based on T matrices of the form $\langle \vec{k}+\vec{q}, -(\vec{k}+\vec{q})/A | t(\omega_0) | \vec{k}, -\vec{k}/A \rangle$. In this case the implied violation of momentum and energy conservation is clearly incompatible with the use of Lorentz invariance to specify the kinematical transformations.

In a later work,[14] a new scheme was devised to remedy the violation of three-momentum conservation in the above version of the FSA (version 1). In this new version[14] of the FSA (version 2), one evaluates the πN scattering amplitude Eq. (2.1) at values of nucleon momentum chosen as follows:

$$\vec{p}_0 = -\vec{k}/A + \vec{q}(A-1)/A = [(A-1)\vec{k}' - (A+1)\vec{k}]/2A \quad (2.11)$$

and

$$\vec{p}_0 - \vec{q} = [(A-1)\vec{k} - (A+1)\vec{k}']/2A. \quad (2.12)$$

Furthermore, Eqs. (2.5), (2.6), (2.8), and (2.9) are respectively replaced by

$$\omega_0 = E_{\pi,\vec{k}_0} + E_{N,\vec{p}_0} = E_{\pi,\vec{k}_0} + E_{N,\vec{p}_0 - \vec{q}}, \quad (2.13)$$

$$\tilde{\omega}_0^2 = (E_{\pi,\vec{\kappa}_0} + E_{N,\vec{\kappa}_0})^2 = (E_{\pi,\vec{k}_0} + E_{N,\vec{p}_0})^2 - (\vec{k}_0 + \vec{p}_0)^2, \quad (2.14)$$

$$\gamma = \left(\frac{E_{\pi,\vec{k}_c} E_{\pi,\vec{k}'_c} E_{N,\vec{k}_c} E_{N,\vec{k}'_c}}{E_{\pi,\vec{k}} E_{\pi,\vec{k}'} E_{N,\vec{p}_0} E_{N,\vec{p}_0 - \vec{q}}} \right)^{1/2}, \quad (2.15)$$

and

$$(E_{\pi,\vec{k}_c} + E_{N,\vec{k}_c})^2 = (E_{\pi,\vec{k}} + E_{N,\vec{p}_0})^2 - (\vec{k} + \vec{p}_0)^2. \quad (2.16)$$

Similarly, the magnitude of the relative momentum $\vec{k}'_c$ is determined by

$$(E_{\pi,\vec{k}'_c} + E_{N,\vec{k}'_c})^2 = (E_{\pi,\vec{k}'} + E_{N,\vec{p}_0 - \vec{q}})^2 - (\vec{k}' + \vec{p}_0 - \vec{q})^2. \quad (2.17)$$

The angle transformation is still given by Eq. (2.10) with the center-of-mass momenta calculated according to Eqs. (2.16) and (2.17). The choice of $\vec{p}_0$ and $\vec{p}_0 - \vec{q}$ in this scheme depends, therefore, on the scattering angle. Although such a choice restores four-momentum conservation for the πN scattering whenever $|\vec{k}_c| = |\vec{k}'_c|$, it does not improve the calculational scheme from a fundamental point of view. In particular, we observe that:

(1) As in the previous scheme, the energy parameter $\tilde{\omega}_0$ is again determined once and for all by Eq. (2.14), and is independent of the actual πN collision parameters.

(2) For the off-shell pion-nucleus scattering ($|\vec{k}| \neq |\vec{k}'|$), energy is not conserved in the elementary πN amplitude. Therefore, this scheme again becomes incompatible with the use of Lorentz invariance to specify the angle transformation.

(3) This scheme will lead to a pion-nucleus optical potential $\langle \vec{k}' | V | \vec{k} \rangle$, which is not symmetric under the interchange of $\vec{k}$ and $\vec{k}'$ when $|\vec{k}| \neq |\vec{k}'|$. This happens because the energy parameters ω_0 and $\tilde{\omega}_0$ are noninvariant under this interchange. There is therefore a violation of the general symmetry prop-

erty that an optical potential must possess.

Since there is no clear way of avoiding the asymmetry mentioned above, we will not make *full* numerical analysis of this scheme here and only restrict our discussion to the implied angle transformations.

Various other kinematical transformations were proposed in the literature.[10,15] These transformations were based on Lorentz invariance and an on-mass-shell description for the particles. (We should note that the work of Phatak[15] resolves some of the problems associated with the transformation originally proposed in Ref. 8.)

Of the published works, we analyze one typical FSA scheme[10] (version 3). On the whole, this scheme follows version 1 described above except that the angle transformation is not calculated with Eq. (2.10). To avoid the occurence of unphysical values of $\cos\bar{\theta}_\pi$, one uses the relation[10]

$$\cos\theta_\pi^{\rm lab} = \frac{E_{\pi,\vec{\kappa}_0} + E_{N,\vec{\kappa}_0}\cos\bar{\theta}_\pi}{(E_{\pi,\vec{\kappa}_0}{}^2 + M_N{}^2 + 2E_{\pi,\vec{\kappa}_0}E_{N,\vec{\kappa}_0}\cos\bar{\theta}_\pi + \vec{\kappa}_0{}^2\cos^2\bar{\theta}_\pi)^{1/2}}\,, \tag{2.18}$$

where $\vec{k}_0$ is given by Eq. (2.6). The quantity $\cos\theta_\pi^{\rm lab}$ can be further related to $\cos\theta_\pi$ in the c.m. frame of the pion-nucleus system by an on-shell Lorentz transformation. Equation (2.18) guarantees physical values of $\cos\bar{\theta}_\pi$; however, it is a fixed relation for the scattering of a on-mass-shell pion incident upon a nucleon *at rest* in the laboratory frame. Thus it can not be generalized to off-shell scattering processes, nor can the subsequent change in the magnitude of the pion momentum (in the case of finite nucleon recoil) be included. Therefore this scheme is somewhat limited in application.

We conclude this section by emphasizing that the kinematical ambiguities in noncovariant treatments are due to the introduction of relativistic kinematics into nonrelativistic dynamical schemes. We remark that four-momentum conservation for the scattering of on-mass-shell particles implies a fixed scattering angle for a definite magnitude of the final momentum of one of the particles. For the calculation of the optical potential, the magnitude of the pion momentum and angle of scattering are independent variables. Only in a covariant theory involving off-mass-shell particles, such as that discussed in the next section, does the use of relativistic kinematics produce no ambiguity.

III. OFF-MASS-SHELL CALCULATIONS IN THE COVARIANT THEORY

A. Unambiguous kinematical transformation and the covariant optical potential

In this work we apply the methods of Ref. 6 to the calculation of the optical potential. We evaluate the triangle diagram (Fig. 2) using the rules outlined in the appendix of that reference. In this method, each internal particle of four-momentum p_j has its mass defined as $M_j^{*2} \equiv p_j{}^2$. In general, M_j^* is different from the on-shell mass M_j. This is therefore an "off-mass-shell" description of the phenomenon.

It has been shown that the first-order covariant optical potential involves the integration of the product of an invariant off-mass-shell scattering amplitude and an invariant density matrix for the target nucleus,[6]

$$\langle \vec{k}'|\bar{K}_6(W)|\vec{k}\rangle = \sum_{nIL}\int d\vec{Q}\sum_{s''s'}\bar{u}^{s''}(P'-Q)[(2\pi)^3\langle p',P'-Q|M_{\pi N}(s)|p,P-Q\rangle]u^{s'}(P-Q) \times (M_{nI}/E_{nI,\vec{Q}})^{1/2}\rho_{s's''}^{(+)n(I\frac{1}{2})L}(\vec{Q}_R,\vec{Q}_R')(M_{nI}/E_{nI,\vec{Q}})^{1/2}. \tag{3.1}$$

[We refer to Ref. 6 for a detailed discussion of the quantities appearing in Eq. (3.1).] This potential may be used in the three-dimensional covariant integral equation:

$$\langle \vec{k}'|\bar{M}_6(W)|\vec{k}\rangle = \langle \vec{k}'|\bar{K}_6(W)|\vec{k}\rangle + \int d\vec{k}''\langle \vec{k}'|\bar{K}_6(W)|\vec{k}''\rangle\frac{R_6(\vec{k}'')}{2W - E_{\pi,\vec{k}''} - E_{A,\vec{k}''} + i\eta}\langle \vec{k}''|\bar{M}_6(W)|\vec{k}\rangle. \tag{3.2}$$

Here and in Eq. (3.1) the subindices of M, K, and R refer to the specific covariant reduction scheme employed.[3,6] The right-hand side of Eq. (3.1) is a relativistic invariant: the quantity $(M_{nI}/E_{nI,\vec{Q}})d\vec{Q}$ is an invariant volume element, $\rho^{(+)}$ is the invariant nuclear density matrix, and the rest of the expression is the invariant off-shell pion-nucleon scattering amplitude. Denoting this invariant πN amplitude by $\mathfrak{F}_{fi}^{s''s'}$, with the indices f and i referring to the isospin states of the πN system, we find[6]

$$\mathfrak{F}_{fi}^{s''s'} = \bar{u}^{s''}(P'-Q)[(2\pi)^3\langle p'P'-Q|M_{\pi N}(s)|p,P-Q\rangle]u^{s'}(P-Q) \tag{3.3}$$

$$= -(2\pi)^{-3}\bar{u}^{s''}(P'-Q)\{[A^{(+)}+\tfrac{1}{2}B^{(+)}\gamma\cdot(p'+p)]+(\vec{I}\cdot\vec{\tau})_{fi}[A^{(-)}+\tfrac{1}{2}B^{(-)}\gamma\cdot(p'+p)]\}u^{s'}(P-Q), \tag{3.4}$$

where (see Fig. 2)

$$A^{(+)}=\tfrac{1}{3}\{2A_{T=3/2}[s,t,p^2,p'^2,(P-Q)^2,(P'-Q)^2]+A_{T=1/2}[s,t,p^2,p'^2,(P-Q)^2,(P'-Q)^2]\}, \tag{3.5}$$

$$A^{(-)}=\tfrac{1}{3}\{A_{T=3/2}[s,t,p^2,p'^2,(P-Q)^2,(P'-Q)^2]-A_{T=1/2}[s,t,p^2,p'^2,(P-Q)^2,(P'-Q)^2]\}, \tag{3.6}$$

$$s=(p+P-Q)^2=(p'+P'-Q)^2, \tag{3.7}$$

and

$$t=(p-p')^2. \tag{3.8}$$

There are analogous relations for $B^{(+)}$ and $B^{(-)}$. A novel feature of Eqs. (3.3) and (3.4) is the appearance of off-mass-shell nucleon spinors[6] $\bar{u}^{s''}(P'-Q)$ and $u^{s'}(P-Q)$. Here $\bar{u}^{s''}(P'-Q)$ is a function of $\vec{P}'-\vec{Q}$, $P'^0-Q^0=\{(\vec{P}'-\vec{Q})^2+M'^{*2}_N\}^{1/2}$, and M'^*_N, while $u^{s'}(P-Q)$ is a function of $\vec{P}-\vec{Q}$, $P^0-Q^0=\{(\vec{P}-\vec{Q})^2+M^{*2}_N\}^{1/2}$, and M^*_N. The quantities M'^*_N and M^*_N can be termed the "off-shell-masses," and are ultimately obtained from the application of four-momentum conservation for each diagrammatic element. Together with $\bar{v}^{s''}(P'-Q)$ and $v^{s'}(P-Q)$, these off-shell spinors have the same completeness and orthogonality relations as the conventional (on-shell) spinors, which we write as $\bar{u}^{s''}(\vec{P}'-\vec{Q})$, $u^{s'}(\vec{P}-\vec{Q})$, $\bar{v}^{s''}(\vec{P}'-\vec{Q})$, and $v^{s'}(\vec{P}-\vec{Q})$. We refer to our previous work for detailed discussions.[6] It is the introduction of these off-mass-shell nucleon spinors that enables us to consider Eq. (3.4) to be the most general Lorentz-invariant off-shell πN amplitude. Also, the invariant amplitudes $A^{(\pm)}$, defined by Eqs. (3.5) and (3.6), and $B^{(\pm)}$ depend on more invariants than the on-shell invariant amplitudes used to describe free πN scattering. (The relation between our general off-mass-shell πN amplitude and various phenomenological off-shell πN amplitudes was discussed in Ref. 6.)

The great advantage of using Eq. (3.4) is its invariance with respect to frame transformation. In the c.m. frame of the πN system, we obtain simply

$$\mathfrak{F}^{s''s'}_{fi}=\frac{-\sqrt{s}}{2\pi^2[M^*_N M'^*_N]^{1/2}}\chi^\dagger_{s''}\{[F_1^{(+)}+F_2^{(+)}(\vec{\sigma}\cdot\hat{k}'_c)(\vec{\sigma}\cdot\hat{k}_c)]\delta_{fi}+(\vec{I}\cdot\vec{\tau})_{fi}[F_1^{(-)}+F_2^{(-)}(\vec{\sigma}\cdot\hat{k}'_c)(\vec{\sigma}\cdot\hat{k}_c)]\}\chi_{s'}. \tag{3.9}$$

In Eq. (3.9), F_1 and F_2 are functionals of $A^{(\pm)}$ and $B^{(\pm)}$, and $\hat{k}'_c$ and $\hat{k}_c$ are the unit vectors along the relative momenta.[6] The product $\hat{k}'_c\cdot\hat{k}_c$ contained in this equation is related to the scalar product of the pion momenta in π-*nucleus* c.m. frame by

$$\hat{k}'_c\cdot\hat{k}_c=(k^0_c k'^0_c-p^0p'^0+\vec{k}\cdot\vec{k}')/|\vec{k}'_c||\vec{k}_c| \tag{3.10}$$

with p^0 and p'^0 being the zeroth components of the pion momenta,

$$k^0_c=\tfrac{1}{2}\sqrt{s}+[p^2-(P-Q)^2]/2\sqrt{s}, \tag{3.11}$$

$$k'^0_c=\tfrac{1}{2}\sqrt{s}+[p'^2-(P'-Q)^2]/2\sqrt{s}, \tag{3.12}$$

$$\vec{k}_c^{\,2}=\{[p\cdot(P-Q)]^2-p^2(P-Q)^2\}/s \tag{3.13}$$

$$=\{s^2-2s[p^2+(P-Q)^2]+[p^2-(P-Q)^2]^2\}/4s, \tag{3.14}$$

and

$$\vec{k}'^{2}_c=\{[p'\cdot(P'-Q)]^2-p'^2(P'-Q)^2\}/s \tag{3.15}$$

$$=\{s^2-2s[p'^2+(P'-Q)^2]+[p'^2-(P'-Q)^2]^2\}/4s. \tag{3.16}$$

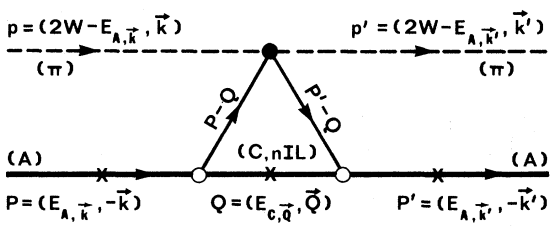

FIG. 2. The single-scattering diagram considered in the convariant scattering theory. This is a true Feynman diagram with the lines having the same significance as in Fig. 1. The letters (nIL) denote the unoccupied orbit in the nucleus C. The black circle denotes an invariant scattering amplitude and the open circles are vertices. Only the heavy nuclei are placed on their mass shells.

Equation (3.10) is the covariant angle transformation we have proposed. This transformation is obtained from calculating the invariant scalar product $(p'_\pi\cdot p_\pi)$ in both the πN c.m. frame and the π-nucleus c.m. frame. In contrast to Eq. (2.10), the off-shell masses of the pion, p^2, and p'^2, and of the nucleon, $(P-Q)^2$, and $(P'-Q)^2$, enter into the calculation of Eq. (3.10) via Eqs. (3.11)–(3.16). In this case we have therefore guaranteed that the angle cosines given by $\hat{k}'_c\cdot\hat{k}_c$ and $\hat{k}'\cdot\hat{k}$ always remain between −1 and +1. We may also note that the scalars $(P-Q)^2$ and $(P'-Q)^2$ are determined at the nuclear vertices, where their values are influenced by the binding of the target nucleon.

B. Covariant fixed scatterer approximation

Since the noncovariant analyses were all based upon the FSA, it is useful to introduce a *covariant* FSA for the purpose of comparing the covariant and noncovariant schemes. The covariant FSA we propose can be introduced by the following prescriptions. Replace s in Eq. (3.1) by $\bar{s}$ of Eq. (2.6) and factorize the πN amplitude from the $d\vec{Q}$ integration to obtain

$$\langle \vec{k}' | \bar{K}_6(W) | \vec{k} \rangle \simeq \sum_{nILs''s'} \bar{u}^{s''}(P'-\tilde{Q})[(2\pi)^3 \langle p', P'-\tilde{Q} | M_{\pi N}(\bar{s}) | p, P-Q \rangle] u^{s'}(P-\tilde{Q}) \rho_{s's''}^{(+)\eta(I\frac{1}{2})L}(\vec{k}-\vec{k}') \tag{3.17}$$

with $\eta = (A-1)/A$ and (see Fig. 2), and

$$P-\tilde{Q} \equiv (E_{A,\vec{k}} - E_{C,\eta\vec{k}}, -\vec{k}/A), \tag{3.18}$$

$$P'-\tilde{Q} \equiv (E_{A,\vec{k}'} - E_{C,\eta\vec{k}'}, \eta\vec{k} - \vec{k}'). \tag{3.19}$$

Note that $(P-\tilde{Q})^2 \equiv M_N^{*2} = (M_N - \Delta_{nIL})^2$ and $(P'-\tilde{Q})^2 \equiv M_N'^{*2}$, with Δ_{nIL} representing the binding energy of the nucleon in the orbit (nIL). Also we have, in general, $p^2 \neq M_\pi^2$ and $p'^2 \neq M_\pi^2$. Equations (3.18) and (3.19) correspond to the situation where before the scattering event, the nucleon is at rest in the rest frame of the target nucleus and after scattering the nucleon is moving in the rest frame of the target.

The πN amplitude in Eq. (3.17) depends on $v_{2T,2j}^{l}(|\vec{k}_c'|) v_{2T,2j}^{l}(|\vec{k}_c|)/D_{2T,2j}^{l}(\bar{s})$. The values of $\vec{k}_c'^2$, $\vec{k}_c^2$ and $\vec{k}_c' \cdot \vec{k}_c$ are calculated in the covariant FSA according to Eqs. (3.10)–(3.14) with $P-\tilde{Q}$, $P'-\tilde{Q}$, and s (*not* $\bar{s}$). Note that we only use the approximation $s \to \bar{s}$ in the evaluation of D.

The covariant optical potential in the FSA [Eq. (3.17)], calculated by using Eqs. (3.18) and (3.19), does not have the desired off-shell symmetry property with respect to the interchange of $\vec{k}$ and $\vec{k}'$. The geometry connected with this FSA is not symmetric; namely, the initial nucleon is at rest and the final nucleon is moving in the target nucleus. We *symmetrize the result* by averaging the result obtained in the scheme characterized by Eqs. (3.18) and (3.19) and that obtained with the specifications:

$$P-\tilde{Q} \equiv (E_{A,\vec{k}} - E_{C,\eta\vec{k}'}, \eta\vec{k}' - \vec{k}), \tag{3.20}$$

$$P'-\tilde{Q} \equiv (E_{A,\vec{k}'} - E_{C,\eta\vec{k}'}, -\vec{k}'/A). \tag{3.21}$$

Equations (3.20) and (3.21) correspond to the situation where a nucleon moving in the initial nucleus has its momentum changed so that it is at rest in the *final* nucleus. The averaging procedure is illustrated in Fig. 3. The resulting optical potential now possesses the desired symmetry property.

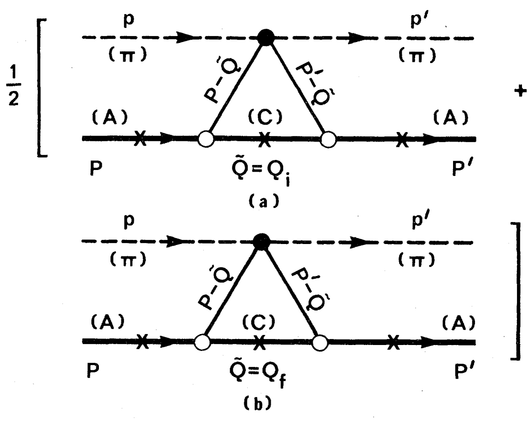

FIG. 3. A diagram representing the covariant fixed scatterer approximation. The elements have the same significance as in Fig. 2. The four-momenta Q_i and Q_f are fixed at $(E_{C,\eta\vec{k}}, -\eta\vec{k})$ and $(E_{C,\eta\vec{k}'}, -\eta\vec{k}')$ as discussed in the text. These two diagrams are averaged to obtain a symmetric optical potential for the covariant FSA.

IV. COMPARISON OF COVARIANT AND NONCOVARIANT CALCULATIONS

A. Comparison of angle transformations

Owing to the fundamental ambiguities existing in the use of relativistic kinematics within the frame work of a nonrelativistic dynamical calculation, the number of *ad hoc* transformation schemes which may be conceived is virtually unlimited. We choose to compare in Fig. 4 the covariant angle transformation defined by Eq. (3.10) with the noncovariant schemes used in Refs. 8, 10, and 14.

For on-shell pion-nucleus scattering [Fig. 4(a)] the covariant result (solid curve) is hardly distinguishable from the result of the procedure advocated in Refs. 10 and 14. (The latter results are not shown graphically.) However, the covariant result does differ from that of Ref. 8 (dot-dashed curve).

In the case of off-shell pion-nucleus scattering [Fig. 4(b)], the transformation defined in Ref. 10 (dashed curve) remains unchanged with respect to the on-shell calculation, [Fig. 4(a)], as discussed in Sec. II, that is, the dashed curve in Fig. 4(b) coincides with the solid curve in Fig. 4(a). It may be seen that the curves resulting from the covariant theory change their position but stay entirely in the physical region; the curves based upon Refs. 8 and 14 move out of the physical region. The difference between the four schemes is most marked in the off-shell case.

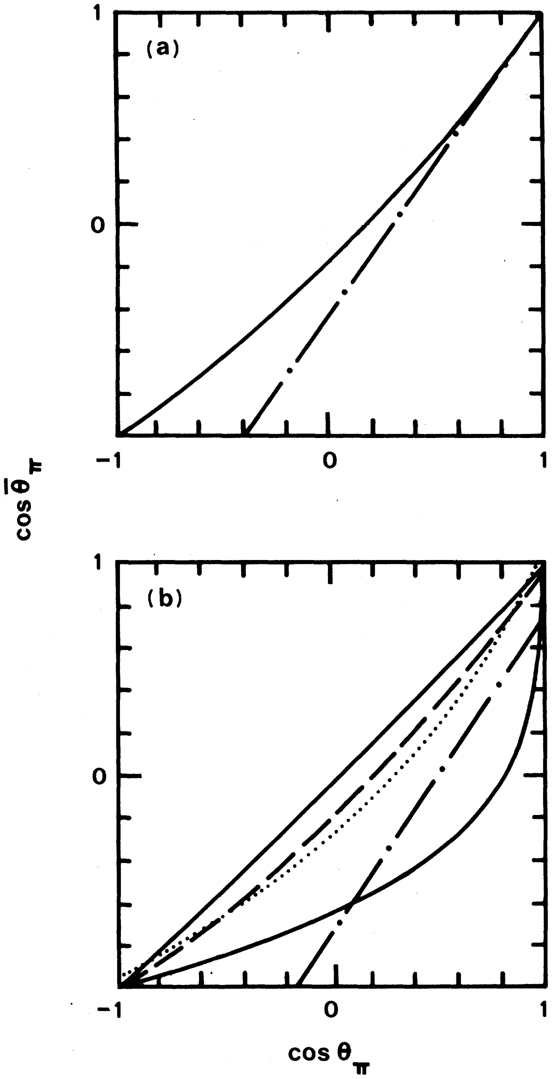

FIG. 4. Relation between the cosine of the pion scattering angle in the c.m. frame of the π-^{12}C system, θ_π, and the pion scattering angle in the c.m. frame of the πN system $\bar{\theta}_\pi$, for $T_\pi^{\rm lab}=80$ MeV. (a) On-shell scattering $k=k'=k_0=166$ MeV/c. (b) Off-shell scattering with $k_0=166$ MeV/c, $k=333$ MeV/c, and $k'=72$ MeV/c. The solid, dot-dashed, dotted, and dashed curves represent, respectively, the results given by the covariant calculation, and versions 1, 2, and 3 of the noncovariant FSA. The upper solid curve corresponds to having the target nucleon at rest initially, while the lower solid curve corresponds to having this nucleon at rest in the target after the collision.

B. Comparison of Born amplitudes (off-mass-shell effect)

The difference between the "off-mass-shell" description we have used in the covariant analysis and the "off-energy-shell" description used in nonrelativistic approaches is already marked in the Born term of the multiple scattering series. (Note that in the Born term, the pions are on their mass shells).

From Eq. (2.9) or Eq. (2.16) we see that $|\vec{k}_c|=|\vec{k}_c'|$ in the schemes of Refs. 8, 10, and 14. It is worth pointing out while the value of $|\vec{k}_c|$ ($=|\vec{k}_c'|$) is completely determined by the incoming (beam) pion momentum $|\vec{k}|$, once and for all, in the schemes of Refs. 8 and 10, the value of $|\vec{k}_c|$ ($=|\vec{k}_c'|$) depends on the pion-nucleus scattering angle in Ref. 14.

In contrast, we may use Eqs. (3.14) and (3.16) to determine the values of $|\vec{k}_c|$ and $|\vec{k}_c'|$ in the covariant theory. These values are different, that is $|\vec{k}_c|\neq|\vec{k}_c'|$, (except in the case of forward scattering) as may be seen from the following considerations. Owing to the conservation of four-momentum in the πN scattering process (in the covariant approach) the off-shell mass of the struck nucleon changes after the collision with the pion. We have, therefore, $M_N'^*\neq M_N^*$; that is, $(P'-Q)^2\neq(P-Q)^2$, for nonforward scattering. From Eqs. (3.14) and (3.16) we therefore find $|\vec{k}_c|\neq|\vec{k}_c'|$ in general (see Fig. 5)—in marked contrast to the situation that obtains in the noncovariant FSA.

Also in the covariant FSA, the struck nucleon, which is at rest in the target, has an off-shell mass $M_N^*=M_N-\Delta$, where Δ represents the effect of nuclear binding. This mass difference is illustrated in Fig. 5, where the value of $M_N'^*$ as a function of the angle of scattering of the pion is also exhibited.

In summary, with regard to $M_N'^*$ and $|\vec{k}_c'|$, we note that they differ from the corresponding noncovariant (constant) FSA values most dramatically for large scattering angles (Fig. 5). These large differences are not as important as might appear from

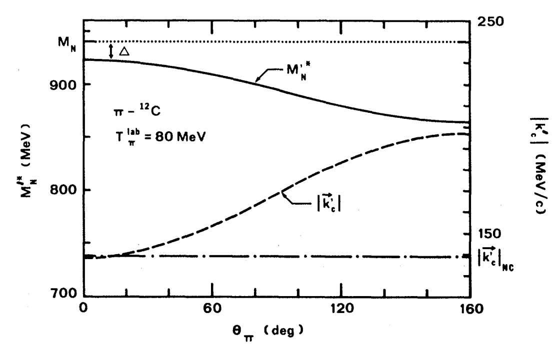

FIG. 5. Off-shell effects influencing the calculation of the Born term of the covariant theory. The solid and dashed curves, respectively, show the off-shell mass, $M_N'^*$, and the final relative momentum in the πN c.m. frame, $|\vec{k}_c'|$, as functions of the pion scattering angle, θ_π, in the π-nucleus c.m. frame. The calculation made corresponds to the physical situation of Fig. 3(a) and is for pion-carbon elastic scattering at $T_\pi^{\rm lab}=80$ MeV. The dotted and dot-dashed lines are the nucleon mass M_N and the relative momentum $|\vec{k}_c|$ used in the noncovariant FSA versions 1 and 3.

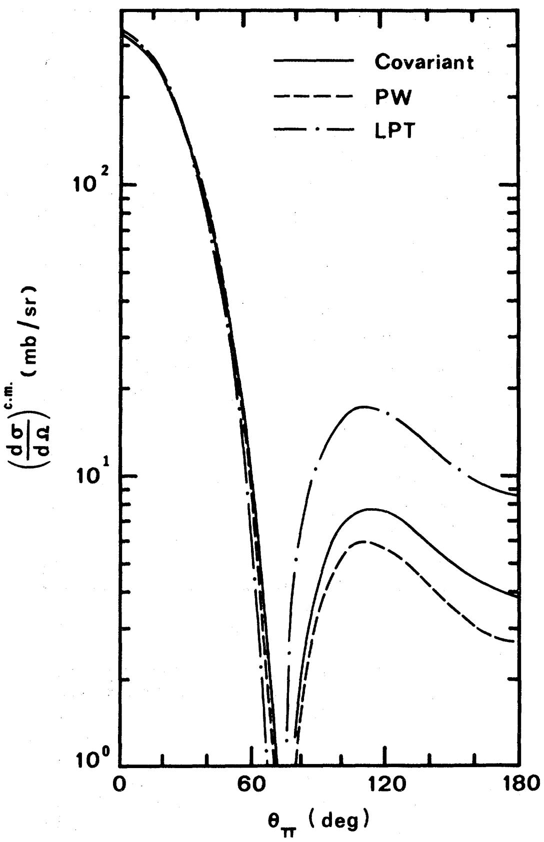

FIG. 6. *Born* cross sections for π-^{12}C scattering at T_π^{lab}= 80 MeV. The solid, dashed and dot-dashed curves are, respectively, the covariant, (Ref. 6) LPT (Ref. 8), and PW (Ref. 10) schemes. In all the calculations the *same* off-shell πN interaction, taken from Ref. 11, was used.

the study of Fig. 5, since the nuclear form factor is a rapidly decreasing function of momentum transfer.

The Born cross sections for the covariant and two noncovariant schemes are compared in Fig. 6. In each case the πN interaction of Londergan, McVoy, and Moniz[11] was used. For scattering angles less than 20°, the results of the Landau, Phatak, and Tabakin (LPT) (dot-dashed curve) and the Piepho and Walker (PW) schemes (dashed curve) are not distinguishable on the figure and both are slightly higher than the covariant result (solid curve). This difference at forward scattering angles is a reflection of the binding effect taken into consideration in the covariant theory. At large angles, the three curves differ significantly. The unusually large (Born) cross section indicated by the dot-dashed curve can be related to the very unusual angle transformation (see Fig. 4) used in the LPT scheme. The angle transformations of the covariant theory and the PW scheme are very similar in the calculation of the Born term. The difference between the Born amplitudes at large angles obtained in these two approaches is due to the following: For nonforward scattering, the covariant theory gives the result $|\vec{k}_c| \neq |\vec{k}_c'|$, while the PW scheme has $|\vec{k}_c| = |\vec{k}_c'|$. This means that in the calculation of the Born term of the pion-*nucleus* scattering process the elementary πN interaction is off shell in the covariant theory but on shell in the PW scheme. This affects the numerical value of the amplitude.

C. Comparison of calculated potentials and cross section for π-^{12}C scattering

It was shown[6] that Eq. (3.2) could be converted to a covariant three-dimensional equation for the π-nucleus T matrix of the form

$$\langle \vec{k}' | T_6(W) | \vec{k} \rangle = \langle \vec{k}' | V_6(W) | \vec{k} \rangle + \int d\vec{k}'' \frac{\langle \vec{k}' | V_6(W) | \vec{k}'' \rangle \langle \vec{k}'' | T_6(W) | \vec{k} \rangle}{2W - E_{\pi,\vec{k}''} - E_{A,\vec{k}''} + i\eta}, \tag{4.1}$$

where

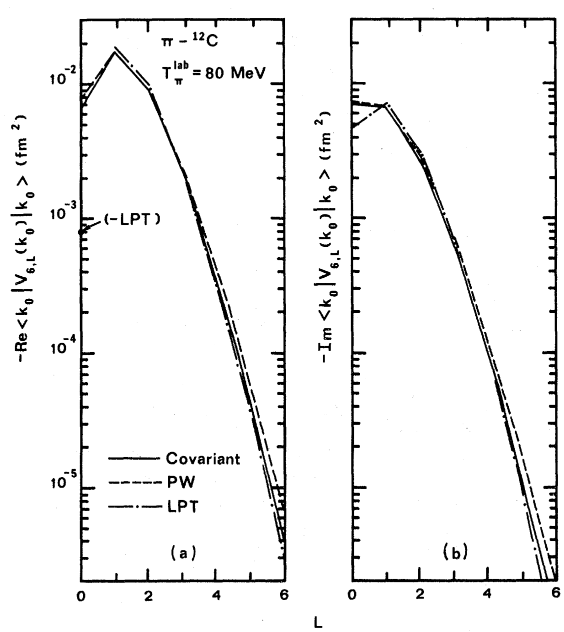

FIG. 7. On-shell values $\langle k_0 | V_{6,L}(W) | k_0 \rangle$ of the optical potential defined in Eq. (4.2). (a) The real part (note that the isolated LPT point has a sign opposite to that shown on the figure). (b) The imaginary part; the calculation is done for π-^{12}C scattering at 80 MeV. Note $2W = (k_0^2 + M_\pi^2)^{1/2} + (k_0^2 + M_A^2)^{1/2}$ and k_0 =166 MeV/c. Also, 1 fm^2 equals 197.3 MeV fm^3.

$$\langle \vec{k}' | V_6(W) | \vec{k}\rangle = R_6^{1/2}(\vec{k}')\langle \vec{k}' | \bar{K}_6(W) | \vec{k}\rangle R_6^{1/2}(\vec{k})$$
$$= 4\pi \sum_{L,M} Y_{LM}(\hat{k}')\langle k' | V_{6,L}(W) | k\rangle Y_{LM}(\hat{k}) \quad (4.2)$$

and

$$R_6(\vec{k}) = (2E_{\pi,\vec{k}}2E_{A,\vec{k}})^{-1}. \quad (4.3)$$

[Equations (4.1) and (4.2) represent approximations to the more general equations obtained from the covariant reduction schemes of Refs. 2 and 6.]

In this section we consider the partial wave decomposition given in Eq. (4.2), and also compare matrix elements of $V(W)$ for various kinematical schemes. These elements are calculated with the interaction of Ref. 11, [Eq. (2.4)]. In these calculations $D(s)$ is replaced by $D(\bar{s})$ and $|\vec{k}_c|^2$, $|\vec{k}_c'|^2$, and $(\hat{k}_c \cdot \hat{k}_c')$ are calculated using the procedures of Refs. 6, 8 and 10. In Fig. 7 we exhibit the real and imaginary parts of the on-shell matrix elements, $\langle k_0 | V_{6,L}(k_0) | k_0\rangle$ for π-^{12}C scattering at 80 MeV. We see that only elements with $L < 5$ are significant. In Figs. 8 and 9 we present some half-off-shell and completely off-shell matrix elements of $\langle k | V_{6,L}(W) | k'\rangle$ for $L = 1$.

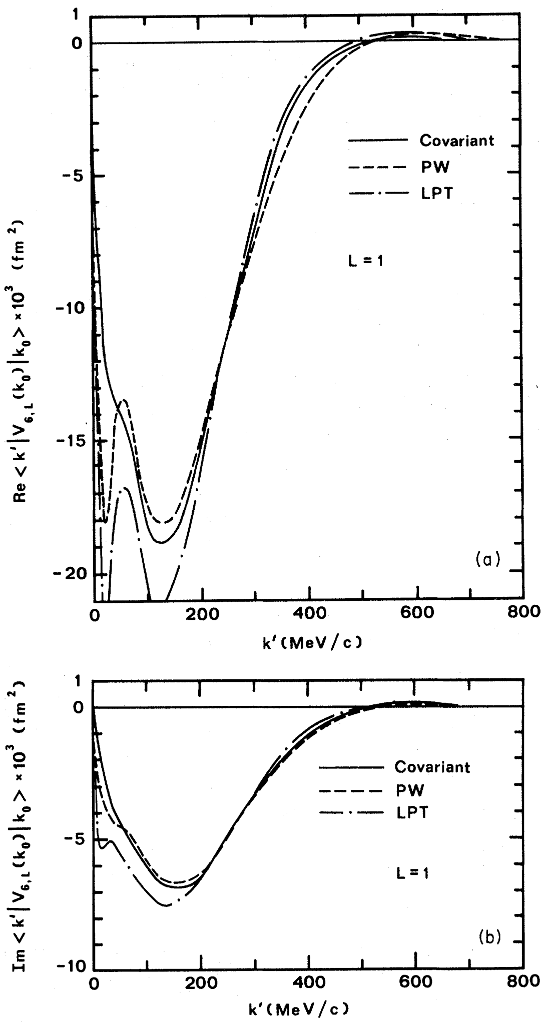

FIG. 8. Values of the half-off-shell optical potential: $\langle k' | V_{6,L}(W) | k_0\rangle$ for $L=1$. See caption of Fig. 7 for values of k_0 and W.

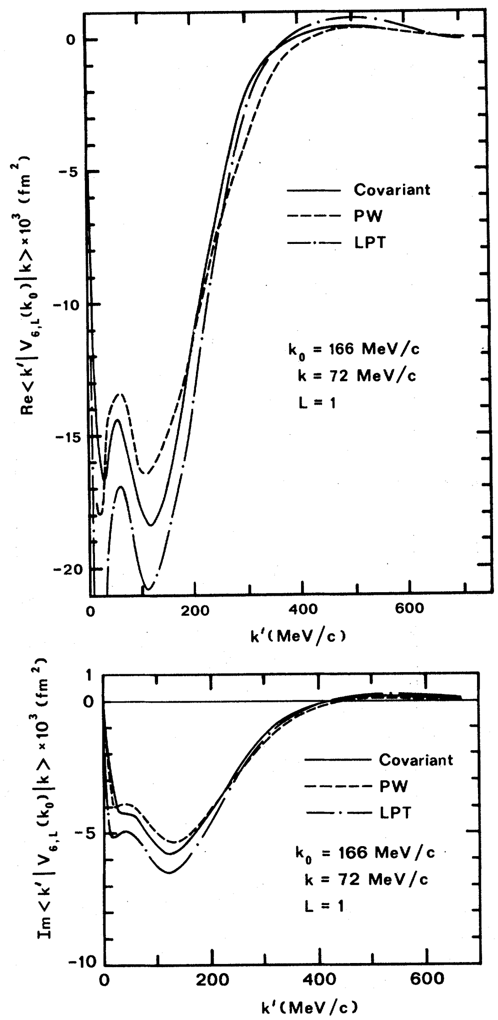

FIG. 9. Values of the fully off-shell optical potentials: $\langle k' | V_{6,L}(W) | k\rangle$ for $L = 1$, $k_0 = 166$ MeV/c, and $k = 72$ MeV/c. See caption of Fig. 7 for the value of W.

Finally, we present the differential cross sections resulting from the solution of Eq. (4.1). The results of the various kinematical schemes are compared with the data[16] in Fig. 10. In the calculations, harmonic-oscillator nuclear wave functions[17] were used to construct the nuclear form factor. Since the same separable model for the πN

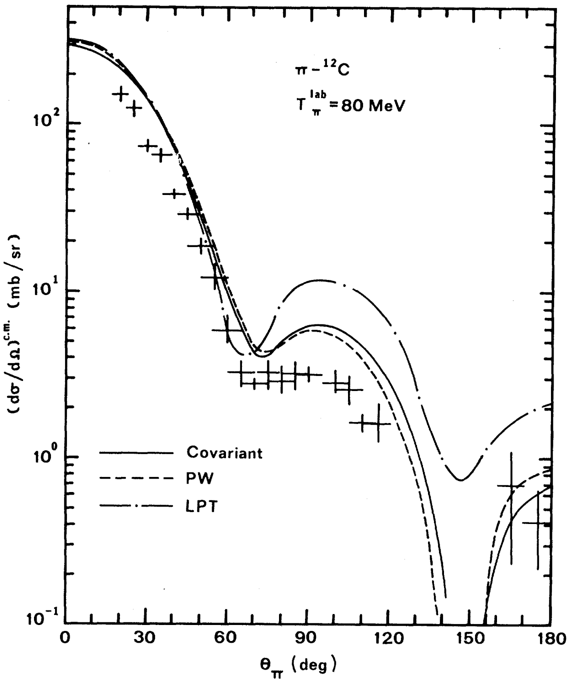

FIG. 10. Differential cross sections for π-^{12}C elastic scattering at $T_\pi^{\rm lab}=80$ MeV. The data are from Ref. 16. The curves represent the results for various *kinematical* schemes as discussed in the text.

interaction[11] was used as input in all three optical potentials, the difference in the differential cross sections provides an estimate of the effect of using various kinematical transformations. It can be seen from Fig. 10 that the LPT scheme yields results which differ significantly from the covariant and PW schemes.

Finally, we wish to emphasize that the FSA is not needed in the application of the covariant theory. The introduction of the covariant FSA in this work is for the sole purpose of comparing the *kinematical* aspects of different theories. (The theories of LPT and PW, in their original application, employ different interactions or different *dynamical* equations from those considered here.) Further, it is clear that the energy dependence of the πN amplitude in the resonance region will invalidate the FSA. In the covariant theory this energy dependence, as well as the Fermi motion of the target nucleons, can be taken into account properly. A detailed numerical analysis including these important features will be reported elsewhere.

V. DISCUSSIONS AND CONCLUSIONS

We have used the FSA to compare the covariant optical potential of Ref. 6 with two noncovariant optical potentials in order to illustrate the effects of using different kinematical schemes. Since the use of relativistic *kinematics* is unambiguous in the covariant theory, we consider the results of the covariant calculations least open to question. It is necessary to emphasize that in the covariant theory the kinematical transformation is determined from first principles, and therefore the covariant optical potential contains only true dynamical off-shell effects. This is clearly a desirable result since it allows one to construct dynamical off-shell models for the invariant πN scattering amplitude without being confused by the presence of kinematical ambiguities. In fact, the so-called "off-shell extensions" discussed for noncovariant theories in the literature often have their origins in the use of different *ad hoc* kinematical transformations. "Off-shell effects" generated in this manner have no relation with true dynamical off-shell considerations.

The dynamical assumptions of the relativistic theory are fixed by the specification of the covariant scattering amplitudes used in the comparison of the different kinematical schemes. In particular, we have used the separable model of Ref. 11 after making use of the approximation that the form factors, $v^{l}_{2T,2J}$, depend only on the single invariants $|\vec{k}_c|^2$ or $|\vec{k}'_c|^2$. As noted in our previous work, the off-mass-shell invariant πN amplitude depends on more variables than the amplitude of the noncovariant theories. Thus, the use of the phenomenological amplitudes of Ref. 11 requires that some approximations be made. These are described in Ref. 6. Further studies of the off-mass-shell amplitudes would be of value in providing further justification of the approximations we have used in obtaining a covariant amplitude via a particular interpretation[6] of the phenomenological amplitudes.

We wish also to emphasize that the use of an invariant πN scattering amplitude involving only spinors for *on-mass-shell* nucleons[18] does not resolve the kinematical ambiguities discussed in this work. This is because the invariant amplitude to be used in multiple scattering theory, as we have pointed out in Sec. II, requires an *off-mass-shell* treatment of the target nucleon. (The physical meaning of the off-shell-masses M_N^* and $M_N'^*$ of the struck nucleon and their evaluation were discussed previously.[6]) The introduction of the spinor for an off-mass-shell nucleon is therefore an essential ingredient in obtaining a convenient and well defined kinematical description.

The calculated differential cross sections obtained from the covariant and noncovariant optical potentials may show important differences at large scattering angles. We note that at these large angles various *ad hoc* kinematical transformations may not be reliable. Although some nonrelativistic theories may provide good fits to the data, we feel that only in the covariant theory can the full kinematical complications of off-shell effects arising from nuclear Fermi motion and nuclear binding receive an adequate theoretical description. The difference between covariant and noncovariant calculations for pion-nucleus elastic scattering should be more marked if we drop the FSA approximation. In particular, the inclusion of the effects of the motion of the target nucleons will cause the πN off-shell amplitudes to be evaluated in kinematical domains different from those of the noncovariant theories.[4] Detailed numerical studies of these novel aspects of our approach will be presented in future work.

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