High-energy expansion for nuclear multiple scattering*

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(Received 27 January 1975)

The Watson multiple scattering series is expanded to develop the Glauber approximation plus systematic corrections arising from three effects: (1) deviations from eikonal propagation between scatterings, (2) Fermi motion of struck nucleons, and (3) the kinematic transformation which relates the many-body scattering operators of the Watson series to the physical two-body scattering amplitude. Operators which express effects ignored at the outset to obtain the Glauber approximation are subsequently reintroduced via perturbation expansions. Hence a particular set of approximations is developed which renders the sum of the Watson series to the Glauber form in the center of mass system, and an expansion is carried out to find leading order corrections to that summation. Although their physical origins are quite distinct, the eikonal, Fermi motion, and kinematic corrections produce strikingly similar contributions to the scattering amplitude. It is shown that there is substantial cancellation between their effects and hence the Glauber approximation is more accurate than the individual approximations used in its derivation. It is shown that the leading corrections produce effects of order $(2kR_c)^{-1}$ relative to the *double scattering* term in the uncorrected Glauber amplitude, $\hbar k$ being momentum and R_c the nuclear charge radius. The leading order corrections are found to be small enough to validate quantitative analyses of experimental data for many intermediate to high energy cases and for scattering angles not limited to the very forward region. In a Gaussian model, the leading corrections to the Glauber amplitude are given as convenient analytic expressions.

[NUCLEAR REACTIONS Multiple scattering theory at intermediate to high energy; leading corrections to Glauber approximation due to eikonal, Fermi motion, and kinematic effects.]

I. INTRODUCTION

Whether or not nuclear structure information can be reliably extracted from precise hadron-nucleus scattering experiments depends to a large extent on the quantitative accuracy of nuclear multiple scattering theory used to interpret the data.[1-3] On the grounds of accuracy, the very convenient Glauber approximation analysis[4] is questionable[5,6] in the intermediate energy range although it is generally accepted as asymptotically correct at fixed momentum transfer. The theory has been successful in predicting total cross sections at intermediate energies[7] which hints that the corrections should not be very large until the energy gets quite low.

In this paper a high energy expansion of the Watson multiple scattering series[8] is given which reproduces the Glauber approximation as the leading (asymptotic) term and which produces systematic corrections to the particular combination of eikonal and sudden passage approximations inherent in Glauber theory. To be specific, the Glauber approximation is developed as the leading term in a particular form of eikonal perturbation theory. For elastic scattering the method is free of such approximations as setting the longitudinal momentum transfer (q_z) zero. For potential scattering the eikonal perturbation theory provides systematic improvement over the eikonal approximation in numerical calculations for forward hemisphere scattering. Assuming the scattering amplitude has no essential singularities as $k\to\infty$ at fixed momentum transfer q, the eikonal perturbation theory generates a unique asymptotic $1/k$ expansion of the scattering amplitude, and hence reliable corrections to the high energy limit. This expansion is thus ideally suited for multiple diffraction scattering by high energy hadrons.

The objective of the high energy expansion is to introduce corrections to the sudden passage approximation as well as corrections to the eikonal approximation. Sudden passage exploits the idea that the multiple scattering process is dominated by many small momentum transfer two-body scatterings in each of which the projectile encounters an essentially free constituent nucleon.

The picture further presumes the projectile *velocity* to be much greater than that which constituent nucleons have due to either their Fermi motion or the most probable momentum transferred to any one of them in the scattering process. Under such conditions the passage of the projectile is quite sudden and the constituent nucleons do not have time to move much during the scattering as indicated in Fig. 1. Hence it is common practice to regard the target nucleons as fixed scatterers. However, it is important to appreciate the difference between the sudden passage picture and the distinct approximation (fixed scatterers approximation) in which one assumes the projectile scatters from rigidly interconnected scatterers, later performing an average over the nuclear wave function in either case.

The fixed scatterer approximation arises from applying a closure approximation to the propagator of a projectile passing through the nuclear medium. One assumes the excitation energy of the nucleus between scatterings is a negligible fraction of the projectile energy, and can thus be ignored even when summing over all possible nuclear states. What remains is then a two-particle propagator with kinematics relevant to projectile-nucleus scattering. The resultant multiple scattering theory involves a physical picture of rigidly interconnected scatterers; each recoiling as if it possessed the entire inertia of the nucleus. In contrast, the sudden passage case involves two-body recoil kinematics as indicated in Fig. 1 and this is clearly preferable for high energy scattering as the most prominent inelastic channels involve quasielastic knockouts rather than rigid body rotations.

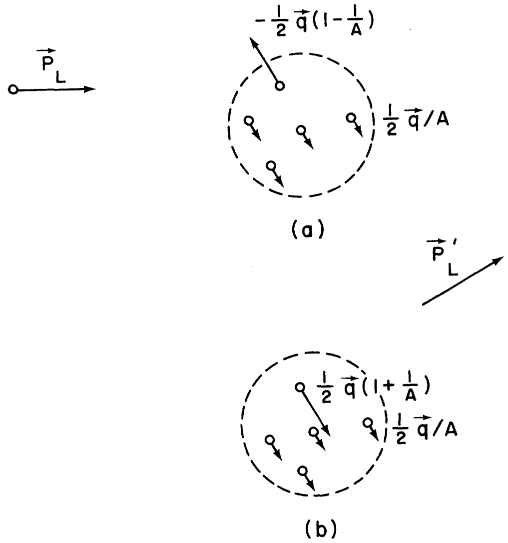

FIG. 1. *Sudden passage approximation.* For elastic scattering via the single scattering mechanism (a) indicates a projectile with laboratory momentum P_L just before scattering from a nucleus (particles inside dashed circle) which has zero center-of-mass momentum, while (b) shows the system just after scattering. Momentum $\vec{q}$ is lost by the projectile and gained by one nucleon in such a way that the nuclear c.m. recoils with total momentum $\vec{q}$ as required by momentum conservation. The internal kinetic energy of the nucleus does not change. In the weak binding limit, elastic momentum transfer $\vec{q}$ measures the probability of finding the nucleus essentially as in (a), other distributions of initial nucleon momenta leading to different final states (inelastic, breakup, etc.). Although the target nucleons remain fixed in position in (a) and (b), the interaction is a two-body quasifree on-energy-shell scattering with a single nucleon absorbing the recoil momentum. To the extent that Fermi motion and momentum transfer are negligible, the same laboratory momentum applies to the two-body and nuclear scattering.

At very high energy, such distinctions are not crucial because the recoil effects tend to become negligible. The reason is because the eikonal approximation scattering amplitude dominates at very high energy and it in turn depends only on the beam velocity, not the recoil kinematics. However, when corrections to the eikonal limit become important, then the question of recoil kinematics enters as was dramatized by Kujawski and Lambert.[9] The point we wish to emphasize is that the Glauber approximation presumes two-body recoil kinematics, not rigidly fixed scatterers as is often implied. For sudden passage the scatterers are fixed only in the sense of a flash photograph; they are assumed to recoil as unbound nucleons and generally do not have zero momentum relative to target nucleus center of mass (c.m.). To the extent the target nucleons are free, the individual scatterings are on energy shell.

Two previous analyses of multiple scattering starting from the fixed scatterer approximation and nonrelativistic kinematics have been carried out using the eikonal perturbation theory to introduce corrections to the eikonal approximation[10] and also the Fermi motion.[11] The result of these analyses is that the multiple scattering amplitude separates into two parts; one of which is very similar to the Glauber form, however, with a kinematical difference because the recoil kinematics in the fixed scatterer analysis necessarily involve the total nuclear mass. Hence the zero order (i.e., before corrections) multiple scattering phase shift of Refs. 10 and 11 does not bear the same relationship to the free two-body scattering amplitude as is appropriate for the Glauber approximation, although the difference is slight at high energy for reasons already given. Because the rigidly fixed scatterer model does not introduce the physical two-body kinematics to individual scatterings, it becomes clear that a new starting point is needed for developing corrections for use at intermediate energies. Hence the purpose of the present paper is to carefully formulate a high energy expansion starting from the Watson multiple scattering expansion and not using the fixed scatterer approximation. Close attention is paid to Fermi motion effects and two-body recoil kinematic effects in addition to the deviations from eikonal propagation between scatterings.

A significant finding of the analysis is that there is substantial cancellation between leading order corrections to the Glauber approximation arising from noneikonal, kinematic, and Fermi motion effects. Stated another way, the finding is that the use of two-body on-energy-shell amplitudes plus neglect of Fermi motion plus use of a simple eikonal propagator between scatterings tends to be a more optimal combination of approximations than any one of these approximations taken alone. This result is important because it suggests that some straightforward improvements over the Glauber derivation based on removing one of the approximations can be *less* accurate rather than more accurate.

Each correction considered in this paper produces in leading order a scattering amplitude change of order $(2kR_c)^{-1}$ relative to the double scattering term in the Glauber theory, $\hbar k$ being c.m. momentum and R_c the charge radius of the target nucleus. For calculations at intermediate to high energies, the first order corrections are small and they can be reliably calculated using perturbation methods. The result is a more precise approximation to the full Watson series. When two-particle scattering data and nuclear form factors are parametrized by Gaussian forms, the leading correction to the Glauber approximation can be analytically evaluated as can the Glauber amplitude itself. Thus the enormous computational convenience of the theory is not sacrificed. The leading correction developed in this paper provides a simple test of when the Glauber approximation is reliable.

The present development of the high energy expansion is based on inserting relativistic kinematics into the Schrödinger propagator of the Watson theory in a noncovariant way. To justify this approach, we have separately examined a covariant multiple scattering model employing relativistic eikonal expansion techniques. That analysis shows that within kinematical factors arising from Lorentz contraction, the leading terms of an eikonal expansion take the same form in the fully covariant model as in the present one. The extra kinematical factors are inserted in our final formulas to respect Lorentz covariance although the net effect is generally small.

A novel feature of the analyses is that a two-body t matrix is introduced whose *nuclear* elastic matrix element involves just the physical *two-body* scattering information. This two-body t matrix is convenient for multiple scattering because it automatically accomplishes the kinematic transformation from two-body to nuclear center-of-mass systems. The error made when this two-body t matrix is substituted for Watson's many-body operators can then be recovered by an eikonal-like perturbation expansion in just the same fashion that noneikonal effects and Fermi motion effects are recovered.

A parametrization of two-body amplitudes is made in terms of an effective eikonal potential related directly to the observable scattering amplitude. The effective eikonal potential is so defined that it is local, energy dependent, and when it is used together with an eikonal propagator in a Lippmann-Schwinger integral equation, one obtains a t matrix whose on-energy-shell matrix elements agree with the physical amplitude. By using this device to parametrize the fundamental interaction via a two-body t matrix, the high energy expansion is rendered tractable and the corrections to the high energy limit are determined directly from on-energy-shell information.

II. HIGH ENERGY EXPANSION

For hadron-nucleus elastic scattering, let the initial and final c.m. momenta be $\vec{k}_i$ and $\vec{k}_f$, respectively (units: $\hbar = c = 1$). The c.m. scattering amplitude is expressed in terms of a T matrix as

$$F(\vec{q}) = \langle T\rangle \equiv -\epsilon(2\pi)^{-1}\langle \vec{k}_f; E_0|T|\vec{k}_i; E_0\rangle; \qquad \vec{q} = \vec{k}_i - \vec{k}_f \,. \tag{1}$$

In what follows the notation $\langle\ \rangle$ is consistently used to denote the same nuclear elastic scattering matrix element as here, E_0 being the target nucleus ground state energy.

The Watson multiple scattering expansion provides the following model for T assuming that each two-particle scattering proceeds through the elastic channel

$$T = \sum_{i=1}^{A}\tau_i + \sum_{i\neq j=1}^{A}\tau_i G\tau_j + \sum_{i\neq j\neq k}^{A}\tau_i G\tau_j G\tau_k + \cdots \,, \tag{2}$$

where

$$G^{-1} = \frac{k^2 - p^2}{2\epsilon} - H_T;$$

$$H_T = \sum_{n=1}^{A}\frac{[\vec{p}_n - (m/M)\vec{P}]^2}{2m} + V_T\,,$$

$$\tau_i = V_i + V_i G\tau_i\,. \tag{3}$$

The kinematic parameters for the projectile nucleus c.m. are

$$\epsilon = E_L(M/\sqrt{s}), \quad k = P_L(M/\sqrt{s}),$$
$$s = m_0{}^2 + M^2 + 2ME_L, \quad E_L{}^2 = m_0{}^2 + P_L{}^2, \qquad (4a)$$

while those for the projectile nucleon c.m. are

$$\epsilon_2 = E_L(m/\sqrt{s_2}), \quad \kappa = P_L(m/\sqrt{s_2}),$$
$$s_2 = m_0{}^2 + m^2 + 2mE_L. \qquad (4b)$$

In these expressions, E_L, P_L, and m_0 are the laboratory energy, momentum, and rest mass of the projectile, while ϵ and k are the nuclear c.m. energy and momentum; ϵ_2 and κ are the two-body c.m. energy and momentum; $\vec{p}$ is the nuclear c.m. momentum operator; H_T is a nonrelativistic internal Hamiltonian for the nucleus of A nucleons each with momentum $\vec{p}_n$ and mass m, while the total nuclear mass and momentum are M and $\vec{P} = \sum_n^A \vec{p}_n$. The operators τ_i are similar to two-particle t matrices with the important difference that G is a many-body propagator. The interaction V_i is regarded as known only through the effects it produces in physical two-particle scattering.

In the nonrelativistic limit, Eqs. (3) and (4) represent standard kinematics as ϵ and ϵ_2 become reduced masses $m_0M/(m_0+M)$ and $m_0m/(m_0+m)$ for nuclear and two-body scattering, respectively. Relativistically, the kinematics are chosen so that the eikonal (high energy) limit *and* the leading $1/k$ corrections to the high energy limit are the same as in a covariant model of two-particle scattering based on summing the generalized ladder set of Feynman diagrams. Hence the departure from the more commonly used Watson kinematic prescription is intentional and is necessary for agreement with covariant high energy results.

For large $k = |\vec{k}_i| = |\vec{k}_f|$, the Glauber approximation has been successfully used to approximate

$$\langle T\rangle \simeq \langle T_G\rangle \equiv F_0(\vec{q}), \qquad (5)$$

where the matrix element $\langle T_G\rangle$ can be expanded into a finite multiple scattering series when expressed as the following integral over impact parameters

$$F_0(\vec{q}) = \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}} \times \langle E_0| \left\{ \prod_{i=1}^{A} [1 - \Gamma(\vec{b} - \vec{b}_i)] - 1 \right\} |E_0\rangle. \qquad (6)$$

The profile functions Γ are assumed to be related to physical two-particle elastic scattering amplitudes at the same laboratory energy E_L by Fourier transformation

$$\Gamma(b) = i(2\pi\kappa)^{-1} \int d^2q\, e^{-i\vec{q}\cdot\vec{b}} f_{\text{c.m.}}(q; \kappa, \epsilon_2); \qquad (7)$$

κ and ϵ_2 being the c.m. momentum and energy for two-particle scattering. Although it is not explicitly shown, $\Gamma(b)$ obviously depends parametrically on kinematic parameters κ and ϵ_2 for two-particle scattering. For simplicity, the spin and isospin dependence of Γ is neglected in the present analysis.

Because the Glauber theory is tractable, it is enormously useful. Our goal is to perform a high energy expansion of the Watson series which produces Eq. (6) plus corrections. Step 1 is to expand the many-body operators τ_i about the following two-particle operator

$$\tau_i' = V_i + V_i G_2 \tau_i', \qquad (8)$$

where G_2 is a two-particle propagator which we divide into two parts as follows

$$G_2{}^{-1} = g^{-1} - N_2; \qquad (9)$$

g is an eikonal propagator for nuclear scattering defined by[12]

$$g^{-1} = \vec{v}\cdot[(\vec{k}_i + \vec{k}_f)/2 - \vec{p}]; \quad \vec{v} = \frac{k}{\epsilon}\frac{\vec{k}_i + \vec{k}_f}{|\vec{k}_i + \vec{k}_f|}; \qquad (10)$$

and N_2 contains two-body recoil effects:

$$N_2 = (\vec{p} - \vec{k}_f)\cdot(\vec{p} - \vec{k}_i)/(2\epsilon_2) + \lambda_2 g^{-1};$$
$$\lambda_2 = 1 - [1 - q^2\epsilon^2/(4k^2\epsilon_2{}^2)]^{1/2}. \qquad (11)$$

The operator N_2 is very similar to the operator used in Refs. 10 and 11 to introduce corrections to the eikonal approximation. The so defined G_2 and τ_i' are not the usual ones for free two-particle scattering (e.g., $\vec{k}_i$ and $\vec{k}_f$ are projectile-nucleus relative momenta and the struck nucleon momentum $\vec{p}_i$ does not enter); however, they are very suitable to our purpose which is *nuclear* scattering. If the interaction V_i is any local potential, then the operator τ_i' has the property that its *nuclear* matrix element involves (exactly) the physical *two-particle* phase shift. In impact parameter representation, the statement is the following where Γ is the same function appearing in Eq. (7):

$$\langle\tau_i'\rangle = ik(2\pi)^{-1}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\langle E_0|\Gamma(\vec{b} - \vec{b}_i)|E_0\rangle. \qquad (12)$$

Hence approximating τ_i by τ_i' causes the single scattering terms in Eqs. (2) and (6) to be identical. A demonstration that G_2 is the propagator needed to accomplish this result is given in Appendix A. We emphasize that the result does not involve use of the eikonal approximation but does assume a local potential.

The error in the approximation $\tau_i \approx \tau_i'$ is recoverable as follows. We express the difference between G_2 and the full propagator G by

$$G_2^{-1} - G^{-1} = N - N_2 + H_T - E_0\,, \tag{13}$$

where

$$N = (\vec{p} - \vec{k}_f)\cdot(\vec{p} - \vec{k}_i)/(2\epsilon) + \lambda g^{-1}\,;$$
$$\lambda = 1 - [1 - q^2/(4k^2)]^{1/2}\,. \tag{14}$$

To obtain Eq. (13), we have used $G^{-1} = g^{-1} - N - (H_T - E_0)$ together with Eq. (9). The eikonal term g^{-1} is common to both G^{-1} and G_2^{-1} and hence cancels in Eq. (13). Note that N involves projectile-nucleus kinematics and hence differs from N_2 which involves two-particle kinematics. Even though N and N_2 are operators which express deviations from eikonal propagation, the portion $N - N_2$ of Eq. (13) is not a correction to the eikonal approximation but rather expresses the error in the kinematic transformation from projectile-nucleus to two-body systems. Because this kinematic effect can be so described, it is obviously comparable in importance to deviations from the eikonal approximation. One deduces from Eqs. (3), (8), and (13) that τ_i differs from τ_i' as follows

$$\tau_i = \tau_i' + \tau_i' G_2(N - N_2 + H_T - E_0)G\tau_i\,. \tag{15}$$

Equation (15) provides a basis for recovering effects missing from τ_i' via a perturbation expansion. Generally the three corrections cancel each other to some degree.

Step 2 is to introduce an effective two-body eikonal potential which is local and energy dependent as follows:

$$U_E(r) = i\pi^{-1}\int_{-\infty}^{\infty} du \frac{d}{du^2}\ln[1 - \Gamma(r^2 + u^2)^{1/2}]\,, \tag{16}$$

where $\Gamma(b)$ is the two-particle profile function of Eq. (7). Once given $\Gamma(b)$, this equation defines a local eikonal potential whether or not the underlying interaction V_i responsible for the phase shift is local. The eikonal potential is so defined that when it is used together with the eikonal propagator (10) to define a new scattering operator t_i by the integral equation

$$t_i = V_{Ei} + V_{Ei} g t_i, \qquad V_{Ei} \equiv (k/\epsilon)U_E(|\vec{r} - \vec{r}_i|), \tag{17}$$

then the new operator t_i has the same *nuclear* matrix element as does τ_i':

$$\langle t_i\rangle = ik(2\pi)^{-1}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\langle E_0|\left\{1 - \exp\left[-i\int_{-\infty}^{\infty} dz\, U_E(|\vec{r} - \vec{r}_i|)\right]\right\}|E_0\rangle\,. \tag{18}$$

This is the same as Eq. (12) because Eq. (16) represents the well-known result of equating Eqs. (12) and (18) and effecting the solution for $U_E(r)$ in terms of $\Gamma(b)$ by an Abel integral transformation.

Hence for on-energy-shell momenta, the nuclear matrix elements of τ_i' and t_i are equivalent by construction; for off-energy-shell momenta they may differ. The utility of t_i is that it provides a convenient off-shell extrapolation of the physical phase shift. If we approximate τ_i' by t_i, the approximation involved is analogous to that of using a separable potential to extrapolate known physical phase shifts off the energy shell.[13] Step 2 is to so approximate τ_i' by t_i in Eq. (15), obtaining

$$\tau_i \approx t_i + t_i G_2(N - N_2 + H_T - E_0)G\tau_i\,. \tag{19}$$

The third step is to expand the propagators G and G_2 about the eikonal propagator g using

$$G = g + g(N + H_T - E_0)G, \tag{20}$$

$$G_2 = g + gN_2G_2\,. \tag{21}$$

The integral equation (19) expresses what is often called the off-energy-shell effect in that it relates the many-body operators τ_i of the Watson expansion to a two-body operator t_i constructed from on-energy-shell information. In Eq. (20), the operator N generates corrections to eikonal propagation while $H_T - E_0$ generates Fermi motion effects. If H_T is set equal to E_0 (for example, by projecting onto the nuclear ground state), then Eq. (20) reduces to a two-body fixed scatterer (or optical model) propagator. If furthermore N is set to zero then Eq. (20) reduces to the eikonal propagator g. Because the same operators appear in Eqs. (20) and (19), one can see that the approximations $\tau_i \approx t_i$ and $G \approx g$ which underlie the Glauber approximation are of comparable validity.

When perturbation expansions based on Eqs. (19) and (20) are introduced to the Watson series (2) (see Appendix B for details), one finds that T reduces to the Glauber approximation plus corrections. To leading order in k^{-1} but all orders in multiple scattering, the result is

$$\langle T\rangle = \langle T_G\rangle + \langle(1 + T_G g)\Big(N + H_T - E_0 - \sum_i V_{Ei} g N_2 g V_{Ei}\Big)$$
$$\times (T_G g + 1)\rangle + O(1/k^2), \tag{22}$$

where

$$T_G = \sum_i t_i + \sum_{i\neq j} t_i g t_j + \sum_{i\neq j\neq k} t_i g t_j g t_k + \cdots \quad (23)$$

is the Glauber approximation to the Watson series.[14] It is worth noting that the operator N_2 appearing in the correction term (22) arises from the difference of $N+H_T-E_0$ from Eq. (20) and $N-N_2+H_T-E_0$ from Eq. (19). Thus there is partial cancellation of the corrections from Eqs. (19) and (20) which is already incorporated into Eq. (22).

III. EVALUATION OF THE CORRECTION

Due to the simple properties of the eikonal propagator and the fact that U_E is local, Eq. (21) can be reduced to calculable expressions. The scattering amplitude is written as a perturbation series in k^{-1}

$$F(\vec{q}) = F_0(\vec{q}) + F_1(\vec{q}) + \cdots , \quad (24)$$

where the correction to the Glauber approximation (6) involves eikonal potentials as follows (see Appendix B for details):

$$F_1(\vec{q}) = \left(\frac{m_0+M}{4\pi\sqrt{s}}\right) \int d^2b\, e^{i\vec{q}\cdot\vec{b}} \langle E_0| \left\{\prod_{i=1}^{A} [1-\Gamma(\vec{b}-\vec{b}_i)]\right\} \sum_{i,j} [1-\epsilon/M - \delta_{ij}\epsilon(\epsilon_2^{-1} - m^{-1})]$$
$$\times \int_{-\infty}^{\infty} dz \left[\vec{\nabla}\int_z^{\infty} dz' U_E(|\vec{r}\,'-\vec{r}_i|)\right]\left[\vec{\nabla}\int_{-\infty}^{z} dz' U_E(|\vec{r}\,'-\vec{r}_j|)\right]|E_0\rangle . \quad (25)$$

The factor $(m_0+M)/\sqrt{s}$ evident in Eq. (25) is the aforementioned factor inserted to respect Lorentz covariance.

To obtain Eq. (25), the Fermi motion operator $H_T - E_0$ in Eq. (22) has been commuted to one side until it acts on the nuclear ground state $|E_0\rangle$ and thereby annihilates itself. Hence Eq. (25) only involves the noncommuting (kinetic energy) portion of the H_T and one finds that the Fermi motion correction reduces to the same form as the corrections arising from N and N_2. In consequence, the kinematic factors in Eq. (25) combine the effects of all three sources of correction.

In the limit of nonrelativistic kinematics, $\epsilon^{-1} = m_0^{-1} + M^{-1}$ and $\epsilon_2^{-1} = m_0^{-1} + AM^{-1}$ become the reduced masses for projectile-nucleus and projectile-nucleon systems, respectively. Then the diagonal $(i=j)$ terms in Eq. (25) vanish altogether due to the cancellations. The coefficient of the off-diagonal $(i\neq j)$ terms becomes $M/(m_0+M)\approx 1$ in place of $1-\epsilon/M$. Thus the off-diagonal terms in Eq. (25) produce the major correction and as previously discussed, this correction is due to the "overlapping potentials" effect. The fact that kinematic, Fermi motion and eikonal corrections all cancel in the diagonal portion of Eq. (25) suggests that the Glauber theory embodies an optimal combination of eikonal and sudden passage approximations. It is worth noting that this cancellation would not show up in the expansion of Ref. 11 due to omission of the kinematic corrections due to Eq. (19).

The off-diagonal (overlapping potentials) portion of Eq. (25) resembles the result found in a previous analysis[10] based on the fixed scatterer approximation in which the diagonal terms do not arise. Because the diagonal terms cancel (in the nonrelativistic limit), Eq. (25) is essentially the fixed scatterer result. However, an important difference is that Eq. (25) provides a well-defined and calculable correction starting from only on-energy-shell information introduced via the eikonal potential U_E.

The relative size of the correction $F_1(q)$ in Eq. (24) gives a specific indication of when the Glauber approximation is accurate. To calculate $F_1(q)$, we employ the often used Gaussian model in which the target nucleus wave function is written

$$|\langle E_0|\vec{r}_1, \ldots, \vec{r}_A\rangle|^2 = \prod_{i=1}^{A} \rho_0(r_i)(2\pi)^3\delta^{(3)}(\vec{r}_1 + \cdots + \vec{r}_A) \quad (26)$$

with each $\rho_0(r)$ represented by a Gaussian term

$$\rho_0(r) = (4\pi B)^{-3/2} e^{-r^2/4B}, \quad \rho_0(q^2) = e^{-Bq^2} \quad (27)$$

together with the standard approximation to the two-particle scattering amplitude

$$f(q) = \frac{\kappa\sigma(i+\rho)}{4\pi} e^{-\beta q^2} . \quad (28)$$

In this model, the Glauber amplitude takes a well-known analytic form[15]:

$$F_0(q) = -2ike^{Bq^2/A}(B+\beta)\sum_{j=1}^{A}\binom{A}{j}\frac{1}{j}\left[\frac{-\sigma(1-i\rho)}{8\pi(B+\beta)}\right]^j e^{-(B+\beta)q^2/j} . \quad (29)$$

Although it is more complicated, the leading correction obtained from Eq. (25) can be reduced to analytic expressions. To do this, it is first necessary to determine the eikonal potential $U_E(r)$, or equivalently, its Fourier transform $U_E(q)$. From Eq. (16) one can show that the Fourier transform takes the form

$$U_E(q) = 2i\pi \int_0^\infty db\, bJ_0(qb) \ln[1 - \Gamma(b)] \tag{30a}$$

$$= \frac{-i\pi}{\beta q} \int_0^\infty db\, b^2 J_1(qb) \frac{\sigma_1 e^{-b^2/4B}}{1 - \sigma_1 e^{-b^2/4B}}, \tag{30b}$$

where the second line follows after inserting

$$\Gamma(b) = \sigma_1 e^{-b^2/4\beta}; \quad \sigma_1 \equiv \frac{\sigma(1 - i\rho)}{8\pi\beta} \tag{31}$$

in accordance with Eqs. (28) and (7). It is an important numerical detail that $|\sigma_1|$ is often bigger than unity for nucleon-nucleon diffraction scattering and in such cases an expansion of Eq. (30b) in powers of σ_1 diverges. However, the quantity needed in evaluating Eq. (25) is not directly $U_E(q)$ but rather the product $U_E(q)\rho_0(q)$ because in coordinate space $\int d\vec{r}\,' U_E(\vec{r} - \vec{r}\,')\rho_0(\vec{r}\,')$ enters. Since the density $\rho_0(q)$ falls off rapidly with q, we need only determine $U_E(q)$ for relatively small q values.

A numerical calculation of $U_E(q)$ for diffractive scattering is shown in Fig. 2 based on (30b) with parameter values $\sigma = 44$ mb, $\rho = -0.3$, and $\beta = 2.72$ $(\text{GeV}/c)^{-2}$ chosen to represent the 1 GeV nucleon-nucleon diffraction scattering via Eq. (28). In this case $|\sigma_1| = 1.67$. As shown by the dashed lines in Fig. 2, the magnitude and phase of $U_E(q)$ can be well approximated by a Gaussian form within the range where $\rho_0(q)$ for ^{4}He is significant. The dashed lines in Fig. 2 are based on

$$U_E(q) = -i4\pi\beta\sigma_1 Z e^{-\gamma q^2} \tag{32}$$

with complex parameter values $Z = 1.20 + i0.73$, $\gamma = 2.21 - i0.89$. The form of Eq. (32) is chosen to facilitate comparison with the simple approximation $Z = 1$, $\gamma = \beta$ which would follow from ignoring the denominator in Eq. (30b). Note that for a target nucleus larger than ^{4}He, $\rho_0(q)$ would fall faster with q thereby improving the approximation (32).

Once $U_E(q)$ has been approximated by the Gaussian form (32), it is possible to carry out *all* the remaining integrations in Eq. (25) to obtain an analytic approximation to $F_1(q)$ in the Gaussian model including center-of-mass correlations. The analytic result can be easily generalized to the case where $U_E(q)$ is represented by a sum of Gaussian terms although there is no need to do so in the present case.

The analytic expressions are based on separating $F_1(q)$ into two components:

$$F_1(q) = F_1^O(q) + F_1^D(q) . \tag{33}$$

The part $F_1^O(q)$ represents the off diagonal $(i \neq j)$ terms of Eq. (25) which contain the overlapping potentials effect in nuclear multiple scattering, while $F_1^D(q)$ represents the diagonal $(i = j)$ terms of Eq. (25) and is related to corrections to single scattering. Due to the cancellations already noted, $F_1^D(q)$ tends to be small. The results are expres-

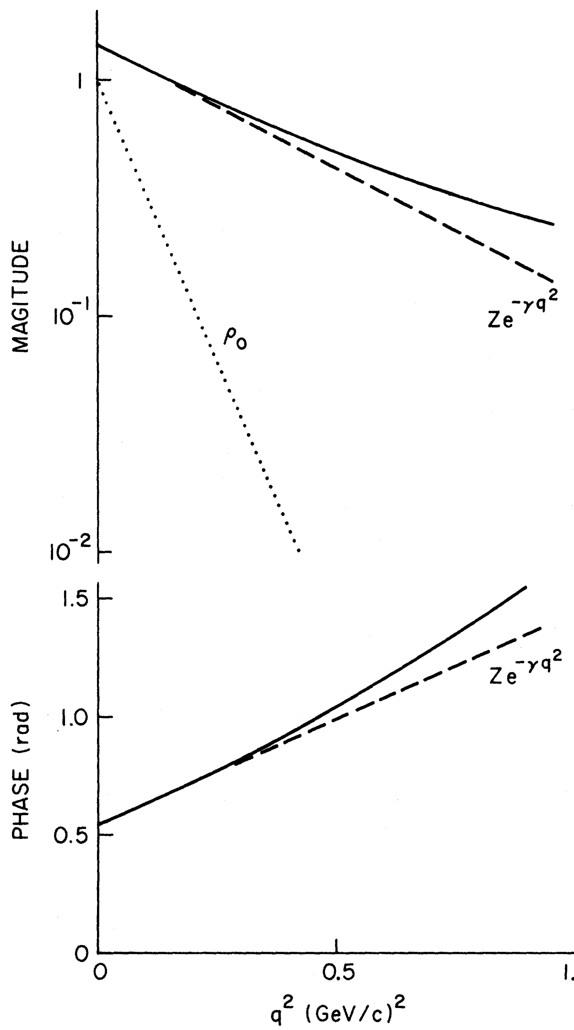

FIG. 2. The amplitude and phase of the complex eikonal potential $U_E(q)$ are compared to a Gaussian approximation by plotting $Z(q) = U_E(q)/[-i4\pi\beta\sigma_1]$ (solid lines) and comparing to a simple fit $Ze^{-\gamma q^2}$ as per Eq. (32) (dashed lines). The dotted curve shows the rapid fall off of $\rho_0(q) = e^{-Bq^2}$ representative of a ^{4}He target nucleus. At the point where the Gaussian approximation $Ze^{-\gamma q^2}$ deviates by 10% from $Z(q)$, the density factor ρ_0 has increased to 2% of its maximum value. Hence the product $\rho_0(q)U_E(q)$ is accurately represented by a single Gaussian term throughout the range where it is significant.

sed as sums of Gaussian terms as follows[16]:

$$F_1^O(q)=A(A-1)F_{\text{c.m.}}(q)\eta_O\frac{(\beta\sigma_1 Z)^2}{2[2\pi(B+\beta)]^{1/2}}\sum_{l=0}^{A-2}\binom{A-2}{l}\left(-\frac{\sigma_1\beta}{B+\beta}\right)^l\sum_{n=1}^{3}\alpha_{nl}[a_n-4\alpha_{nl}b_n(1-\alpha_{nl}q^2)]e^{-\alpha_{nl}q^2}, \quad (34)$$

$$F_1^D(q)=AF_{\text{c.m.}}(q)\eta_D\frac{(\beta\sigma_1 Z)^2}{2\gamma\sqrt{2\pi\gamma}}\sum_{l=0}^{A-1}\binom{A-1}{l}\left(-\frac{\sigma_1\beta}{B+\beta}\right)^l\sum_{n=1}^{3}\beta_{nl}[c_n-4\beta_{nl}d_n(1-\beta_{nl}q^2)]e^{-\beta_{nl}q^2}, \quad (35)$$

where

$$F_{\text{c.m.}}(q)=e^{Bq^2/A} \quad (36)$$

is the well-known factor arising from the center-of-mass constraint of Eq. (26). The kinematic factors η_O and η_D are given by

$$\eta_O=\left(1-\frac{\epsilon}{M}\right)\left(\frac{m_0+M}{\sqrt{s}}\right)$$

$$\eta_D=\left[1-\frac{\epsilon}{\epsilon_2}+(A-1)\frac{\epsilon}{M}\right]\left(\frac{m_0+M}{\sqrt{s}}\right) \quad (37)$$

and expressions for the remaining parameters in Eqs. (34) and (35) are given in Table I in terms of previously defined parameters B Eq. (27), β Eq. (28), and σ_1 Eq. (31). The parameters Z and γ of Eq. (32) must be determined numerically as described above if $|\sigma_1|>1$. The index l in Table I takes the values 0 through 4 as required by the sums in Eqs. (34) and (35).

Equations (34) and (35) provide useful forms for the leading corrections to the Glauber approximation. It is instructive to compare these results to the double scattering terms of the Glauber amplitude (29):

$$F_d(q)=-ikA(A-1)\frac{\beta+B}{2}F_{\text{c.m.}}(q)\left(\frac{\beta\sigma_1}{\beta+B}\right)^2e^{-(B+\beta)q^2/2}, \quad (38)$$

a comparison which is more transparent in terms of the following approximate ratios:

$$F_1^O(q)/F_d(q)\approx -i\eta_O Z^2[1-q^2R^2/(8\pi)]/(kR), \quad (39a)$$

$$F_1^D(q)/F_d(q)\approx -i\eta_D Z^2(1-\beta q^2)e^{-Bq^2/2}/[(A-1)kR]. \quad (39b)$$

A length parameter R has been defined by

$$R^2\equiv 8\pi(B+\beta). \quad (40)$$

TABLE I. Parameter definitions for the analytic results (34) and (35).

$\gamma_1=\gamma\beta/(\gamma+\beta)$
$\alpha_{1l}^{-1}=2(\gamma+B)^{-1}+l(\beta+B)^{-1}$
$\alpha_{2l}^{-1}=(\gamma+B)^{-1}+(\gamma_1+B)^{-1}+l(\beta+B)^{-1}$
$\alpha_{3l}^{-1}=2(\gamma_1+B)^{-1}+l(\beta+B)^{-1}$
$a_1=(\gamma+B)^{-2}$
$a_2=-2\sigma_1(\gamma_1/\gamma)(\gamma+B)^{-1}(\gamma_1+B)^{-1}$
$a_3=\sigma_1^2(\gamma_1/\gamma)^2(\gamma_1+B)^{-2}$
$b_1=(\gamma+B)^{-3}$
$b_2=-\sigma_1(\gamma_1/\gamma)(\gamma_1+B)^{-1}(\gamma+B)^{-1}\left\{\left[1-\frac{B\gamma_1}{\beta(B+\gamma_1)}\right]\frac{1}{\gamma+B}+\left(1-\frac{\gamma_1}{\beta}\right)\frac{1}{\gamma_1+B}\right\}$
$b_3=\sigma_1^2(\gamma_1/\gamma)^2(\gamma_1+B)^{-3}\left[1-\frac{B\gamma_1}{\beta(\gamma_1+B)}\right]\left(1-\frac{\gamma_1}{\beta}\right)$
$\gamma_2=\gamma\beta/(\beta+2\gamma)$
$\beta_{1l}=(\gamma/2+B)^{-1}+l(\beta+B)^{-1}$
$\beta_{2l}=(\gamma_2/2+B)^{-1}+l(\beta+B)^{-1}$
$c_1=(\gamma-2B)(\gamma+2B)^{-2}$
$c_2=-\sigma_1(\gamma_2/\gamma_1)(\gamma_2+2B)^{-1}\left[1-\frac{2\gamma_2}{\gamma}+\frac{2\gamma_2^2}{\gamma(\gamma_2+2B)}\right]$
$d_1=\gamma(\gamma+2B)^{-3}$
$d_2=-\sigma_1\gamma^{-2}[\gamma_2/(\gamma_2+2B)]^3$

First one notes the appearance of a dimensionless ordering parameter $(kR)^{-1}$. When this parameter is small compared to unity, the subject corrections are small relative to the *double* scattering term of the Glauber approximation, at least for qR not large. The essential point is that the projectile wavelength need only be small compared to the rms charge radius R_c of the target nucleus for $(kR)^{-1}$ to be a good expansion parameter. This follows from noting that $B=\frac{1}{6}R_c^2$ and hence an upper bound is

$$(kR)^{-1} \leqslant (2kR_c)^{-1}. \tag{41}$$

Thus the high energy expansion can be convergent at rather low momenta for a variety of projectile-nucleon interactions. Notice that this statement is made for highly overlapping interactions which is the case relevant to nuclear multiple scattering. It is the spread out nature of each target nucleon wave function which gives enhanced covergence of the high energy expansion. For contrast consider scattering by rigidly fixed centers in which binding effects are absent. Then there is no cutoff from the target form factor (i.e., $R_c \to 0$) radically altering the above estimates. Our point is that at energies for which the Glauber approximation can be shown to be inadequate for rigidly fixed scatterers,[17] the high energy expansion remains effective for the physically interesting case of weakly bound scatterers.

Typically $R_c \geqslant 1.4$ fm, and thus $2kR_c$ is less than unity for c.m. momenta k greater than 60 MeV/c. Although the present expansion may very well not converge at such low momenta, it appears well suited to the intermediate energy range ($2kR_c \sim 4$ for 200 MeV pions, $2kR_c \sim 7$ for 200 MeV protons on helium). At sufficiently high energy, multiple scattering can be accurately calculated using the perturbation expansion (24) with leading corrections given very simply in the Gaussian model by Eqs. (34) and (35). The higher order terms [$F_2(q)$ and so on] in the high energy expansion (24) tend to be of order $(kR)^{-1}$ or smaller compared to the *triple* scattering term in the Glauber amplitude provided $qR \lesssim 1$.

There remains the essential question of the angular range of validity of the approximations made, or to say the same thing, the limits on q. Because an eikonal expansion can be arranged in several forms, we mention here an accumulation of results[12,18] which, in sum, show that the eikonal perturbation theory in the form presently used does give reliable extension of the angular range of validity. To illustrate the results, differential cross sections for proton-^{4}He elastic scattering have been computed at 1 GeV for which standard Glauber calculations and experimental data exist. Although there are other well defined corrections to the Glauber approximation which need be considered before one compares the theory in a detailed way with experimental data (e.g., spin[19] and isospin effects, N^* virtual production[20] nuclear structure effects,[21] and phase indeterminancy of two-particle amplitudes) they are all logically distinct from the ones we consider here.

The present purpose is to establish the relative size of eikonal, kinematic, and Fermi motion corrections to the Glauber approximation at energies which are high enough for the leading corrections to be taken seriously. We employ the Gaussian model for scattering by ^{4}He.

At 1 GeV (Fig. 3), the expansion parameter $\eta_0|Z|^2/(2kR_c)$ is 0.08, naively implying a 16% correction to the differential cross section in the double scattering region from Eq. (39). At the double scattering maximum in Fig. 3 ($\theta = 28°$), the correction is in fact 15%. Near the first diffraction minimum, the corrections are quite small due to cancellations and they do not become larger than 31% in the range $\theta < 60°$ shown. Evidently the leading eikonal and sudden passage corrections do

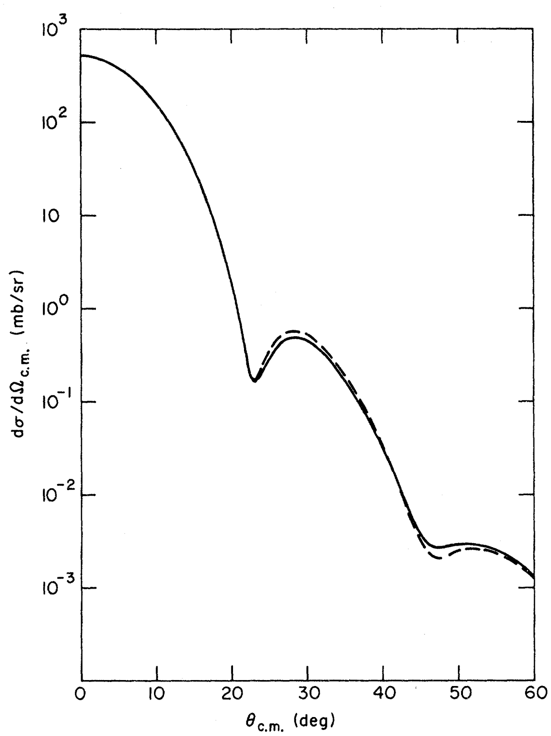

FIG. 3. For elastic scattering of 1.05 GeV protons by ^{4}He, the Glauber approximation $|F_0|^2$ (dashed line) is compared to the high energy expansion result $|F_0+F_1|^2$ (solid line) based on Eqs. (26) to (37). The term F_1 incorporates effects of eikonal, Fermi motion, and kinematic corrections to leading order in the high energy expansion.

in fact represent small effects in this test case. For a beam kinetic energy of 290 MeV instead of 1 GeV, k^{-1} becomes twice as large and the leading corrections would be about double those shown in Fig. 3. The effectiveness of the high energy expansion thus overlaps most of the intermediate energy range and is not limited to extremely high energy scattering.

The high energy expansion provides a framework in which to gauge the effectiveness of other approximations commonly used in multiple scattering calculations. For this purpose we assume that the high energy expansion result (solid line of Fig. 3) has converged very close to the exact result for the 1 GeV p-^{4}He scattering and then compare other approximations with this result.

In the simplest form of fixed scatterer approximation $H_T - E_0$, which represents the nuclear excitation energy, is set to zero everywhere and therefore only the projectile nucleus kinematics enter via Eq. (2). Even though the τ_i do not involve the two-body kinematics, they can be related to two-body physical amplitudes via Eq. (19). The key feature is that the eikonal approximation is not used for the propagator G of Eq. (20) and it is therefore often assumed that the fixed scatterer approximation is for this reason alone better founded than the Glauber approximation.

A way to test this notion is to keep just the effects of N and N_2 operators in Eqs. (19) and (20). The change is simply to alter the kinematic parameters (37) to $\eta_O = 1$ and $\eta_D = 1 - \epsilon/\epsilon_2$. Figure 4 compares the high energy expansion result of Fig. 3 to the corresponding fixed scatterer result calculated as just described.

The fixed scatterer result is no more effective than Glauber approximation in such a comparison, the reason being that fewer cancellations have occurred among the leading corrections. If nothing else, such a result suggests that it is subtle matter to realize improvements over relatively simple Glauber model calculations.

For elastic scattering, the optical model provides a theory which is in principle free of approx-

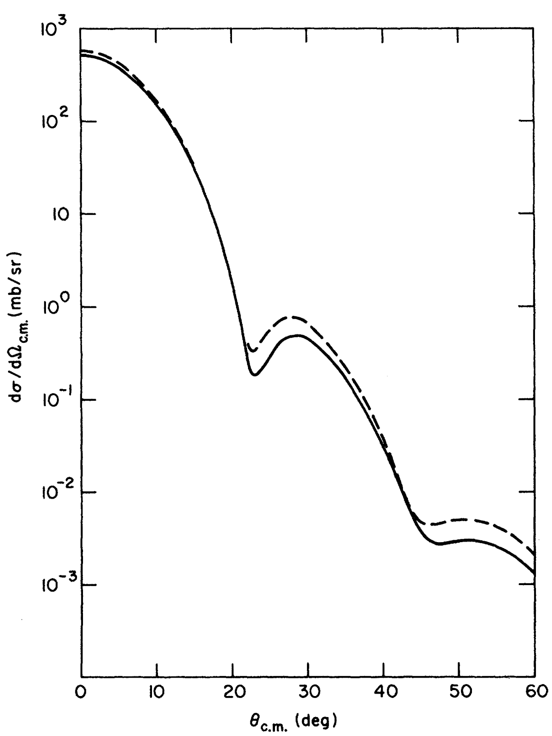

FIG. 4. The full high energy amplitude $|F_0+F_1|^2$ (solid line) of Fig. 3 is compared to the amplitude $|F_0+F_1'|^2$ (dashed line) which omits corrections due to the Fermi motion, i.e., $H_T - E_0 \rightarrow 0$. Thus F_0+F_1' represents (to leading order) the results a so-called exact fixed scatterer calculations in which only projectile-nucleus kinematics enter but in which the interaction is taken to be the physical two-particle scattering amplitude.

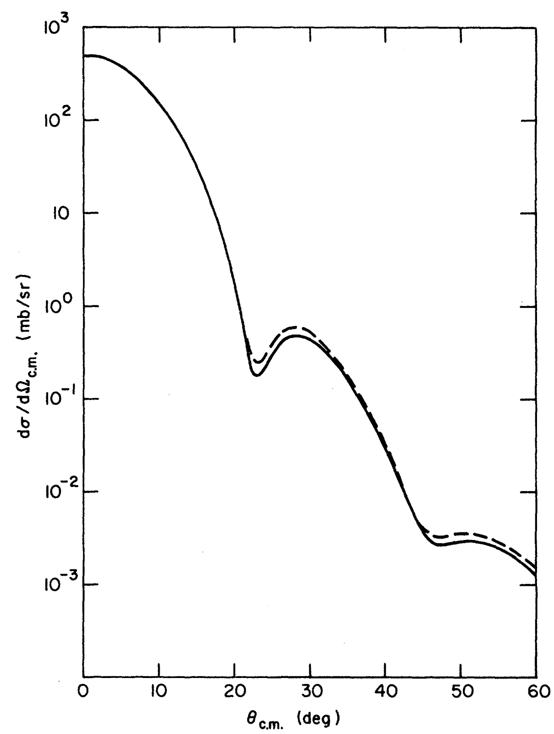

FIG. 5. The full high energy expansion result $|F_0+F_1|^2$ (solid line) of Fig. 3 is compared to the amplitude $|F_0+F_1''|^2$ (dashed line) which incorporates only the correction for non eikonal propagation between scatterings. Thus F_0+F_1'' represents (to leading order) the results of using free two-particle amplitudes in place of the operators τ_i of the Watson series and ignoring Fermi motion terms in the propagator G, both of which are common practice in optical model calculations.

imations and is limited only by the fact that exact optical model *calculations* are intractable. It is common practice to simplify matters by employing the impulse approximation to the τ_i operators of the Watson series together with a projection in the propagator G which excludes all but the nuclear ground state between scatterings. Usually Fermi-motion and kinematic transformation effects are ignored. However, the eikonal approximation is avoided by solving a Schrödinger equation. The effectiveness of this scheme can be estimated within the high energy expansion by making corresponding approximations, namely, $\tau_i \simeq t_i$ which corresponds to the impulse approximation plus ignoring $H_T - E_0$ in Eq. (20) which corresponds to the nucleus always being in its ground state between scatterings. In this case the high energy expansion proceeds with only the correction from noneikonal propagation between scatterings. The change is to alter the kinematic parameters (37) to $\eta_O=1$, $\eta_D=0$. Figure 5 compares the full high energy expansion result to the result based on the typical optical model approximations just described. A comparison with Fig. 3 indicates that such optical model calculations are no more accurate than Glauber calculations for the 1 GeV p-^{4}He scattering, assuming the solid line in either figure represents the correct result.

IV. CONCLUDING REMARKS

For nuclear multiple diffraction scattering the high energy expansion provides a systematic framework within which the Glauber approximation emerges as the leading term at asymptotic energies. It should be noted that the version of the Glauber approximation which emerges applies to the center-of-momentum system (as necessary for momentum conservation) rather than the laboratory frame. Also we carefully distinguish between fixed scatterer and sudden passage type kinematics. Although a nonrelativistic framework is employed, relativistic kinematics have been introduced in such a way that all kinematic factors are in agreement with those of a covariant model.

Several qualifications on the analysis should be noted. First and most important, the eikonal perturbation theory is basically an asymptotic expansion in k^{-1} which is most convergent for large k and forward scattering angles. For large scattering angles or too low energy, the convergence may be poor, and hence the high energy expansion is most relevant to forward hemisphere ($\theta_{\text{c.m.}} < 90°$) scattering. Within the forward hemisphere region there are several approximations implicit in the analysis which are believed to involve negligible errors but which are noted for completeness. For example, the relevant two-body scattering amplitudes have been assumed to be diffractive in nature, at least to the extent that exchange channel effects have been ignored in writing the simple impact parameter representation of Eq. (7). This assumption reinforces the forward hemisphere limitation. Also, note that the kinematic transformation involved in showing that the nuclear matrix elements of the operator τ_i' involves the two-body physical phase shift is only exact if the interaction is through a local potential. If a nonlocal potential is required, there is a presumably small error involved. Finally, the off-energy-shell matrix elements of the two-particle scattering operator have been rather arbitrarily parametrized through use of the eikonal potential. Other means of extrapolating the physical phase shift to construct off-energy-shell matrix elements may provide different results, but only to the extent that off-shell effects are important at all. Both of these sources of potential error are small when the two-particle scattering amplitudes depend primarily on momentum transfer and are relatively insensitive to the beam energy as is characteristic of intermediate to high energy diffractive amplitudes. Hence we believe that the high energy expansion provides reliable improvement over the Glauber approximation for forward hemisphere nuclear multiple diffraction scattering.

The results suggest that the very successful Glauber approximation works well because the dominant corrections involve some subtle cancellations and because the overlapping potentials effect is small when the projectile wavelength is much shorter than the charge radius of the bound nucleons. In conclusion, the results lead us to believe that nuclear structure and/or other effects can be reliably extracted from precise hadron nucleus scattering experiments using the Glauber approximation plus leading corrections.

A portion of this work was performed at the Los Alamos Scientific Laboratory during a visit to the Theoretical Division. The author thanks the Intermediate Energy Theory group for their hospitality.

APPENDIX A

It is to be shown that the nuclear c.m. matrix element of the operator τ' defined in Eq. (8) involves the physical two-particle phase shift. To specify $\langle\tau'\rangle$, consider the scattering wave function for a local potential $V(\vec{r})$ satisfying

$$[G_2^{-1} - V(\vec{r})]\Psi^{(+)}(\vec{r};\vec{k}_i)=0 \tag{A1}$$

and the boundary condition of incoming plane waves

$$\Psi^{(+)}(\vec{r};\vec{k}_i)\sim e^{i\vec{k}_i\cdot\vec{r}}. \tag{A2}$$

G_2 is given by Eq. (9). If we express the scattering wave function in the form

$$\Psi^{(+)}(\vec{r};\vec{k}_i)=e^{i\vec{k}_i\cdot\vec{r}}\phi(\vec{r}), \tag{A3}$$

the nuclear scattering matrix element of τ' takes the form

$$-\epsilon(2\pi)^{-1}\langle\vec{k}_f|\tau'|\vec{k}_i\rangle=-\epsilon(2\pi)^{-1}\int d\vec{r}\,e^{i\vec{q}\cdot\vec{r}}V(\vec{r})\phi(\vec{r}), \tag{A4}$$

where from (A1) and (A3) it follows that $\phi(r)$ is the solution of

$$[-\vec{v}\cdot\vec{p}(1-\lambda_2)-\vec{p}\cdot(\vec{p}+\vec{q})/(2\epsilon_2)-V(\vec{r})]\phi(\vec{r})=0 \tag{A5}$$

and the boundary condition

$$\phi(\vec{r})\sim 1. \tag{A6}$$

Use has been made of the fact that $\vec{v}\cdot\vec{q}=0$. From Eq. (A5), the function $\phi(r)$ also depends parametrically on $\vec{v}$, $\vec{q}$, ϵ_2, and λ_2. Now define a phase shift function by

$$\Gamma(b)\equiv i(\epsilon/k)\int_{-\infty}^{\infty}dz\,V(\vec{r})\phi(\vec{r}),\quad (z\parallel\vec{k}_i+\vec{k}_f); \tag{A7}$$

to render (A4) to the form

$$-\epsilon(2\pi)^{-1}\langle\vec{k}_f|\tau'|k_i\rangle=ik(2\pi)^{-1}\int d^2b\,e^{i\vec{q}\cdot\vec{b}}\Gamma(\vec{b}). \tag{A8}$$

The z axis is parallel to the velocity $\vec{v}$, thus perpendicular to $\vec{q}$. To this point nothing has been accomplished other than to define $\Gamma(\vec{b})$ via (A7).

To relate $\Gamma(\vec{b})$ to the physical two-particle scattering, introduce a momentum vector

$$\begin{aligned}&\vec{\kappa}_i=\tfrac{1}{2}\vec{q}+\vec{v}\epsilon_2(1-\lambda_2);\\ &|\vec{\kappa}_i|\equiv\kappa=k(\epsilon_2/\epsilon)=P_L m/\sqrt{s_2}.\end{aligned} \tag{A9}$$

Inspection of Eqs. (4) shows that $\vec{\kappa}_i$ has the correct magnitude for the c.m. momentum in two-particle scattering at the same lab energy we consider for the nuclear scattering.

Observe that Eq. (A5) rewritten in terms of $\vec{\kappa}_i$ takes the form

$$\{[\kappa^2-(\vec{p}+\vec{\kappa}_i)^2](2\epsilon_2)^{-1}-V(\vec{r})\}\,\phi(\vec{r})=0 \tag{A10}$$

or equivalently,

$$[(\kappa^2-p^2)(2\epsilon_2)^{-1}-V(\vec{r})]\,\psi^{(+)}(\vec{r};\vec{\kappa}_i)=0 \tag{A11}$$

if we define

$$\psi^{(+)}(\vec{r};\vec{\kappa}_i)=e^{i\vec{\kappa}_i\cdot\vec{r}}\phi(\vec{r}). \tag{A12}$$

Equation (A11) is the wave equation for two particle scattering and Eq. (A12) thus relates $\phi(\vec{r})$ to the two-particle scattering wave function for initial momentum $\vec{\kappa}_i$. Hence the two-particle c.m. scattering amplitude takes the form

$$\begin{aligned}f_{\text{c.m.}}(\vec{q})&=-\epsilon_2(2\pi)^{-1}\langle\vec{\kappa}_f|V|\psi^{(+)}(\vec{r};\vec{\kappa}_i)\rangle\\&=-\epsilon_2(2\pi)^{-1}\int d\vec{r}\,e^{i(\vec{\kappa}_i-\vec{\kappa}_f)\cdot\vec{r}}V(\vec{r})\phi(\vec{r}).\end{aligned} \tag{A13}$$

To evaluate Eq. (A13) at the same momentum transfer $\vec{q}$ which is considered for the nuclear matrix element, let $\vec{\kappa}_f=\vec{\kappa}_i-\vec{q}$. Note that $\epsilon_2/\kappa=\epsilon/k$, $|\vec{\kappa}_f|=|\vec{\kappa}_i|=\kappa$, and that $\frac{1}{2}(\vec{\kappa}_i+\vec{\kappa}_f)$ is parallel to $\vec{v}$ and thus also perpendicular to $\vec{q}$. Integration over z then produces

$$f_{\text{c.m.}}(\vec{q})=i\kappa(2\pi)^{-1}\int d^2b\,e^{i\vec{q}\cdot\vec{b}}\Gamma(\vec{b}), \tag{A14}$$

where $\Gamma(\vec{b})$ is exactly the same phase shift function previously defined in Eq. (A7). Since Eq. (A14) is the on-energy-shell scattering amplitude, we conclude that the nuclear matrix element of τ' involves the physical two-particle phase shifts embodied in $\Gamma(\vec{b})$. Equation (7) of the paper involves the further assumption that Eq. (A14) may be inverted by the standard method to obtain $\Gamma(\vec{b})$ once given $f_{\text{c.m.}}(\vec{q})$.

APPENDIX B

If in the Watson series, we approximate $\tau_i\approx t_i$ and $G\approx g$ [see Eqs. (19) and (20)] then the Glauber approximation is obtained in the form

$$\begin{aligned}T_G&=\sum_i t_i(1+gt_i)^{-1}(1+gT_G)\\&=\sum_i t_i+\sum_{i\neq j}t_igt_j+\sum_{i\neq j\neq k}t_igt_jgt_k+\cdots.\end{aligned} \tag{B1}$$

To verify this, note from the definition (17) of t_i that

$$V_{Ei}=t_i(1+gt_i)^{-1} \tag{B2}$$

and hence the Glauber T matrix satisfies

$$T_G=\sum_i V_{Ei}(1+gT_G). \tag{B3}$$

When operating on a plane wave $|\vec{k}_i\rangle$ in the nuclear c.m., the operator $1+gT_G$ creates an eikonal scattering wave function

$$\langle\vec{r}|\Psi^{(+)}_{\vec{k}_i}\rangle\equiv\langle\vec{r}|(1+gT_G)|\vec{k}_i\rangle=e^{i\vec{k}_i\cdot\vec{r}}e^{i\chi_+} \tag{B4}$$

which is a plane wave phase shifted by

$$\chi_+=-\int_{-\infty}^{\infty}dz'\sum_i U_{Ei}(|\vec{r}\,'-\vec{r}_i|),\quad \vec{r}\,'=(\vec{b},z'). \tag{B5}$$

The z direction lies parallel to the average momentum $\vec{k}_i + \vec{k}_f$. Similarly,

$$\langle \Psi^{(-)}_{\vec{k}_i} | \vec{r} \rangle \equiv \langle \vec{k}_f | (1 + T_G g) | \vec{r} \rangle = e^{-i \vec{k}_f \cdot \vec{r}} e^{i\chi_-}, \tag{B6}$$

where

$$\chi_- = -\int_z^\infty dz' \sum_i U_{Ei}(|\vec{r}' - \vec{r}_i|). \tag{B7}$$

When either of these expressions is used, the matrix element $\langle T_G \rangle$ defined in Eq. (1) is easily verified to take the form given in Eq. (6) after an integration over the z direction is carried out.

To display corrections to T_G, let

$$N_1 \equiv N + H_T - E_0; \tag{B8}$$

$$\tau_i = t_i + t_i', \tag{B9}$$

where Eq. (19) gives

$$t_i' = t_i G_2 (N_1 - N_2) G \tau_i$$
$$= [1 - t_i G_2 (N_1 - N_2) G]^{-1} t_i G_2 (N_1 - N_2) G t_i \tag{B10}$$

and

$$G = g + G', \tag{B11}$$

where Eq. (20) gives

$$G' = g N_1 G = (1 - g N_1)^{-1} g N_1 g. \tag{B12}$$

Then the Watson series can be expressed as

$$T = T_G + T'$$
$$= \sum_i (t_i + t_i')[1 + (g + G')(t_i + t_i')]^{-1}$$
$$\times [1 + (g + G')(T_G + T')] \tag{B13}$$

which defines the complete correction T' to the Glauber approximation in this model. If we retain just first order deviations (primed objects), the expression for T' is

$$T' \approx \sum_i \{ t_i'(1 + g t_i)^{-1}(1 + g T_G)$$
$$\times - t_i (1 + g t_i)^{-1}(G t_i + g t_i')(1 + g t_i)^{-1}$$
$$\times (1 + g T_G) + t_i (1 + g t_i)^{-1}(g T' + G' T_G) \} \tag{B14}$$

and the leading order terms in (B10) and (B12) yield

$$G' \approx g N_1 g, \quad t_i' \approx t_i g (N_1 - N_2) g t_i.$$

These perturbations introduce objects considered negligible in the Glauber approximation. Using $t_i(1 + g t_i)^{-1} = V_{Ei}$ and rearranging leads to

$$\left(1 - \sum_i V_{Ei} g\right) T' \approx \sum_i [t_i g (N_1 - N_2) g V_{Ei} - V_{Ei} g N_1 g V_{Ei} - V_{Ei} g t_i g (N_1 - N_2) g V_{Ei}](g T_G + 1) + \sum_i V_{Ei} g N_1 g T_G \tag{B15}$$

which defines T' in terms of known objects g, V_{Ei}, and t_i. Now from (B3) it can be shown that

$$\left(1 - \sum_i V_{Ei} g\right)^{-1} = 1 + T_G g.$$

Hence applying this operator from the left to both sides of Eq. (B15) we arrive at

$$T' \approx (1 + T_G g) \sum_i [- V_{Ei} g N_2 g V_{Ei} (g T_G + 1)$$
$$+ V_{Ei} g N_1 g T_G], \tag{B16}$$

where the N_2 term is the sole survivor of cancellations amongst terms in the first sum of Eq. (B15), after using $(1 - V_{Ei} g) t_i = V_{Ei}$.

Because of the simple relations (B4) and (B6), the relevant matrix element of T' takes a convenient form involving the phase shifted plane waves

$$\langle T' \rangle = \langle \Psi^{(-)}_{k_f}; E_0 | \left[N_1 - \sum_i V_{Ei} g N_2 g V_{Ei} \right] | \Psi^{(+)}_{k_i}; E_0 \rangle. \tag{B17}$$

To obtain Eq. (B17) we have used the fact that

$$\langle \Psi^{(-)}_{k_f} | \sum_i V_{Ei} g = \langle \Psi^{(-)}_{k_f} |$$

to simplify the N_1 term.

Explicit forms for the corrections can be obtained by using

$$N_1 = (\vec{p} - \vec{k}_f) \cdot (\vec{p} - \vec{k}_i)(2\epsilon)^{-1}$$
$$+ \sum_n [\vec{p}_n - (m/M)\vec{P}]^2 (2m)^{-1} + V_T - E_0, \tag{B18}$$

$$N_2 = (\vec{p} - \vec{k}_f) \cdot (\vec{p} - \vec{k}_i)(2\epsilon_2)^{-1}, \tag{B19}$$

where $\vec{p} = -i\vec{\nabla}$ acts on the projectile coordinate $\vec{r}$ while $\vec{p}_n = -i\vec{\nabla}_n$, $\vec{P} = \sum_n \vec{p}_n$ acts on the target nucleon coordinates. Terms involving $1 - \lambda$ and $1 - \lambda_2$ from Eqs. (11) and (14) are of order k^{-2} and are not included in the leading order (k^{-1}) calculation. Lest this omission appear arbitrary, the reader is referred to previous work which exhaustively justifies such a procedure.[18] Also the eikonal Green function is required

$$g(\vec{r}, \vec{r}') = -i(\epsilon/k)\delta^{(2)}(\vec{b}-\vec{b}')\theta(z-z')e^{i\vec{k}_i\cdot(\vec{r}-\vec{r}')} . \tag{B20}$$

We find that Eq. (B17) then takes the form

$$\langle T'\rangle = -\epsilon(2\pi)^{-1}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\langle E_0| \Big\{(2\epsilon)^{-1}[e^{i\chi_-}, \vec{p}]\cdot[\vec{p}, e^{i\chi_+}] + \sum_n (2m)^{-1}[e^{i\chi_-}, \vec{p}_n - (m/M)\vec{P}]\cdot[\vec{p}_n - (m/M)\vec{P}, e^{i\chi_+}]$$
$$+ \sum_i (2\epsilon_2)^{-1}\left[\int_z^\infty dz' e^{i\chi_-}U_{Ei}, \vec{p}\right]\cdot\left[\vec{p}, \int_{-\infty}^z dz' U_{Ei}e^{i\chi_+}\right]\Big\}|E_0\rangle . \tag{B21}$$

Because $H_T - E_0$ has been commuted to act on $|E_0\rangle$, it no longer appears in Eq. (B21). As a consequence, the Fermi motion effects arise solely from the kinetic energy part of H_T and are model independent, assuming that the binding interaction V_T commutes with the phase $e^{i\chi_\pm}$. Finally, because

$$[\vec{p}_n - (m/M)\vec{P}, e^{i\chi_+}] = -\vec{\nabla}\int_{-\infty}^z dz'\, U_E(\vec{r}' - \vec{r}_n)e^{i\chi_+} + (m/M)\vec{\nabla}\chi_+ e^{i\chi_+} ,$$

the Fermi motion effects take a form similar to the correction to eikonal propagation.

To proceed, it is necessary to approximate the N_2 type correction due to the intractable z integrations in Eq. (B21). To this end we first calculate to leading order in a multiple scattering expansion to find that the N_2 term gives

$$(4\pi)^{-1}(\epsilon/\epsilon_2)\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\langle E_0| \sum_{i=1}^{A}\int_{-\infty}^\infty dz\, \vec{\nabla}\chi_-(i)\cdot\vec{\nabla}\chi_+(i)|E_0\rangle , \tag{B22}$$

where $\chi_+(i) = \int_{-\infty}^z dz' U_E(\vec{r}' - \vec{r}_i)$ etc., and where extra phase factors $e^{i\chi_\pm}$ have been set to unity. On the other hand, if U_{Ei} in one of the z integrals of Eq. (B21) is replaced by the average $A^{-1}\sum_{i=1}^A U_{Ei}$, then both z integrals can be done via

$$-i\int_{-\infty}^z dz' \sum_i U_{Ei}e^{i\chi_+} = e^{i\chi_+} - 1 ,$$
$$-i\int_z^\infty dz' \sum_i U_{Ei}e^{i\chi_-} = e^{i\chi_-} - 1 .$$

Hence with the average approximation, we find that to all orders in multiple scattering, the N_2 term gives just

$$(4\pi)^{-1}(\epsilon/\epsilon_2)\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\langle E_0|e^{i\chi_0}A^{-1}\vec{\nabla}\chi_-\cdot\vec{\nabla}\chi_+|E_0\rangle , \tag{B23}$$

where

$$\chi_0 = \chi_+ + \chi_- = \int_{-\infty}^\infty dz \sum_{i=1}^A U_E(\vec{r} - \vec{r}_i) .$$

But now setting the Glauber phase factor $e^{i\chi_0}$ to unity does not reproduce the correct leading order result Eq. (B22). To reconcile this difference, we replace $A^{-1}\vec{\nabla}\chi_-\cdot\vec{\nabla}\chi_+$ by $\sum_i \vec{\nabla}\chi(i)\cdot\vec{\nabla}\chi_+(i)$ in Eq. (B23). Then evaluating $\langle T'\rangle$ with this approximation to the ϵ_2^{-1} part gives the following result:

$$\langle T'\rangle = (4\pi)^{-1}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\langle E_0|e^{i\chi_0}\Big\{[1-\epsilon/M]\sum_{i\neq j=1}^{A}\int_{-\infty}^\infty dz\, \vec{\nabla}\chi_-(i)\cdot\vec{\nabla}\chi_+(j) + [1-\epsilon/\epsilon_2 + (\epsilon/m)(1-m/M)]$$
$$\times\sum_{i=1}^{A}\int_{\infty}^\infty dz\, \vec{\nabla}\chi_-(i)\cdot\vec{\nabla}\chi_+(i)\Big\}|E_0\rangle \tag{B24}$$

which is equivalent to the form used in Eq. (25) of the paper.

*Work supported in part by the U. S. Air Force Office of Scientific Research under Contract No. F44620-70-C-0030 and in part by U. S. Atomic Energy Commission.

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