Backbending feature of rotational spectra in the generalized variable-moment-of-inertia model and its equivalence with the Harris model

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The equivalence of Harris model equations with those of the generalized variable-moment-of-inertia (GVMI) model given by Das *et al.* is examined in the light of the backbending feature of the rotational states. It is shown that this feature is absent in the Harris model taken to any order. The GVMI model equations are found to be consistent and in one-to-one correspondence with an expansion of the square of the angular velocity in terms of a polynomial in the moment of inertia rather than with the Harris expansion and may give a backbending feature in some cases depending on the relative values of the parameters appearing in the potential energy term.

[NUCLEAR STRUCTURE Variable moment of inertia, angular velocity, cranking model, backbending, high spin rotational states.]

One of the successful descriptions of low-lying rotational states of the ground-state band of even-even nuclei was proposed by Harris.[1] Retaining higher-order terms in ω (nuclear angular velocity) in the cranking model[2] Harris obtained the following expansions (we use the units $\hbar=1$):

$$E_I=\frac{\omega^2}{2}(\mathscr{I}_0+3C\omega^2+5D\omega^4+\cdots) \qquad (1)$$

and

$$[I(I+1)]^{1/2}=\omega(\mathscr{I}_0+2C\omega^2+3D\omega^4+\cdots)\,. \qquad (2)$$

Self-consistency of these equations is through the relation

$$\frac{dE_I}{d\omega}=\omega\frac{d}{d\omega}[I(I+1)]^{1/2}\,. \qquad (3)$$

One may eliminate ω up to any desired degree of approximation, and energy can be expressed in terms of the parameters $\mathscr{I}_0$, C, $D,\ldots$, etc. Another remarkably successful description is the variable-moment-of-inertia (VMI) model proposed by Mariscotti, Scharff-Goldhaber, and Buck.[3] The model makes use of the energy expression

$$E_I=\frac{I(I+1)}{2\mathscr{I}}+V(\mathscr{I})\,, \qquad (4)$$

with the potential energy term

$$V(\mathscr{I})=\frac{C_2}{2!}(\mathscr{I}-\mathscr{I}_0)^2\,, \qquad (5)$$

and the moment of inertia $\mathscr{I}$ being determined by the equilibrium condition

$$\left(\frac{\partial E_I}{\partial\mathscr{I}}\right)_I=0\,. \qquad (6)$$

A link between the two formulations was found by Mariscotti, Scharff-Goldhaber, and Buck[3] who, employing the semiclassical relation

$$\mathscr{I}\omega=[I(I+1)]^{1/2}\,, \qquad (7)$$

showed that the VMI model and the Harris two-parameter model equations are mathematically equivalent. Later, Klein, Dreizler, and Das[4] generalized the VMI model equations by taking the following expansion for the potential energy $V(\mathscr{I})$ in Eq. (4)

$$V(\mathscr{I})=\sum_{n=2}\frac{C_n}{n!}(\mathscr{I}-\mathscr{I}_0)^n \qquad (8)$$

and showed that the resulting equations of the generalized variable-moment-of-inertia (GVMI) model are formally equivalent to Eqs. (1) and (2) of the Harris model, if taken to *all* orders.

In this paper we reexamine the equivalence of the two descriptions in the light of recent experimental findings[5,6] on the high-spin states. As is now the well-established procedure, the experimental data are represented in terms of an $\mathscr{I}$-ω^2 plot, these quantities being derived from the experimental transition energies. Such plots[5,6] show that as the high-spin states are reached, $\mathscr{I}$ as a function of ω^2 deviates from a straight line (VMI behavior) and becomes a multivalued quantity, thus producing a "backbending" curve.

In the Harris description, the moment of inertia [Eqs. (2) and (7)] is given by

$$\mathscr{I}=\mathscr{I}_0+2C\omega^2+3D\omega^4+\cdots \qquad (9)$$

and is a single-valued function of ω^2. For the existence of backbending the derivative $d\mathscr{I}/d\omega^2$ should equal infinity at some point, and it is easily seen that such a point does not exist for any parameter values. Thus one can *never* describe the experimental feature of high-spin states in such an approach by including any number of terms. This is also demonstrated in a recent exhaustive calculation by Saethre *et al.*[7] using the Harris-model equations wherein they restrict themselves to low-spin ($I \leq 10$) states and show that increasing the number of parameters from two to four improves significantly the fit to these states, but as higher-spin states (multivalued region of $\mathscr{I}$) are reached even the four-parameter results are worse off.

We now show that the GVMI model equations have an altogether different $\mathscr{I}$-ω^2 behavior and in some cases can give rise to backbending. Firstly, we find that in the VMI approach, without specifying the dependence of the potential function $V(\mathscr{I})$, Eqs. (4), (6), and (7) give the following relationship between the square of the frequency and the potential function:

$$\omega^2 = 2\frac{dV}{d\mathscr{I}} . \qquad (10)$$

Secondly, we remark about an interesting observation that even in the region where $\mathscr{I}$ becomes a multivalued function of ω^2, the inverse relationship, i.e., ω^2 as a function of $\mathscr{I}$, remains single-valued (except in a case or two where downbending occurs as still more high-spin states are reached resulting in multivaluedness of ω^2 also). A study of ω^2-$\mathscr{I}$ plots[8] for several nuclei in the rare-earth region suggests that ω^2 can be nicely expressed in terms of a polynomial in $\mathscr{I}$, and backbending occurs at the points where $d\omega^2/d\mathscr{I}$ equals zero. Hence purely from a phenomenological point of view it is more appropriate to try for such a polynomial relationship. In fact, combining Eqs. (8) and (10) we get for the GVMI model equations:

$$\omega^2 = 2\sum_{n=0} \frac{C_n}{(n-1)!}(\mathscr{I}-\mathscr{I}_0)^{n-1} . \qquad (11)$$

The inverse procedure of getting the potential function from integration of ω^2 over $\mathscr{I}$ using Eq. (10), as long as ω^2 remains a single-valued function, has been suggested earlier by Thieberger.[9] Our conclusion is that the GVMI model equations are consistent and in one-to-one correspondence with the expansion (11) rather than with the Harris expansion (9). Equation (11) shows that for the VMI model ($n=2$) we have the relation $\omega^2 = 2C_2(\mathscr{I}-\mathscr{I}_0)$, which can be rewritten as $\mathscr{I} = \mathscr{I}_0 + \omega^2/2C_2$ and is therefore identical with the two-parameter cutoff of Harris, Eq. (9), if $C_2 = 1/4C$.

We do not present the results of any numerical calculations using Eq. (11), which is the same as using Eq. (8) for potential energy in the VMI approach. Varshni and Bose[10] have done extensive calculations (referred to as the VMI23 model) retaining two terms ($n=2$ and 3) in the expansion (8), and recently Das and Banerjee[11] have improved the calculations by including terms up to $n=4$. Here we discuss only the backbending feature of $\mathscr{I}$-ω^2 plots in the GVMI model which, as mentioned earlier, occurs at $\mathscr{I}$-ω^2 points for which $d\omega^2/d\mathscr{I}=0$. Using Eq. (11), this takes place for the GVMI model at $\mathscr{I}$ values given by the solutions of the equation

$$\sum_{n=0} \frac{C_n}{(n-2)!}(\mathscr{I}-\mathscr{I}_0)^{n-2} = 0 . \qquad (12)$$

Equation (12) may or may not have a physically acceptable solution depending on the relative values of the parameters, but it is clear that retaining m number of terms in the potential-energy expansion (8) we have at the most $(m-1)$ number of $\mathscr{I}$ points at which backbending may occur. Thus backbending is ruled out in the VMI model for which $m=1$. It must be noted, however, that the acceptability of a solution is finally to be decided by the energy equation. We show this for the case of the VMI23 model which retains the first two terms (involving C_2 and C_3) in the potential-energy expansion (8). Equation (12) then shows that backbending occurs in the VMI23 model for the $\mathscr{I}$ values given by

$$\mathscr{I} = \mathscr{I}_0 - C_2/C_3 . \qquad (13)$$

The parameter C_2 must be positive as a condition for stability, and, therefore, backbending occurs at physically acceptable values of $\mathscr{I}$ in *only* those nuclei for which C_3 is negative. However, as observed by Varshni and Bose,[10] in such cases the energies become complex beyond a certain spin level and are interpreted as providing a natural cutoff to the rotational band, which is not supported, however, by the experimental data. Extension of the VMI model by Das and Banerjee[11] retains three terms in the expansion (9), and Eq. (12) shows that in this case backbending may occur for two $\mathscr{I}$ values given by the roots of the quadratic equation

$$C_2 + C_3(\mathscr{I}-\mathscr{I}_0) + \tfrac{1}{2}C_4(\mathscr{I}-\mathscr{I}_0)^2 = 0 . \qquad (14)$$

Again the equations do not remain meaningful[11] in giving energies beyond a certain spin level for nuclei with parameter C_4 negative, but calculations performed for a representative set of nuclei show that the observed backbending is satisfactorily reproduced.

Thus we have shown that the equations of the GVMI model may produce a backbending feature, are consistent, and are in one-to-one correspondence with an expansion of the square of the angular velocity in a polynomial in the moment of inertia rather than with the Harris expansion, which can never give backbending. Incidentally, our derivation also shows the futility of the attempts of

Varshni and Bose[10] who, in order to show the correspondence of a special case of the VMI23 model with an expansion similar to that of Harris, included terms which are noninvariant with respect to time reversal.

Finally, we may mention that, whereas the theoretical foundation of the GVMI model is given by several workers,[12,13] it is also possible to derive Eq. (11) in the framework of a simple classical model given by Thieberger[14] to show the classical analog of the VMI model. Thieberger observed that in this model the deviations from the VMI predictions can be interpreted by modification of the relation involving potential-energy change and the mass term, and, in fact, higher-order terms ($n>2$) in Eq. (11) correspond to taking higher-order mass terms in the basic relation (3) of Ref. 14.

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