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Effective channel approach to nuclear scattering at high energies. I

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The elastic scattering of high-energy particles by composite target nuclei is formulated in terms of coupled equations in which the effect of inelastic processes is represented by an average inelastic channel. The average fluctuation energy and fluctuation potentials which specify this channel are explicitly constructed and shown to be related to the two- and three-particle correlation functions. The theory incorporates approximately the effects of nonlocality, energy-dependence, rescattering, and absorption of all the inelastic channels. Systematic improvements of the method as well as several extensions of the formalism to target excitations and particle exchanges are discussed.

[NUCLEAR REACTIONS Scattering theory of protons and pions by nuclei at high energies. Construction of the effective inelastic channels.]

I. INTRODUCTION

High energy scatterings of protons and pions off complex nuclei have been treated in the past mainly by two seemingly distinct approaches, the multiple diffraction theory (MDT) of Glauber[1] and the multiple-scattering-optical-potential theory (OPT) of Watson.[2,3] These theories attempt to correctly incorporate the distortion effect arising from the presence of inelastic channels, which is, of course, the principal difficulty of the scattering theory of composite systems. The simplifying assumption common to both approaches is the impulse picture in which the target particle binding and Fermi motion are negligible during the scattering.

The MDT is based essentially on the eikonal approximation on each of the target particle-projectile amplitudes, without the overlapping phases. The target particles are assumed "fixed" during the scattering, thus reducing the many-body problem to a sum of effective two-body situations. As a result, the theory is extremely simple to apply and numerous analyses[4,5] have been carried out with the MDT, with excellent results for high energy, small angle data. The theory often works well also at much lower energies and larger angles than that expected from the theoretical estimates, and as a consequence much effort has gone into understanding the MDT and obtaining various correction terms. There are some indications[6] that the corrections to the MDT arising from the linearization,[7] the Fermi motion,[8] and others cancel with each other to some extent.

The OPT approach requires a careful construction of the optical potential which contains all the effect of inelastic channels. Once this potential is available in some approximation, the amplitude can be calculated rather simply. In the course of simplifying the optical potential calculation, Feshbach and collaborators[9] converted the earlier result of Kerman, McManus, and Thaler[10] into a *finite* set of coupled equations which are to represent the elastic and all the inelastic channels. The result of their study strongly suggests that, with further refinements of the theory, it may be possible to extract dynamical information about the target system and about the interaction among the particles involved.

In this paper, we reexamine the effective channel approach of Ref. 9, and formulate a more consistent procedure of evaluating the various averaged quantities which are needed to specify the effective inelastic channels.[11] In its simplest form, all the inelastic channels are being replaced by a single effective channel. Therefore it is most critical that the average fluctuation potentials and average excitation energy are carefully evaluated. These points are discussed in Secs. II and III. An intuitive and physical discussion is first presented in order to bring out more clearly the content of the effective channel approach, which is then followed in Sec. IV by a more rigorous formulation using an effective inelastic excitation function. The connection between these average potentials and the many-particle correlation functions is also worked out explicitly. Section IV contains a brief discussion of various improvements and extensions of the formulation.

The representation of all the inelastic channel effect by a single effective channel may not be sufficiently accurate for some purposes and may be strongly model-dependent, but this problem can be systematically improved, if desired. The theory outlined for the elastic scattering is then extended to target excitation and particle rearrangement

and exchange processes. The actual application of the formalism to the proton-helium system will be given in paper II.[12] The pion scattering can also be treated with little modifications so long as we incorporate the pion absorptions phenomenologically.

II. EFFECTIVE CHANNEL THEORY

For simplicity of discussion, we consider the scattering of distinguishable particles by nuclei and neglect the exchange and spin effects as well as the relativistic kinematic corrections. Then, for the elastic scattering, we have the scattering equations[13]

$$P(H-E)P\Psi = -PVQ\Psi \ , \tag{2.1a}$$

$$Q(H-E)Q\Psi = -QVP\Psi \tag{2.1b}$$

with

$$\Psi = P\Psi + Q\Psi, \quad H = T(\vec{R}) + H_T(\vec{r}) + V(\vec{r},\vec{R}) \ ,$$

and the elastic channel is projected out by the projection operator P given by

$$P = |\psi_0(\vec{r})\rangle\langle\psi_0^*(\vec{r}')| \ . \tag{2.2}$$

The target is assumed to be in its ground state ψ_0 initially, where

$$H_T(\vec{r})\psi_n(\vec{r}) = E_n\psi_n(\vec{r}) \ , \tag{2.3}$$

while all the inelastic channels are contained in the space projected by

$$Q = 1 - P \ .$$

As usual, (2.1) may be formally uncoupled to obtain

$$P(H - E + VG^QV)P\Psi = 0 \ , \tag{2.4}$$

where

$$G^Q = [Q(E + i\epsilon - H)Q]^{-1} \ , \tag{2.5}$$

and similarly using (2.5),

$$Q\Psi = G^QVP\Psi \ . \tag{2.5a}$$

Obviously, the complication of the original scattering problem with H is now entirely contained in the "effective channel" Green's function G^Q. Once a simple and explicit form for G^Q is given, (2.4) can be solved immediately. Thus, it is of special interest in developing proper procedures to consider some simple approximations on G^Q which are used in high energy scattering problems.

A. Closure approximation

We start with the simplest approximation in which[14]

$$G^Q \to \frac{Q}{E - \bar{E}_c} \ , \tag{2.6}$$

where $\bar{E}_c$ could be energy-dependent. For a projectile energy large compared with typical interaction energies between particles, the scattering becomes semiclassical, and the contribution of many *soft* collisions may be effectively described by a smooth local interaction generated by (2.6) for small angle scattering, as

$$P\left(H - E + \frac{VQV}{E - \bar{E}_c}\right)P\Psi_c = 0 \tag{2.7}$$

and

$$Q\Psi_c = \frac{1}{E - \bar{E}_c} QVP\Psi_c \ . \tag{2.8}$$

Both in (2.7) and (2.8), the proper evaluation of $\bar{E}_c$ is essential in estimating the amplitude, where $\bar{E}_c$ is in general expected to be complex when the Q channels are open. A form such as (2.6) has been used previously for bound states[15,16] and also for low energy scattering[17,18,14] problems. When the effect of the G^Q term in (2.4) is large, (2.6) can at best be only a crude first approximation.

B. High energy (slow) collisions of heavy particles

Although (2.6) reduces (2.4) to a simple form (2.7), the replacement of the entire QHQ by a single constant $\bar{E}_c$ may be too drastic an approximation. When high energy heavy ions are involved, the collision may take place with little change in the projectile kinetic energy, but *slow* enough to significantly affect the target structure. Then, an improved approximation would retain the $H_T(\vec{r})$ term in QHQ but replace $Q(T + V - E_0')Q$ by its average $\bar{U}(\vec{r}) + \bar{E}_s$, with $E_0' \equiv E - E_0$, i.e.,

$$G^Q \to \frac{1}{Q(E_0 - H_T(\vec{r}) - \bar{U}(\vec{r}) - \bar{E}_s)Q} \ . \tag{2.9}$$

Equation (2.4) reduces to

$$P\left[H - E + V\frac{Q}{Q(E_0 - H_T - \bar{U} - \bar{E}_s)Q}V\right]P\Psi_s = 0 \tag{2.10}$$

and similarly for $Q\Psi_s$, with

$$E \equiv E_n + E_n' \quad (n = 0, 1, 2, \ldots).$$

Presumably, (2.9) is also an improvement over the usual *adiabatic* approximation without the average quantities $\bar{U}$ and $\bar{E}_s$, where

$$G^Q \to \frac{1}{Q(E_0 - H_T)Q} \quad \text{or} \quad \frac{1}{Q(E_0 - H_T - V)Q} \ . \tag{2.11}$$

The effect of the projectile motion on the distortion potential is approximately taken into account in (2.9) through $\bar{U}(\vec{r})$ and $\bar{E}_s$. Note that both (2.9) and (2.11) are local in the projectile coordinate $\vec{R}$ and related to the polarizability of the target nu-

clei. (If the projectile has its own structure as in the heavy ion reactions, distortions of both systems are involved.)

C. Fast collisions

In complete analogy to (2.9), (2.6) may be improved for high energy fast collisions by[9]

$$Q(H_T+V-E_0)Q \to \overline{V}(\vec{R})+\overline{E}_F-E_0 \tag{2.12}$$

and thus

$$G^Q \to \frac{Q}{E-T(\vec{R})-\overline{V}(\vec{R})-\overline{E}_F+i\epsilon} \ . \tag{2.13}$$

Except for the projection Q, the target coordinate $\vec{r}$ is completely eliminated in (2.13), so that we have effectively a two-body scattering problem for G^Q. That is, the target is providing a distortion $\overline{V}$ and energy shift $\overline{E}_F$ during the collision. The resulting scattering equation is

$$P\left[H-E+V\,\frac{Q}{E+i\epsilon-T(\vec{R})-\overline{V}(\vec{R})-\overline{E}_F}\,V\right]P\Psi_F=0 \tag{2.14}$$

and (2.1) reduces to a simple set of coupled equations

$$P(H-E)P\Psi_F=-PVQ\Psi_F \ , \tag{2.15a}$$

$$Q(T+\overline{V}-E'_F)Q\Psi_F=-QVP\Psi_F \ , \tag{2.15b}$$

where $E'_F \equiv E-\overline{E}_F$. Relations (2.15) are the basic equations we will analyze in detail in the following sections, with regard to the evaluation of $\overline{V}$ and E'_F.

The elastic amplitude is given by

$$\mathcal{T}_{el}=(\Phi_f, V\Psi_i)=(\Phi_f, V[P\Psi_i+Q\Psi_i]) \ , \tag{2.16}$$

where Φ_f is an unscattered plane wave with the target in its ground state. We now rewrite (2.16) in a form more convenient for the discussion of $\overline{V}$, $\overline{U}$, $\overline{E}_F$, and $\overline{E}_c$ given in Sec. III. From (2.1a)

$$P\Psi_i=P\Psi_i^P+G_i^PVQ\Psi_i \ . \tag{2.17}$$

where

$$\begin{aligned} &P(H-E)P\Psi_i^P=0 \ , \\ &P(H-E)G_i^P=-P \end{aligned} \tag{2.18}$$

and similarly for $P\Psi_f$. Then, we have from the two-potential formula[3]

$$\begin{aligned} \mathcal{T}_{el}&=(\Phi_f, VP\Psi_i^P)+(P\Psi_f^P, VG^QVP\Psi_i) \\ &=\mathcal{T}_{el}^P+(P\Psi_f^P, V\mathcal{G}^QVP\Psi_i^P) \ , \end{aligned} \tag{2.19}$$

where

$$\mathcal{T}_{el}^P \equiv (\Phi_f, VP\Psi_i^P) \tag{2.20}$$

and

$$\begin{aligned} \mathcal{G}^Q&=[Q(E+i\epsilon-H-VG_i^PV)Q]^{-1} \\ &\equiv[Q(E+i\epsilon-\mathcal{H})Q]^{-1} \ . \end{aligned}$$

The form (2.19) is especially useful, because the second term on the right hand side is *symmetric* and involves the simple functions $P\Psi_i^P$ and $P\Psi_f^P$. For convenience we denote the "energy-weighted" excitation functions related to them as

$$\begin{aligned} &\chi_i \equiv QVP\Psi_i^P=(H-E)P\Psi_i^P \ , \\ &\chi_f \equiv QV^*P\Psi_f^P=(H^*-E)P\Psi_f^P \ . \end{aligned} \tag{2.21}$$

Then, (2.19) becomes

$$\mathcal{T}_{el}=\mathcal{T}_{el}^P+(\chi_f, \mathcal{G}^Q\chi_i) \ . \tag{2.22}$$

III. AVERAGE FLUCTUATION POTENTIALS

The approximations on G^Q introduced in (2.6), (2.9), and (2.13) involve average quantities $\overline{E}_s$, $\overline{E}_c$, $\overline{E}_F$, $\overline{U}$, and $\overline{V}$, which are left unspecified so far. The effective channel formulation is incomplete without a proper way of evaluating these quantities, and we consider in this section a variational procedure to estimate them.[14] For this purpose, we return to the expression (2.24) for the elastic amplitude and consider the closure approximation

$$(\chi_f, \mathcal{G}^Q\chi_i) \to (\chi_f, \chi_i)/(E-\bar{\mathcal{E}}) \ . \tag{3.1}$$

If we were to approximate $\mathcal{G}^Q$ by a simple, separable, *variational form*[18] as an alternative to (3.1), i.e.,

$$\mathcal{G}^Q=\frac{1}{Q(E+i\epsilon-\mathcal{H})Q} \approx \frac{Q\varphi_t)(Q\tilde{\varphi}_t}{(Q\tilde{\varphi}, [E+i\epsilon-\mathcal{H}]Q\varphi)} \ , \tag{3.2}$$

then (3.1) and (3.2) give a relationship between $\bar{\mathcal{E}}$ and the trial functions $Q\varphi_t$ and $Q\tilde{\varphi}_t$, i.e.,

$$\frac{(\chi_f, \varphi_t)(\tilde{\varphi}_t, \chi_i)}{(Q\tilde{\varphi}_t, [E+i\epsilon-\mathcal{H}]Q\varphi_t)} \approx \frac{(\chi_f, \chi_i)}{E-\bar{\mathcal{E}}} \ . \tag{3.3}$$

By rearranging the terms in (3.3), we finally obtain

$$\begin{aligned} \bar{\mathcal{E}} \approx E\left[1-\frac{(\chi_f, \chi_i)(Q\tilde{\varphi}_t, Q\varphi_t)}{(\chi_f, \varphi_t)(\tilde{\varphi}_t, \chi_i)}\right] \\ +\frac{(\chi_f, \chi_i)}{(\chi_f, \varphi_t)(\tilde{\varphi}_t, \chi_i)}\,(\tilde{\varphi}_t, \mathcal{H}\varphi_t) \ . \end{aligned} \tag{3.4}$$

Obviously, by choosing a higher rank approximation to $\mathcal{G}^Q$ in (3.2), it should be possible to improve the accuracy of $\bar{\mathcal{E}}$. However, we limit our discus-

sion to the simple form (3.2) and (3.4), which further simplifies by the choice[16,17,14]

$$Q\varphi_t = \chi_i \quad \text{and} \quad Q\bar{\varphi}_t = \chi_f \ . \tag{3.5}$$

(3.4) then reduces to[14]

$$\bar{g} \approx \frac{(\chi_f, \mathcal{H}\chi_i)}{(\chi_f, \chi_i)} = \frac{(P\Psi_f^P, PVQ[T+V+H_T+VG^PV]QVP\Psi_i^P)}{(P\Psi_f^P, PVQVP\Psi_i^P)} \ . \tag{3.6}$$

The various terms in (3.6) immediately suggest that we may also set for $\bar{E}_c$ in (2.6),

$$\bar{E}_c \approx \frac{(\chi_f, QHQ\chi_i)}{(\chi_f, \chi_i)} \equiv \frac{A(H)}{B} \tag{3.7}$$

which is obtained by omitting the shift operator $VG^PV = \mathcal{H} - QHQ$ in (3.6).

Writing out (3.7) explicitly, we have

$$B = (P\Psi_f^P[\,V^2 - \langle V\rangle^2\,]\,P\Psi_i^P) \tag{3.8}$$

and

$$A(H) \equiv A(T) + A(V) + A(H_T) \ , \tag{3.9}$$

where

$$A(T) = (P\Psi_f^P, [\,VTV - \langle V\rangle T\langle V\rangle\,]P\Psi_i^P) \ , \tag{3.10}$$

$$A(V) = (P\Psi_f^P, [\,V^3 - 2V^2\langle V\rangle + \langle V\rangle^3]P\Psi_i^P), \tag{3.11}$$

$$A(H_T) = (P\Psi_f^P, [VH_TV - E_0\langle V\rangle^2]P\Psi_i^P) \ , \tag{3.12}$$

and where

$$\langle V^n\rangle \equiv \int d\vec{r}\,|\psi_0(\vec{r})|^2 V^n(\vec{r},\vec{R}) \ . \tag{3.13}$$

Note that $\langle\ \rangle$ in (3.13) denotes the $d\vec{r}$ integration only with the weight $|\psi_0|^2$ included. We will also use $\langle\ \rangle_{\vec{r}}$, with the subscript $\vec{r}$, for the $d\vec{r}$ integration when ψ_0 is explicitly indicated. Although the form (3.7) is perhaps the simplest, its accuracy is not yet fully investigated. However, in view of its usefulness in a wide range of high energy scattering problems involving electrons, pions, and nucleons, it warrants a more extensive analysis, both in formalism and in applications.

If, instead of the approximation (2.6), we choose (2.12), then the construction of the Q channel requires the evaluation of $\overline{V}$ and $\overline{E}_F$. As the approximation (2.12) requires quantities which are averaged over the target variables *only*, we may write for $\overline{V}$ and $\overline{E}_F$

$$\overline{V} + \overline{E}_F - E_0 \approx \frac{\langle\psi_0, VQ\,[H_T+V-E_0]QV\psi_0\rangle_{\vec{r}}}{\langle\psi_0, VQV\psi_0\rangle_{\vec{r}}} \xrightarrow[r\to\infty]{} \overline{E}_F - E_0 \ . \tag{3.14}$$

This form is suggested by (3.6) and by comparing the approximations involved in (2.6) and (2.13). As will be shown in the next section, (3.14) is correct except for the factor $(J_{\varphi\varphi})$ which arises from the noncommutativity of the averaging procedure and the operator $T(\vec{R})$. More explicitly, the denominator of (3.14) can be written as

$$\begin{aligned}\langle\psi_0, VQV\psi_0\rangle_{\vec{r}} &= \langle\psi_0, V(1-P)V\psi_0\rangle_{\vec{r}} \\ &= \langle\psi_0, V^2\psi_0\rangle_{\vec{r}} - \langle\psi_0, V\psi_0\rangle_{\vec{r}}^2 \\ &\equiv \langle V^2\rangle - \langle V\rangle^2 \ ,\end{aligned} \tag{3.14a}$$

while the numerator becomes

$$\begin{aligned}&\langle\psi_0, VQ\,[H_T+V-E_0\,]QV\psi_0\rangle_{\vec{r}} \\ &\quad = \langle\psi_0, VH_T(1-P)V\psi_0\rangle_{\vec{r}} + \langle\psi_0, VQVQV\psi_0\rangle_{\vec{r}} \\ &\qquad - E_0\langle\psi_0, VQV\psi_0\rangle_{\vec{r}} \\ &\quad = [\langle VH_TV\rangle - E_0\langle V\rangle^2] + [\langle V^3\rangle - 2\langle V^2\rangle\langle V\rangle \\ &\qquad + \langle V\rangle^3] - E_0[\langle V^2\rangle - \langle V\rangle^2] \ .\end{aligned} \tag{3.14b}$$

The input for the evaluation of the average potentials and energies are closely related to the correlation functions among the nucleons inside the nucleus.[3,9] The $\tilde{C}$'s defined by (3.18) and (3.21) below carry the correlation information. From the definition for V

$$V(\vec{r},\vec{R}) = \sum_{i=1}^{N} v_i(\vec{r}_i - \vec{R}) \to Nv(\vec{r}_1 - \vec{R}) \ ,$$

for v's inside the integrations involving the properly symmetrized function ψ_0, we have

$$\tilde{V}(\vec{q}) \equiv \frac{N}{(2\pi)^{3/2}} \int d\vec{x}\, e^{-i\vec{q}\cdot\vec{x}} v(\vec{x}) \equiv N\tilde{v}(\vec{q}) \ . \tag{3.15}$$

If we define

$$W_2 \equiv \langle V^2\rangle - \langle V\rangle^2 \ , \qquad W_3 \equiv \langle V^3\rangle - 2\langle V\rangle\langle V^2\rangle + \langle V\rangle^3 \ \text{ etc.}, \tag{3.16}$$

then

$$\tilde{W}_2(\vec{q},\vec{q}') \equiv \Big\langle\psi_0, \sum_{i=1}^{N}\sum_{j=1}^{N} \exp(i\vec{q}\cdot\vec{r}_i + i\vec{q}'\cdot\vec{r}_j)\tilde{v}(\vec{q})\tilde{v}(\vec{q}')\psi_0\Big\rangle_{\vec{r}} - \Big\langle\psi_0, \sum_{i=1}^{N} e^{i\vec{q}\cdot\vec{r}_i}\tilde{v}(\vec{q})\psi_0\Big\rangle\Big\langle\psi_0, \sum_{j=1}^{N} e^{i\vec{q}'\cdot\vec{r}_j}\tilde{v}(\vec{q}')\psi_0\Big\rangle_{\vec{r}}$$

$$= N^2\tilde{v}(\vec{q})\tilde{v}(\vec{q}')\tilde{C}_2(\vec{q},\vec{q}') \ , \tag{3.17}$$

where

$$\tilde{C}_2(\vec{q},\vec{q}')=\tilde{D}_2(\vec{q},\vec{q}')-\tilde{D}_1(\vec{q})\tilde{D}_1(\vec{q}') \tag{3.18}$$

and

$$\tilde{D}_1(\vec{q})=\frac{1}{N}\sum_{i=1}^{N}\langle\psi_0, e^{i\vec{q}\cdot\vec{r}_i}\psi_0\rangle_{\vec{r}}=\tilde{C}_1(\vec{q})\ , \tag{3.19}$$

$$\tilde{D}_2(\vec{q},\vec{q}')=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\langle\psi_0,\exp(i\vec{q}\cdot\vec{r}_i+i\vec{q}'\cdot\vec{r}_j)\psi_0\rangle_{\vec{r}}\ \text{ etc.} \tag{3.20}$$

Note that $\tilde{C}_2(\vec{q},\vec{q}')$, from its definition, contains "self-correlation" terms of the form

$$\frac{1}{N}\langle\psi_0, e^{i(\vec{q}+\vec{q}')\cdot\vec{r}}\psi_0\rangle_{\vec{r}}\ ,$$

as well as the correlations which are of dynamical origin. We have not taken them out in our definition of the $\tilde{C}$'s. Obviously, $\tilde{D}_n(\vec{q},\ldots,\vec{q}_n)$ contains both connected and disconnected parts, and can be expressed entirely in terms of the fully connected correlation functions $\tilde{C}_m(\vec{q},\ldots,\vec{q}_m)$, for $1\leq m\leq n$. The cluster expansion of this kind has been given previously,[19] and we have for example

$$\tilde{C}_3(\vec{q}_1\vec{q}_2\vec{q}_3)=\tilde{D}_3(\vec{q}_1\vec{q}_2\vec{q}_3)-\tilde{D}_2(\vec{q}_1\vec{q}_2)\tilde{D}_1(\vec{q}_3)-\tilde{D}_2(\vec{q}_2\vec{q}_3)\tilde{D}_1(\vec{q}_1)-\tilde{D}_2(\vec{q}_3\vec{q}_1)\tilde{D}_1(\vec{q}_2)+2\tilde{D}_1(\vec{q}_1)D_1(\vec{q}_2)\tilde{D}_1(\vec{q}_3)\ , \tag{3.21}$$

where

$$\tilde{D}_3(\vec{q}_1\vec{q}_2\vec{q}_3)\equiv\frac{1}{N^3}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\langle\psi_0,\exp(i\vec{q}_1\cdot\vec{r}_i+i\vec{q}_2\cdot\vec{r}_j+i\vec{q}_3\cdot\vec{r}_k)\psi_0\rangle_{\vec{r}}\ . \tag{3.22}$$

The Fourier transform of the W_3 function which appears in (3.16) may be written as

$$\begin{aligned}\tilde{W}_3(\vec{q}_1\vec{q}_2\vec{q}_3)&=N^3\tilde{v}(\vec{q}_1)\tilde{v}(\vec{q}_2)\tilde{v}(\vec{q}_3)\{\tilde{D}_3(\vec{q}_1\vec{q}_2\vec{q}_3)-\tfrac{2}{3}[\tilde{D}_1(\vec{q}_1)\tilde{D}_2(\vec{q}_2\vec{q}_3)+\tilde{D}_1(\vec{q}_2)\tilde{D}_2(\vec{q}_1\vec{q}_3)+\tilde{D}_1(\vec{q}_3)\tilde{D}_2(\vec{q}_2\vec{q}_1)]\\&\qquad+\tilde{D}_1(\vec{q}_1)\tilde{D}_1(\vec{q}_2)\tilde{D}_1(\vec{q}_3)\}\\&=N^3\tilde{v}(\vec{q}_1)\tilde{v}(\vec{q}_2)\tilde{v}(\vec{q}_3)\{\tilde{C}_3(\vec{q}_1\vec{q}_2\vec{q}_3)+\tfrac{1}{3}[\tilde{C}_1(\vec{q}_1)\tilde{C}_2(\vec{q}_2\vec{q}_3)+\tilde{C}_1(\vec{q}_2)\tilde{C}_2(\vec{q}_1\vec{q}_3)+\tilde{C}_1(\vec{q}_3)\tilde{C}_2(\vec{q}_2\vec{q}_1)]\}\ ,\end{aligned} \tag{3.23}$$

which has been made symmetric in all three $\vec{q}$'s by averaging.

The evaluation of (3.14) for $\overline{E}_F$ is more involved because of the appearance of the target Hamiltonian H_T. This requires that we construct a specific Hamiltonian H_T from which ψ_0 and E_0 are generated. Thus, a simple spin-averaged form for H_T is, e.g.,

$$\begin{aligned}H_T&=\sum_{i=1}^{N}\nabla_i^2\left(-\frac{\hbar^2}{2m}\right)+\sum_{i=1}^{N}v_c(\vec{r}_i)+\sum_{i>j}v(\vec{r}_i-\vec{r}_j)\\&\equiv t_T+U_T\ ,\end{aligned} \tag{3.24}$$

where v_c is the particle-core interaction for the model assumed here. Although H_T with v_c and v appears explicitly in (3.14), the actual form for U_T is not required in the evaluation of $\overline{E}_F+\overline{V}$ because of a partial cancellation between the first and third terms on the right hand side of (3.14b) when (2.3) is used. Of course, the effect of U_T is reflected in ψ_0 and E_0. We define

$$Y_H(\vec{R})\equiv\langle\psi_0, VH_TV\psi_0\rangle_{\vec{r}}-E_0\langle V\rangle^2 \tag{3.25}$$

from which, of course,

$$A(H_T)=(P\Psi_f^P, Y_H P\Psi_i^P)\ . \tag{3.26}$$

Let

$$\tilde{M}(\vec{q},\vec{q}')\equiv\frac{1}{N^3}\sum_{i=1}^{N}\sum_{j=1}^{N}\left\langle\exp(i\vec{q}\cdot\vec{r}_i+i\vec{q}'\cdot\vec{r}_j)\left(-\frac{\hbar^2}{2m}\right)[-\tfrac{1}{2}\psi_0^*(\vec{q}^2+\vec{q}'^2)\psi_0+i\psi_0\vec{q}'\cdot\vec{\nabla}_j\psi_0^*+i\psi_0^*\vec{q}\cdot\vec{\nabla}_i\psi_0]\right\rangle_{\vec{r}}\ . \tag{3.27}$$

Then,

$$\tilde{Y}_H(\vec{q},\vec{q}')=N^3\tilde{v}(\vec{q})\tilde{v}(\vec{q}')[\tilde{M}(\vec{q},\vec{q}')+\frac{1}{N}E_0\tilde{C}_2(\vec{q},\vec{q}')]\ . \tag{3.28}$$

In terms of these functions, we finally have

$$\overline{E}_F\equiv\lim_{R\to\infty}[Y_H(\vec{R})/W_2(\vec{R})], \tag{3.29}$$

provided the limit $R\to\infty$ exists, and

$$\overline{V}(\vec{R}) \equiv \frac{W_3(\vec{R}) + Y_H(\vec{R})}{W_2(\vec{R})} - \overline{E}_F + J_{\varphi\varphi} , \qquad (3.30)$$

where we have added the term $J_{\varphi\varphi}$, the presence of which will be discussed in the next section. This additional term $J_{\varphi\varphi}$ is missing in the above derivation because the averaging of the operator QHQ over the function φ, as we have done in (3.7) for example, does not in general commute with the kinetic energy operator $T(\vec{R})$ contained in QHQ. This point is often overlooked in many previous formulations, and we present in the next section a derivation which is mathematically more consistent.

IV. IMPROVEMENTS AND EXTENSIONS

In this section, we consider several aspects of the elastic scattering formulation presented in Secs. II and III which may be improved and extended.

A. Variational formulation of the effective channel

The intuitive derivation of (2.15) and (3.14) overlooked one important aspect of the averaging procedure carried out over the target variable $\vec{r}$. Since the replacement of the operator $(H_T + V - E_0)$ by its averaged form should depend on the scattering solution, the spectral averaging procedure does *not* in general *commute* with the projectile kinetic energy operator $T(\vec{R})$. To correct for this, we may start with an ansatz for the form of the wave function Ψ. This will give a mathematically more rigorous derivation of the coupled equations, although the relevant physical picture is obscured somewhat. Thus, we write[11]

$$\Psi(\vec{r}, \vec{R}) = P\Psi + Q\Psi \approx \psi_0(\vec{r}) u_0(\vec{R}) + \varphi(\vec{r}, \vec{R}) w(\vec{R}) \equiv \Psi_t , \qquad (4.1)$$

with the choice for the average excitation function φ of the form

$$\varphi(\vec{r}, \vec{R}) = N_R Q V \psi_0 = N_R (V - \langle V \rangle) \psi_0(\vec{r}) . \qquad (4.2)$$

The normalization factor N_R is given explicitly by

$$N_R^{-2} = \langle \psi_0, VQV\psi_0 \rangle_{\vec{r}} \equiv W_2(\vec{R}) .$$

Such a normalization factor can always be absorbed into $w(\vec{R})$ but the inclusion of N_R in φ simplifies the resulting equations as will be seen below.

Now, we construct a variational functional of the form

$$I = \mathcal{T}_t + (\tilde{\Psi}_t, [H - E] \Psi_t) , \qquad (4.3)$$

where $\mathcal{T}_t$ denotes the asymptotic amplitude of the trial function Ψ_t and

$$\tilde{\Psi}_t = \psi_0(\vec{r}) \tilde{u}_0(\vec{R}) + \tilde{\varphi}(\vec{r}, \vec{R}) \tilde{w}(\vec{R}),$$

in which $\tilde{\varphi} = \varphi^*$. Taking the variations with respect to $\tilde{u}_0^*$ and $\tilde{w}^*$, as

$$\delta I / \delta \tilde{u}_0^* = 0 \quad \text{and} \quad \delta I / \delta \tilde{w}^* = 0 , \qquad (4.4)$$

we obtain the set of coupled equations

$$[T + \langle V \rangle - E_0'] u_0(\vec{R}) = - V_{0\varphi} w , \qquad (4.5a)$$

$$[T + V_{\varphi\varphi} + E_F + J_{\varphi\varphi} - E] w = - V_{\varphi 0} u_0 , \qquad (4.5b)$$

where

$$V_{\varphi\varphi} = \langle \tilde{\varphi}, V\varphi \rangle_{\vec{r}} ,$$
$$E_F = \langle \tilde{\varphi}, H_T \varphi \rangle_{\vec{r}} \xrightarrow[R \to \infty]{} \overline{E}_F , \qquad (4.6)$$
$$J_{\varphi\varphi} = \langle \tilde{\varphi}, [T\varphi] \rangle_{\vec{r}} = \int d\vec{r}\, \tilde{\varphi}^* [T\varphi] ,$$

and

$$V_{\varphi 0} = V_{0\varphi} = \langle \tilde{\varphi}, V\psi_0 \rangle_{\vec{r}} = N_R W_2 = W_2^{1/2} . \qquad (4.7)$$

Comparing (4.6) with (3.30), we can identify

$$\overline{V} = V_{\varphi\varphi} + E_F + J_{\varphi\varphi} - \overline{E}_F ,$$

thus further exhibiting the mutual consistency of the choice (4.2) for φ and the origin of $J_{\varphi\varphi}$. For the perturbation theory argument for the choice of φ, we refer to Ref. 11.

We now briefly compare (4.5) with (2.15). The operators on the left hand sides of the two sets are exactly the same; on the other hand, the coupling potentials in (2.15) are still dependent on the variable $\vec{r}$, while $V_{0\varphi}$ in (4.5) is dependent on the variable $\vec{R}$ only. Therefore, (4.5) is definitely simpler to apply and will be studied in great detail in connection with the proton-helium scattering as reported on subsequently.

Incidentally, (2.15) can still be solved for $P\Psi_F$ and $Q\Psi_F$ using the Green's function $(E_F' + i\epsilon - T - \overline{V})^{-1}$, although the operator Q is still present explicitly.

B. Average excitation functions

The derivation of (4.5) does not depend on the specific choice (4.2) of the excitation function φ, so long as it represents a function in the Q space. Thus, we can generalize the form to be[11,12]

$$\varphi = N_R Q \hat{V} \psi_0 , \qquad (4.8)$$

where $\hat{V}$ may in general contain one or more variational parameters. Some of these parameters could be determined by additional experimental information, such as the total cross section through the optical theorem.[11,12]

When the projectile-nucleon interaction is very short-ranged, the form (4.2) will contain an excessive amount of high momentum component,

while we expect physically that the φ's should be spread out more in the coordinate space than the ground state ψ_0. This unsatisfactory situation is often brought about by the closure approximation introduced in (2.12) for the target Hamiltonian H_T. That is, the approximation

$$[Q(E_0 - H_T)Q]^{-1} \approx Q(E_0 - \bar{E}_F)^{-1}Q$$

automatically places too much weight on the higher excited states of H_T. A smoother $\hat{V}$ than V often compensates partially for this error.

The elastic amplitude calculated may turn out to be sensitive to the particular form of φ chosen. This difficulty can easily be treated by simply including additional excitation functions in Ψ_t. Thus, we may write

$$\Psi_t = \psi_0(\vec{r})u_0(\vec{R}) + X_M(\vec{r},\vec{R}) , \tag{4.9}$$

where

$$X_M = \sum_{\alpha=1}^{M} \varphi_\alpha(\vec{r},\vec{R})w_\alpha(\vec{R}) . \tag{4.10}$$

To preserve the Q-space character of φ_α, we still take the form

$$\varphi_\alpha = N_\alpha Q\hat{V}_\alpha\psi_0 , \tag{4.11}$$

where the $Q\hat{V}_\alpha P$ may be chosen to be approximately orthogonal to each other in order to optimize the efficiency and minimize cancellations.[18] As M increases, the sensitivity of the amplitude on the form for φ_α should decrease.

C. Inelastic excitations and rearrangement scatterings

When the particular final state of interest is weakly coupled to the initial channel i, the Ψ_t of the form (4.1) or (4.9) would be sufficient to obtain the transition amplitude for $i \to f$. The well-known distorted wave Born approximation procedure can also be used. Thus, we consider here only those reactions in which the channels i and f are strongly coupled.

For the inelastic excitations $i \to f$, we may set simply

$$\Psi_t \approx \psi_i u_i + \psi_f u_f + X_M , \tag{4.12}$$

where X_M is now in the Q space defined by $Q = 1 - P_i - P_f$ with $P_iP_f = 0$ and $P_iQ = P_fQ = 0$. (Again, the orthogonality requirement of X_M is not necessary but convenient in improving the accuracy of the theory.) Experimental data have improved sufficiently in recent years to make such analysis meaningful.

The rearrangement and exchange channels are more difficult to incorporate because of the nonorthogonality between these channels. Although the exchange effect of the projectile nucleon and the target nucleons is probably negligible at high energies,[20] the situation is bound to change as the energy is lowered, and also for larger angles. The processes such as (p,d) with neutron pickup should be useful in probing the neutron distribution inside the target. As a typical example, we consider the (p,d) reaction, and write the reduced matrix equations[21,22] for it, as

$$\begin{aligned} (H_p - E)\Psi_p &= -V_d\Psi_d , \\ (H_d - E)\Psi_d &= -V_p\Psi_p , \end{aligned} \tag{4.13}$$

where, in the obvious notation,

$$\Psi = \Psi_p + \Psi_d \tag{4.14}$$

and

$$H = H_p + V_p = H_d + V_d .$$

This form avoids the nonorthogonality problem,[22] $P_dP_p \neq 0$. Following (4.9), we set

$$\begin{aligned} \Psi_p &\approx \Psi_{pt} = \psi_{p0}u_{p0} + Q_pX_{Mp} , \quad Q_p = 1 - P_p , \\ \Psi_d &\approx \Psi_{dt} = \psi_{d0}u_{d0} + Q_dX_{Nd} , \quad Q_d = 1 - P_d . \end{aligned} \tag{4.15}$$

The detailed discussions of these equations will be omitted, as they have already been given elsewhere.[22]

D. v-t_F conversion

The theoretical description of scattering processes in terms of the two-particle potential v is basically a low energy approach and is on a less solid foundation at higher energies mainly because of the importance of the particle creations and the coupling to many other species of particles, as well as for the fact that the two-nucleon interaction can be singular at small distances. Therefore, the formalism with the v's is often rewritten such that the v's are replaced by the two-particle transition operator t_F. The use of the t_F has additional advantages in that the on-shell information of t_F can be obtained directly from the experimental data and that the t_F already contains all the multiple scattering information pertaining to a given scattering pair. However, this modification necessarily involves approximations, as will be discussed below.

We start with the Schrödinger equation:

$$(T + H_T + V - E)\Psi = 0, \text{ with } V = \sum_{i=1}^{N} v_i , \tag{4.16}$$

and assume that the target states are properly antisymmetrized. (For simplicity, we neglect here the complication of the antisymmetrization of operators in the intermediate states.[10]) By defining the scattering operator t by

$$t = v + vG_0t , \tag{4.17}$$

where

$$G_0 = (E + i\epsilon - T - H_T)^{-1} ,$$

we can rewrite (4.16) into an analogous form[9]

$$[T + H_T + (N-1)t - E]\Psi' = 0 , \tag{4.18}$$

where Ψ' is related to the original Ψ by

$$v\Psi = t\Psi' . \tag{4.19}$$

The amplitude t of (4.17) is still an $(N+1)$-body operator and requires further approximations before it can be identified with the two-body operator t_F, i.e.,

$$t \to t_F = v + vgt_F , \tag{4.20}$$

where g is purely a two-particle free Green's function. In actual applications of t_F in (4.18), however, we make the additional approximation

$$t_F \to t_{ex} , \tag{4.21}$$

where t_{ex} is the experimental on-shell amplitude. Taking a particular analytic form for t_{ex} amounts to assuming a specific off-shell extension. Presumably, t_{ex} already contains the correct relativistic kinematic and other corrections, and these should be consistently accounted for when (4.17) and (4.18) are modified. Some efforts are being made recently[23] to correct for the errors involved in (4.20) and (4.21).

Alternatively, as has been done in Secs. II and III, we may solve (4.16) for Ψ, with the v's obtained from t_{ex} via (4.20). The *effective* v in this case contains a particular off-shell extension, but we avoid the introduction of the operator t altogether. Thus, provided G_0 and g have a "mutually compatible" structure, with the same relativistic extensions, the use of v seems more direct. Undoubtedly, the off-shell behavior of v and resulting t_F should be model-dependent, but this is one of the points which the study of high energy scattering is to explore. For pion projectiles in the energy region below approximately 500 MeV and for proton beams of roughly 3 GeV or less, we expect that the effective potential description for the projectile-target nucleon interaction should be reasonably effective. Thus, the potential approach will be adopted in our analysis of the proton-helium scattering at 1 GeV, to be reported on in the second paper, with the proper kinematic corrections incorporated both in G_0 and g. Provided the corrections mentioned above, in connection with (4.20) and (4.21), are properly taken into account, the calculation with the t operator should give a result which is consistent with the v operator formulation if both the t and v operators used have in fact the same off-shell extensions. This problem is being carefully analyzed.

V. DISCUSSION

We have reexamined the effective channel theory of high energy scattering and explicitly constructed the average fluctuation potentials and average excitation energy, all of which are needed to completely specify the theory. Although they involve the two- and three-particle correlation functions and the related quantities $\tilde{Y}_H$ and $J_{\varphi\varphi}$, the resulting set of coupled equations (4.5) for the elastic scattering is extremely simple and should be useful in a wide range of physical problems. The average excitation function φ chosen in (4.2) is basically a quasiadiabatic form, but the discussion given in Sec. II, (2.12) and in Sec. III, (3.3), strongly suggests that the set (4.5) with (4.6), (4.7), and (4.8) should be reasonably effective at high energies as well. A detailed examination of the range of validity of the theory will be the subject of the second report[12] on proton-helium scattering.

As noted earlier, the various kinematic[10] and relativistic corrections[3] to the formulation presented here using the Schrödinger equation may be incorporated as usual, and this will be discussed explicitly when the theory is applied to specific scattering systems. The theory assumes that the input information on the interaction v and the target function ψ_0 is *a priori* known. Since the actual situation is often far from this, analyses of nuclear scattering with an approximate v and ψ_0 should shed further light on these input quantities.

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