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Field-theoretic Low equation approach to pion-nucleus scattering*

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From the pion-nucleus Low equation in the one-meson approximation we obtain an uncoupled, nonlinear, singular integral equation to determine the crossing symmetric elastic pion-nucleus scattering amplitude. The driving term of this equation is evaluated in an impulse single-scattering approximation for ($J^\pi = 0^+$, $I = 0$) ground state nuclei. It is found that unitarity requires that we include the effects of the strong absorption of the low partial waves in the entrance and exit channels. Thus, we introduce distortion of the pion waves by a procedure used in the inelastic peripheral scattering of elementary particles. Numerical results are presented for the iterative solution of this equation for π-^{12}C scattering in the energy region of the (3, 3) resonance.

[NUCLEAR REACTIONS. π-nucleus scattering theory with field-theoretic Low equation. $^{12}C(\pi,\pi)$, E in (3, 3) resonance region; calculated $\sigma, \sigma(\theta)$.]

I. INTRODUCTION

There have been a large number of different approaches to the theoretical analysis of the elastic scattering of pions by nuclei in the energy region of the (3, 3) resonance.[1] Prominent among these have been optical models (both relativistic and nonrelativistic), Glauber theory, and direct applications of multiple scattering theory. These analyses have generally been successful in interpreting the pion-nucleus interaction in terms of the elementary pion-nucleon interaction.

In this work we consider another approach to the analysis of elastic π-nucleus scattering which is based on the field-theoretic equation of Low.[2] This approach exhibits several useful features. First, this equation explicitly displays both the crossed π-nucleus processes and the elastic nuclear rescattering of intermediate state pions. Thus, the importance of these processes can be readily determined. And second, this equation can easily be generalized to determine any inelastic scattering amplitude or even a coupled-channel-type set of equations for evaluating both elastic and inelastic amplitudes.[3]

Here we apply the Low equation to the elastic scattering of pions by ^{12}C in the energy region of the (3, 3) resonance. In the context of this problem we discuss in detail the features of this equation which are of importance to the unitarity condition on scattering amplitudes and the general problem of its solvability, both of which will be important considerations in further applications of the Low equation.

II. PION-NUCLEUS LOW EQUATION

We consider the process

$$\pi_\alpha(\vec{k}) + A \to \pi_\beta(\vec{k}') + B\,, \tag{1}$$

where an initial state pion with isospin α, space momentum $\vec{k}$ scatters from a nucleus whose initial state is completely specified by the symbol A, going into a final state of a pion ($\beta, \vec{k}'$) and nucleus B. By reducing out the initial state pion, the S-matrix element for this process can be written

$$\langle \beta\vec{k}', B; \text{in}\,|S|\,\alpha\vec{k}, A; \text{in}\rangle = \langle B|A\rangle\langle\beta\vec{k}'\,|\,\alpha\vec{k}\rangle - 2\pi i\,\delta(E_B + \omega_{k'} - E_A - \omega_k) \times \langle\beta\vec{k}', B; \text{out}\,|J^\dagger_{\alpha\vec{k}}|A\rangle\,, \tag{2}$$

where $\omega_k{}^2 = \vec{k}^2 + m_\pi{}^2$ and the pion current $J^\dagger_{\alpha\vec{k}}$ is expressed in terms of the interpolating pion field by

$$J^\dagger_{\alpha\vec{k}} = -\frac{1}{(2\omega_k)^{1/2}}\int d^3x\, e^{i\vec{k}\cdot\vec{x}}(\Box + m_\pi{}^2)\phi_\alpha(\vec{x}, 0)\,. \tag{3}$$

The matrix element of this current, given in Eq. (2), can be shown to satisfy the Low equation by reducing out the final state pion and inserting a complete set of states into the resulting causal commutator of currents. We then find

$$\langle\beta\vec{k}', B; \text{out}\,|J^\dagger_{\alpha\vec{k}}|A\rangle = \text{ST} + \sum_n\left(\frac{\langle B|J_{\beta\vec{k}'}\,|n;\text{out}\rangle\langle n;\text{out}\,|J^\dagger_{\alpha\vec{k}}|A\rangle}{\omega_{k'} + E_B - E_n + i\epsilon} - \frac{\langle B|J^\dagger_{\alpha\vec{k}}|n;\text{out}\rangle\langle n;\text{out}\,|J_{\beta\vec{k}'}|A\rangle}{\omega_{k'} + E_n - E_A + i\epsilon}\right), \tag{4}$$

where ST denotes the so-called seagull term given by the integral

$$\frac{1}{(2\omega_{k'})^{1/2}}\int d^3x\, dt\, \delta(t) \times [e^{i(\omega_k t-\vec{k}\cdot\vec{x})}(i\overleftrightarrow{\partial}_t)\langle B|[\phi_\beta(\vec{x},t), J^\dagger_{\alpha\vec{k}}]|A\rangle], \tag{5}$$

where $[\Gamma_1(i\overleftrightarrow{\partial}_t)\Gamma_2] \equiv i[\Gamma_1(\partial_t\Gamma_2) - (\partial_t\Gamma_1)\Gamma_2]$. This term has been shown to describe the contribution from diagrams in which two pions interact at the same space-time point, such as occurs in pion-pion interactions and pion-nucleon interactions which are quadratic in the pion field.[4] With the view that the main features of medium energy pion-nucleus scattering are a consequence of the strongly attractive p-wave pion-nucleon interaction, and assuming the validity of an impulse approximation treatment of this term, we can conclude from the study of the pion-nucleon problem that the seagull term will be small in comparison to other terms in Eq. (4). In the following we will therefore neglect this term.

III. SPECIALIZATION OF THE LOW EQUATION TO DETERMINE THE ELASTIC SCATTERING AMPLITUDE FOR ($J^\pi=0^+, I=0$) NUCLEI

We now consider a method for evaluation of (4) to determine the elastic scattering amplitude for pions on nuclei in the state ($J^\pi=0^+$, $I=0$) with the scattering energy in the region of the (3, 3) resonance. (For numerical illustration we will examine π-^{12}C scattering.) Since we expect that nuclear recoil will be negligible for this problem, our analysis will reflect this fact.

As in the Chew-Low theory of pion-nucleon scattering,[5] we neglect the multimeson states in the completeness sum in (4). The neglect of these states will be valid if the cross sections for inelastic (π production) processes are small compared to the elastic ones for all values of the energy. We then have for the ground state to ground state amplitude in the one-meson approximation (ignoring "in" and "out" labels and the seagull term)

$$\begin{aligned}\langle\Phi, \beta\vec{k}'|J^\dagger_{\alpha\vec{k}}|\Phi\rangle = &\sum_A \left(\frac{\langle\Phi|J_{\beta\vec{k}'}|A\rangle\langle A|J^\dagger_{\alpha\vec{k}}|\Phi\rangle}{\omega_{k'}+E_\Phi-E_A+i\epsilon} - \frac{\langle\Phi|J^\dagger_{\alpha\vec{k}}|A\rangle\langle A|J_{\beta\vec{k}'}|\Phi\rangle}{\omega_{k'}+E_A-E_\Phi+i\epsilon}\right)\\ &+\sum_{A\neq\Phi}\sum_\nu\int\frac{d^3p}{(2\pi)^3}\left(\frac{\langle\Phi|J_{\beta\vec{k}'}|A,\nu\vec{p}\rangle\langle A,\nu\vec{p}|J^\dagger_{\alpha\vec{k}}|\Phi\rangle}{\omega_{k'}+E_\Phi-\omega_p-E_A+i\epsilon} - \frac{\langle\Phi|J^\dagger_{\alpha\vec{k}}|A,\nu\vec{p}\rangle\langle A,\nu\vec{p}|J_{\beta\vec{k}'}|\Phi\rangle}{\omega_{k'}+\omega_p+E_A-E_\Phi+i\epsilon}\right)\\ &+\sum_\nu\int\frac{d^3p}{(2\pi)^3}\left(\frac{\langle\Phi|J_{\beta\vec{k}'}|\Phi,\nu\vec{p}\rangle\langle\Phi,\nu\vec{p}|J^\dagger_{\alpha\vec{k}}|\Phi\rangle}{\omega_{k'}-\omega_p+i\epsilon} - \frac{\langle\Phi|J^\dagger_{\alpha\vec{k}}|\Phi,\nu\vec{p}\rangle\langle\Phi,\nu\vec{p}|J_{\beta\vec{k}'}|\Phi\rangle}{\omega_{k'}+\omega_p+i\epsilon}\right),\end{aligned} \tag{6}$$

where Φ denotes the nuclear ground state and the sum over A denotes a sum over the nuclear states. We shall now consider models for the evaluation of the first two terms on the right of this equation. These terms will then serve as the driving term of the resulting nonlinear singular integral equation for the π-nucleus elastic scattering amplitude.

A. Plane-wave impulse approximation

To develop models for the evaluation of the inelastic matrix elements appearing on the right of (6), we will first ignore absorption effects due to competition from other inelastic channels and treat the basic πN interaction in the generalized impulse approximation.[6] Thus, we shall be describing the incoming and outgoing pions by plane waves. However, the use of this plane-wave impulse approximation (PWIA) will be seen to result in a violation of the limits imposed on the elastic amplitude by the unitarity condition of the S matrix. In particular, we will find that it is the lowest partial waves which exceed the unitarity limit. Consequently, to include the effects of the strong absorption of the low partial waves in the entrance and exit channels into the numerous competing channels, we introduce distortion of the pion waves. This will be discussed in the text subsection where we consider a procedure which has been successful in describing the distortion due to strong absorption in the inelastic "peripheral scattering" of elementary particles.

Proceeding with the evaluation of the first term on the right of (6) using the plane wave impulse approximation, we assume that $(E_A - E_\Phi)$ averaged over nuclear states satisfies $(\overline{E_A - E_\Phi}) \ll \omega_{k'}$. We can then use closure on this term and get

$$\frac{1}{\omega_{k'}} \langle \Phi \,|[\hat{J}_{\beta\vec{k}'}, \hat{J}^{\dagger}_{\alpha\vec{k}}]|\Phi\rangle , \tag{7}$$

where the circumflex denotes that only the part of the current which conserves pion number is to be considered. This Born term can be evaluated in impulse single-scattering approximation assuming that the πN interaction is described by the pseudoscalar coupling, so that

$$\hat{J}^{\dagger}_{\alpha\vec{k}} = \frac{iv(k^2)}{(2\omega_k)^{1/2}} \frac{f}{m_\pi} \sum_{n=1}^{A} \vec{\sigma}_n \cdot \vec{k}\, \tau_\alpha^{(n)} e^{i\vec{k}\cdot\vec{r}_n} . \tag{8}$$

Here, $v(k^2)$ is the form factor (cutoff) of the πN interaction, f is the renormalized unrationalized πN coupling constant ($f^2 \cong 0.08$), and for the nth nucleon $\vec{r}_n$ is the position vector, $\tau_\alpha^{(n)}$ is the αth component of the isospin operator, and $\vec{\sigma}_n$ is the spin operator. We then have

$$\frac{1}{\omega_{k'}} \langle \Phi \,|[\hat{J}_{\beta\vec{k}'}, \hat{J}^{\dagger}_{\alpha\vec{k}}]|\Phi\rangle = -\frac{1}{\omega_{k'}} \frac{v(k^2)v(k'^2)}{[4\omega_k\omega_{k'}]^{1/2}} \langle \Phi \,|\sum_{n=1}^{A} e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_n} \sum_{\mu=1}^{4} \lambda_\mu P_\mu^{(n)}(\beta\vec{k}', \alpha\vec{k})|\Phi\rangle , \tag{9}$$

where λ_μ are the Born amplitudes

$$\lambda_\mu = \frac{2}{3}\left(\frac{f}{m_\pi}\right)^2 \begin{pmatrix} -4 \\ -1 \\ -1 \\ +2 \end{pmatrix} , \tag{10}$$

and the $P_\mu^{(n)}$ are projection operators onto states of definite angular momentum and isospin acting with respect to the variables of the nth nucleon

$$\begin{aligned} P_1^{(n)}(\beta\vec{k}', \alpha\vec{k}) &= \tfrac{1}{3}\tau_\beta{}^{(n)}\tau_\alpha{}^{(n)}\vec{\sigma}_n\cdot\vec{k}'\vec{\sigma}_n\cdot\vec{k} , \\ P_2^{(n)}(\beta\vec{k}', \alpha\vec{k}) &= \tfrac{1}{3}\tau_\beta{}^{(n)}\tau_\alpha{}^{(n)}[3\vec{k}'\cdot\vec{k} - \vec{\sigma}_n\cdot\vec{k}'\vec{\sigma}_n\cdot\vec{k}] , \\ P_3^{(n)}(\beta\vec{k}', \alpha\vec{k}) &= [\delta_{\alpha\beta} - \tfrac{1}{3}\tau_\beta{}^{(n)}\tau_\alpha{}^{(n)}]\vec{\sigma}_n\cdot\vec{k}'\vec{\sigma}_n\cdot\vec{k} , \\ P_4^{(n)}(\beta\vec{k}', \alpha\vec{k}) &= [\delta_{\alpha\beta} - \tfrac{1}{3}\tau_\beta{}^{(n)}\tau_\alpha{}^{(n)}] \\ &\quad\times[3\vec{k}'\cdot\vec{k} - \vec{\sigma}_n\cdot\vec{k}'\vec{\sigma}_n\cdot\vec{k}] . \end{aligned} \tag{11}$$

With the ground state nucleus defined to have $(J^\pi, I) = (0^+, 0)$, only the scalar, isoscalar, and even-parity part of the operator shown in Eq. (9) will contribute to the nuclear expectation value. The three nucleon vector operators $\vec{\tau}_n$, $\vec{\sigma}_n$, and $\vec{r}_n$ which are present in this expression can be coupled to obtain the relevant part using standard techniques, as described in the Appendix. Defining

$$a_\mu = \langle \Phi \,|\sum_n e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_n} P_\mu^{(n)}(\beta\vec{k}', \alpha\vec{k})|\Phi\rangle \tag{12}$$

gives

$$\begin{aligned} a_1 &= \tfrac{1}{3}\delta_{\alpha\beta}\tilde{\rho}(q)\,\vec{k}'\cdot\vec{k} , \\ a_2 &= \tfrac{2}{3}\delta_{\alpha\beta}\tilde{\rho}(q)\,\vec{k}'\cdot\vec{k} = a_3 , \\ a_4 &= \tfrac{4}{3}\delta_{\alpha\beta}\tilde{\rho}(q)\,\vec{k}'\cdot\vec{k} , \\ \tilde{\rho}(q) &= \int j_0(qr)\rho(\vec{r})d^3r , \end{aligned} \tag{13}$$

where $q = |\vec{k}' - \vec{k}|$, and $\rho(\vec{r})$ is the nuclear density which is normalized to the number of nucleons.

We therefore find

$$\begin{aligned} \frac{1}{\omega_{k'}} \langle \Phi \,|[\hat{J}_{\beta\vec{k}'}, \hat{J}^{\dagger}_{\alpha\vec{k}}]|\Phi\rangle &= -\frac{1}{\omega_{k'}} \frac{v(k^2)v(k'^2)}{[4\omega_k\omega_{k'}]^{1/2}} \sum_{\mu=1}^{4} \lambda_\mu a_\mu \\ &= 0 . \end{aligned} \tag{14}$$

We now consider the evaluation of the second term on the right of Eq. (6). This requires a model for the nuclear ground state to excited state matrix elements. Again assuming that the pion current acts as a single-particle operator we can evaluate these matrix elements in impulse approximation using the transition operator of the Chew-Low theory. Writing

$$\langle A, \nu\vec{p}\,|J^{\dagger}_{\alpha\vec{k}}|\Phi\rangle = \langle A\,|W(\nu\vec{p}, \alpha\vec{k})|\Phi\rangle , \tag{15}$$

the nuclear transition operator W is expressed as

$$W(\nu\vec{p}, \alpha\vec{k}) = -4\pi \frac{v(p^2)v(k^2)}{[4\omega_p\omega_k]^{1/2}} \sum_{n=1}^{A}\sum_{\mu=1}^{4} e^{-i(\vec{p}-\vec{k})\cdot\vec{r}_n} P_\mu^{(n)}(\nu\vec{p}, \alpha\vec{k}) h_\mu(\omega_p) , \tag{16}$$

where $h_\mu(\omega_p)$ are the Chew-Low amplitudes for πN elastic scattering. We then have for the second term of Eq. (6)

$$\sum_{A\neq\Phi}\sum_\nu \int \frac{p^2dp\,d\Omega_p}{(2\pi)^3}\left[\frac{\langle \Phi\,|W^\dagger(\nu\vec{p}, \beta\vec{k}')|A\rangle\langle A\,|W(\nu\vec{p}, \alpha\vec{k})|\Phi\rangle}{\omega_{k'} - \omega_p - \Delta + i\epsilon} - \frac{\langle \Phi\,|W^\dagger(\nu\vec{p}, \alpha - \vec{k})|A\rangle\langle A\,|W(\nu\vec{p}, \beta - \vec{k}')|\Phi\rangle}{\omega_{k'} + \omega_p + \Delta + i\epsilon}\right] , \tag{17}$$

where we have used closure on the denominators; and the constant $\Delta = (\overline{E_A - E_\Phi})$ is again assumed to satisfy

$\Delta << \omega_{k'}$. (In the following we use $\Delta = 0.1\, m_\pi$.)

Ignoring Pauli principle effects, we approximate the sum over excited nuclear states

$$\sum_{A \neq \Phi} |A\rangle\langle A| \cong 1\,. \tag{18}$$

Then, using the orthogonality property of the projection operators

$$\sum_\nu \int d\Omega_p P_\mu^{(n)\dagger}(\nu\vec{p}, \beta\vec{k}')P_{\mu'}^{(n')}(\nu\vec{p}, \alpha\vec{k}) = 4\pi p^2 \delta_{\mu\mu'}\delta_{nn'}P_\mu^{(n)}(\beta\vec{k}', \alpha\vec{k})\,, \tag{19}$$

we readily obtain

$$\frac{(4\pi)^3}{(2\pi)^3}\frac{v(k^2)v(k'^2)}{(4\omega_k\omega_{k'})^{1/2}}\int \frac{p^4 dp}{2\omega_p} v^2(p^2)\sum_n\sum_\mu \left[\frac{\langle\Phi|e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_n}P_\mu^{(n)}(\beta\vec{k}', \alpha\vec{k})|\Phi\rangle|h_\mu(\omega_p)|^2}{\omega_{k'} - \omega_p - \Delta + i\epsilon} - \frac{\langle\Phi|e^{i(\vec{k}'-\vec{k})\cdot\vec{r}_n}P_\mu^{(n)}(\alpha - \vec{k}, \beta - \vec{k}')|\Phi\rangle|h_\mu(\omega_p)|^2}{\omega_{k'} + \omega_p + \Delta + i\epsilon}\right]. \tag{20}$$

Using Eq. (13) then yields

$$4\frac{v(k^2)v(k'^2)}{[4\omega_k\omega_{k'}]^{1/2}}\int p^3 d\omega_p v^2(p^2)\tfrac{1}{3}[|h_1(\omega_p)|^2 + 4|h_2(\omega_p)|^2 + 4|h_4(\omega_p)|^2] \times\left(\frac{1}{\omega_{k'} - \omega_p - \Delta + i\epsilon} - \frac{1}{\omega_{k'} + \omega_p + \Delta + i\epsilon}\right)\tilde{\rho}(q)\delta_{\alpha\beta}\vec{k}'\cdot\vec{k}\,, \tag{21}$$

where we have used the fact that $h_2 = h_3$. This equation can be further simplified by considering the defining equation for the Chew-Low amplitudes

$$h_\mu(z) = \frac{\lambda_\mu}{z} + \frac{1}{\pi}\int d\omega_p p^3 v^2(p^2)\left[\frac{|h_\mu(\omega_p)|^2}{\omega_p - z} + \sum_\nu A_{\mu\nu}\frac{|h_\nu(\omega_p)|^2}{\omega_p + z}\right], \tag{22}$$

where $A_{\mu\nu}$ is the crossing matrix

$$(A_{\mu\nu}) = \tfrac{1}{9}\begin{pmatrix} 1 & -4 & -4 & 16 \\ -2 & -1 & 8 & 4 \\ -2 & 8 & -1 & 4 \\ 4 & 2 & 2 & 1 \end{pmatrix}. \tag{23}$$

From Eq. (22) we obtain

$$[h_1(z) + 2h_2(z) + 2h_3(z) + 4h_4(z)] = \frac{1}{\pi}\int p^3 d\omega_p v^2(p^2)[|h_1(\omega_p)|^2 + 2|h_2(\omega_p)|^2 + 2|h_3(\omega_p)|^2 + 4|h_4(\omega_p)|^2]\left(\frac{1}{\omega_p - z} + \frac{1}{\omega_p + z}\right), \tag{24}$$

where we have used the relation

$$\sum_\nu [A_{1\nu}|h_\nu|^2 + 2A_{2\nu}|h_\nu|^2 + 2A_{3\nu}|h_\nu|^2 + 4A_{4\nu}|h_\nu|^2] = |h_1|^2 + 2|h_2|^2 + 2|h_3|^2 + 4|h_4|^2 \tag{25}$$

and the fact that

$$\lambda_1 + 2\lambda_2 + 2\lambda_3 + 4\lambda_4 = 0\,. \tag{26}$$

Using Eq. (24) in (21) then gives

$$-4\pi\frac{v(k^2)v(k'^2)}{(4\omega_k\omega_{k'})^{1/2}}\frac{1}{3}H^{(+)}(\omega_{k'} - \Delta)\tilde{\rho}(q)\delta_{\alpha\beta}\vec{k}'\cdot\vec{k}, \qquad H^{(+)} \equiv h_1 + 2h_2 + 2h_3 + 4h_4\,, \tag{27}$$

where corrections of relative order Δ are ignored.

Equation (27) is thus the complete expression for the driving term of the pion-nucleus Low equation for $(J^\pi, I) = (0^+, 0)$ nuclei. Denoting this term by $V(\vec{k}', \vec{k}; \omega_{k'})$, we consider the partial wave expansion

$$V(\vec{k}', \vec{k}; \omega_{k'}) = -4\pi \frac{v(k^2)v(k'^2)}{(4\omega_k\omega_{k'})^{1/2}} \sum_l (2l+1)V_l(k', k; \omega_{k'})P_l(\hat{k}'\cdot\hat{k})\delta_{\alpha\beta}. \tag{28}$$

Similarly, expanding the elastic amplitude

$$\langle\Phi, \beta\vec{k}'|J^\dagger_{\alpha\vec{k}}|\Phi\rangle = -4\pi \frac{v(k^2)v(k'^2)}{(4\omega_k\omega_{k'})^{1/2}} \sum_l (2l+1)T_l(k', k; \omega_{k'})P_l(\hat{k}'\cdot\hat{k})\delta_{\alpha\beta}, \tag{29}$$

the Low equation (6) can be written[7]

$$T_l(k', k; \omega_{k'}) = V_l(k', k; \omega_{k'}) + \frac{1}{\pi}\int p\, d\omega_p v^2(p^2)\left[\frac{T_l^*(p, k'; \omega_p)T_l(p, k; \omega_p)}{\omega_p - \omega_{k'} - i\epsilon} + \frac{T_l^*(p, k; \omega_p)T_l(p, k'; \omega_p)}{\omega_p + \omega_{k'} + i\epsilon}\right], \tag{30}$$

where

$$V_l(k', k; \omega_{k'}) = \frac{4\pi}{6}\int_{-1}^{+1} dx\, P_l(x) x\tilde{\rho}(q)k'kH^{(+)}(\omega_{k'} - \Delta), \quad x \equiv \hat{k}'\cdot\hat{k}. \tag{31}$$

We now specialize the driving term (31) to ^{12}C by choosing for the ground state density

$$\rho(\vec{r}) = \alpha(1 + \tfrac{4}{3}r^2/b^2)e^{-r^2/a^2}, \quad \alpha = 12[a^2\pi^{3/2}(1 + 2a^2/b^2)]^{-1}, \tag{32}$$

which is the form used to parametrize electron scattering,[8] with values of a and b as corrected for the nucleon's spatial extension[9]

$$a = 1.59 \text{ fm}, \quad b = 1.66 \text{ fm}. \tag{33}$$

This density yields the nuclear form factor

$$\tilde{\rho}(q) = 4\alpha\pi^{3/2}e^{-(1/4)a^2q^2}\left(\frac{1}{4}a^3 + \frac{a^5}{2b^2} - \frac{a^7q^2}{12b^2}\right). \tag{34}$$

Expanding this in partial waves according to

$$\tilde{\rho}(q) = 4\pi\sum_{l=0}^{\infty} \xi_l(k', k)P_l(\hat{k}'\cdot\hat{k}), \tag{35}$$

gives

$$\xi_l(k', k) = \alpha\sqrt{\pi}\, e^{-y}\left\{\left[\frac{a^3}{4} + \frac{a^5}{b^2}\left(\frac{1}{2} - \frac{1}{3}y\right)\right](2l+1)i_l(z) + \frac{a^5 z}{3b^2}[l i_{l-1}(z)(1 - \delta_{l0}) + (l+1)i_{l+1}(z)]\right\}, \tag{36}$$

where

$$z = \tfrac{1}{2}k'ka^2, \quad y = \tfrac{1}{4}(k^2 + k'^2)a^2, \tag{37}$$

and $i_l(z)$ is a modified spherical Bessel function of the first kind.[10] From Eq. (31) the driving term is finally obtained as

$$V_l(k', k; \omega_{k'}) = \frac{4\pi}{3}\left[\frac{(l+1)\xi_{l+1}(k', k)}{(2l+1)(2l+3)} + \frac{l\xi_{l-1}(k', k)}{(2l-1)(2l+1)}(1 - \delta_{l0})\right]k'kH^{(+)}(\omega_{k'} - \Delta). \tag{38}$$

This completes the evaluation of the driving term in the plane wave impulse approximation.

From the structure of the Low equation, the on-shell π-nucleus amplitude $T_l(k', k'; \omega_{k'})$ can be related to phase shifts as follows. Using Eq. (2) and the unitarity relation for the S matrix

$$S^\dagger S = 1, \tag{39}$$

the elastic scattering amplitude can be shown to satisfy the equation

$$\langle\Phi, \beta\vec{k}'|J^\dagger_{\alpha\vec{k}}|\Phi\rangle - \langle\Phi, \alpha\vec{k}|J^\dagger_{\beta\vec{k}'}|\Phi\rangle^* = -2\pi i\sum_n \delta(E_\Phi + \omega_{k'} - E_n)\langle n|J^\dagger_{\beta\vec{k}'}|\Phi\rangle^*\langle n|J^\dagger_{\alpha\vec{k}}|\Phi\rangle \tag{40}$$

for $\omega_k = \omega_{k'}$. In the one-meson approximation this gives

$$\begin{aligned}\langle\Phi, \beta\vec{k}'|J^\dagger_{\alpha\vec{k}}|\Phi\rangle - \langle\Phi, \alpha\vec{k}|J^\dagger_{\beta\vec{k}'}|\Phi\rangle^* = &-2\pi i\sum_\nu \int\frac{d^3p}{(2\pi)^3}\delta(\omega_{k'} - \omega_p)\langle\Phi, \nu\vec{p}|J^\dagger_{\beta\vec{k}'}|\Phi\rangle^*\langle\Phi, \nu\vec{p}|J^\dagger_{\alpha\vec{k}}|\Phi\rangle \\ &-2\pi i\sum_{A\neq\Phi}\sum_\nu\int\frac{d^3p}{(2\pi)^3}\delta(\omega_{k'} - \omega_p - E_A + E_\Phi)\langle A, \nu\vec{p}|J^\dagger_{\beta\vec{k}'}|\Phi\rangle^*\langle A, \nu\vec{p}|J^\dagger_{\alpha\vec{k}}|\Phi\rangle.\end{aligned} \tag{41}$$

Using Eq. (29) to expand the ground-state to ground-state amplitude and Eq. (15) *et. seq.* to evaluate the ground-state to excited-state amplitude, we obtain

$$T_l(k',k;\omega_{k'}) - T_l^*(k',k;\omega_{k'}) = 2ik'v^2(k'^2)T_l^*(k',k';\omega_{k'})T_l(k',k;\omega_{k'}) + i\tfrac{4}{3}\pi\{p^3v^2(p^2)[|h_1(\omega_p)|^2 + 2|h_2(\omega_p)|^2 + 2|h_3(\omega_p)|^2 + 4|h_4(\omega_p)|^2]\}\Big|_{\omega_p=\omega_{k'}-\Delta} \times \int_{-1}^{+1} dx\, P_l(x) x\tilde{\rho}(q)k'k\,. \tag{42}$$

Setting $\vec{k}' = \vec{k}$ and using Eq. (31) and the unitarity relation for the Chew-Low amplitudes

$$\mathrm{Im}\, h_\mu(\omega_p) = p^3v^2(p^2)|h_\mu(\omega_p)|^2 \tag{43}$$

then yields

$$\mathrm{Im}\, T_l(k',k';\omega_{k'}) = k'v^2(k'^2)|T_l(k',k';\omega_{k'})|^2 + \mathrm{Im}\, V_l(k',k';\omega_{k'})\,. \tag{44}$$

We note that Eq. (44) can also be obtained quite trivially by taking the imaginary part of Eq. (30), after setting $k' = k$. This latter result is a consequence of the fact that any amplitude which satisfies the Low equation (6) will identically satisfy the unitarity relation (40), and that we have applied an identical set of approximations to these equations.

Equation (44) is the statement of inelastic unitarity for the partial-wave amplitudes T_l. It implies that we may write

$$T_l(k',k';\omega_{k'}) = \frac{n_l(\omega_{k'})e^{2i\delta_l(\omega_{k'})} - 1}{2ik'v^2(k'^2)}\,, \tag{45}$$

where δ_l are real phase shifts and the inelasticity parameter n_l is given by

$$n_l^2(\omega_{k'}) = 1 \quad \text{for } \omega_{k'} < m_\pi + \Delta$$

$$n_l^2(\omega_{k'}) = 1 - 4k'v^2(k'^2)\,\mathrm{Im}\, V_l(k',k';\omega_{k'}) \quad \text{for } \omega_{k'} \geq m_\pi + \Delta\,, \tag{46}$$

where, as usual, the unitarity condition requires that $0 \leq n \leq 1$. This result then establishes that the π-nucleus inelastic cross section is determined directly from the driving term of the Low equation according to

$$\sigma_{\text{inel}}^{(l)}(\omega_{k'}) = \frac{4\pi}{k'}v^2(k'^2)(2l+1)\,\mathrm{Im}\, V_l(k',k';\omega_{k'})\,. \tag{47}$$

Using for the πN form factor the same expression which has been successfully used in the Low equation analysis of πN scattering,[11]

$$v^2(k'^2) = e^{-k'^2/49m_\pi^2}\,, \tag{48}$$

we find that the PWIA model for the driving term leads to a violation of the unitarity limits on the inelasticity parameter. In Table I we show the values of $4k'v^2(k'^2)\mathrm{Im}\, V_l(k',k';\omega_{k'})$ for several energies in the range from 57 to 290 MeV with all relevant partial waves included. Using the classical correspondence between impact parameter and angular momentum given by $kb \cong l + \frac{1}{2}$, and a radius for ^{12}C equal to the equivalent charge radius $R = 3.2$ fm,[8] we find that the unitarity limits are violated when (roughly) the pion's classical impact parameter is less than or equal to the nuclear radius. (In Table I an asterisk is used to denote that $b \leq R$ for the number displayed.) However, when the unitarity limits are not violated (corresponding to the higher partial waves), we find that the inelastic partial cross sections computed using Eq. (47) agree quite well with those determined from an optical potential calculation.

As mentioned earlier, the PWIA treatment of the driving term is unrealistic in that it ignores the many competing open inelastic channels which

TABLE I. Values of $4k'v^2(k'^2)\,\mathrm{Im}\, V_l(k',k';\omega_{k'})$ for several energies in the range from 57 to 290 MeV with all relevant partial waves included. An asterisk denotes that the classical impact parameter is less than or equal to the equivalent charge radius.

Partial wave	Pion kinetic energy (MeV) 56.987	82.591	106.47	125.39	153.77	172.69	196.57	222.17	264.96	290.25
0	0.279 76*	1.5045*	4.3994*	6.9110*	8.3304*	7.6363*	6.5143*	5.4585*	4.2325*	3.6975*
1	0.334 18*	1.3292*	3.4346*	5.2307*	6.3932*	6.0253*	5.3487*	4.6618*	3.7980*	3.3858*
2	0.115 02	0.642 74*	1.9831*	3.2873*	4.3501*	4.2589*	3.9404*	3.5753*	3.0897*	2.8359*
3	0.020 20	0.172 61	0.704 25	1.3796*	2.1976*	2.3581*	2.3877*	2.3352*	2.2172*	2.1258*
4	0.002 32	0.030 85	0.170 76	0.406 10	0.816 13	0.991 46*	1.1420*	1.2477*	1.3620*	1.3911*
5	0.000 20	0.004 09	0.030 93	0.090 15	0.233 05	0.325 58	0.435 83	0.544 62	0.708 71*	0.785 38*
6	0.000 01	0.000 43	0.004 44	0.015 94	0.053 47	0.086 57	0.136 25	0.197 46	0.313 98	0.382 90*
7		0.000 04	0.000 52	0.002 34	0.010 20	0.019 23	0.035 80	0.060 69	0.119 84	0.162 47
8			0.000 05	0.000 29	0.001 66	0.003 66	0.008 09	0.016 11	0.039 94	0.060 63

exist at the energies we consider. The single inelastic channel (i.e., nuclear excitation) represented by the driving term (38) should actually constitute only a fraction of the total inelastic cross section. In particular, numerous reactions with more complex final-state configurations are also expected to contribute to this cross section. On intuitive grounds, one expects these complex reactions to be initiated by collisions with impact parameters which are small (low partial waves). Thus, the existence of competing open channels is expected to reduce the driving term in low partial waves below the values given by Eq. (38).

To describe the effects of the competing open inelastic channels on the driving term of the π-nucleus Low equation, we consider the modification of this term which results from the use of distorted waves for the incoming and outgoing pions. In addition to simulating the loss of pion flux into the various inelastic channels, the use of distorted waves should also approximately take into account the shadowing or absorption effect caused by the fact that each quasifree collision of the pion with a nucleon reduces the undeflected pion flux seen by the other nucleons. Consequently, this modification should further improve on the generalized impulse approximation treatment of the driving term.

B. Distorted wave impulse approximation

The necessary information concerning absorption in the entrance or exit channels will be inferred from the elastic scattering in the initial and final state in a manner first proposed by Sopkovich,[12] and subsequently discussed in detail by Gottfried and Jackson.[13] The distortion of the pion waves resulting from this absorption will be introduced in the form of a modified Born approximation suggested by the solution of a two-channel potential scattering problem with absorptive potentials introduced in each channel to represent the net effect of all other channels.

We first consider an inelastic process which, in lowest nonvanishing order, is caused by the interaction V. Letting $U^{(+)}$ and $U^{(-)}$ be the *remaining* interaction between the colliding systems in the initial and final state, respectively, the scattering amplitude to first order in V and all orders in $U^{(+)}$ is given by

$$\mathfrak{M}_{fi} = \langle \psi_f^{(-)} | V | \psi_i^{(+)} \rangle , \tag{49}$$

where

$$\begin{aligned} \psi_i^{(+)} &= \phi_i + [E - (H - V) + i\epsilon]^{-1} U^{(+)} \phi_i , \\ \psi_f^{(-)} &= \phi_f + [E - (H - V) - i\epsilon]^{-1} U^{(-)} \phi_f . \end{aligned} \tag{50}$$

The interactions $U^{(\pm)}$ are taken to be complex (optical) potentials; and at sufficiently high energies and small momentum transfers the wave functions can be determined using the Glauber approximation,[14] which gives

$$\psi_k^{(\pm)}(\vec{b}, z) \cong e^{i\vec{k}\cdot\vec{r}} \exp\left[-\frac{i}{v_\pm} \int_{\pm\infty}^{z} U^{(\pm)}(\vec{b} + \hat{\kappa} z') dz' \right]. \tag{51}$$

In obtaining this expression, the z axis was chosen along the vector $\vec{\kappa} = \vec{k}' + \vec{k}$, where $\vec{k}$ and $\vec{k}'$ are the initial and final projectile momenta; the relative velocity is v, and $\vec{b}$ is the impact parameter vector chosen such that it is perpendicular to $\vec{\kappa}$. We then have for the scattering amplitude

$$\mathfrak{M}_{fi} \cong \int d^2 b \int_{-\infty}^{+\infty} dz\, e^{i\vec{q}\cdot\vec{r}} V(\vec{b} + \hat{\kappa} z) \exp\left[-\frac{i}{v_-} \int_z^{\infty} U^{(-)*}(\vec{b} + \hat{\kappa} z') dz' \right] \exp\left[-\frac{i}{v_+} \int_{-\infty}^{z} U^{(+)}(\vec{b} + \hat{\kappa} z'') dz'' \right] , \tag{52}$$

where $\vec{q} = \vec{k}' - \vec{k}$ is the three-momentum transfer. Assuming that the interactions in the initial and final channels are the same, so that $U^{(-)*} = U^{(+)} \equiv U$, and $v_+ \cong v_- \equiv v$, and using the integral representation for the Bessel function[10]

$$\int_0^{2\pi} e^{iqb\cos\theta} d\theta = 2\pi J_0(qb) , \tag{53}$$

the scattering amplitude can be expressed as

$$\mathfrak{M}_{fi} = 2\pi \int_0^{\infty} J_0(qb) e^{i\chi(b)} B(b) b\, db , \tag{54}$$

where

$$\begin{aligned} B(b) &\equiv \int_{-\infty}^{+\infty} V(\vec{b} + \hat{\kappa} z) dz , \\ \chi(b) &\equiv -\frac{1}{v} \int_{-\infty}^{+\infty} U(\vec{b} + \hat{\kappa} z) dz . \end{aligned} \tag{55}$$

We note that the ordinary Born approximation is obtained from Eq. (54) by setting $\exp[i\chi(b)]$ equal to unity. Thus, in this approximation the effects of absorption in the entrance or exit channel are introduced by multiplying the Born amplitude by the factor $\exp[\frac{1}{2} i\chi(b)]$ for each channel.

To determine this distortion factor we use the fact that χ is related to the elastic scattering amplitude in the initial and final channels according to[14]

$$f(q, k) = -ik \int_0^{\infty} J_0(qb) [e^{i\chi(b)} - 1] b\, db , \tag{56}$$

so that

$$e^{i\chi(b)} = 1 + \frac{i}{k} \int_0^{\infty} J_0(qb) f(q, k) q\, dq . \tag{57}$$

We also use the fact that, from Eq. (56), the total inelastic cross section follows in the form[14]

$$\sigma_{\text{inel}} = 2\pi \int [1 - e^{-2Im\,\chi(b)}]\, b\, db\,. \tag{58}$$

As pointed out by Gottfried and Jackson,[13] the amplitude f is not the actual elastic amplitude observed when the constituents of the initial (or final) states are allowed to collide, as the true elastic amplitude also includes terms where V acts an even number of times. If, however, the inelastic channel in question has a cross section which is small compared to the total inelastic cross section, the true elastic amplitude will not differ appreciably from that given by Eq. (56). Our previous discussion suggests that this condition can be assumed to be true (at least for the low partial waves), so we shall henceforth identify f with the amplitude which describes the elastic scattering of the initial or final system.

At pion energies near the (3, 3) resonance region, the π-nucleus elastic cross sections are found to extrapolate to the optical theorem point at zero momentum transfer.[15] Hence, at least at small angles, the elastic amplitudes are essentially imaginary, implying that the elastic scattering is almost entirely the shadow of the inelastic processes. It then follows from Eq. (57) that we may neglect the real part of χ compared to its imaginary part, at least for near forward scattering.

To proceed with the evaluation of χ we transform from the impact parameter representation used above to the partial wave representation. This is easily accomplished using the classical correspondence between impact parameter and partial wave

$$b \cong (l + \tfrac{1}{2})/k\,, \tag{59}$$

and the asymptotic formula (valid for small scattering angles θ) between the Legendre and Bessel functions

$$P_l(\cos\theta) \cong J_0(2(l+\tfrac{1}{2})\sin\tfrac{1}{2}\theta)\,. \tag{60}$$

Equation (54) establishes that to include the effects of absorption on an incoming or outgoing projectile of momentum k, simply multiply the partial wave Born amplitude by $\exp[\frac{1}{2}i\chi_l(k)]$. Thus the driving term of the Low equation in the distorted wave impulse approximation (DWIA) is given by

$$\tilde{V}_l(k', k; \omega_{k'}) \equiv e^{-(1/2)\,\text{Im}\,\chi_l(k')}\, V_l(k', k; \omega_{k'}) \times e^{-(1/2)\,\text{Im}\,\chi_l(k)}\,, \tag{61}$$

where V_l is given by Eq. (38) and, as discussed above, we have neglected the real part of χ compared to its imaginary part.

Note that we are here distorting only the initial and final state pions, while allowing the intermediate state pion to propagate freely. In the present approximation the distortion of the intermediate state pion can be effected by a modification of the phase-space factors in Eq. (21). Thus, provided that the πN cutoff $v^2(p^2)$ remains the dominant phase-space factor, the distortion of the intermediate state pion can reasonably be neglected. For the present analysis we will assume that this latter condition is satisfied.

The expression Eq. (47) for the π-nucleus inelastic partial cross section then becomes, in the DWIA,

$$\begin{aligned}\sigma_{\text{inel}}^{(l)}(\omega_{k'}) &= \frac{4\pi}{k'}\, v^2(k'^2)(2l+1)\,\text{Im}\,\tilde{V}_l(k', k'; \omega_{k'}) \\ &= \frac{4\pi}{k'}\, v^2(k'^2)(2l+1)\,\text{Im}\,V_l(k', k'; \omega_{k'}) \\ &\quad \times e^{-\text{Im}\,\chi_l(k')}\,.\end{aligned} \tag{62}$$

Comparing this with the corresponding expression obtained from Eq. (58),

$$\sigma_{\text{inel}}^{(l)}(\omega_{k'}) = \frac{\pi}{k'^2}(2l+1)[1 - e^{-2\text{Im}\,\chi_l(k')}]\,, \tag{63}$$

allows the evaluation of the distortion factor in terms of V_l. We find

$$\begin{aligned} e^{-\text{Im}\,\chi_l(k')} &= (\lambda_{k'}^2 + 1)^{1/2} - \lambda_{k'}\,, \\ \lambda_{k'} &= 2k' v^2(k'^2)\,\text{Im}\,V_l(k', k'; \omega_{k'})\,. \end{aligned} \tag{64}$$

The expression for the inelasticity parameter in the DWIA takes the form

$$\begin{aligned} n_l^2(\omega_{k'}) &= 1 \quad \text{for } \omega_{k'} < m_\pi + \Delta \\ n_l^2(\omega_{k'}) &= 1 - 4k' v^2(k'^2)\,\text{Im}\,\tilde{V}_l(k', k'; \omega_{k'}) \\ &\qquad \text{for } \omega_{k'} \geq m_\pi + \Delta\,. \end{aligned} \tag{65}$$

Using for the πN form factor the expression (48), we find that the inelasticity parameter is within the unitarity limits for all energies and all partial waves considered. In Table II we show the values of $4k' v^2(k'^2)\text{Im}\,\tilde{V}_l(k', k'; \omega_{k'})$ for several energies in the range 57 to 290 MeV with all relevant partial waves included. We find that the distortion factor succeeds in reducing the magnitude of the driving term sufficiently to bring the inelasticity parameter within the unitarity limits, and that for the lower partial waves the inelastic partial cross sections determined from Eq. (62) are in good agreement with those obtained from optical potential calculations. However, the higher partial cross sections are found to be significantly smaller than those determined from optical potential calculations. This results in the magnitude of the total inelastic cross section generally falling below the experimental values, as shown in Fig. 1. Though several expressions differing substantially from Eq. (64) have been proposed for the evaluation of the distortion factor,[12, 13, 16] all such expres-

TABLE II. Values of $4k'v^2(k'^2)\,\mathrm{Im}\tilde{V}_l(k',k';\omega_{k'})$ for several energies in the range from 57 to 290 MeV with all relevant partial waves included.

Partial wave	Pion kinetic energy (MeV) 56.987	82.591	106.47	125.39	153.77	172.69	196.57	222.17	164.96	290.25
0	0.243 35	0.750 89	0.953 07	0.979 90	0.985 99	0.983 42	0.977 48	0.968 52	0.949 66	0.935 93
1	0.282 97	0.712 60	0.927 13	0.965 90	0.976 66	0.973 88	0.967 29	0.957 79	0.938 89	0.925 31
2	0.108 60	0.468 56	0.826 37	0.921 43	0.952 10	0.950 22	0.942 76	0.932 04	0.912 73	0.899 42
3	0.020 00	0.158 36	0.498 65	0.724 35	0.850 29	0.865 33	0.867 89	0.863 32	0.852 25	0.842 81
4	0.002 32	0.030 38	0.156 80	0.331 93	0.548 43	0.615 10	0.662 97	0.692 21	0.720 30	0.726 93
5	0.000 20	0.004 08	0.030 45	0.086 18	0.207 47	0.276 86	0.351 08	0.416 14	0.500 75	0.535 35
6	0.000 01	0.000 43	0.004 43	0.015 81	0.052 06	0.082 91	0.127 28	0.178 93	0.268 53	0.316 55
7		0.000 04	0.000 52	0.002 33	0.010 15	0.019 04	0.035 17	0.058 88	0.112 87	0.149 81
8			0.000 05	0.000 29	0.001 66	0.003 65	0.008 05	0.015 98	0.039 15	0.058 82

sions are found to result in an even stronger suppression in the magnitude of the higher partial cross sections than results from the use of this expression. This reduction in the magnitude of the higher partial waves clearly results from our identification of the amplitude f of Eq. (56) with the amplitude which describes the elastic scattering of the initial or final system. In a more careful treatment, we would expect the higher partial wave components of f to differ from the corresponding components of the true elastic amplitude, so that the effects of distortion on the higher partial waves would be lessened.

IV. ITERATIVE SOLUTION OF THE PION-NUCLEUS LOW EQUATION

With the driving term evaluated in the distorted wave impulse approximation, the partial-wave pion-nucleus Low equation for the elastic scattering amplitude takes the form

$$T_l(k',k;\omega_{k'}) = \tilde{V}_l(k',k;\omega_{k'}) + \frac{1}{\pi}\int_{m_\pi}^{\infty} p\,d\omega_p v^2(p^2)\,\frac{T_l^*(p,k';\omega_p)T_l(p,k;\omega_p)}{\omega_p-\omega_{k'}-i\epsilon} + \frac{1}{\pi}\int_{m_\pi}^{\infty} p\,d\omega_p v^2(p^2)\,\frac{T_l^*(p,k;\omega_p)T_l(p,k';\omega_p)}{\omega_p+\omega_{k'}+i\epsilon}\,, \qquad (66)$$

where $\tilde{V}_l$ is given by Eq. (61). The diagrammatic structure of the π-nucleus Low equation is similar to that of the πN Low equation. The final two terms on the right of Eq. (66) are the rescattering integrals; the first integral corresponds to the direct π-nucleus processes and the second corresponds to the crossed processes. Thus, for example, the role of crossing in the π-nucleus interaction can be determined by solving Eq. (66) for the elastic scattering amplitude both with and without the final integral on the right.

Nonlinear singular integral equations of the type shown in Eq. (66) are not uncommon in strong-interaction physics where, as in our problem, they arise quite naturally through the combination of crossing symmetry and unitarity. Some of the techniques which have been used to solve such equations have been reviewed by Warnock.[17] One finds that often a solution is obtained only after some mutilation of crossing symmetry. Though fixed-point theorems which give sufficient conditions for the solvability of such equations have

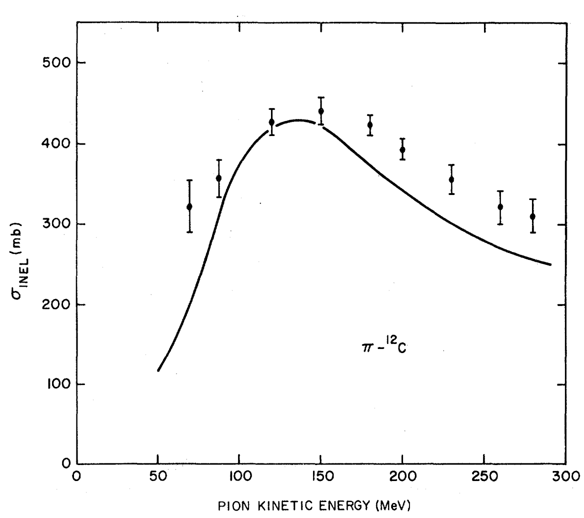

FIG. 1. π-^{12}C inelastic cross section as determined from the pion-nucleus Low equation (66), with driving term evaluated in DWIA. The data points are from Binon *et al.*, Ref. 15.

been proved, the necessary conditions for solvability are unknown. Furthermore, all of the work with these theorems only proves the existence of solutions which are unphysically small; for the theorems to be applicable, the coupling constant (or some other measure of the magnitude of amplitudes) must be somewhat smaller than its known experimental value.

Of the several fixed-point theorems on the solvability of nonlinear singular integral equations, only one is constructive and so gives a solution algorithm. This is the Banach-Cacciopoli theorem (also known as the contraction mapping principle), which establishes sufficient conditions for the solution of these equations by iteration. When applied to equations of the form Eq. (66), the Banach-Cacciopoli theorem has been shown to predict that a solution can be obtained by iteration provided the driving term is sufficiently small and the expression $k^3v^2(k^2)\omega_k$ is Hölder-continuous. (This latter condition is discussed further in Ref. 17, where it can be seen that the form factor (48) meets this constraint.)

Guided by this analysis, we have studied the iterative solution of the π-nucleus Low equation (66) using the πN form factor of Eq. (48). The integrals appearing in this equation were evaluated using Gauss-Legendre quadrature. To evaluate the principal-value integral we added and subtracted

$$\frac{1}{\pi}\int_{m_\pi}^{\infty} d\omega_p \frac{k'v^2(k'^2)T_l^*(k',k';\omega_{k'})T_l(k',k;\omega_{k'})}{\omega_p^2(\omega_p-\omega_{k'})} = -\frac{1}{\pi}k'v^2(k'^2)T_l^*(k',k';\omega_{k'})T_l(k',k;\omega_{k'})\left[\frac{1}{\omega_{k'}m_\pi}+\frac{1}{\omega_{k'}{}^2}\ln\left(\frac{\omega_{k'}-m_\pi}{m_\pi}\right)\right], \quad (67)$$

so that at the principal value point the integrand of the first integral in Eq. (66) becomes a well-defined derivative.

For pion energies below 125 MeV the iterative series diverges for the lower partial waves. We find that the rate of divergence is inversely related to the pion energy, and that the cause of this divergence can be traced to the large magnitude of the driving term and the small energy denominators in the rescattering integrals at these lower pion energies. For pion energies greater than 125 MeV, the iterative series is found to converge for all partial waves. For the lower partial waves the rate of convergence is slow, requiring as many as 15 iterations. However, the higher partial waves prove to be strongly convergent, requiring in general only two or three iterations.

As with the inelastic cross sections determined from the driving term of Eq. (66), the π-^{12}C total cross sections are found to fall below the experimental values. In Table III we show our calculated values for the π-^{12}C total cross sections at 150 and 200 MeV. The term "with crossing" here refers to the iterative solution of Eq. (66) as shown, while the term "without crossing" refers to the iterative solution obtained when the final integral in Eq. (66) (corresponding to crossed π-nucleus processes) is omitted. In these figures we see confirmed one of the results of our optical potential study of crossing[18]: at high energies the crossed π-nucleus processes have a negligible effect on the cross sections.

Finally, in Figs. 2 and 3 we show the π-^{12}C differential cross sections at pion energies of 150 and 200 MeV, respectively, obtained from the iterative solution of Eq. (66). The corresponding cross sections obtained when the crossed π-nucleus processes are not included differ negligibly from those shown, and are therefore omitted.

V. DISCUSSION

Our treatment of the pion-nucleus Low equation should be considered as only a preliminary approach. In our model of the driving term of the equation we used the generalized impulse approximation. However, our optical potential studies[19] indicate that there are many corrections to this approximation which individually have substantial effects on the elastic amplitude. Each of these corrections, in addition to such features as the multinucleon processes, should therefore be included in the evaluation of the driving term. The fact that the driving term alone completely deter-

TABLE III. Calculated values for the π-^{12}C total cross sections at 150 and 200 MeV. The term "with crossing" refers to the iterative solution of Eq. (66) as shown, while the term "without crossing" refers to the iterative solution obtained when the final integral in Eq. (66) (corresponding to crossed π-nucleus processes) is omitted. The experimental values are from Ref. 15.

Pion energy (MeV)	σ^{total} (mb) With crossing	Without crossing	Exp.
150	593.68	594.34	696 ± 7
200	501.82	508.51	637 ± 7

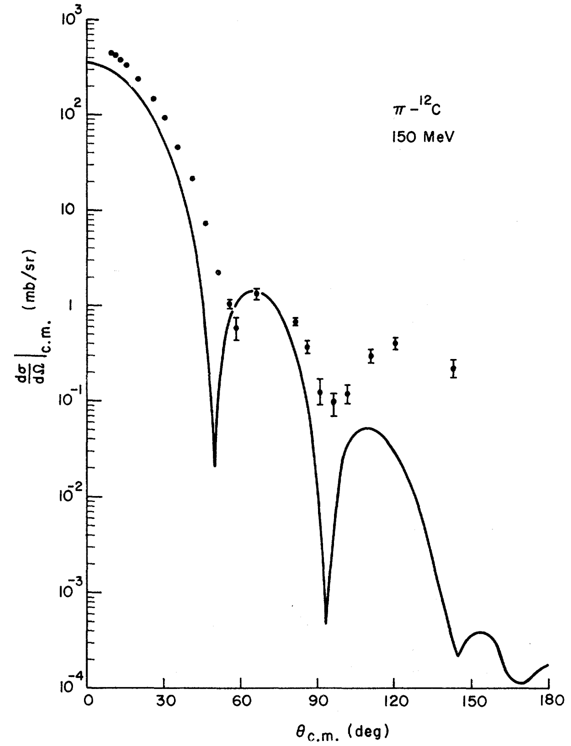

FIG. 2. The π-^{12}C differential cross section at 150 MeV as determined from the iterative solution of the pion-nucleus Low equation (66). The data points are from Binon *et al.*, Ref. 15.

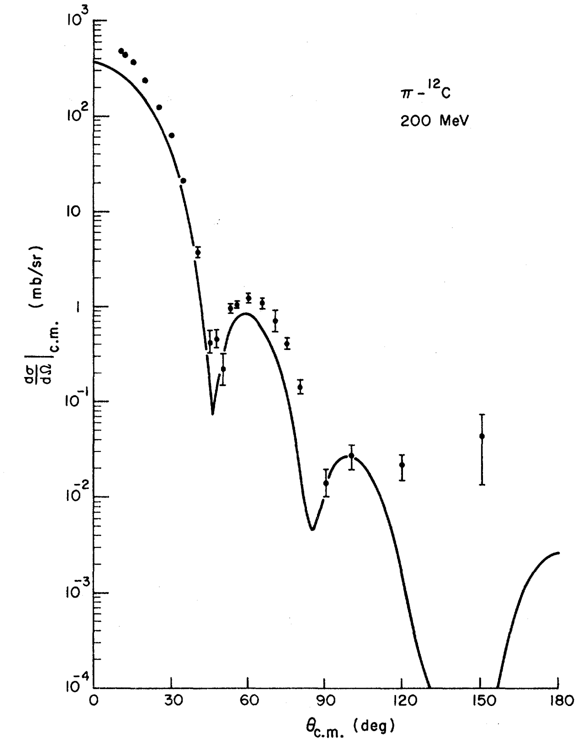

FIG. 3. Same as Fig. 2, except at a pion energy of 200 MeV.

mines the total inelastic cross section further requires a careful treatment of the effects of all the open inelastic channels. And finally, if one desires a reasonably accurate solution of the Low equation over a wide energy range there is evidently a need to study further the techniques for the solution of such nonlinear singular integral equations. We note that the most promising technique, that of matrix N/D,[20] requires a careful treatment of the left-hand cut discontinuity in view of the importance of the crossed π-nucleus processes at low energies.[18]

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APPENDIX

In relation to Eq. (12) we wish to evaluate the nuclear expectation value

$$a_\mu=\left\langle\Phi\left|\sum_{n=1}^{A}e^{i(\vec k-\vec k')\cdot\vec r_n}P_\mu^{(n)}(\beta\vec k',\alpha\vec k)\right|\Phi\right\rangle \qquad \text{(A1)}$$

for $|\Phi\rangle$ having the quantum numbers $(J^P,I)=(0^+,0)$, so that only the part of the operator shown which is an even parity scalar and isoscalar will contribute. We illustrate the technique by considering

$$a_1=\frac{1}{3}\sum_{n=1}^{A}\langle\Phi|e^{i(\vec k-\vec k')\cdot\vec r_n}\tau_\beta^{(n)}\tau_\alpha^{(n)}\vec\sigma_n\cdot\vec k'\,\vec\sigma_n\cdot\vec k|\Phi\rangle, \qquad \text{(A2)}$$

where Eq. (11) has been used to determined $P_1^{(n)}$.

As the coupling of the nucleon isospin operator $\vec\tau$ to the position or spin vectors is not physically defined, the only contribution from the isospin factor will occur when $\alpha=\beta$ so that $\tau_\beta^{(n)}\tau_{\alpha=\beta}^{(n)}=1$. Thus

$$a_1=\tfrac{1}{3}\delta_{\alpha\beta}\vec k'\cdot\vec k\sum_{n=1}^{A}\langle\Phi|e^{i(\vec k-\vec k')\cdot\vec r_n}|\Phi\rangle + \tfrac{1}{3}i\delta_{\alpha\beta}\sum_{n=1}^{A}\langle\Phi|e^{i(\vec k-\vec k')\cdot\vec r_n}\vec\sigma_n\cdot(\vec k'\times\vec k)|\Phi\rangle, \qquad \text{(A3)}$$

where $\vec\sigma_n\cdot\vec k'\,\vec\sigma_n\cdot\vec k=\vec k'\cdot\vec k+i\vec\sigma_n\cdot(\vec k'\times\vec k)$ has been used. This expression is evaluated using the expansion

$$e^{i(\vec k-\vec k')\cdot\vec r_n}=4\pi\sum_{lm}i^l j_l(qr_n)Y_{lm}^*(\hat q)Y_{lm}(\hat r_n), \qquad \text{(A4)}$$

where $q = |\vec{k}' - \vec{k}|$ and $\hat{q} = (\vec{k}' - \vec{k})/q$. In the first term of (A3) the only contribution will result from the scalar term in (A4) involving $Y_{00}(\hat{r}_n)$. As there is no way to couple $Y_{lm}(\hat{r}_n)$ and $\vec{\sigma}_n$ to form an even-parity configuration space scalar, the second term in (A3) vanishes. Therefore

$$\begin{aligned} a_1 &= \tfrac{1}{3}\delta_{\alpha\beta}\vec{k}' \cdot \vec{k} \sum_{n=1}^{A} \langle \Phi | j_0(qr_n) | \Phi \rangle \\ &= \tfrac{1}{3}\delta_{\alpha\beta}\vec{k}' \cdot \vec{k} \int d^3r\, j_0(qr) \sum_{n=1}^{A} \langle \Phi | \delta(\vec{r} - \vec{r}_n) | \Phi \rangle \\ &= \tfrac{1}{3}\delta_{\alpha\beta}\vec{k}' \cdot \vec{k} \int d^3r\, j_0(qr)\rho(\vec{r}) \\ &= \tfrac{1}{3}\delta_{\alpha\beta}\vec{k}' \cdot \vec{k}\,\tilde{\rho}(q), \end{aligned} \tag{A5}$$

$$\tilde{\rho}(q) \equiv \int d^3r\, j_0(qr)\rho(\vec{r}),$$

where the nuclear density is normalized to the nucleon number

$$\int d^3r\, \rho(\vec{r}) = A. \tag{A6}$$

Proceeding with a similar analysis for the remaining terms, we find

$$\begin{aligned} a_2 &= \tfrac{2}{3}\delta_{\alpha\beta}\vec{k}' \cdot \vec{k}\tilde{\rho}(q) = a_3, \\ a_4 &= \tfrac{4}{3}\delta_{\alpha\beta}\vec{k}' \cdot \vec{k}\,\tilde{\rho}(q). \end{aligned} \tag{A7}$$

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†Present address.

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