Nonperturbative three-dimensional formalism for the study of the nucleon-nucleon interaction*

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(Received 22 May 1975)

As an alternative to the dispersion theoretic approach to the two-nucleon interaction, we outline an improved version of an older method which derives both an interaction and the wave equation of which it is a part on an equal footing. This is done by reducing the Bethe-Salpeter equation to an equal time equation. The main improvement is that the interaction is given in nonperturbative form rather than as a sum of Feynman diagrams. As an illustration of the formalism, we test the validity of a simple version of the isobar model for the nucleon-nucleon problem and find it quantitatively inadequate. We discuss parenthetically how the existence of classes of reduction schemes is reflected as an ambiguity in the definition of the interaction.

[NUCLEAR REACTIONS Theory of nucleon-nucleon interaction. Non-perturbative field theoretical method.]

I. INTRODUCTION

Because it is so fundamental to nuclear physics, the problem of deriving the two-nucleon interaction from a field theory of strong interactions continues to be a worthy intellectual challenge. For the specifically two-nucleon problem, present consensus is that the most powerful approach[1-6] consists of the following steps: (i) Compute the nucleon-nucleon (N-N) S matrix as a sum of one-boson exchange and two-pion exchange contributions. The latter is calculated using dispersion-theoretic machinery with experimental input of S and P wave π-π phase shifts and π-N scattering amplitudes. This includes the effect of the ρ meson so that the one-boson exchange is restricted mainly, though not necessarily exclusively, to π and ω exchange. (ii) To permit calculation of all partial waves of the N-N interaction a unitarization procedure is carried out. For application to nuclear physics, this is done by introduction of a potential, in general nonlocal and energy dependent, within the framework of a covariant Lippmann-Schwinger equation (LSE), chosen so that the computed S matrix is the solution of the latter to the appropriate order of approximation.

It has been noted repeatedly[7] that there are an infinite number of ways of carrying out this unitarization procedure, corresponding to the lack of uniqueness in defining off-shell behavior of the scattering. This is made evident by studying procedures for obtaining the LSE from the Bethe-Salpeter equation (BSE) by "reduction" from an individual time to a single-time description. Based on some recent test calculations,[8] it is now believed[5] that the theoretical ambiguity alluded to here is without serious numerical consequence, as long as the potential properly includes two pion exchange effects.

The test calculations were made with a scalar model, however. It is an historic *cause célèbre* that, at least in the static approximation, the situation is quite otherwise for pseudoscalar mesons.[9] Whether this ancient difficulty is an artifact of the static approximation or an intrinsic property of the theory is an unsettled matter. In our view, it represents a major, if not the major, uncertainty in recent work and requires renewed attention. (This is not, however, the aim of the present work.)

Recent work[1-5] has uniformly followed the procedures described by Partovi and Lomon.[10] For reasons (or prejudices) that have remained constant for two decades,[9] we shall follow the distinct procedure of Lévy, Klein, and Macke (LKM) reviewed recently.[7] Transformation of the two results into each other can be done, however.[7] An illustration of the quantitative differences involved will be included in our results.

But we must not turn to the main stimulus for the present work. The ambiguity just discussed has nothing to do with the S-matrix part of the calculation. However, the latter calculation itself may involve too many difficulties for direct study of the many-body problem (or even the three- and four-body problem) from "first principles." To make progress here, it appears adviseable to make some compromises which we shall first test out on the two-body problem.

One such compromise that has received con-

siderable attention recently is the isobar model.[11-15] Out interest in this model has been kindled by the conviction—which remains to be substantiated—that the compromises here have been more severe than necessary. For example, the treatment of the $\Delta(1236)$, the 3-3 resonance, as a discrete isobar, may in many instances be an unnecessary as well as an unwarranted approximation. In this paper, we develop a formulation of the two-nucleon problem which contains the usual isobar model as a limiting case and should therefore allow a study of its validity.

Since our first aim is to study the effect on nuclear processes of excited states of the nucleon, we shall assume, for the time being, that the major effects of the π-π interaction can be accounted for by the exchange of the well-known boson resonances, treated as fundamental particles. This is the major theoretical weakness of the approach described in Sec. II, since it is subject to the same criticism we have leveled above against previous treatments of the nucleon resonances. The finite lifetime effects can be put in, but problems of possible double counting (or omissions) remain to be resolved within this framework.

With the admitted limitation, however, it is then possible to give a systematic and nonperturbative form for the kernel of the BSE.[16] In Sec. III this is then reduced to a relativistic Schrödinger or LSE equation, using the method of LKM. An advantage of this method, which pertains even to the nonperturbative form derived in this paper is that diagrams can be associated with contributions and systematic rules for construction given. *This is the major new result for this paper.*

In the remaining sections we undertake a preliminary calculation of the 2π exchange potential, deriving formulas for the potential in coordinate space for each eigenchannel. As a preliminary result, repeating much previous work, we include only nucleon and zero width Δ intermediate states in calculation of the potential, but include all two-pion exchange diagrams, some of which have been omitted in recent applications.[11,14,15] A numerical comparison is carried out in Sec. V, together with some discussions and plans for future work.

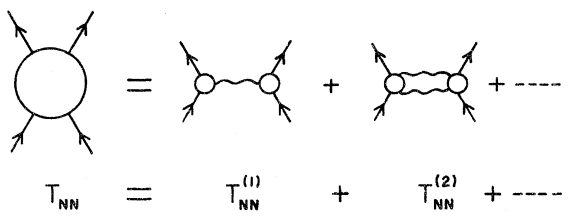

FIG. 1. Expansion of the nucleon-nucleon scattering amplitude in terms of the number of exchanged mesons (neglecting meson-meson interaction).

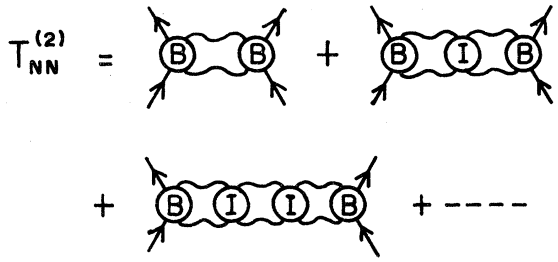

FIG. 2. Diagrammatic decomposition of the two-boson exchange part of the two-nucleon scattering amplitude, taking boson-boson interaction into account. The definitions of B and I are given in the text.

II. NONPERTURBATIVE KERNEL FOR THE BETHE-SALPETER EQUATION IN THE LINEAR MESON APPROXIMATION

Let us assume, to start with, that the strongly interacting bosons and, in particular, pions do not interact with each other. Then, as illustrated in Fig. 1, the amplitude for nucleon-nucleon (N-N) scattering is decomposable uniquely into a sum of diagrams (terms) involving the *exchange* of one, two, ..., mesons; furthermore, by a *single* cut in the momentum transfer channel each diagram is divided into pieces related in a simple way to single-nucleon processes. These processes are in turn the pion emission form factor, virtual pion-nucleon scattering, etc.

As soon as meson-meson interactions are taken into account this simple analysis fails. If the wavy line represents the exact boson propagator, $T^{(1)}_{N-N}$ is still as we have described it. On the other hand, $T^{(2)}_{N-N}$ is replaced by the sum of diagrams shown in Fig. 2. This figure together with Fig. 3 illustrates the difficulty associated with straightforward off-shell diagram analysis. In these figures, the bubble I represents the irreducible (with respect to two-pion cuts) pion-pion interaction. The bubble B is not, by itself, the continuation off shell of any physical process.

It follows that the sum illustrated in Fig. 2 cannot be written off shell as a convolution of π-nucleon scattering processes associated with each nucleon because the counting of diagrams is wrong.

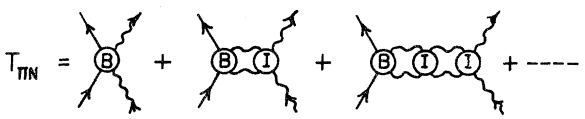

FIG. 3. Diagram analysis of the π-nucleon scattering amplitude, exhibiting its two-pion momentum transfer channel singularities.

It is precisely *this* difficulty which is solved by analyzing the sum in Fig. 2, in terms of its t channel (momentum transfer) singularities and writing a dispersion relation in this variable for the on-shell scattering.[1-6] This beautiful solution encounters problems only later, if one wishes to embed it into a complete off-shell formalism.

Because the latter is also our aim, we have decided to admit uncertainties at an earlier stage. In effect, we shall assume that the B bubble still represents a good approximation to off-shell π-N scattering through the simplest accessible nucleon and isobar intermediate states, namely, that nonlinear interactions do not play a decisive role in the formation of these states. The effect of these pion-pion interactions is to be taken into account in a generalized version of the one-boson exchange model, in a form which can be derived from an effective field theory.

In such a field theory, one might, for example, include on an equal footing the coupling of the π, ρ, ω, and σ mesons with the nucleon, as well as the couplings of these various mesons with each other in order to account for their widths. The full resulting model for the nucleon-nucleon interaction will be reserved for a future publication. In the present one we shall be concerned mainly with the theoretical formulation of nucleon isobar excitation by pion emission and absorption.

In Ref. 16 two nonperturbative forms were given for the S matrix for nucleon-nucleon scattering from an initial state p_1s_1, p_2s_2 to a final state $p_1's_1'$, $p_2's_2'$, under the assumption that nucleons interact only with pions and that the pion-pion interaction can be neglected. We refer the reader to this previous publication for the derivation. Ignoring antisymmetrization, the general structure of either result is

$$(p_1's_1', p_2's_2'|S|p_1s_1, p_2s_2) = \delta(s_1s_1')\delta(s_2s_2')\delta^4(p_1'-p_1)\delta^4(p_2'-p_2) + \sum_{n=1}^{\infty} (p_1's_1', p_2's_2'|T^{(2n)}|p_1s_1, p_2s_2), \tag{2.1}$$

where $T^{(2n)}$ is the contribution from the exchange of n mesons. (We shall deal here explicitly only with pions, but all results and remarks are extended readily to include other bosons.)

Of the two forms of $T^{(2n)}$, one is related directly to its structure in terms of Feynman diagrams. For this form we write

$$(p_1's_1', p_2's_2'|T^{(2n)}|p_1s_1, p_2s_2) = -C(p_1')C(p_2')C(p_1)C(p_2)\int (dx_1')(dx_2')(dx_1)(dx_2)\exp[-ip_1'x_1' - ip_2'x_2' + ip_1x_1 + ip_2x_2]$$
$$\times \bar{u}(p_1's_1')\bar{u}(p_2's_2')I^{(2n)}(x_1'x_2'; x_1x_2)u(p_1s_1)u(p_2s_2), \tag{2.2}$$

where $u(ps)$ is a positive-energy Dirac spinor normalized so that $\bar{u}u$ is invariant, $px = \vec{p}\cdot\vec{x} - p_0x_0$, and

$$C(p) = [M/(2\pi)^3E(p)]^{1/2}, \quad E(p) = (p^2+m^2)^{1/2}. \tag{2.3}$$

Though a more formal definition can be given, for our purposes we may view $I^{(2n)}$ simply as the sum of all Feynman diagrams involving the exchange of n mesons. From this quantity, we select its two-particle irreducible part as kernel of the BSE.

The second form of $T^{(2n)}$ and the one we shall utilize in this work is (suppressing spin indices)

$$(p_1'p_2'|T^{(2n)}|p_1p_2) = (i^n/n!)\int (d1)(d2)\cdots(dn)(d1')\cdots(dn')(p_1'|T(\hat{j}_R(1)\cdots\hat{j}_R(n))|p_1)(p_2'|T(\hat{j}_R(1')\cdots\hat{j}_R(n'))|p_2)$$
$$\times \Delta(11')\cdots\Delta(nn'). \tag{2.4}$$

Here $\Delta(11')$ is the complete meson propagator and the currents $\hat{j}_R$ which appear in the Wick products are defined by the equation (omitting renormalization constants)

$$\hat{j}_R(1) = \int \Delta^{-1}(11')\hat{\varphi}(1')(d1'), \tag{2.5}$$

where $\hat{\varphi}$ is the meson field operator. [Because the full inverse propagator appears in (2.5), the matrix elements of $\hat{j}_R$ differ from those of the source operator appearing in the fundamental field equation for the operator $\hat{\varphi}$ by the *removal* of certain vacuum polarization effects. In so far as the latter may be assumed not be dominate our considerations, this distinction will later become blurred, especially for other than single-meson-exchange effects.]

We wish to extract from (2.2), a contribution to a nonperturbative kernel for a BSE in momentum space. Since in (2.2) we are on the mass shell, and considering the center-of-mass system, the

momenta there satisfy

(a) $\vec{p}_1+\vec{p}_2=\vec{p}_1'+\vec{p}_2'=0$,

(b) $p_{10}+p_{20}=p_{10}'+p_{20}'\equiv W$,

(c) $p_{10}=(p_1^2+m^2)^{\frac{1}{2}}$, etc.

In the BSE (a) and (b) remain, but (c) is replaced by

(c′) $p_{10}=\frac{1}{2}W+p_0$, $p_{20}=\frac{1}{2}W-p_0$,

and similarly for the primed coordinates. This defines the analytic continuation off the mass shell of the energy variables that occur in the BSE. In addition we strip away the spinors u, $\bar{u}$, fore and aft, since in the BSE we consider both positive- and negative-energy components on an equal footing.

For the low energy two-nucleon problem all our experience indicates that this latter feature is of no particular merit. In fact, we use the BSE only as an intermediate step on the way to a covariant Schrödinger or Lippmann-Schwinger formalism containing only positive-energy components in the wave function. As described in Ref. 17, for this process we require only the positive-energy part of the full $I^{(2n)}$. This observation is useful in connection with (2.4) because it means that for our purposes we do not have to consider an extension of definition of the latter to include negative-energy components, i.e., antiparticles.

We still have to consider for (2.4) the continuation described by the requirements (a), (b), and (c)′ above. It is convenient to first rewrite (2.4) in terms of time-ordered currents, namely,

$$(p_1'p_2'|T^{(2n)}|p_1p_2)=i^n\int(d1)\cdots(dn)(d1')\cdots(dn')\theta(1-2)\theta(2-3)\cdots\theta((n-1)-n)\,\theta(1'-2')\cdots\theta((n-1)'-n')$$
$$\times(p_1'|\hat{j}_R(1)\cdots\hat{j}_R(n)|p_1)(p_2'|\hat{j}_r(1')\cdots\hat{j}_R(n')|p_2)\sum_{P(i_1'\cdots i_n')}\Delta(1i_1')\cdots\Delta(ni_n'),\qquad(2.6)$$

where the θ are step functions in time and the P are the permutations of $(1'\cdots n')$.

We now identify 1 with x_1', 1′ with x_2', n with x_1, and n' with x_2, where the x's are the variables of (2.2). For the integrand of (2.6) we write the following trivial identity

$$(p_1'|\hat{j}_R(x_1')\hat{j}_R(2)\cdots\hat{j}_R(x_1)|p_1)(p_2'|\hat{j}_R(x_2')\hat{j}_R(2')\cdots\hat{j}_R(x_2)|p_2)$$
$$=\exp[-ip_1'x_1'-ip_2'x_2'+ip_1x_1+ip_2x_2](p_1'|\exp(ip_1'x_1')j(x_1')\cdots j(x_1)\exp(-ip_1x_1|p_1)$$
$$\times(p_2'|\exp(ip_2'x_2')j(x_2')\cdots j(x_2)\exp(-ip_2x_2)|p_2).\qquad(2.7)$$

The first factor is the one which is continued according to the recipe given above. Comparison of (2.6) and (2.7) with (2.2) now allows us to make the plausible identification

$$\bar{u}(p_1')\bar{u}(p_2')I^{(2n)}(\xi_1'\xi_2';\xi_1\xi_2)u(p_1)u(p_2)$$
$$=-i^n[C(p_1')C(p_2')C(p_1)C(p_2)]^{-1}\int(d2)\cdots(d(n-1))(d2')\cdots(d(n-1)')\,\theta(\xi_{10}'-t_2)\theta(t_2-t_3)\cdots\theta(t_{n-1}-\xi_{10})$$
$$\times\theta(\xi_{20}'-t_2')\theta(t_2'-t_3')\cdots\theta(t_{n-1}'-\xi_{20})$$
$$\times(p_1'|\exp(ip_1'\xi_1')\hat{j}_R(2)\cdots\hat{j}_R(n-1)\hat{j}_R(\xi_1)\exp(-ip_1\xi_1)|p_1)$$
$$\times(p_2'|\exp(ip_2'\xi_2')\hat{j}_R(\xi_2')\hat{j}_R(2')\cdots\hat{j}_R((n-1)')\hat{j}_R(\xi_2)\exp(-ip_2\xi_2)|p_2)$$
$$\times\sum_P\Delta(\xi_1'-i_1')\Delta(2-i_2')\cdots\Delta(\xi_1-i_n').\qquad(2.8)$$

The factors $C(p)$ will now be dropped as we revert to a corresponding noncovariant normalization of $\bar{u}u$. In conjunction with the method of Ref. 17, (2.8) suffices to provide us with a new basis for the study of the two nucleon problem.

III. REDUCTION OF THE BETHE-SALPETER EQUATION TO A NEW SINGLE-TIME EQUATION

The essential result of Ref. 17 is a compact form for the (homogeneous) iterated single-time equation associated by the method of LKM with the BSE. In coordinate space in the nth order of approximation this

equation takes the form

$$\varphi(\vec{x}_1,\vec{x}_2)=\int \Omega^{(2n)}(\vec{x}_1\vec{x}_2;\vec{x}_1'\vec{x}_2')\varphi(\vec{x}_1',\vec{x}_2')d\vec{x}_1'd\vec{x}_2', \tag{3.1}$$

where[18]

$$\begin{aligned}\Omega^{(2n)}(\vec{x}_1\vec{x}_2;\vec{x}_1'\vec{x}_2')=iT^{-1}[E(p_1)+E(p_2)-W]^{-1}\int d\xi_{10}d\xi_{20}dt_1'dt_2'd\vec{\xi}_1d\vec{\xi}_2\exp[i(\xi_{10}+\xi_{20}-t_1'-t_2')\tfrac{1}{2}W]\\ \times\Big(\theta(\xi_{20}-\xi_{10})\Lambda_+^{(1)}(\vec{p}_1)\beta^{(1)}\exp\{-i(\xi_{20}-\xi_{10})[E(p_1)-\tfrac{1}{2}W]\}\\ +\theta(\xi_{10}-\xi_{20})\Lambda_+^{(2)}(\vec{p}_2)\beta^{(2)}\exp\{-i(\xi_{10}-\xi_{20})[E(p_2)-\tfrac{1}{2}W]\}\Big)\\ \times\delta_{p_1}(\vec{x}_1-\vec{\xi}_1)\delta_{p_2}(\vec{x}_2-\vec{\xi}_2)I^{(2n)}(\xi_1\xi_2;x_1'x_2')\\ \times\Big(\theta(t_1'-t_2')\Lambda_+^{(1)}(\vec{q}_1)\beta^{(1)}\exp\{-i(t_1'-t_2')[E(q_1)-\tfrac{1}{2}W]\}\\ +\theta(t_2'-t_1')\Lambda_+^{(2)}(\vec{q}_2)\beta^{(2)}\exp\{-i(t_2'-t_1')[E(q_2)-\tfrac{1}{2}W]\}\Big).\end{aligned} \tag{3.2}$$

In this expression $\vec{p}_1$, $\vec{p}_2$, $\vec{q}_1$, and $\vec{q}_2$ are differential operators acting on coordinates 1 and 2, respectively, $\vec{p}_1$ and $\vec{p}_2$ acting on the δ functions as indicated, and q_1 and q_2 acting on $\varphi(\vec{x}_1',\vec{x}_2')$. T is an infinite time which will cancel out after all time integrations have been performed.

The evaluation of (3.2), with $I^{(2n)}$ given by (2.8) yields a result for $\Omega^{(2n)}$ which can be described systematically. The derivation will be given in barest outline. The basic technique is to consider (3.2) as a sum of all possible time-ordered contributions. This allows the time integrals to be carried out explicitly if we decompose a typical operator product (suppressing the subscript R on $\hat{j}_R$)

$$\begin{aligned}(p_1'|\hat{j}(x_1)\cdots\hat{j}(x_n)|p_1)&=\sum_{I_1\cdots I_{n-1}}(p_1'|\hat{j}(x_1)|I_1)(I_1|\hat{j}(x_2)|I_2)\cdots(I_{n-1}|\hat{j}(x_n)|\vec{p}_1)\\ &=\sum_{I_1\cdots I_{n-1}}\exp\{i[E(p_1')t_1-E(I_1)(t_1-t_2)-\cdots-E(I_{n-1})(t_{n-1}-t_n)-E(p_1)t]\}\\ &\quad\times(\vec{p}_1'|\hat{j}(\vec{x}_1)|I_1)(I_1|\hat{j}(\vec{x}_2)|I_2)\cdots(I_{n-1}|\hat{j}(\vec{x}_n)|\vec{p}_1),\end{aligned} \tag{3.3}$$

and represent the (free) meson propagator in the form

$$\Delta(x_1-x_2)=\frac{i}{(2\pi)^3}\int\frac{d\vec{k}_1}{2\omega_1}\exp[i\vec{k}_1\cdot(\vec{x}_1-\vec{x}_2)-i\omega_1|t_1-t_2|]. \tag{3.4}$$

It is most convenient to express the results of these integrations in momentum space. With

$$\tilde{\varphi}(\vec{p}_1,\vec{p}_2)=\frac{1}{(2\pi)^3}\int\exp(-i\vec{p}_1\cdot\vec{x}_1-i\vec{p}_2\cdot\vec{x}_2)\varphi(\vec{x}_1,\vec{x}_2), \tag{3.5}$$

we write

$$[W-E(\vec{p}_1)-E(\vec{p}_2)]\tilde{\varphi}(\vec{p}_1,\vec{p}_2)=\frac{1}{(2\pi)^6}\int\Omega^{(2n)}(\vec{p}_1,\vec{p}_2;\vec{p}_1'\vec{p}_2')\tilde{\varphi}(\vec{p}_1',\vec{p}_2'), \tag{3.6}$$

and give the rules for constructing $\Omega^{(2n)}$. These are summarized in the formula

$$\begin{aligned}\Omega^{(2n)}(\vec{p}_1\vec{p}_2;\vec{p}_1'\vec{p}_2')=\sum_{P(\alpha)}\sum_{O(\beta)}\int\prod_{i=1}^{n}\left(\frac{d\vec{k}_i}{2\omega_i}\right)\prod_{j=1}^{n+1}[D_j]^{-1}\sum_{I_1\cdots I_{n-1}}(\vec{p}_1|\tilde{j}(\vec{k}_1)|I_1)\cdots(I_{n-1}|\tilde{j}(\vec{k}_n)|\vec{p}_1')\\ \times\sum_{I_1'\cdots I_{n-1}'}(\vec{p}_2|\tilde{j}(-\vec{k}_{1\alpha})|I_1')\cdots(I_{n-1}'|\tilde{j}(-\vec{k}_{n\alpha})|\vec{p}_2'),\end{aligned} \tag{3.7}$$

where

$$\tilde{j}(\vec{k})=\frac{1}{(2\pi)^{3/2}}\int\exp(i\vec{k}\cdot\vec{x})\hat{j}(\vec{x})(d\vec{x}). \tag{3.8}$$

Here $\vec{k}_{1\alpha}\cdots\vec{k}_{n\alpha}$ is a permutation $P(\alpha)$ of $\vec{k}_1\cdots\vec{k}_n$, and we sum (as in (2.6)) over all such permutations. Each different permutation determines a different order of "absorption" by nucleon 2 of the mesons which have been "emitted" by nucleon 1.

The second sum is over time orderings $O(\beta)$

which must be specified before the product of energy denominators D_i can be computed. Each ordering is defined by associating with each of the $\hat{j}$ operators its original time variable from the sets $t_1 \cdots t_n$ and $t_1' \cdots t_n'$ for nucleons 1 and 2, respectively, as in (2.6). The sum then includes independently each possible over-all time ordering of these $2n$ variables which maintains the constraints $t_1 > t_2 > \cdots > t_n$, $t_1' > t_2' > \cdots > t_n'$. For each such time ordering, there is a contribution which can be associated with a diagram, which only in its labeling differs from standard time-ordered diagrams *without* any nucleon pairs. For example for $n=2$, we have the diagrams of Fig. 4 plus another six crossed diagrams obtained by switching the meson line terminal points on the second nucleon. The energy denominators are now given by the standard rule: Draw a horizontal line cutting the diagram between each pair of interactions. The associated energy denominator is $W - \sum E - \sum \omega$, where the sum is over intersected nucleon and meson lines. Here nucleon is to be understood in a generalized sense, to include any intermediate state I, representing an eigenstate of energy $E(I)$. Thus for diagram (4c), we find

$$\begin{aligned} D_1 &= W - E(\vec{p}_1) - E(I_1') - \omega_1 , \\ D_2 &= W - E(\vec{p}_1) - E(\vec{p}_2') - \omega_1 - \omega_2 , \\ D_3 &= W - E(I_1) - E(p_2') - \omega_2 . \end{aligned} \tag{3.9}$$

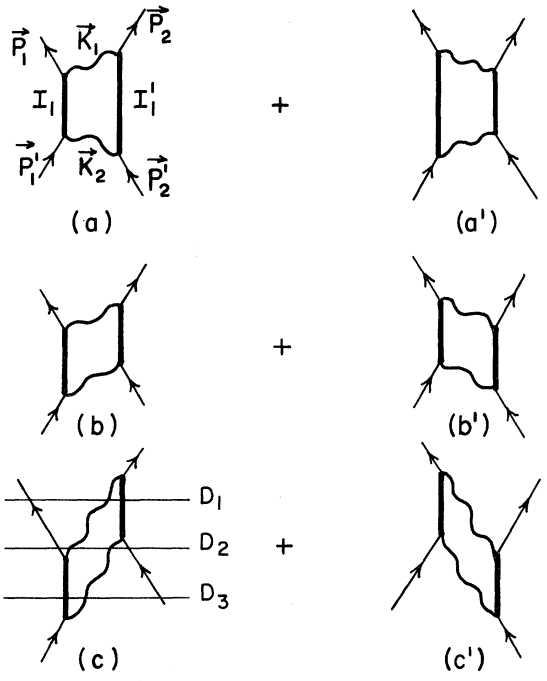

FIG. 4. Uncrossed general time-ordered two-pion exchange diagrams representing contributions to (3.7).

As we have remarked above, $\Omega^{(2n)}$ is the kernel of an iterated equation and therefore contains reducible parts. In the conventional three-dimensional definition a diagram is reducible if it contains at least one intermediate state with precisely two nucleons (i.e., no mesons). For instance, from Fig. 4, diagrams (a), (a′), (b), (b′), are reducible if I_1 and I_1' are single-nucleon intermediate states. Let $\Omega_r^{(2n)}$ be the residual part of $\Omega^{(2n)}$ after all terms which are reducible have been removed. Our final (conventional) two-nucleon equation is obtained in the form of Eq. (3.6) with $\Omega^{(2n)}$ replaced by $V^{(2n)}$, where

$$V^{(2n)} = \sum_{i=1}^{n} \Omega_r^{(2i)} . \tag{3.10}$$

Further reductions can be carried out, leading for instance to the isobar model[11] if additional intermediate states are approximated by energy eigenstates. These are represented by introducing amplitudes in addition to the two-nucleon amplitude $\varphi(\vec{p}_1, \vec{p}_2)$ and coupled equations are written such that elimination by iteration of the new amplitudes leads us back to (3.6). This matter will be pursued in more detail on a future occasion, but here we shall confine our further attention to the study of (3.10).

IV. TWO-PION EXCHANGE N-N POTENTIAL

In this section, we present a calculation of the 2π exchange potential to illustrate the application of our formalism. For simplicity, only the N and $N^*(1236)$ will be included as intermediate states $|I\rangle$ of (3.7). Furthermore the $N^*(1236)$ will be treated as an elementary particle. The accuracy

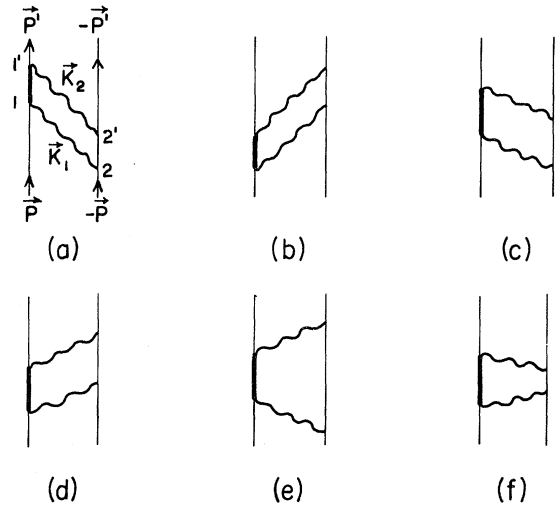

FIG. 5. Direct two-pion exchange time-ordered diagrams involving (N^*-N) intermediate baryon states.

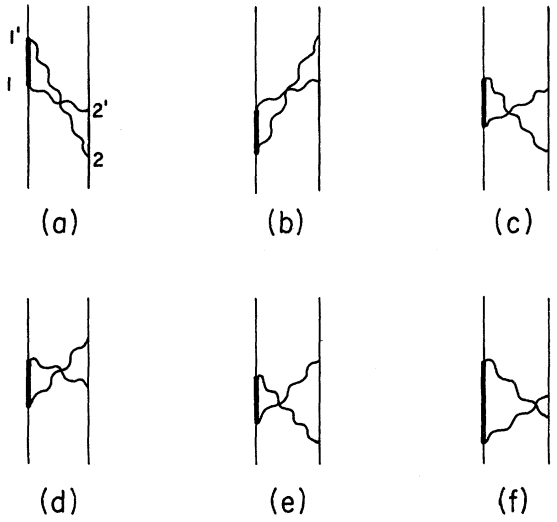

FIG. 6. The crossed analogs of Fig. 5.

of this zero-width approximation will hopefully be examined in a future publication.

In the center-of-mass system, the coordinate space N-N potential $V^{(2n)}(\vec{r}', \vec{r})$ is related to $V^{(2n)}$ of (3.10) by

$$V^{(2n)}(\vec{r}', \vec{r}) = \frac{1}{(2\pi)^9} \int d\vec{p}\, d\vec{p} \exp(-i\vec{p}' \cdot r' + i\vec{p} \cdot \vec{r}) \times \overline{V}^{(2n)}(\vec{p}', \vec{p}), \tag{4.1}$$

where $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$ are the relative momentum and coordinate, respectively, and

$$V^{(2n)}(\vec{p}_1', \vec{p}_2'; \vec{p}_1, \vec{p}_2) = \delta(\vec{p}_1' + \vec{p}_2' - \vec{p}_1 - \vec{p}_2)\overline{V}^{(2n)}(\vec{p}', \vec{p}). \tag{4.2}$$

The relevant kinematic variables for $n=2$ are shown in Fig. 5(a).

For the low-energy N-N interaction, the kernel $V^{(2n)}(\vec{p}', \vec{p})$ (dropping the bar) can be expanded in powers of (p/M) and the above nonlocal form of the potential can be transformed in a well-known way into a local but momentum-dependent form: By introducing the variables $\vec{K} = \vec{p}' - \vec{p}$, $\vec{q} = \frac{1}{2}(\vec{p}' + \vec{p})$, we can write, with $V^{(2n)}(\vec{p}', \vec{p}) \to V^{(2n)}(\vec{K}, \vec{q})$,

$$V^{(2n)}(\vec{r}', \vec{r}) = \tilde{V}^{(2n)}[\tfrac{1}{2}(\vec{r} + \vec{r}'),\ i\vec{\nabla}_r]\,\delta(\vec{r} - \vec{r}'),$$
$$\tilde{V}^{(2n)}(\vec{r}, \vec{q}) = \frac{1}{(2\pi)^6} \int d\vec{k} \exp(i\vec{k} \cdot \vec{r}) V^{(2n)}(\vec{k}, \vec{q}). \tag{4.3}$$

The local limit corresponds to setting $\vec{q} = 0$.

To calculate $\tilde{V}^{(4)}(\vec{r}, 0) \equiv V^{(4)}(\vec{r})$, we utilize the rules for constructing $\Omega^{(4)}$ given in Sec. III. Among these diagrams, we illustrate in Fig. 5 the uncrossed diagrams in which one nucleon goes through the N^* intermediate state and the other through the intermediate state N. Figure 6 shows the corresponding crossed diagrams.

For computational purposes, each diagram consists of two kinds of elements, the vertex function for each vertex, $N \to N' + \pi$ (where $N' = N$ or N^*), and the energy denominator for each independent intermediate state. A vertex $N \to N' + \pi$ corresponds to a matrix element $\langle \vec{p}' | j(\vec{k}) | \vec{p} \rangle$ where $\vec{p}$ and $\vec{p}'$ are, respectively, the incoming and outgoing fermion momenta and $\vec{k}$ is the momentum of the meson emitted from the vertex. We write, in a noncovariant notation

$$\langle \vec{p}' | j(\vec{k}) | \vec{p} \rangle = \delta(\vec{p}' + \vec{k} - \vec{p}) F(\vec{p}', \vec{k}, \vec{\Sigma}, \vec{T}, \vec{I}), \tag{4.4}$$

where F is a scalar function constructed from the momenta $\vec{k}, \vec{p}$, a fermion spin operator $\vec{\Sigma}$, a fermion isospin operator $\vec{T}$, and the pion isospin $\vec{I}$. If N and N' are both nucleons ($s = t = \frac{1}{2}$), then $\vec{\Sigma}$ and $\vec{T}$ are just the ordinary Pauli operators. If N^* occurs as an intermediate state, the operators $\vec{\Sigma}$ and $\vec{T}$ are transition operators and are defined by their reduced matrix elements $\langle S_1 \| \vec{\Sigma} \| S_2^* \rangle$ and $\langle t_1 \| \vec{T} \| t_2^* \rangle$ in the baryonic spin and isospin space.

In the static limit to which we confine our present attention, we shall write[13,14]

$$F(\vec{p}', \vec{k}, \vec{\Sigma}, \vec{T}, \vec{I}) \cong f(k)[(i g_l / 2m^{[l]})(\vec{\Sigma} \cdot \vec{K}^{[l]})(\vec{T} \cdot \vec{I})], \tag{4.5}$$

where l denotes the internal π-N relative orbital angular momentum for the N^* and g_l and $m^{[l]}$ are the coupling constant and the intermediate fermion mass, respectively. The cutoff function $f(k)$ is discussed below. The angular function $K^{[l]}$ is the spherical tensor defined by

$$K_m^{[l]} = k^l [(2l+1)/4\pi]^{1/2} Y_m^{(l)}(\hat{K}). \tag{4.6}$$

We note that all direct or ladder-type diagrams with a given set of intermediate states yield the same product of vertex functions, namely,

$$V_D(\vec{k}_1, \vec{k}_2) = f(k_1) f^*(k_1) f(k_2) f^*(k_2) (i g_{l_1} / 2m^{[l_1]})^2 (i g_{l_2} / 2m^{[l_2]})^2 (\vec{\Sigma}_1 \cdot \vec{K}_1^{[l_1]})(\vec{\Sigma}_{1'} \cdot \vec{K}_2^{[l_1]}) \times [\vec{\Sigma}_2 \cdot (-\vec{K}_1)^{[l_2]}][\vec{\Sigma}_{2'} \cdot (-\vec{K}_2)^{[l_2]}](\vec{T}_1 \cdot \vec{I}_1)(\vec{T}_2 \cdot \vec{I}_1)(\vec{T}_{1'} \cdot \vec{I}_2)(\vec{T}_{2'} \cdot \vec{I}_2). \tag{4.7}$$

The corresponding vertex function for the crossed diagrams, V_c, is obtained from (4.7) by the interchanges $\Sigma_2 \to \Sigma_{2'}$, $T_2 \to T_{2'}$.

In this paper, the cutoff function has been chosen simply as

$$f(k)=\Lambda^2/(\Lambda^2+k^2). \tag{4.8}$$

Since our results are sensitive to the choice of Λ, it would be well if some theoretical basis were available for the choice of both the form of the cutoff and the parameters in it. Recent suggestions[5] directed toward the solution of this problem may prove of great value, but take us beyond the scope of the present work.

The rule for evaluation of the energy denominators has been given in Sec. III. Static limits are to be taken as previously described. This yields the quantity $V^{(4)}(\vec{K},\vec{q}=0)$ [(4.3)], where

$$V^{(4)}(\vec{K},\vec{q}=0)=\int \frac{d\vec{k}_1}{2\omega_1}\frac{d\vec{k}_2}{2\omega_2}\delta(\vec{K}-\vec{k}_1-\vec{k}_2)[V_D(\vec{k}_1,\vec{k}_2)\Pi_D+V_C(\vec{k}_1,\vec{k}_2)\Pi_C], \tag{4.9}$$

where V_D and V_C have been defined above, as in (4.7), and Π_D, Π_C are the appropriate sums of products of energy denominators. Substituting (4.9) into (4.3), the integrations cannot be done analytically in their entirety, and therefore the potential for each eigenchannel has been calculated numerically. The potential $V_{l'l}^{jsT}(r)$ in each eigenchannel $|(ls)jT\rangle$ is defined by

$$V(\vec{r})=\sum_{ll',s,jm,Tm_T}\mathcal{Y}_{l's}^{jmTm_T*}(r)V_{l'l}^{jsT}(r)\mathcal{Y}_{ls}^{jmTm_T}(r), \tag{4.10}$$

where $\mathcal{Y}_{ls}^{jmTm_T}$ are the usual spin-angle isospin functions,

$$\mathcal{Y}_{ls}^{jmTm_T}=\sum_{m_sm_l}\langle jm\,|\,lsm_lm_s\rangle\,\mathcal{Y}_{m_l}^{(l)}(\hat{r})\,\chi_{m_s}^{(s)}\chi_{m_T}^{(T)}, \tag{4.11}$$

and χ are spin or isospin functions. From Eqs. (4.9) and (4.10), we have

$$V_{l'l}^{jsT}(r)=\frac{1}{(2\pi)^6}\int\frac{d\vec{k}_1d\vec{k}_2}{2\omega_12\omega_2}\langle\mathcal{Y}_{l's}^{jmTm_T}(\hat{r})|\,e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{r}}[V_D(\vec{k}_1,\vec{k}_2)\Pi_D+V_C(\vec{k}_1,\vec{k}_2)\Pi_C]\,|\mathcal{Y}_{ls}^{jmTm_T}(\hat{r})\rangle. \tag{4.12}$$

Expanding plane waves in spherical harmonics, one can carry out the angular integrations over $\hat{k}_1$, $\hat{k}_2$, and $\hat{r}$ by standard use of Racah algebra. After a straightforward but lengthy derivation, we get for either the direct or crossed terms and a *definite choice of intermediate states* $(\bar{l}_1\bar{t}_1\bar{s}_1;\bar{l}_2\bar{t}_2\bar{s}_2)$,

$$\begin{aligned}V_{l'l}^{jsT}=&\left(\frac{g_1}{2m_1^{[\bar{l}_1]}}\right)^2\left(\frac{g_2}{2m_2^{[\bar{l}_2]}}\right)^2(-)^{j-s}\frac{1}{4\pi^4}\frac{1}{(2\bar{l}_1+1)(2\bar{l}_2+1)}\\
&\times\sum_{LL'L_1L_2J}(i)^{-(L_1+L_2)}\left[\frac{(2L_1+1)(2L_2+1)(2l'+1)(2l+1)(2S+1)}{2J+1}\right]^{1/2}\begin{Bmatrix}\bar{l}_1&\bar{l}_2&L_1\\ \bar{l}_1&\bar{l}_2&L_2\\ L&L'&J\end{Bmatrix}W(l'lSS;Jj)\\
&\times\begin{pmatrix}\bar{l}_1&\bar{l}_2&L_1\\0&0&0\end{pmatrix}\begin{pmatrix}\bar{l}_1&\bar{l}_2&L_2\\0&0&0\end{pmatrix}\begin{pmatrix}L_1&L_2&J\\0&0&0\end{pmatrix}\begin{pmatrix}l'&l&J\\0&0&0\end{pmatrix}A_{LL'}^{SJ}(\bar{l}_1\bar{s}_1,\bar{l}_2\bar{s}_2)B^T(\bar{t}_1\bar{t}_2)\\
&\times\int k_1^2dk_1\int k_2^2dk_2k_1^{\bar{l}_1+\bar{l}_2}k_2^{\bar{l}_1+\bar{l}_2}j_{L_1}(k_1r)j_{L_2}(k_2r)f^2(k_1)f^2(k_2)\Pi,\end{aligned} \tag{4.13}$$

where

$$A_{LL'}^{SJ}(\bar{l}_1\bar{s}_1,\bar{l}_2\bar{s}_2)=C_p{}^2[(2S+1)(2J+1)(2L+1)(2L'+1)]^{1/2}(-1)^{L+L'}\begin{Bmatrix}S&S&J\\ \frac{1}{2}&\frac{1}{2}&L\\ \frac{1}{2}&\frac{1}{2}&L'\end{Bmatrix}W(\tfrac{1}{2}\bar{l}_1\tfrac{1}{2}\bar{l}_1;\bar{s}_1L)W(\tfrac{1}{2}\bar{l}_2\tfrac{1}{2}\bar{l}_2;\bar{s}_2L')$$

$$\times\langle\tfrac{1}{2}\|\Sigma_1^{(\bar{l}_1)}\|\bar{s}_1\rangle\langle\bar{s}_1\|\Sigma_1^{(\bar{l}_1)}\|\tfrac{1}{2}\rangle\langle\tfrac{1}{2}\|\Sigma_2^{\bar{l}_2}\|\bar{s}_2\rangle\langle\bar{s}_2\|\Sigma_2^{\bar{l}_2}\|\tfrac{1}{2}\rangle \tag{4.14}$$

is the spin matrix element,

$$B^T(\bar{t}_1\bar{t}_2)=-\tfrac{2}{3}(2T+1)^{1/2}\,(-1)^T\sum_I(-1)^IC_q(2I+1)^{3/2}W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2};IT)W(\tfrac{1}{2}1\tfrac{1}{2}1;\bar{t}_1I)W(\tfrac{1}{2}1\tfrac{1}{2}1,\bar{t}_2I)$$

$$\times\langle\tfrac{1}{2}\|T_1\|\bar{t}_1\rangle\langle\bar{t}_1\|T_1\|\tfrac{1}{2}\rangle\,\langle\tfrac{1}{2}\|T_2\|\bar{t}_2\rangle\langle\bar{t}_2\|T_2\|\tfrac{1}{2}\rangle \tag{4.15}$$

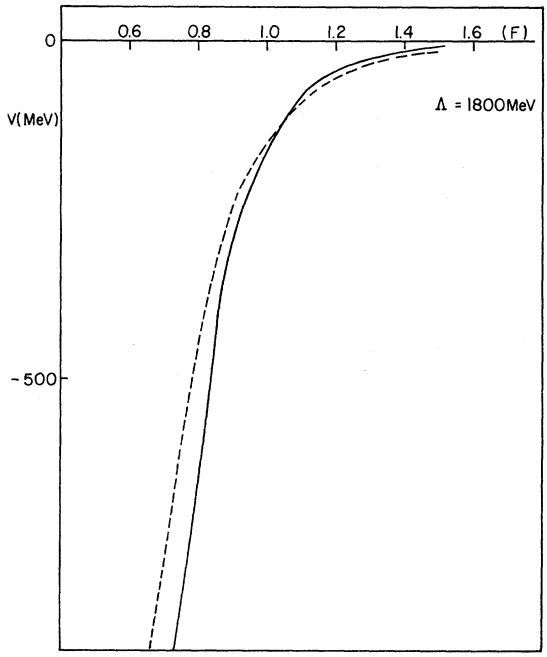

FIG. 7. Comparison of the coupled-channel theory's two-pion exchange potential in the $^1S_0$ state (solid line) with the sum of all two-pion exchange terms with (N-N), (N^*-N), and (N^*-N) intermediate states normally omitted by the former theory (dashed line). The cutoff parameter Λ defined by (4.8) has been taken as $\Lambda = 1800$ MeV.

is the isospin matrix element, and Π is again the appropriate product of energy denominators. Direct and crossed diagrams are also distinguished by a different choice of phase factors C_p and C_q. We have

$$C_p=(-1)^J, \quad C_q=1 \text{ (direct)},$$
$$C_p=(-1)^{J+L'}, \quad C_q=(-1)^J \text{ (crossed)}. \tag{4.16}$$

For the final potential, we must sum over direct and crossed diagrams and over all allowed types of intermediate states. For the choice of coupling constants, we follow Jena and Kisslinger.[14] The radial integrations are carried out numerically. It is found that for a cutoff parameter $\Lambda \gtrsim 1000$ MeV, 50 gaussian grid points are sufficient to do accurate numerical integrations in the region $r \lesssim 2$ fm.

We emphasize that (4.13) is for the static limit. Inclusion of nucleon recoil by a systematic expansion in powers of $(\vec{p}/M)$ can be dealt with by the same calculational techniques. Here, we confine ourselves to the static limit and use the resultant potentials as the basis for the observations made in the next section.

V. RESULTS AND DISCUSSION

To motivate future application of the formalism presented in this paper, we first use (4.13) to briefly investigate within the LKM formalism the extent to which the usual coupled-channel ap-

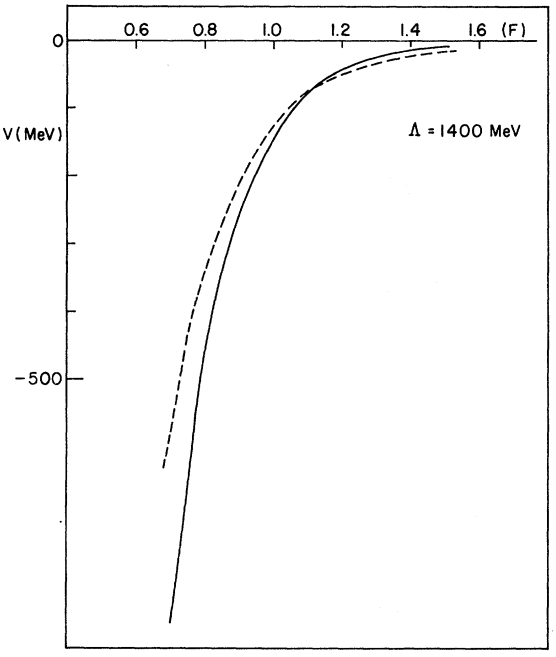

FIG. 8. Same as Fig. 7 with $\Lambda = 1400$ MeV.

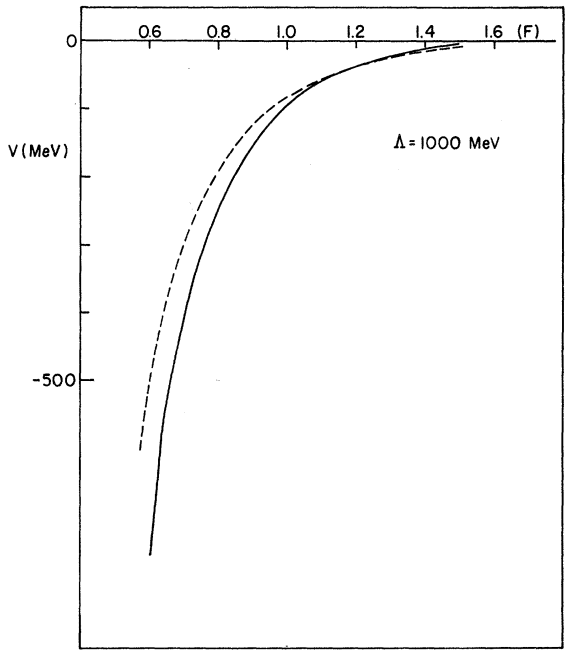

FIG. 9. Same as Fig. 7 with $\Lambda = 1000$ MeV.

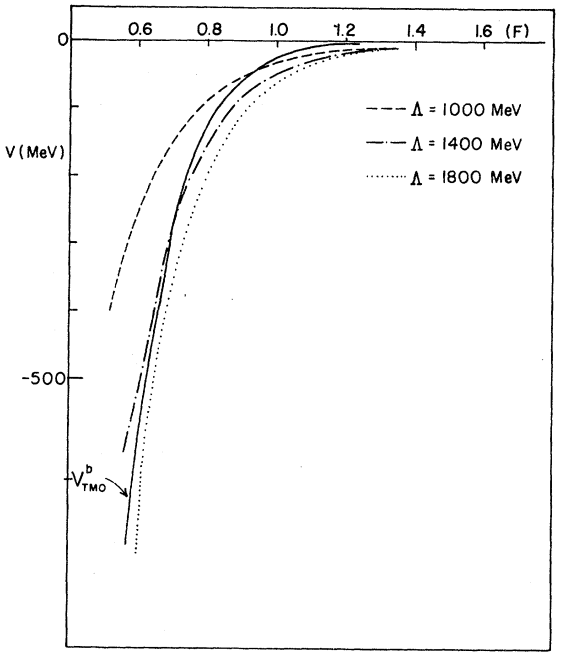

FIG. 10. This figure illustrates the uncertainty in the fourth-order static potential which is implied by different three-dimensional formalisms. The curves marked by different values of Λ reproduce the corresponding dashed curves of Figs. 7–9, whereas the curve marked $(-)V^b_{TMO}$ represents the specific difference between the formalisms, as described in the text.

proach is an adequate approximation to the 2π exchange N-N potential.

Within the framework of the coupled-channel formalism, Sugawara and Von Hippel[11] and Riska and Brown[19] first showed that the 2π exchange involving intermediate isobar states can possibly generate the attractive force otherwise accounted for in one-boson exchange models by introducing scalar-meson exchange. Jena and Kisslinger,[14] in an elaborate coupled-channel calculation (involving only one-meson exchange terms in the potential) reached a similar conclusion but found that the resulting intermediate-range attraction is not as large as that generated by the scalar-meson exchange. Within the LKM formalism, the 2π exchange N-N potential involved in those calculations include only the contributions of time-ordered diagrams which have an intermediate state without mesons, i.e., Figs. 5(c)–(f). Hence, in order to justify the coupled-channel approach, it is important to investigate the role of the remaining diagrams [Figs. 5(a), (b), and 6].

In Figs. 7–9, we compare the $^1S_0$ 2π exchange N-N potentials, calculated for several choices of cutoff parameter Λ, from the coupled-channel scheme (CCS) and from the remaining correction diagrams. Only N and $N^*(1236)$ are included in the calculations to describe the intermediate states. There are three possible intermediate two-baryon states which are N-N, N-N^*, and N^*-N^*. For the case of N-N intermediate states, Figs. 5(c)–(f) must be excluded. Cutoff parameters Λ = 1800, 1400, and 1000 MeV have been used for the vertices in Figs. 7, 8, and 9, respectively. We see that the correction terms are of the same sign as that derived from CCS. More importantly, the correction terms are about 100% of the CCS for all choices of cutoff. Therefore, the 2π exchange potential calculated from the usual coupled-channel scheme does not account for all of the attraction included in the single-channel LKM formalism. It is clear that the correction terms tend to cure the insufficient attractions found in the coupled-channel calculation of Jena and Kisslinger.[14] Quantitative results are, however, cutoff dependent.

We should like finally to illustrate the kind of quantitative differences that arise from different choices of reduction procedure from the BSE. For example, the method of Lomon-Partovi[10] leads in the static approximation to an additional contribution to the 2π exchange potential. As illustrated in Fig. 10, the negative of this term is marked[9] $(-)V^b_{TM0}$ and has been calculated analytically without cutoff. The remaining curves of this figure represent the corrections due to N^* and N^*-N^* intermediate states omitted in the usual isobar model, but do not include the contributions due to N-N intermediate states, the so-called BW terms. An essential cancellation between these two sets of curves is thus indicated. In this theory then the main corrections to CCS come from the so-called BW terms, Figs. 5(a), (b), and 6 for N-N intermediate states only, which have simply been omitted in Fig. 10. We have thus illustrated the ambiguity in the 2π exchange potential discussed in the introduction to this paper and overlooked in all recent work.

In conclusion, we have shown by some examples how the nonperturbative formalism described in this paper may be utilized to calculate the N-N potential. It remains for us to carry out a comprehensive theoretical investigation and calculation based on our approach. However, before such a program can be of significance, the ambiguity described above must be settled. It is to this matter that we plan to turn our immediate attention.

*Work supported in part by Energy Research and Development Agency.

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