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Minimal three-body equations with finite-range effects

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In this paper it is shown how the fundamental quantum mechanical constraints of unitarity and analyticity in three-body final states can be realized when finite-range effects, like centrifugal barriers, are important. The resulting equations are remarkably similar to the usual separable-potential scattering equations.

[NUCLEAR REACTIONS Three-body scattering theory, implementation of unitarity and analyticity constraints with finite-range effects.]

Unitarity forces rapid variation and interrelationship on the quasi-two-body amplitudes that appear in the sequential theory or isobar model of three-body final states.[1-3] We have seen that adding subenergy analyticity to unitarity, that is using the discontinuities provided by unitarity in a dispersion relation, gives a simple form (the zero-range form) of the usual separable interaction scattering equations.[1,3] In this note we reexamine the question of implementing analyticity with particular attention to problems associated with threshold behavior and total-energy analyticity. We show that, at least in a given form of the analytic continuation, the full separable interaction equation is obtained, and that all other continuations that we have been able to study that deal with the threshold question lead to equations with serious flaws.

The isobar or sequential-decay formalism begins by decomposing the amplitude leading to a three-body final state into a sum of three terms each "ending" with the interaction of a particular pair. These pair interactions are typically assumed to proceed via a resonance so that each of the three terms may be further decomposed into a product of a quasi-two-body amplitude for forming the resonance (treated as a particle) and the third particle times a factor giving the subsequent propagation and decay of that resonance. It is the dependence of that quasi-two-body amplitude on the center-of-mass energy of the resonating pair (or on the energy of the third particle which is the same thing because of total-energy conservation) that we wish to study. The factor carrying the subsequent propagation and decay of the resonating pair is neither the pair's two-body t matrix nor the propagator or D function, but rather something halfway in between because of threshold effects. The two-body t matrix for an interacting pair of relative momentum q and center-of-mass energy ϵ in the lth partial wave may be written

$$\langle q|t_l(\epsilon)|q\rangle = N_l(\epsilon,q)/D_l(\epsilon)\ , \tag{1}$$

for small q

$$\langle q|t_l(\epsilon)|q\rangle \approx q^{2l}\ . \tag{2}$$

A term representing the propagation and decay of an l-wave pair goes only like q^l for small q. Hence the factor giving the propagation and subsequent decay of the resonant pair in a three-body formalism should be

$$\frac{\langle q|t_l(\epsilon)|q\rangle}{q^l} \quad\text{or}\quad \frac{q^l}{D_l(\epsilon)} \quad\text{or}\quad \frac{(N_l(\epsilon,q))^{1/2}}{D_l(\epsilon)}\ . \tag{3}$$

Following Ref. 1, we therefore write the amplitude for going from an initial state labeled by $\vec{k}$ to a final state of three spinless bosons (for simplicity) of momentum $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ in the over-all center-of-mass system with total energy E as

$$\langle \vec{k} | T | \vec{p}_1, \vec{p}_2, \vec{p}_3 \rangle = (2\pi)^3 \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \frac{1}{2} \sum_{\substack{i,j,k \\ m}} \langle \vec{k} | F(E) | \vec{p}_i, l \rangle \frac{|\vec{q}_{jk}|^l}{D_l(2q_{jk}^2)} Y_{lm}(\hat{q}_{jk}) , \tag{4}$$

where l is the partial wave of the dominant pairwise interaction or resonance, $\vec{q}_{jk} = \frac{1}{2}(\vec{p}_j - \vec{p}_k)$, $\hat{q}$ is a unit vector and $\hbar = 2m = 1$. It is the p dependence of F we wish to study.

The existence of the q^l in (3) or the q^{2l} factor in (2) is a finite range effect. It is really qR, where R is some range parameter, that we are using as expansion parameter. Of course, if the resonance under consideration is narrow and far from threshold, the threshold factors are irrelevant, but so then are the three-body effects we are studying. In cases where the resonance bands are broad and/or threshold effects are important, we need the two-body propagation and decay factors far from threshold as well as near threshold. If one tries to include analyticity and obtain dispersion relations, one needs this factor outside the physical scattering region as well. Hence we cannot simply use a form like q^l but must use (or guess at) the analytic function to which q^l is the small-q approximation. The finite-range origins of the q^l, or alternatively its connection to N in (3), make it clear that the function should have left-hand cuts only. One can think of this factor as a penetrability factor familiar in nuclear physics. Hence we replace the q^l in (4) by

$$q^l \rightarrow [N_l(\epsilon, q)]^{1/2} \equiv v_l(q^2) , \tag{5}$$

with the conditions

$$v_l(q^2) \rightarrow q^l \quad \text{for small } q ,$$

$$v_l(q^2) < M \quad \text{for } q^2 \rightarrow \infty ,$$

and $v_l(q^2)$ has only left-hand cuts in q^2. We are making the simplifying assumption that the ϵ dependence of N in (3) may be dropped. The alert reader may suspect that we are sneaking separable interactions into the formalism under a new guise. In a sense we are, but only because we see no other way of dealing with the threshold factor or left-hand cut problem. If we make the replacement (5) and follow the discussion of Refs. 1 and 2, we obtain for the discontinuity of F across the pair subenergy cut

$$\begin{aligned} \mathrm{Disc} \langle \vec{k} | F(E) | \vec{p}, l \rangle = & -2\pi v_l(E - \tfrac{3}{2}p^2) \\ & \times \sum_m \int \frac{d^3p'}{(2\pi)^3} \langle \vec{k} | F(E) | \vec{p}\,', l \rangle \frac{v_l(E - \frac{3}{2}p'^2)}{D_l(E - \frac{3}{2}p'^2)} Y_{l,m}(\hat{P}) Y_{l,m'}(\hat{P}') \delta(E - 2p^2 - 2p'^2 - 2\vec{p}\cdot\vec{p}\,') , \end{aligned} \tag{6}$$

where $\hat{P} = (\vec{p} + \frac{1}{2}\vec{p}\,')/|\vec{p} + \frac{1}{2}\vec{p}\,'|$ and $\hat{P}' = (\vec{p}\,' + \frac{1}{2}\vec{p})/|\vec{p}\,' + \frac{1}{2}\vec{p}|$. To further simplify the argument, let us take $l = 0$ and assume the factor v (we drop the subscript l) is then associated with some dynamical barrier like the Coulomb barrier, or a nonmonotonic potential, or is simply a manifestation of left-hand cuts.

If we use the technique discussed in Ref. 1 to implement analyticity starting with (6) as the discontinuity across the subenergy cut of $\langle \vec{k} | F(E) | \vec{p} \rangle$ going from $E - \frac{3}{2}p^2 = 0$ to $E - \frac{3}{2}p^2 = \infty$, we run into problems with total-energy analyticity. Forgetting for a moment about the effect of the $v(E - \frac{3}{2}p^2)$ in front of the integral in (6), which only adds to the complications, and using (6) as the subenergy discontinuity in a dispersion relation in the subenergy leads to an integral equation for $\langle \vec{k} | F(E) | \vec{p} \rangle$ of the form

$$\langle \vec{k} | F(E) | \vec{p} \rangle = \langle \vec{k} | R(E) | \vec{p} \rangle + \frac{1}{(2\pi)^4} \int \frac{d^3p' \langle \vec{k} | F(E) | \vec{p}\,' \rangle v(E - \frac{3}{2}p'^2)}{D(E - \frac{3}{2}p'^2)(E - 2p^2 - 2p'^2 - 2\vec{p}\cdot\vec{p}\,')} , \tag{7}$$

where R is the inhomogeneous part of the equation or the part of F without subenergy cuts. Because v has only left-hand cuts, its presence under the integral will introduce singularities in F as a function of E even when E is negative. Strictly speaking, these could be absorbed by compensatory terms in R, but that is awkward since R has to be given, (7) does not determine it, and we know very little about the structure of these singularities. This was just the problem we encountered in Ref. 1 when discussing the zero-range form of the integral equation obtained for F [Eq. (40) of Ref. 1]. If we try to use a finite-range form for the t matrix, the left-hand cuts of t give extra E singularities we do not want. Of course the pair subenergy singularities of (7) are correct, that is $\langle \vec{k}|F(E)|\vec{p}\rangle$ obtained from (7) clearly satisfies (6). It is the "extra" or "new" condition of total-energy analyticity that we are now imposing that gives us difficulty. Let us examine the origin of this difficulty in more detail. We are studying $\langle \vec{k}|F(E)|\vec{p}\rangle$, or in its partial-wave form $\langle k|F_l(E)|p\rangle$. (Here l is the three-body angular momentum.) We know it has a cut in the pair subenergy $E-\frac{3}{2}p^2$ in the interval $0<E-\frac{3}{2}p^2<\infty$. If there are finite-range effects (like two-body threshold behavior), it also has left-hand cuts in p^2 starting at $p^2=-\mu^2$, where μ is a momentum-space range parameter. Finally, it has cuts in E beginning at the lowest scattering threshold and running to ∞. Clearly, E, p^2, and $E-\frac{3}{2}p^2$ are not all independent variables. If we study F for fixed E as a function of $E-\frac{3}{2}p^2$ we eventually find that the subenergy cut going from $p^2=\frac{2}{3}E$ to $p^2=-\infty$ and the finite-range singularities get intertwined. This is the problem we are encountering. The way out is to take (6) to be giving us information about the discontinuity across the $E-\frac{3}{2}p^2$ cut for fixed p, in other words we can disperse in E. Because the essential feature of the discontinuity expression (6) is the δ function, it is now very easy to disperse in E. The fact discovered in Ref. 1 that when we disperse we simply put the argument of the δ function into the denominator will now not appear as a technical miracle, but will be trivial. There remains, however, questions of analytic continuation. To see this consider a simple example. Suppose we know that the discontinuity of some function $g(x)$ across one of its singularities is given by

$$\operatorname{Disc} g(x)=\pi\rho(x)\delta(x-a)=\pi\rho(a)\delta(x-a)\ , \tag{8}$$

where ρ is some known function. We can realize this analytic structure in two ways. We can write

$$g_1(x)=r_1(x)+\rho(a)/(x-a) \tag{9a}$$

or

$$g_2(x)=r_2(x)+\rho(x)/(x-a)\ , \tag{9b}$$

where r is that part of g that has no pole at $x=a$. Both g_1 and g_2 have the same discontinuity (residue) at the pole at $x=a$, and, in fact, g_1 can be made to equal g_2 by choice of r_1 and r_2, but one form of g, or equivalently of r, may be more "natural" than the other.

In our approach to (6) these same considerations enter. Equation (6) may be written

$$\operatorname{Disc}\langle \vec{k}|F(E)|\vec{p}\rangle =\pi\int d^3p'\,h(\vec{p},\vec{p}\,',E)\delta(E-2p^2-2p'^2-2\vec{p}\cdot\vec{p}\,')\ . \tag{10}$$

This yields

$$\langle \vec{k}|F_1(E)|\vec{p}\rangle=\langle \vec{k}|R_1(E)|\vec{p}\rangle+\int\frac{d^3p'\,h(\vec{p},\vec{p}\,',2p^2+2p'^2+2\vec{p}\cdot\vec{p}\,')}{E-2p^2-2p'^2-2\vec{p}\cdot\vec{p}\,'} \tag{11a}$$

corresponding to (9a) or

$$\langle \vec{k}|F_2(E)|\vec{p}\rangle=\langle \vec{k}|R_2(E)|\vec{p}\rangle+\int\frac{d^3p'\,h(\vec{p},\vec{p}\,',E)}{E-2p^2-2p'^2-2\vec{p}\cdot\vec{p}\,'} \tag{11b}$$

corresponding to (9b), or in fact many things in between, depending on in which factor of h we put $E=2p^2+2p'^2+2\vec{p}\cdot\vec{p}\,'$, and in which we leave it as E. All of these will satisfy (10) or (6) since they all have the same subenergy discontinuity. We need to use our information about the other properties of F, total-energy analyticity for example, to choose among them. In fact, except for the E in F, E enters in (6) always in the combinations $E=\frac{3}{2}p^2$ or $E=\frac{3}{2}p'^2$. For $E=2p^2+2p'^2+2\vec{p}\cdot\vec{p}\,'$ we have

$$E-\tfrac{3}{2}p'^2=2(\vec{p}+\tfrac{1}{2}\vec{p}\,')^2$$

and

$$E-\tfrac{3}{2}p^2=2(\vec{p}\,'+\tfrac{1}{2}\vec{p})^2\ . \tag{12}$$

Clearly we want to make this substitution in the v's in (6) in order to make sure their left-hand cuts do not get involved in the p' integration in the equation corresponding to (7) when E is kept fixed. This is the problem we had before. On the other hand, the E dependence of D is esential to the total-E analyticity and hence we do not make this substitution in D. Furthermore, we do not know the E dependence of F, and hence we do not make the substitution in F. This gives us finally

$$\langle \vec{k}|F(E)|\vec{p}\rangle=\langle \vec{k}|R(E)|\vec{p}\rangle+\int\frac{d^3p'}{(2\pi)^4}\frac{v(2(\vec{p}+\frac{1}{2}\vec{p}\,')^2)v(2(\vec{p}\,'+\frac{1}{2}\vec{p})^2)\langle\vec{k}|F(E)|\vec{p}\,'\rangle}{D(E-\frac{3}{2}p'^2)(E-2p^2-2p'^2-2\vec{p}\cdot\vec{p}\,')}\ , \tag{13}$$

which is essentially the separable-potential form [the difference is that we have not used total-energy analyticity to require that the v's also are used to get D, although some of that is implicit in (5).] It should be stressed that in choosing how we make the substitutions (12) for E in h we are choosing among alternatives all of which give the correct subenergy discontinuity. That is, all satisfy (6). We are using other information (E analyticity) and our desire to keep R simple. It may be that just to get the subenergy dependence we do not need to be so particular, and even forms that include extra E singularities that we do not compensate for by choice of R will give that dependence correctly. As we have seen, the *shape* of the p dependence of F but not its magnitude does seem not to depend on the details of the inner dynamics,[4] but much work remains to understand this. Meanwhile (13) gives a way of avoiding unwanted singularities which include finite-range and barrier effects. It is clear that this method is easily extended beyond identical particles and beyond s-wave pair interactions. The techniques for doing this are implicit in Ref. 1. There is also no apparent obstacle to extending this method to the relativistic case where it will give an equation essentially identical to the Blankenbecler-Sugar three-body equation,[5] but with only the positive-energy part of the exchange propagator.

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[5]A derivation of that equation similar in form to the one presented here, but from a different starting point, is given in R. Aaron, R. D. Amado, and J. E. Young, Phys. Rev. 174, 2022 (1968).