Multiple-diffraction expansion for intermediate-energy reactions

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A multiple-diffraction expansion for heavy-ion reactions is constructed in which the leading term is Glauber's phenomenological multiple-diffraction amplitude. This is achieved with the help of a pseudopotential between elementary particles in ions which in the Glauber approximation gives the empirical elementary scattering amplitude. The leading corrections to Glauber's phenomenological amplitude include (i) a wave-spreading term of Wallace, (ii) an internal-excitation term of Hahn, (iii) a pseudopotential term which contains effects of wave spreading and zero-point motion of bound particles in ions, and (iv) a Pauli term for antisymmetrizing two clusters of identical fermions. Explicit expressions for these corrections are given in the impact-parameter representation. The nature of these corrections is briefly discussed.

[NUCLEAR REACTIONS Multiple-diffraction expansion formalism for intermediate-energy reactions.]

I. INTRODUCTION

Hahn[1] has studied a multiple-diffraction (MD) expansion for intermediate-energy reactions in which the leading term is Glauber's MD amplitude.[2] It is not clear in the development how Glauber's *empirical* scattering amplitude for two isolated scatterers or elementary particles first appears in the theoretical expansion. This paper studies a solution of the problem by using a pseudopotential which reproduces this empirical elementary amplitude in Glauber's high-energy approximation.

The paper is organized as follows. We begin by giving a review of the basic ideas in multiple diffraction expansions for heavy-ion reactions in Sec. II. The pseudopotential is introduced in Sec. III, where how Glauber's phenomenology is treated in the theoretical expansion is also shown. Correction terms to Glauber's phenomenological MD amplitude are discussed in Secs. IV–VIII. Section IX contains brief concluding remarks.

II. MULTIPLE-DIFFRACTION EXPANSION

We first establish notations by briefly reviewing the MD expansion. In Glauber's high-energy method,[2] the elastic amplitude between two bodies A and B has the general form

$$F(\vec{q}) = \frac{ik}{2\pi} \int d^2\vec{b} e^{i\vec{q}\cdot\vec{b}} \Gamma_{AB}(\vec{b}) \tag{1}$$

for a momentum transfer $\vec{q} = \vec{k}_f - \vec{k}_i$. Here $k = |\vec{k}_i| = |\vec{k}_f|$ is the relative momentum, and $\vec{b}$ is the impact vector on a plane perpendicular to $\vec{k}_{av} = \frac{1}{2}(\vec{k}_i + \vec{k}_f)$, the average of the initial and final relative momenta. The profile function $\Gamma_{AB}(\vec{b})$ contains dynamical information of physical interest.

In applying the method to two clusters or ions of N_a and N_b elementary particles of types a and b respectively, Glauber[2] starts with an experimentally determined elementary scattering amplitude $f(\vec{q})$ between a and b for a relative momentum $\vec{k}_{ab}$ corresponding to the same relative speed $|\vec{u}|$. Equation (1) is then inverted to obtain an elementary profile function

$$\Gamma_{ab}(\vec{b}) = (2\pi i k_{ab})^{-1} \int d^2\vec{q} e^{-i\vec{q}\cdot\vec{b}} f(\vec{q})$$
$$= 1 - e^{i\chi_{ab}(\vec{b})} . \tag{2}$$

This can be related to a phase-shift function $\chi_{ab}(\vec{b})$, as indicated in Eq. (2).

For clusters of elementary particles, Glauber[2] points out that under conditions relevant to high-energy scattering, the phase-shift function is approximately given by the sum of elementary phase-shift functions. That is,

$$1 - \Gamma_{AB}(\vec{b}) \approx \langle AB| \prod_{i=1}^{N_a} \prod_{k=1}^{N_b} [1 - \Gamma_{ik}(\vec{b} + \vec{s}_i - \vec{s}_k)] |AB\rangle$$
$$= \langle AB| e^{i\chi(\vec{b};\vec{s})} |AB\rangle . \tag{3}$$

In this expression $|AB\rangle$ is a simple product $|A\rangle|B\rangle$ of properly constructed (symmetrized or anitsymmetrized) internal states $|C\rangle$ of ions $C = A, B$. The additional symmetrizer or antisymmetrizer between ions needed when the elementary particles are identical has been neglected, because the effect is expected to be unimportant at intermediate energies. The vectors $\vec{s}_i$ (referred to collectively as $\vec{s}$) are the perpendicular components of the internal coordinates $\vec{x}_i$, which may be

defined in terms of single-particle (s.p.) coordinates $\vec{r}_i$:

$$\vec{x}_i = \vec{r}_i - \vec{R}_C, \quad \vec{R}_C = \sum_{i=1}^{N_C} \vec{r}_i/N_c; \qquad C = A, B; \quad c = a, b \ . \tag{4}$$

Since the phase functions depend on these perpendicular components only, the integrations in Eq. (3) involving z components of internal coordinates can be done readily.

It is usual to replace the complicated internal state $|AB\rangle$ by a shell-model state $|AB\rangle_{\rm sm}$. The spurious motion of cluster centers of mass is removed by using δ functions in operator matrix elements:

$$\langle AB|\cdots|AB\rangle = {}_{\rm sm}\langle AB|\cdots\delta(\vec{R}_A - M_B\vec{R}/M_{AB}) \times\delta(\vec{R}_B + M_A\vec{R}/M_{AB})|AB\rangle_{\rm sm}, \tag{5}$$

where $M_C = N_c m_c$, $M_{AB} = M_A + M_B$. The vector $\vec{R} = \vec{b} + z\hat{z}$ is the relative coordinate between ions.

We thus see that the MD amplitude depends simply on the elementary amplitude, i.e., on the on-shell properties of the elementary interaction.

It is known[1,3,4] that a many-body scattering amplitude (1) having the MD form (3) can be derived under the following assumptions: (i) the ions are not excited internally[5]; (ii) Glauber's linearized propagator is used[1,2,4,6]; and (iii) the elementary interaction is a local and spin-independent potential.[4] For further discussion, let the Hamiltonian be $H_0 + V$, where $V = \sum_{ik} v_{ik}$ contains the elementary interactions of particle i in one ion and particle k in the other ion, and

$$H_0 = H_A^{\rm int} + H_B^{\rm int} + K_{AB} \tag{6}$$

contains the internal ion Hamiltonians and the relative kinetic energy operator between ions. Let $E_C^{\rm int}$ be the corresponding ground-state energy of ion C. Then Eq. (6) may be written as

$$H_0 = E_A^{\rm int} + E_B^{\rm int} + K_{AB} + \Delta H^{\rm int}\ , \quad \Delta H^{\rm int} = \Delta H_A^{\rm int} + \Delta H_B^{\rm int}\ , \quad \Delta H_C^{\rm int} = H_C^{\rm int} - E_C^{\rm int}\ . \tag{7}$$

The assumption (i) is to ignore the internal excitation operators $\Delta H_C^{\rm int}$. The particles will then not move relative to each other in each ion during the scattering process. The situation may be described as a frozen-ion (more familiarly as a frozen-nucleus) approximation. The many-body scattering problem then reduces to that for a sum of potentials. The correction due to $\Delta H_C^{\rm int}$ has been studied by Hahn.[1] (A similar effect has been discussed earlier by Remler.[16])

Under the second assumption, the nonrelativistic propagator is approximated by Glauber's linearized propagator G_g.[1,2,4,6] We write

$$G = (E^{(+)} - H_0)^{-1} = G_g + G_g \Delta H G\ , \tag{8}$$

where

$$G_g = [\vec{u}\cdot(\hbar\vec{k}_{\rm av} - \vec{P}) + i\eta]^{-1}\ . \tag{9}$$

Here

$$\vec{u} = \hbar k\hat{k}_{\rm av}/\mu, \quad \mu = M_A M_B/M_{AB}; \quad E = E^{\rm tot} - E_A^{\rm int} - E_B^{\rm int} = \hbar^2 k^2/2\mu, \quad E^{(+)} = E + i\eta\ . \tag{10}$$

The correction term $\Delta H = \Delta H^{\rm int} + \Delta H^{\rm w}$ contains, besides $\Delta H^{\rm int}$, a wave-spreading (w) operator

$$\Delta H^{\rm w} = \lambda G_g^{-1} + N_{AI}\ , \quad \lambda = 1 - \cos\tfrac{1}{2}\vartheta \tag{11}$$

of Wallace.[6,7] It contains

$$N_{AI} = (\vec{P} - \hbar\vec{k}_f)\cdot(\vec{P} - \hbar\vec{k}_i)/2\mu\ , \tag{12}$$

the wave-spreading operator for the linearized propagator of Abarbanel and Itzykson.[8] Perturbative corrections due to $\Delta H^{\rm w}$ have been studied by Wallace.[6,7]

The third assumption, that elementary potentials are local and spin-independent, simplifies formulas. It is needed if the elementary profile functions Γ_{ik} are to have the perfect localization implied in Eq. (3). This feature is also responsible for the simple additivity of phase shifts in Glauber's MD amplitude.[4] Spin-dependent and spin-flipping effects will not be considered in the following discussion.

With the correction terms thus enumerated, it is possible to evaluate them perturbatively. This program has been partly carried out by Wallace[6,7] and by Hahn.[1] Related studies have been made by other authors, e.g., Frahn and Schürmann.[9-11]

III. PSEUDOPOTENTIAL AND GLAUBER'S MD AMPLITUDE

If the actual elementary interactions v_{ik} are used in the MD expansion described in Sec. II, the resulting elementary profile functions will differ from the empirical functions used in Glauber's MD method. This is because an approximate (i.e., the linearized) propagator is used in obtaining the MD amplitude. By using empirical profile functions, one removes some of the errors introduced by the approximate propagator. This important feature of the MD method can be incorporated into our theoretical expansion with the help of a pseudopotential v' between the elementary particles a and b. It is so defined as to give the same t matrix with Glauber's linearized propagator g_g (for an isolated ab pair) as the original potential v with the full propagator g. This re-

quires that

$$v' = v + v(g - g_g)v' , \tag{13}$$

where the propagators are to be defined.

Now the initial relative velocity for an isolated ab pair should be the same as for a pair embedded in ions. But there are two distinct groups of momentum operators. The operators $\vec{p} = \vec{p}_a = -\vec{p}_b$ for an elementary pair ab in its c.m. frame, and the operators $\vec{P} = \vec{P}_A = -\vec{P}_B$ for heavy ions in their c.m. frame. The former are differential operators of s.p. coordinates, while the latter are differential operators in ion c.m. coordinates. The relative momentum eigenvalue of $\vec{p}$ is also different: $\hbar\vec{k}_{abi} = \mu_{ab}\vec{u} = \hbar\vec{k}_i\,(\mu_{ab}/\mu)$, where $\mu_{ab} = m_a m_b/m_{ab}$, $m_{ab} = m_a + m_b$, is the reduced mass for the pair ab. Hence, in the c.m. frame of the ab pair,

$$g = (e^{(+)} - h_0)^{-1} , \tag{14}$$

where

$$e^{(+)} = E^{(+)}\mu_{ab}/\mu , \quad h_0 = p^2/2\mu_{ab} .$$

Similarly, g_g in this c.m. frame can be written formally as

$$g_g = [\vec{u}_{ab}\cdot(\hbar\vec{k}_{ab\,\mathrm{av}} - \vec{p}) + i\eta]^{-1} , \tag{15}$$

where $\vec{k}_{ab\,\mathrm{av}} = \frac{1}{2}(\vec{k}_{abi} + \vec{k}_{abf})$ and $\vec{u}_{ab} = \hbar k_{ab}\hat{k}_{ab\,\mathrm{av}}/\mu_{ab}$, with $k_{ab} = |\vec{k}_{abi}| = |\vec{k}_{abf}|$. Therefore

$$\begin{aligned} g &= g_g + g_g \Delta h g , \\ \Delta h &= \lambda_{ab} g_g^{-1} + n_{AI} , \quad \lambda_{ab} = 1 - \cos\tfrac{1}{2}\varphi , \\ n_{AI} &= (\vec{p} - \hbar\vec{k}_{abf})\cdot(\vec{p} - \hbar\vec{k}_{abi})/2\mu_{ab} . \end{aligned} \tag{16}$$

Thus the relationship between v and v' depends on the energy e and on the choice of $\vec{k}_{ab\,\mathrm{av}}$ or $\vec{k}_{abf}$. The energy e is already determined by choosing $|\vec{u}_{ab}| = |\vec{u}|$; we must choose $\vec{k}_{ab\,\mathrm{av}}$ such that the resulting elementary phase-shift function is exactly identical to that appearing in Glauber's phenomenological MD amplitude for heavy-ion reactions.

To be more precise, we should define this MD amplitude as the matrix element of the MD operator

$$T_{\mathrm{MD}} = V' + V' G_g T_{\mathrm{MD}} = V'\Omega_{\mathrm{MD}} \tag{17}$$

calculated with the ion-ion pseudopotential $V'(\vec{R},\vec{x}) = \sum_{ik} v'_{ik}(\vec{R} + \vec{x}_i - \vec{x}_k)$ in the c.m. frame of heavy ions. That is, a Galilean transformation is in general required in order to relate to the scattering between an isolated pair of elementary particles.

Such a Galilean transformation does not change the dynamics of the two-body collision, only its kinematics. Hence we could choose the same $\hat{z}$ direction for the elementary scattering as for the heavy-ion scattering, so that Glauber's phenomenological MD amplitude with additivity of phases is obtained. This choice of $\hat{z}$ means that $\vec{u}_{ab} = \vec{u}$, $\vec{k}_{ab\,\mathrm{av}} = \vec{k}_{\mathrm{av}}\mu_{ab}/\mu$, $\vec{k}_{abf} = \vec{k}_f\mu_{ab}/\mu$, $\varphi = \vartheta$, and $\lambda_{ab} = \lambda$ in the c.m. frame of ab.

To discuss further, we note that the two c.m. frames move with relative velocity

$$\Delta\vec{u} = \vec{u}_{AB} - \vec{u}_{ab} = \left(\frac{\mu}{M_B} - \frac{\mu_{ab}}{m_b}\right)\frac{\hbar\vec{k}_i}{\mu} . \tag{18}$$

The s.p. momenta $\vec{p}_a = -\vec{p}_b = \vec{p}$ transform into the heavy-ion c.m. frame according to the Galilean transformation

$$\tilde{\vec{p}}_c = \mathcal{G}_c\vec{p}_c\mathcal{G}_c^{-1} = \vec{p}_c + m_c\Delta\vec{u} , \tag{19}$$

where

$$\mathcal{G}_c = \exp(-i m_c \Delta\vec{u}\cdot\vec{r}_c/\hbar) .$$

As a result,

$$\tilde{g} = \mathcal{G}_a\mathcal{G}_b g \mathcal{G}_a^{-1}\mathcal{G}_b^{-1} = (e^{(+)} - \tilde{h}_0)^{-1} , \tag{20}$$

where

$$\begin{aligned} \tilde{h}_0 &= \mathcal{G}_a\mathcal{G}_b\left(\frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b}\right)\mathcal{G}_a^{-1}\mathcal{G}_b^{-1} \\ &= \frac{(\vec{p}_a + m_a\Delta\vec{u})^2}{2m_a} + \frac{(\vec{p}_b + m_b\Delta\vec{u})^2}{2m_b} . \end{aligned} \tag{21}$$

Similarly, we can calculate $\tilde{g}_g$, $\Delta\tilde{h}$, and $\tilde{n}_{AI}$. It turns out that these expressions are not needed because we shall do this part of the calculation in the c.m. frame of ab.

IV. CORRECTION TERMS

Having defined the pseudopotential v', we return to the heavy-ion problem. If we use the actual ion-ion potential $V = \sum_{ik} v_{ik}$ with the Glauber propagator G_g, we should get

$$T_g = V + V G_g T_g = V\Omega_g = T_{\mathrm{MD}} + T^{\mathrm{PP}} , \tag{22}$$

which differs from T_{MD} of Eq. (17) by a *pseudopotential* correction

$$T^{\mathrm{PP}} = \Omega_{\mathrm{MD}}^{\dagger}(V - V')\Omega_g . \tag{23}$$

We expect T_{MD} to be closer to the true $T = V + VGT$ than T_g, because T_{MD} is exact when each ion contains only one elementary particle. That is, the term $-T^{\mathrm{PP}}$ gives the improvement which the phenomenological MD amplitude represents over a simple Glauber result for the actual potential under the same frozen-ion approximation. In addition, the phenomenological method is easier to apply because, like the impulse approximation, the actual potential does not have to be constructed.

But, of course, T_{MD} is not exact. The true T matrix is

$$T = T_g + T(G - G_g)T_g = T_{\mathrm{MD}} + T^{\mathrm{PP}} + T^{\Delta H} + R_0 , \tag{24}$$

where

$$T^{\Delta H} = T_{\rm MD}(G - G_g)T_g\,,$$
$$R_0 = (T - T_{\rm MD})(G - G_g)T_g\,. \tag{25}$$

It is convenient to decompose $T^{\Delta H}$ into[1]

$$T^{\Delta H} = T^{\rm w} + T^{\rm int}\,;\quad T^i = (\Omega^\dagger_{\rm MD} - 1)\Delta H^i(\Omega_g - 1)\,,$$
$$i = {\rm w},\ {\rm int}, \tag{26}$$

by using Eqs. (8), (17), and (22). The superscript w denotes wave-spreading (or Fresnel[9]) correction due to the full propagator, while "int" denotes internal excitation of ions, including recoil and binding effects of particles in ions.

To calculate correction terms to the phenomenological MD operator (17) it is necessary to know either the actual potential v or the pseudopotential v'. Of the two, v' is more easily obtainable because of its simple relationship[2] to the elementary profile function Γ_{ab}:

$$v'(r) = \frac{\hbar u}{\pi i}\frac{1}{r}\frac{d}{dr}\int_r^\infty \ln[1 - \Gamma_{ab}(\vec b)]\,(b^2 - r^2)^{-1/2} b\,db \tag{27}$$

if v' is spherically symmetric. In addition, v' is local but v is not. We therefore eliminate v in favor of v', and Ω_g in favor of $\Omega_{\rm MD}$, in T^i (i = pp, w, int). The result will be denoted $T^{i\prime}$. The resulting contributions to the ion-ion scattering amplitude can all be written in the impact-parameter representation:

$$F^i = -\frac{\mu}{2\pi\hbar^2}\int d^2\vec b\, e^{i\vec q\cdot\vec b}\langle AB|\, e^{i\chi(\vec b;\vec s)} f^i(\vec b;\vec x)\,|AB\rangle\,, \tag{28}$$

where $\vec x$ denotes all the internal coordinates $\vec x_i$. Explicit forms for f^i will be derived in Secs. V–VIII.

The result (28) is strictly valid only for elastic scattering. It is possible that a similar expression can be found for inelastic scattering. A possible choice of the relative speed is $u = [(E_i + E_f)/\mu]^{1/2}$, where $E_\alpha = \hbar^2 k_\alpha^2/2\mu$, $\alpha = i$ or f. But since $\vec k_{\rm av} = \frac{1}{2}(\vec k_i + \vec k_f)$ is no longer perpendicular to $\vec q = \vec k_i - \vec k_f$, it is not clear if it is better to choose the z axis to be perpendicular to $\vec q$ or parallel to $\vec k_{\rm av}$. In the absence of a detailed investigation, we prefer the choice $\hat z \perp \hat q$, so that the impact-parameter form (28) is preserved. This is an interesting technical problem which is outside the scope of the present paper.

In addition to the scattering corrections discussed above, there are corrections and effects coming from the internal wave functions. We shall discuss, in Sec. VIII, only one such effect: the antisymmetrization of identical fermions between ions.

V. WAVE-SPREADING CORRECTION

The wave-spreading (or Fresnel) term $T^{\rm w\prime}$ of Eq. (26) has been studied by Wallace[6] for potential scattering. A similar result holds of course for scattering from a "frozen" many-body system.[1,10,11] Wallace[6,7] points out the useful result that, of the terms generated by the perturbation $\Delta H^{\rm w}$, the λ dependent terms tend to cancel each other when ordered in powers of k. (Wallace shows this explicitly in the lowest orders, and conjectures that it may be valid more generally.) Therefore, we just keep

$$T^{\rm w\prime} = (\Omega^\dagger_{\rm MD} - 1)N_{AI}(\Omega_{\rm MD} - 1)\,. \tag{29}$$

The Glauber MD wave functions[2,6] appearing in this expression are

$$\langle \vec R|\,\Omega_{\rm MD}|\,\vec k_i\rangle = e^{i\chi^+(\vec R;\vec x)} e^{i\vec k_i\cdot\vec R}\,,$$
$$\langle \vec k_f|\,\Omega^\dagger_{\rm MD}|\,\vec R\,\rangle = e^{-i\vec k_f\cdot\vec R} e^{i\chi^-(\vec R;\vec x)}\,, \tag{30}$$

where $\vec R = \vec R_A - \vec R_B = \vec b + z\hat z$. The Glauber MD phase-shift functions are calculated from the pseudopotential V' as indicated in Eq. (17):

$$\chi^\pm(\vec R;\vec x) = -\frac{1}{\hbar u}\int_{z_1^\pm}^{z_2^\pm} V'(\vec b + z'\hat z;\vec x)\,dz'$$
$$= \sum_{ik}\chi^\pm_{ik}(\vec R + \vec x_i - \vec x_k)\,, \tag{31}$$

where the limits of integration are $z_1^+ = -\infty$, $z_2^+ = z$ for χ^+ and $z_1^- = z$, $z_2^- = \infty$ for χ^-. Also,

$$\chi^+(\vec R;\vec x) + \chi^-(\vec R;\vec x) = \chi(\vec b;\vec x) \tag{32}$$

is the phase function of Eq. (3). Unlike $\chi^\pm(\vec R;\vec x)$, it does not depend on the z component of the relative coordinate $\vec R$.

Using Eqs. (30) and (12) in Eq. (29), we find that $T^{\rm w\prime}$ contributes a scattering amplitude $F^{\rm w\prime}$ given by Eq. (28) with

$$f^{\rm w\prime} = -\frac{\hbar^2}{2\mu}\int_{-\infty}^{\infty}(\vec\nabla_{\vec R}\chi^-)\cdot(\vec\nabla_{\vec R}\chi^+)\,dz\,.$$

An integration by parts[7] of the resulting Eq. (28) gives

$$f^{\rm w\prime}(\vec b;\vec x) = \left(\frac{\hbar^2}{2\mu}\right)\frac{1}{\hbar u}$$
$$\times\int_{-\infty}^{\infty}\left[\frac{V'^2}{\hbar u} + z\,\vec\nabla_{\vec b}V'\cdot\vec\nabla_{\vec b}(\chi^+ - \chi^-)\right]dz\,. \tag{33}$$

This result is, of course, Eq. (A2) in Appendix A of Wallace.[7] It is reproduced here both for the

sake of completeness and to show how a similar term in the pseudopotential correction (23) may be written down explicitly. The latter turns out to have a sign opposite to that of Eq. (33). This is because the wave-spreading corrections due to isolated pseudopotentials are already taken care of in the definition of these pseudopotentials. These corrections should therefore be subtracted from Eq. (33), leaving only the many-body part of the correction.

VI. PSEUDOPOTENTIAL CORRECTION

The pseudopotential correction has a structure similar to the wave-spreading correction (33):

$$T^{pp'} = -\Omega^{\dagger}_{MD} \sum_{ik} (v' \tilde{g}_g \tilde{n}_{AI} \tilde{g}_g v')_{ik} \Omega_{MD} \tag{34}$$

in the c.m. frame of heavy ions. The resulting scattering amplitude (28) contains the bound-state matrix element

$$\langle AB|\, e^{i\chi} f^{pp'} |\, AB\rangle = -\langle AB|\, e^{i\chi^-} \sum_{ik} (v' \tilde{g}_g \tilde{n}_{AI} \tilde{g}_g v')_{ik}\, e^{i\chi^+} |\, AB\rangle \,. \tag{35}$$

We are interested in the factor

$$-w^{(-)}_{ik} (v' \tilde{g}_g \tilde{n}_{AI} \tilde{g}_g v')_{ik} w^{(+)}_{ik} = -(w^{(-)} v' g_g n_{AI} g_g v' w^{(+)})_{ik}, \tag{36}$$

where $w^{(\pm)}_{ik} = \exp(i\chi^{\pm}_{ik})$, of the ik term. This is a local operator in all coordinates except the z component z_{ik} of $\vec{r}_{ik} = \vec{r}_i - \vec{r}_k = \vec{b}_{ik} + \hat{z} z_{ik}$. It is also invariant under the Galilean transformation $\mathcal{G}_i \mathcal{G}_k$, as shown in Eq. (36).

In the coordinate representation, part of Eq. (36) appears in Eq. (28) as

$$\langle \vec{r}_{ik} = \vec{b}_{ik} + \hat{z} z_{ik}, \cdots |\, (g_g v' w^{(+)})_{ik} \exp(i\vec{k}_i \cdot \vec{R}') |\, AB\rangle \,.$$

Here $\vec{R}'$ is the cluster relative coordinate

$$\begin{aligned} \vec{R}' &= \vec{R}'_A - \vec{R}'_B \\ &= \sum_j \vec{r}'_j N_a^{-1} - \sum_l \vec{r}'_l N_b^{-1} \\ &= \vec{r}'_{ik} \mu_{ab}/\mu + \vec{R}_{rem}, \end{aligned} \tag{37}$$

where

$$\begin{aligned} &\vec{r}'_{ik} = \vec{b}_{ik} + \hat{z} z'_{ik}, \quad \vec{R}_{ik} = (m_a \vec{r}_i + m_b \vec{r}_k)/m_{ab}, \\ &\vec{R}_{rem} = \vec{R}_{ik} (N_a^{-1} - N_b^{-1}) \\ &\quad + \Big(\sum_{j \neq i} \vec{r}_j N_a^{-1} - \sum_{l \neq k} \vec{r}_l N_b^{-1} \Big) . \end{aligned} \tag{38}$$

Because of nonlocality, $\vec{R}'$ differs from the corresponding coordinate $\vec{R}$ on the left, but only in z'_{ik}. The bound state $|AB\rangle$ has the wave function $\Phi_{AB}(\vec{x}_A, \vec{x}_B)$ of the internal coordinates. Since we are here only interested in the dependence on z_{ik}, we shall use the shorthand notation $\Phi_{AB}[(\vec{r}_{ik}, \cdots)]$ for this wave function.

We expect the dependence on z'_{ik} to be weak since it is related to the zero-point motion of nucleons in clusters. This result can be seen as follows. If we approximate $\Phi_{AB}[(\vec{b}_{ik} + \hat{z} z'_{ik}, \cdots)]$ by $\Phi_{AB}[(\vec{b}_{ik} + \hat{z} z_{ik}, \cdots)]$, the operator $(g_g v')_{ik}$ in our matrix element is effectively local, because

$$-\frac{i}{\hbar u} \int_{-\infty}^{z_{ik}} e^{i(k_{abi})_z (z_{ik} - z'_{ik})} v'_{ik}(\vec{r}'_{ik}) w^{(+)}_{ik}(\vec{r}'_{ik}) e^{i(k_{abi})_z z'_{ik}} dz'_{ik} = [w^{(+)}_{ik}(\vec{r}_{ik}) - 1] \exp[i(k_{abi})_z z_{ik}], \tag{39}$$

where $\vec{k}_{abi} = \vec{k}_i \mu_{ab}/\mu$. Hence, we find

$$\begin{aligned} &\langle \vec{r}_{ik}, \cdots |\, (g_g v' w^{(+)})_{ik} \exp(i\vec{k}_i \cdot \vec{R}') |\, AB\rangle \\ &\quad = \exp(i\vec{k}_i \cdot \vec{R}) \langle \vec{r}_{ik}, \cdots |\, (w^{(+)}_{ik} - 1) |\, AB\rangle \\ &\quad + \exp(i\vec{k}_i \cdot \vec{R}) \Big(-\frac{i}{\hbar u} \Big) \int_{-\infty}^{z_{ik}} v'_{ik}(\vec{r}'_{ik}) w^{(+)}_{ik}(\vec{r}'_{ik}) \{ \Phi_{AB}[(\vec{r}'_{ik}, \cdots)] - \Phi_{AB}[(\vec{r}_{ik}, \cdots)] \}\, dz'_{ik} . \end{aligned} \tag{40}$$

In the following development, we shall concentrate on the first, or local, term in Eq. (40), since it leads to a simple expression for $f^{pp'}$. The remaining nonlocal parts can be obtained by directly substituting Eqs. (40) and (34) into Eq. (28). It is not clear at this stage if the nonlocal parts are small in comparison with the local part.

For the local part of Eq. (40), the calculation leading to Eq. (33) can now be repeated, but for the ik pair only. We shall use the notation

$$\begin{aligned} &\psi_\alpha(\vec{r}_{ik}) = \exp(i\vec{k}_{ab\alpha} \cdot \vec{r}_{ik}), \quad \alpha = i, f; \\ &\Phi_{AB}(\vec{x}_A, \vec{x}_B) = \Phi_A(\vec{x}_A) \Phi_B(\vec{x}_B) \,. \end{aligned} \tag{41}$$

To handle the operator n_{AI} of Eq. (16) in Eq. (34), we note that

$$(\vec{p}-\hbar\vec{k}_i\mu_{ab}/\mu)(w_{ik}^{(+)}-1)\psi_i(\vec{r}_{ik})\Phi_A(\vec{x}_A)\Phi_B(\vec{x}_B)$$
$$=\hbar\{[\vec{\kappa}_i(\vec{x}_A)+\vec{\kappa}_k(\vec{x}_B)](w_{ik}^{(+)}-1)+(\vec{\nabla}_{ik}\chi_{ik}^{+})w_{ik}^{(+)}\}$$
$$\times\psi_i\Phi_A\Phi_B\ , \qquad (42a)$$

where $\vec{\nabla}_{ik}$ is the differential operator in $\vec{r}_{ik}$, and

$$\vec{\kappa}_i(\vec{x}_A)=-i\Phi_A^{-1}\vec{\nabla}_{ik}\Phi_A,$$
$$\vec{\kappa}_k(\vec{x}_B)=-i\Phi_B^{-1}\vec{\nabla}_{ik}\Phi_B\ . \qquad (43)$$

Similarly,

$$\Phi_B^*\Phi_A^*\psi_f^*(w_{ik}^{(-)}-1)(\vec{p}-\hbar\vec{k}_f\mu_{ab}/\mu)$$
$$=\hbar\Phi_B^*\Phi_A^*\psi_f^*\{(w_{ik}^{(-)}-1)[\vec{\kappa}_i(\vec{x}_A)+\vec{\kappa}_k(\vec{x}_B)]*$$
$$-w_{ik}^{(-)}(\vec{\nabla}_{ik}\chi_{ik}^{-})\}\ . \qquad (42b)$$

Here $\chi_{ik}^{\pm}$ is a function of $\vec{r}_{ik}=\vec{r}_i-\vec{r}_k=\vec{b}_{ik}+z_{ik}\hat{z}$. Since $\vec{r}_{ik}=\vec{R}+\vec{x}_i-\vec{x}_k$, we also have $\vec{b}_{ik}=\vec{b}+\vec{s}_i-\vec{s}_k$.

There are thus several terms contributing to a matrix element of $t_{ik}^{pp'}$. Among these we recognize one involving $\hbar^2(2\mu_{ab})^{-1}(\vec{\nabla}\chi^{-}\cdot\vec{\nabla}\chi^{+})_{ik}$. The sum of such terms from different elementary pairs gives a wave-spreading function

$$f^{pp(w)'}=-\left(\frac{\hbar^2}{2\mu_{ab}}\right)\frac{1}{\hbar u}$$
$$\times\sum_{ik}\int_{-\infty}^{\infty}\left[\frac{(v'_{ik})^2}{\hbar u}+z_{ik}\frac{\partial v'_{ik}}{\partial b_{ik}}\frac{\partial}{\partial b_{ik}}(\chi_{ik}^{+}-\chi_{ik}^{-})\right]dz_{ik}\ , \qquad (44)$$

to be used in Eq. (28). Equation (44) differs from Eq. (33) by an over-all negative sign, as expected. Indeed, for $N_a=N_b=1$ we find $\mu_{ab}=u$, $\Delta\vec{u}=0$, and therefore $f^{pp(w)'}+f^{w'}=0$. This is, of course, expected since the total wave-spreading correction must vanish by virtue of the definition of the pseudopotential. More generally, we expect $f^{pp(w)'}$ to be of order $-\mu(\mu_{ab}N_aN_b)^{-1}=-m_{ab}/M_{AB}$ of $f^{w'}$. Therefore, the heavier the ions, the poorer the cancellation in $f^{pp(w)'}+f^{w'}$.

The remaining terms from Eq. (34) describe the effects of zero-point motion of bound particles in ions. They appear through the momentum dependence of the actual potential v as defined by Eq. (13). We expect these effects to be small at intermediate energies, but it is nevertheless useful to obtain a closed expression amenable to calculation. For these we find from Eqs. (40), (42a), (42b), and (43) a contribution

$$f^{pp(zpm)'}=\frac{\hbar^2}{2\mu_{ab}}\int_{-\infty}^{\infty}dz\sum_{ik}\ [(1-e^{-i\chi_{ik}^{-}})|\vec{\kappa}_i+\vec{\kappa}_k|^2(1-e^{-i\chi_{ik}^{+}})+(1-e^{-i\chi_{ik}^{-}})(\vec{\kappa}_i+\vec{\kappa}_k)*\cdot(\vec{\nabla}_{ik}\chi_{ik}^{+})$$
$$-(\vec{\nabla}_{ik}\chi_{ik}^{-})\cdot(\vec{\kappa}_i+\vec{\kappa}_k)(1-e^{-i\chi_{ik}^{+}})] \qquad (45)$$

in Eq. (28).

There are, besides Eq. (45), other contributions due to the zero-point motion of elementary particles in ions. They originate from the second term of Eq. (40), as discussed earlier.

VII. INTERNAL-EXCITATION CORRECTIONS

We now turn to the internal excitation term in Eq. (26). This is an important term, especially at high energies, because it contains recoil effects.

We begin by separating the contribution of each ion:

$$T^{int'}=\sum_{C=A,B}T_C^{int'},$$
$$T_C^{int'}=(\Omega_{MD}^{\dagger}-1)\Delta H_C^{int}(\Omega_{MD}-1)\ . \qquad (46)$$

Operating on internal states

$$\Delta H_C^{int}(\Omega_{MD}-1)|C\rangle=[K_C^{int}+V_C^{int},\Omega_{MD}]|C\rangle,$$
$$C=A \text{ or } B\ , \qquad (47)$$

Hahn[1] has shown how the kinetic energy part of Eq. (47) can be handled. If the internal kinetic energy

$$K_C^{int}=\sum_{i=1}^{N_C}K_i-K_C^{c.m.}\ ,$$
$$K_C^{c.m.}=(2N_Cm_C)^{-1}\left(\sum_{i=1}^{N_C}\vec{p}_i\right)^2\ , \qquad (48)$$

is expressed in terms of s.p. operators K_i and the ion C c.m. operator $K_C^{c.m.}$, then the term $[K_C^{int},\Omega_{MD}]$ contributes a part

$$f^{KE'}=\sum_{C=A,B}(f_C^{s.p.KE'}-f_C^{c.m.KE'}) \qquad (49)$$

in the scattering amplitude (28). Of these, Hahn[1] gives the s.p. part $f_C^{s.p.KE'}$. It can be calculated easily from Eq. (47):

$$f_C^{\text{s.p.}KE'} = \frac{\hbar^2}{2m_c}\left(\frac{i}{\hbar u}\right)\int_{-\infty}^{\infty} dz(1-e^{-i\chi^-})\sum_{i=1}^{N_C}\Bigg[\int_{-\infty}^{z}\vec{\nabla}_i^2\sum_k v'_{ik}dz' + 2\left(\int_{-\infty}^{z}\vec{\nabla}_i\sum_k v'_{ik}dz'\right)\cdot\vec{\nabla}_i - \frac{i}{\hbar u}\left|\int_{-\infty}^{z}\vec{\nabla}_i\sum_k v'_{ik}dz'\right|^2\Bigg] . \tag{50}$$

Here $\vec{\nabla}_i$ refers to a s.p. coordinate $\vec{r}_i$ in ion C and m_c is the mass of an elementary particle in ion C. The index k refers to an elementary particle in the other ion. The pseudopotential v'_{ik} is $v'_{ik}[\vec{b}+\vec{s}_i-\vec{s}_k+(z+z_i-z_k)\hat{z}]$, where $\vec{x}_i=\vec{s}_i+z_i\hat{z}$ is an internal coordinate.

The c.m. correction $f_C^{\text{c.m.}KE'}$ has a similar structure

$$f_C^{\text{c.m.}KE'} = N_c^{-1}f_C^{\text{s.p.}KE'} + N_c^{-1}\left(\frac{\hbar^2}{2m_c}\right)\frac{i}{\hbar u}\int_{-\infty}^{\infty} dz(1-e^{-i\chi^-}) \times\sum_{i,j\neq i}\left\{2\left[\int_{-\infty}^{z}\vec{\nabla}_i\sum_k v'_{ik}dz'\right]\cdot\vec{\nabla}_j - \frac{i}{\hbar u}\left[\int_{-\infty}^{z}\vec{\nabla}_i\sum_k v'_{ik}dz'\right]\cdot\left[\int_{-\infty}^{z}\vec{\nabla}_j\sum_l v'_{jl}dz''\right]\right\} . \tag{51}$$

Here the subscripts i,j refer to particles in ion C; while k, l refer to particles in the other ion. Compared with $f_C^{\text{s.p.}KE'}$, this c.m. correction should be of order N_c^{-1} unless the second, or $\vec{p}_1\cdot\vec{p}_2$, term in Eq. (51) is abnormally large. We believe that the correction (51) is important only for lighter ions.

Finally, the potential-energy is described by

$$f_C^{\text{PE}'} = \int_{-\infty}^{\infty} dz(1-e^{-i\chi^-})(e^{-i\chi^+}V_C^{\text{int}}e^{i\chi^+} - V_C^{\text{int}}) . \tag{52}$$

As noted by Remler[16] and by Hahn,[1] this term vanishes if V_C^{int} is local.

VIII. PAULI CORRECTION

Finally, we consider the Pauli correction for heavy ions of identical fermions. Since V' is symmetric in particle coordinates of each ion, we may antisymmetrize just the quantity $\langle AB|$ with the help of an antisymmetrizer $\mathcal{A}$. Thus

$$T^P = (\mathcal{A}-1)V'\Omega_{\text{MD}} . \tag{53}$$

The operator $\mathcal{A}-1$ is nonlocal, with coordinate matrix element $-Q(\vec{R},\vec{R}';\vec{x})$, when expressed in terms of the relative coordinates $\vec{R},\vec{R}'$ between clusters. Explicit formulas for Q for several light nuclei are available in the literature.[12]

Since Q depends on the z components of $\vec{R}$ and $\vec{R}'$, it is no longer possible to integrate z analytically. It is still possible to express the resulting contribution to the scattering amplitude formally in the impact-parameter representation (28). We then find the following expression to be used in Eq. (28):

$$\langle AB|e^{i\chi}f^{P'}|AB\rangle = -\int_{-\infty}^{\infty}dz\langle AB|\int d^3\vec{R}'Q(\vec{R},\vec{R}';\vec{x})V'(\vec{R}';\vec{x})e^{i\chi^+(\vec{R}';\vec{x})+i\vec{k}_i\cdot(\vec{R}'-\vec{R})}|AB\rangle . \tag{54}$$

Expressions of the type (54) are familiar (and have been calculated in the Born approximation) in resonating-group and generator-coordinate theories of heavy-ion reactions.[12] Here at intermediate energies we expect the Pauli correction to decrease rapidly in importance as the momentum transfer increases. This expectation should be checked numerically with the help of Eq. (54), especially in exchange reactions.

IX. DISCUSSIONS

The main results of this paper are (i) the introduction of a pseudopotential v'; (ii) the pseudopotential corrections (44) and (45); (iii) the c.m. correction (51) and the internal potential term (52) describing effects due to the internal excitation of ions, and (iv) the Pauli correction (54). With these refinements, a MD expan-

sion can be made in which the leading term is Glauber's multiple-diffraction (MD) amplitude constructed from the empirical elementary scattering amplitude. It would be interesting to see how important these corrections are, numerically. Calculations of these terms are now in progress.

Still neglected in the expansion are relativistic and spin-dependent corrections. The first appears to be relatively straightforward.[8,13,14] Spin-dependent effects have been studied in the past within the context of the Glauber theory.[15] If these effects are weak, one can treat them perturbatively. If they are strong, their effects in the distorted waves may have to be included. Further work is needed to determine how they can be handled conveniently.

The MD expansion studied here is expected to be particularly useful for p-α scattering. According to Ullo and Feshbach,[17] one may associate the second maximum in the p-α differential cross section with double-scattering and two-nucleon correlations, while triple-scattering and three-nucleon correlational effects are dominant in the third maximum. The quantitative importance of the leading wave-spreading correction, Eq. (33), in the region of the second maximum has been noted by Wallace[7] in potential scattering and by Schürmann[11] in p-nucleus scattering. The results of Wallace[7] also suggests that higher-order wave-spreading corrections can probably be neglected in this second maximum region.

Thus with the addition of many-body corrections discussed by Hahn[1] and in this paper, and of relativistic and spin-dependent corrections, the theoretical description of the second maximum in elastic p-α scattering appears to be quite simple. Therefore, the possibility of extracting information on two-nucleon correlations appears quite favorable if the elementary interaction is sufficiently well determined. (According to a recent calculation of Rule and Hahn,[18] the uncertainties in the elementary interaction are large enough to mask the effects of dynamical correlations and off-shell propagations.)

The situation with respect to larger angles is less promising. According to Wallace,[7] the convergence of the perturbation expansion for wave-spreading corrections tends to be slow, although we may expect the convergence to improve as energy increases. However, as energy increases, the recoil correction of Hahn[1] begins to dominate. The effect is particularly important at large angles where momentum transfers are large. It is then possible that the perturbation expansion does not converge. If this should occur, the divergent series must be summed by special techniques such as that of Padé approximants.[19] The simplicity of the MD expansion and the possibility of generating high-order wave-spreading[7] and recoil corrections suggest that the expansion might still be competitive with, or supplementary to, the multiple-scattering approach,[16] or the optical-potential approach,[13] or the hybrid approach of Hahn,[1] over a rather extended angular range.

We point out in conclusion that the MD expansion considered here explicitly isolates many-body features which must be treated before we can reliably extract certain detailed nuclear information in intermediate-energy reactions.

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[1]Y. Hahn, Phys. Rev. C 10, 585 (1974).
[2]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin and L. G. Dunham (Interscience, New York, 1959), Vol. I, p. 315; R. J. Glauber, in *High Energy Physics and Nuclear Structure*, edited by S. Devons (Plenum, New York, 1970), p. 207.
[3]L. L. Foldy and J. D. Walecka, Ann. Phys. (N.Y.) 54, 447 (1969).
[4]E. Kujawski, Ann. Phys. (N.Y.) 74, 567 (1972).
[5]D. M. Chase, Phys. Rev. 104, 838 (1956).
[6]S. J. Wallace, Phys. Rev. Lett. 27, 622 (1971).
[7]S. J. Wallace, Ann. Phys. (N.Y.) 78, 190 (1973).
[8]H. D. I. Abarbanel and C. Itzykson, Phys. Rev. Lett. 23, 53 (1969).
[9]W. E. Frahn and B. Schürmann, Ann. Phys. (N.Y.) 84, 147 (1974).
[10]B. Schürmann and W. E. Frahn, Nucl. Phys. B62, 365 (1973).
[11]B. Schürmann, Nucl. Phys. B67, 425 (1973).
[12]K. Wildermuth and W. McClure, *Cluster Representation of Nuclei* (Springer Tracts in Modern Physics, Vol. 41, 1966), p. 92; Reviews of the Ljubljana Seminar on Generator-coordinate Method for Nuclear Bound States and Reactions, edited by M. V. Mihailovic and M. Rosina [Suppl. Fiz. 5, 56 (1973)]; C. W. Wong, Phys. Rep. 15C, 283 (1975).
[13]H. Feshbach and J. Hüfner, Ann. Phys. (N.Y.) 56, 268 (1970); H. Feshbach, A. Gal, and J. Hüfner, *ibid.* 66, 20 (1971); E. Lambert and H. Feshbach, *ibid.* 76, 80 (1973).
[14]M. Levy and J. Sucher, Phys. Rev. 186, 1656 (1969).
[15]V. Franco and R. J. Glauber, Phys. Rev. 142, 1195 (1966); R. J. Glauber and V. Franco, *ibid.* 156, 1685 (1967); E. Kujawski, D. Sachs, and J. S. Trefil, Phys. Rev. Lett. 21, 583 (1968); V. Franco, *ibid.* 21, 1360

(1968).
[16]E. A. Remler, Phys. Rev. 176, 2108 (1968).
[17]J. J. Ullo and H. Feshbach, Ann. Phys. (N.Y.) 82, 156 (1974).
[18]D. W. Rule and Y. Hahn (unpublished).
[19]G. A. Baker, Jr., and J. L. Gammel, *The Padê Approximant in Theoretical Physics* (Academic, New York and London, 1970), p. 1.