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Method for three-body equations*

Sadhan K. Adhikari and I. H. Sloan
School of Mathematics, The University of New South Wales, Kensington, New South Wales, 2033 Australia
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A recently proposed degenerate-kernel scheme for solving Fredholm integral equations of the second kind gave very good results for Lippmann-Schwinger equations. The method is here extended to give a very simple and practical method for the three-body equations with separable two-body interactions. Numerical calculations of the three-body amplitudes are carried out at positive energies for two simple models. In spite of the more complicated singularities in the kernel as well as in the solutions of the equation, satisfactory convergence is achieved.

[NUCLEAR REACTIONS Faddeev equation solved. Degenerate kernel scheme used.]

I. INTRODUCTION

A recently proposed method[1] for Lippmann-Schwinger equations is here extended to handle the three-body equations with separable two-body interactions. The kernels and the solutions of the three-body equations at positive energies are rather singular,[2-4] and this makes the solutions of the three-body equations at positive energies more complicated.

The method is based on the recognition[1,5-7] that one can explicitly construct a finite rank approximation to an operator, which is exact when it acts on a set of predetermined functions $u_1, \ldots, u_N$. We seek a rank-N approximation to the kernel K of the Fredholm integral equation of the second kind $y = f + Ky$. We want our rank-N approximation K_N to be a good approximation to K in the context in which K actually occurs in the integral equation, i.e., as an operator on the solution y. And for this to happen the functions $u_1, \ldots, u_N$, used to construct K_N, should be a good set for representing the solution y. Then we solve the integral equation with this finite rank approximation to the kernel.

The methods of Ref. 5 gave good convergence for a wide variety of Fredholm kernels. In particular, the errors were found to be smaller by one or more orders of magnitude than the corresponding errors obtained with the method of moments. The phrase method of moments is not used in the restricted sense of Vorobyev[8] and Harms.[9]

Here we discuss the modifications necessary for dealing with the more complicated singularities[2-4] in both the kernels and the solutions of the three-body equations. The success of the method does not depend on the structure of the kernel, so the singularities of the kernel do not give any additional problem and are avoided by deforming the contour of integration.[3,10] But the singularity in the three-body amplitude is more fundamental and can be incorporated in our method by choosing a set of functions $u_1, \ldots, u_N$ that contain this singularity; as the primary requirement for good convergence is a good choice of the expansion functions $u_1, \ldots, u_N$.

There are already quite a number of methods[1-4,9-11] available for solving singular scattering equations and some of them have successfully been applied to the three-body equations.[2-4,10] But one of the main advantages of the present method is its simplicity. Once the analytic expansion functions $u_1, \ldots, u_N$ are chosen the main numerical task is to invert a matrix of relatively small dimension N ($N = 6$ gave very good convergence for the problems considered).

There is another motivation behind the present work on three-body equations. In few-body problems one is often interested in getting a finite rank approximation for the amplitude, which can be used directly in the kernel of a larger problem. In Ref. 6, we showed how the method of Ref. 1 was extended to find a good separable operator expansion for the two-body t matrix, which might not be very accurate in a pointwise sense, but should serve as a good operator in the context of few-body problems. Similarly, the same work can be extended to the three-body case and we can find a good separable operator expansion for the three-body amplitude at all energies.

Essentially the same technique can be applied to find a separable expansion for the three-body amplitude at negative energy, which will be good in a pointwise sense. This was done for the two-body case[7,12,13] and the technique yielded good separable expansions for coupled[13] and uncoupled[7,12] channels of the two nucleon interactions at all energies.

However, the method does not give a good separable expansion for three-body amplitudes at positive energies due to the presence of complicated logarithmic singularities. Nevertheless, this three-body separable expansion can be used for four-body bound state calculations.

So one motivation of the present work is to find good separable expansions for three-body amplitudes.

The approach of Ref. 1 leads to an infinite family of methods, of which three are considered in this paper. The methods are applied to two simple separable three-body models. The method is considered in detail in Sec. II. In Sec. III we give numerical results for two simple three-body models. In Sec. IV we give a discussion of our method and our conclusions are listed in Sec. V.

II. METHOD

The method is formally the same as that of Ref. 1, because the three-body scattering equation with separable two-body interactions is formally similar to the Lippmann-Schwinger equation. We briefly outline the method here.

In the Amado model[3,14] of three-particle interactions the scattering equation can be formally written for separable two-body interactions as

$$T = A + KT . \tag{1}$$

Here $K = B\tau$ is the kernel, where τ is the two-body propagator, B is the three-body Green's function, T is the three-body amplitude, and A is the Born term.

To start, we construct a rank-N approximation K_N to the kernel K

$$K_N = \sum_{n,\,m=1}^{N} K|u_n\rangle D_{nm}\langle v_m| , \tag{2}$$

where

$$(D^{-1})_{mn} = \langle v_m|u_n\rangle , \tag{3}$$

and $|u_n\rangle$ and $\langle v_m|$ are two sets of suitably chosen analytic functions. The only constraint on the choice of $|u_n\rangle$ and $\langle v_m|$ is that (D^{-1}) is a nonsingular matrix.

With this approximation Eq. (1) can be readily solved to yield

$$T_N = A + \sum_{n,\,m=1}^{N} K|u_n\rangle \Delta_{nm}\langle v_m|A , \tag{4}$$

where

$$(\Delta^{-1})_{mn} = \langle v_m|(1-K)|u_n\rangle . \tag{5}$$

The interesting aspect of approximation (2) is that

$$K_N|u_n\rangle = K|u_n\rangle , \quad n = 1, \ldots, N. \tag{6}$$

This means that if we can choose a good set of functions, $u_1, \ldots, u_N$, K_N will be a good representation of K. Then T_N will be a good representation of T, because this K_N is used to solve Eq. (1).

Once we fix the set of functions $u_1, \ldots, u_N$ there is some arbitrariness in the choice of v_m. In this paper we consider three explicit choices, namely,

$$\begin{aligned} &\text{choice 1} \quad \langle v_m| = \langle u_m| , \\ &\text{choice 2} \quad \langle v_m| = \langle u_m|\tau , \\ &\text{choice 3} \quad \langle v_m| = \langle u_m|\tau B\tau . \end{aligned} \tag{7}$$

The functions u_m form a good basis for representing the solution T. The singular factor τ in choices 2 and 3 is taken because we want [as in Eq. (6)] u_n to be a good basis for representing τ times B. Choice 2 yields a simple symmetric result. Of course many other choices of v_m are possible.

III. NUMERICAL RESULTS

Three-body equations

In this section we apply the present method to two simple three-body systems, with separable Yamaguchi type two-body interactions. We study the model of the neutron-deuteron system, first studied by Aaron, Amado, and Yam.[14] Next, we study the case of weak decay of a particle to three-identical bosons. In this model, studied by Adhikari and Amado,[2] the final particles interact strongly via Yamaguchi type separable potentials. In both the cases the final equations are formally equivalent to Eq. (1). But in the neutron-deuteron problem $A = B$ and in the decay problem $A \neq B$. Now we quote the three-body equations for the two problems.

Neutron-deuteron problem

We study the S-wave quartet case of the neutron-deuteron system. But the method can be easily applied to the doublet case. The S-wave integral equation after antisymmetrization and partial wave analysis takes the form[4]

$$\langle p|T|p'\rangle = \langle p|B|p'\rangle - \frac{3}{2\pi}\int_0^\infty q^2 dq \langle p|B|q\rangle \tau(s - \tfrac{3}{4}q^2)\langle q|T|p'\rangle , \tag{8}$$

where

$$\langle p|B|p'\rangle = -\frac{4\pi^2}{3}\int_{-1}^{+1} \frac{v((\frac{1}{2}\vec{p}+\vec{p}')^2)v((\frac{1}{2}\vec{p}'+\vec{p})^2)}{p^2+p'^2+pp'x-s}dx, \tag{9}$$

and where

$$\tau^{-1}(z) = -\frac{\pi^2}{\nu} - \frac{3}{2\pi}\int_0^\infty q^2dq\, v^2(q^2)\left(-\frac{8\pi^2}{3}\frac{1}{q^2-z}\right), \tag{10}$$

where $s(=E+i\epsilon)$ is the complex energy parameter and $v(q^2)$ has the Yamaguchi form

$$v(q^2) = (q^2+\beta^2)^{-1}. \tag{11}$$

Here $\beta = 1.405\,52$ fm^{-1} and $\nu = 3.767\,541$. Equation (8) is formally similar to Eq. (1) and the solution is

$$\langle p|T|p'\rangle = \langle p|B|p'\rangle + \sum_{n,m=1}^{N}\langle p|B\tau|u_n\rangle\Delta_{nm}\langle v_m|B|p'\rangle, \tag{12}$$

with Δ_{nm} given by Eq. (5).

Decay to three bosons

Here the equation is very similar to the neutron-deuteron problem. The only difference is the Born term. The partial S-wave projection of the equation for the decay amplitude $f(p)$ is[2,15]

$$f(p) = A(p) - \frac{3}{2\pi}\int_0^\infty q^2dq\langle p|B|q\rangle\tau(s-\tfrac{3}{4}q^2)f(q). \tag{13}$$

Here $\langle p|B|q\rangle$ is twice that given by Eq. (9) and

$$A(p) = -\gamma_0\frac{3}{2\pi}\int_0^\infty \frac{q^2dq\,v(q^2)}{(\alpha^2+\frac{3}{4}p^2+q^2)(\frac{3}{4}p^2+q^2-s)} \tag{14}$$

and $\tau^{-1}(z)$ is given by Eq. (10) and $v(q^2)$ given by Eq. (11) with $\beta = 1$ fm^{-1} and $\gamma_0 = -12\pi^2$. The interesting feature of Eq. (13) is that the Born term is not the usual three-body Green's function. In this case the solution can be written as

$$\langle p|f = \langle p|A + \sum_{n,m=1}^{N}\langle p|B\tau|u_n\rangle\Delta_{nm}\langle v_m|A, \tag{15}$$

with Δ_{nm} given by Eq. (5).

Explicit integrals and expansion functions

The various quantities in Eqs. (12) and (15) can easily be written in partial wave integral form, for example

$$\langle p|B\tau|u_n\rangle = -\frac{3}{2\pi}\int_0^\infty q^2dq\langle p|B|q\rangle\tau(s-\tfrac{3}{4}q^2)u_n(q), \tag{16}$$

and

$$\langle v_m|\tau B\tau|u_n\rangle = -\frac{3}{2\pi}\int_0^\infty q^2dq\, v_m(q)\tau(s-\tfrac{3}{4}q^2)\langle q|B\tau|u_n\rangle, \tag{17}$$

and

$$\langle v_m|A = -\frac{3}{2\pi}\int_0^\infty q^2dq\, A(q)v_m(q). \tag{18}$$

The functions $u_n(p) \equiv \langle p|u_n\rangle$ were chosen to be

$$u_n(p) = [p^2+(n+1)^2\tfrac{1}{4}\lambda^2]^{-2}, \quad n = 1,3,5,\ldots,N-1$$
$$= (p_0^2-p^2)^{1/2}u_{n-1}(p), \quad n = 2,4,\ldots,N \tag{19}$$

where $p_0 = [4s/3]^{1/2}$. The term $(p_0^2-p^2)^{1/2}$ is taken to represent the well-known square-root singularity[2,4] in the half-shell three-body amplitude. A detailed discussion on this has already appeared.[4] The parameter λ can be varied to get good convergence and after a small amount of experimentation was chosen to be 0.5 fm^{-1}.

A point of technical importance if the integrals in Eqs. (16)–(18) are to be evaluated numerically is that it is then necessary to use basis functions that are approximately orthogonal in order to prevent the matrix Δ^{-1} from becoming too singular. In the present calculations this approximate orthogonality was achieved by constructing the basis functions in terms of the Sturmian functions for the S-wave Hulthèn potential.

The Sturmian functions satisfy the eigenvalue problem

$$\lambda_n VG_0(E')|\psi_n(E')\rangle = |\psi_n(E')\rangle, \tag{20}$$

where V in this equation is the Hulthèn potential,

$$V(r) = V_0e^{-\lambda r}/(1-e^{-\lambda r}). \tag{21}$$

With E' chosen to be zero, the Sturmian functions satisfy the orthogonality relation[16]

$$\langle\psi_n|G_0(0)|\psi_m\rangle = -\delta_{nm}, \tag{22}$$

and are given analytically by[16]

$$\psi_n(p) = \sum_{m=1}^{n} b_{nm}(p^2+m^2\lambda^2)^{-1}, \tag{23}$$

where

$$b_{nm} = \pi^{-1}\lambda^{3/2}(-n)_m(n)_m[(m-1)!]^{-2}n^{-1/2}, \tag{24}$$

with

$$(x)_m = x(x+1)(x+2)\cdots(x+m-1).$$

In practice the calculations were not directly done with the basis Eq. (19), but rather with the equivalent orthogonalized form

$$u_n(p) = [p^2 + (n+1)^2\lambda^2/4]^{-1}\psi_{[(n+1)/2]}\,, \qquad n = 1, 3, 5, \ldots, N-1$$

$$= (p_0^2 - p^2)^{1/2} u_{n-1}(p)\,, \qquad n = 2, 4, \ldots, N.$$

This makes the Δ^{-1} matrix well conditioned and eliminates numerical problems arising from inverting an ill conditioned matrix but otherwise has no effect on the results.

Numerical calculations

Evaluation of integrals (16), (17), and (18) along the real axis is difficult due to presence of complicated logarithmic singularities along that particular axis. We use the contour deformation technique originally proposed by Hetherington and Schick,[10] adapted to this problem by Aaron and Amado,[3] and later refined by Cahill and Sloan.[10] The caution introduced by Cahill and Sloan is important in case of Eq. (16) and for $p < p_0$.

First we give numerical results for the neutron-deuteron system. The calculations were carried out for the three choices of the expansion functions v_m of Sec. II. In Fig. 1 we show the real and imaginary parts of the three-body amplitude $T(p, p'; E)$ at $E = 41.47$ MeV and $p' = 1.341$ fm^{-1} for the three choices of v_m. Then we consider the three-body amplitude $T(p, p'; E)$ at $E = 41.47$ MeV and $p' = 0.4066$ fm^{-1}. The square-root singularity in the half-shell amplitude T occurs at $p_0 = 1.1547$ fm^{-1}. Here $p' < p_0$ and consequently the Born term B in Eq. (12) develops a logarithmic singularity. The quantity plotted in Fig. 2 is U, which is the amplitude T minus the Born term B; so that we do not have to show the discontinuity in T arising from the logarithmic singularity in the Born term (which is, of course, treated exactly). From Figs. 1 and 2 we see that the convergence is very good.

Next we consider the decay problem. Here we take $\alpha = 1$ fm^{-1}, $\nu = 0.98$, $E = 1$ fm^{-2}. The real and imaginary parts of the amplitude f are plotted in Fig. 3 for the simplest choice of v_m (choice 1 of Sec. II). Here $f(p)$ is not plotted against p but against $k = [3(p_0^2 - p^2)/4]^{1/2}$. The physical range of k is $E^{1/2}$ to 0. This particular variable will easily allow us to compare our results with those by a direct solution of Eq. (13) by Adhikari and Amado.[2] For this particular case the low-rank result is very close to the converged result, which agrees with Ref. 2.

In all the cases considered the convergence is very good and $N = 6, 7$ result is within 1 ~ 2% of the converged result. The three-body amplitude also converges equally rapidly at other values of energy and momentum. But one interesting aspect of the results was that choices 2 and 3 consistently gave better results than choice 1. This is due to the fact that in the case of choice 1 in the solution, B directly acts on u_m [see Eq. (4) with $A = B$], and $u_m(p)$ given by Eq. (19) falls off too slowly as $p \to \infty$

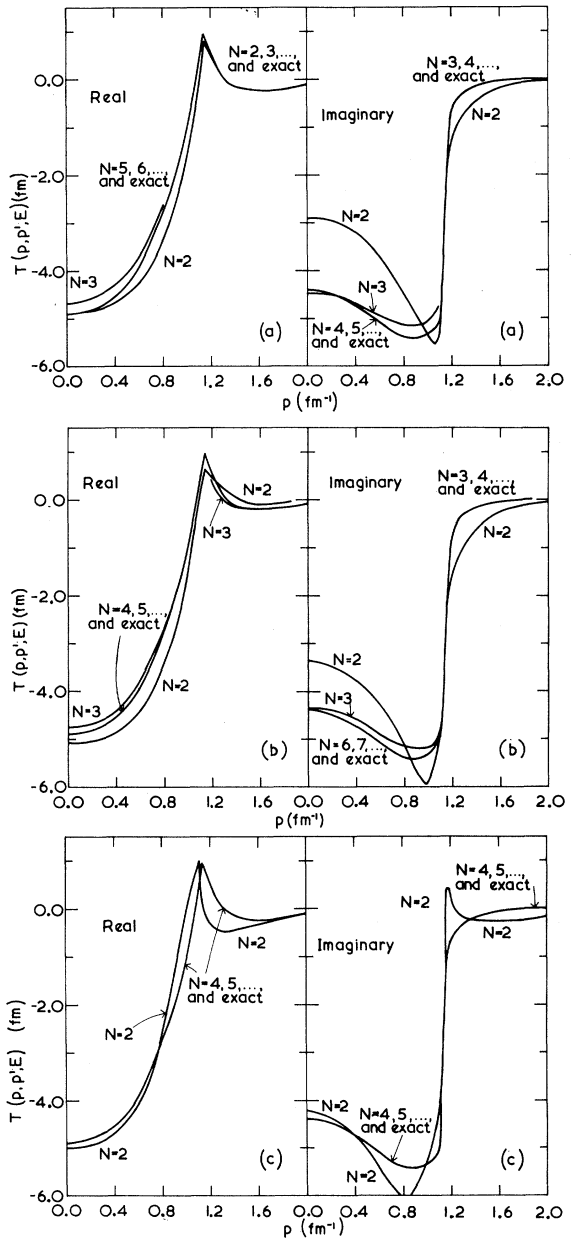

FIG. 1. The real and imaginary parts of the three-body amplitude $T(p, p'; E)$ at $E = 41.47$ MeV and $p' = 1.341$ fm^{-1} for (a) choice 1, (b) choice 2, and (c) choice 3 of v_m. The imaginary part has been multiplied by -2.

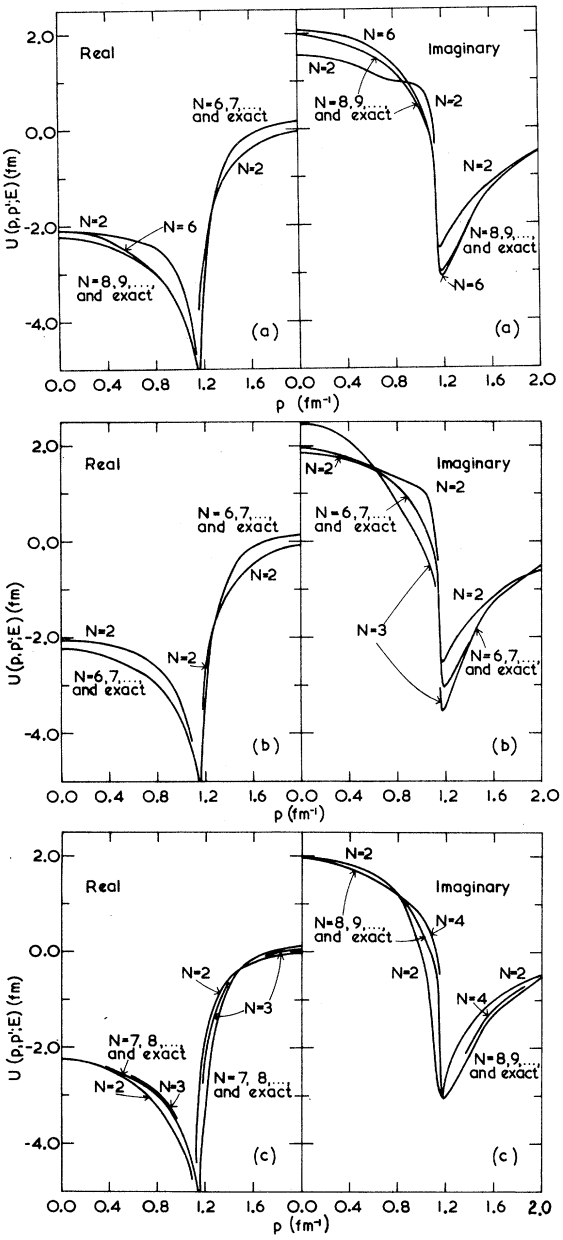

FIG. 2. The real and imaginary parts of $U(p,p';E)$, the three-body amplitude minus the corresponding Born term, at $E=41.47$ MeV and $p'=0.4066$ fm^{-1} for (a) choice 1, (b) choice 2, and (c) choice 3 of v_m. The sign of the imaginary part has been changed.

for that purpose.

In all these cases $N=1$ is not good. This is expected because $N=1$ does not include the famous square-root branch point in the half-shell three-body amplitude; consequently, it will be a poor representation of the actual three-body amplitude. But for $N=2$ this square-root branch point is included and in all the cases considered, $N=2$ result is reasonable.

Choice 1 is the simplest and gives very good results in the decay problem, but unfortunately for reasons discussed above it does not give good convergence for the neutron-deuteron problem. But in case of neutron-deuteron problem choice 2 gives a simple symmetric result and is as good as any. This turns out to be a very effective and practical method for the neutron-deuteron problem. Further results for this method and for the doublet case are given in Ref. 4 (but with a different form for u_n and described there as the variational results).

IV. DISCUSSION

The present method is very similar to some other methods for singular scattering equations. For the particular case of choice 2, discussed in Sec. II, the approximation is the same as one obtained previously from a variational principle and is called variational results in Ref. 4.

A recently proposed method of moment by Harms for singular scattering equations[9] is formally equivalent to choice 2 in Sec. II. The only difference is in the choice of expansion functions. Harms applied his method to the Lippmann-Schwinger equation. He uses a set of momentum dependent expansion functions, which are in fact the different terms in the Born series. It is tedious to calculate these expansion functions at each momentum; whereas, in the present method a simple momentum independent orthogonal set of analytic expansion functions are chosen, yet the numerical convergence is very satisfactory. Another technical difference, when the integrals are evaluated

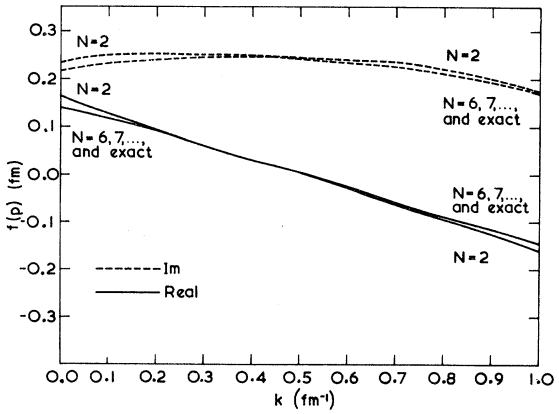

FIG. 3. The real and imaginary parts of the amplitude f at $E=1$ fm^{-2} for choice 1 of v_m. The exact result is taken from Ref. 2 and agrees with the converged result.

numerically, is that the functions of Ref. 9 are not orthogonalized in some sense and for large N this might give rise to numerical trouble in inverting the Δ^{-1} matrix, which may become too singular.

Another recent method for singular scattering equations, studied by Osborn[11] for the Lippmann-Schwinger equations, is similar to the present method and uses a set of analytic functions. He also solves the equation with a finite rank approximate kernel K_N. But his approximate kernel is essentially a numerical quadrature operator and not a degenerate kernel operator as in the present work.

All these methods for the singular scattering equations significantly simplify the numerical problem associated with inverting a very large matrix. Once we can find good low rank separable expansions for the few-body amplitudes by using our formalism, few-body problems beyond three can, in principle, be solved by the present method. But we have to know the analytic structure of the solution so that we can make a good choice of expansion functions which will give rapid convergence.

V. CONCLUSION

The numerical calculations of Sec. III were carried out using Eqs. (12) and (15), with v_m given by Eq. (7). The final results—Figs. 1–3—show that the approximation converges very well in all the cases. The present method also gave very good results for Lippmann-Schwinger equations.[1] Essentially choice 2 [Eq. (7)] of the present method has been successfully applied to the doublet case.[4]

As discussed in the Introduction the present method also provides a foundation for finding a good separable expansion for three-body amplitude which will be very useful in the kernel of the four-body problem.

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