Numerical investigation of minimal three-body equations*

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We compare numerical solution of the "minimal" three-body equations obtained from unitarity and analyticity with the full Schrödinger solution in a simple three-boson model. We find that for short-range forces the minimal equations give the correct shape, but not magnitude, for the three-body amplitudes. This is true even for a singular form with no mathematical credentials.

[NUCLEAR REACTIONS Numerical investigation of unitarity and analyticity constraints in a three-boson model.]

In a recent letter[1] and paper[2] we showed that the general principles of quantum mechanics (rather than questions of detailed dynamics) determine the major features of three-body final states. In particular we demonstrated that implementing the constraints of unitarity and analyticity in a "minimal" way in the sequential decay or isobar language leads to a set of equations for three-body systems that are remarkably similar to the usual three-body scattering integral equations with separable potentials.[3] In order to explore the similarity and develop some experience in the question of how much is determined by general principles and how much left to detailed dynamics, we have undertaken a program of solving the equations for various simple numerical cases. In this note we report results for a simple three-boson s-wave model with Yamaguchi interactions. The more complicated, but potentially more interesting case of resonant final state interactions is still under study.

In our study of the consequences of unitarity and analyticity for three-body amplitudes we found that the simplest manifestation of these constraints leads, in the three-boson case, to a one-variable scattering integral equation of a particularly simple form[2] but that the equation is strictly only valid in the zero-range approximation and even then is not a Fredholm equation.[3] A somewhat more sophisticated treatment of the constraints leads to a one-variable scattering integral equation that does take the finite range of interactions into account and is a Fredholm equation,[2] but as one expects in such an approach, involves only on-shell information. In this note we compare numerical solutions of each of these equations with the "exact" Schrödinger solution; that is with numerical solution of the full off-shell scattering integral equation with separable potentials. The numerical techniques involved in solving the off-shell separable potential equation and the finite-range unitarity equation are the same and are by now well known. For the zero-range equation, which is not Fredholm, we have simply ignored the mathematical difficulty and solved the equation as if it were Fredholm. Since we never actually carry momentum integrals to infinity, all numerical operations are well defined. Of course, there is no guarantee that such a scheme converges to the true solution of the equation, or even that the equation has a solution; hence, we call the method "illegal." As we shall see, it nevertheless gives surprisingly useful results.[4] In general, we find that the nearer we are to zero range, the better the Schrödinger and finite-range unitarity equations agree, at least in shape, while the illegal answers are never close to either. However, if we study not the full solution of the integral equation, but normalized solutions, all three methods agree remarkably well for short range and even at finite range there is moderate agreement. Our philosophy in studying the normalized solutions is that most phenomenology of three-body final states studies the shape of momentum distributions and the like, but not the over-all magnitude of the cross section. For such a study only the normalized form is relevant. Hence, we find that the shape of the three-body amplitudes is almost entirely determined by the general constraints of quantum mechanics, so long as the momenta in question are small compared with the momenta associated with the range of the force. This is so even if the general constraints are expressed in an illegal form. However the full force of the dynamics is required for determining the magnitude of the amplitude.

The numerical model we chose to study is the weak decay of a spinless particle into three identical bosons ($\hbar = m = 1$). The final particles interact via the s-wave separable Yamaguchi potential

$$\langle \vec{q}\,|V|\,\vec{q}\,'\rangle = \lambda v(q^2)v(q'^2) = \lambda\beta^4/(q^2+\beta^2)(q'^2+\beta^2)\,, \tag{1}$$

where λ is the interaction strength, β the momentum space range, and q and q' are two-body relative momenta. For the primary decay vertex from an initial state at rest to three final particles of momenta $\vec{p}$, $-\frac{1}{2}\vec{p}+\vec{q}$, and $-\frac{1}{2}\vec{p}-\vec{q}$, we take the simple symmetric form

$$\Gamma(\vec{p},\, -\tfrac{1}{2}\vec{p}+\vec{q},\, -\tfrac{1}{2}\vec{p}-\vec{q}) = \gamma_0(\alpha^2+\tfrac{3}{4}p^2+q^2)^{-1}\,. \tag{2}$$

α is the momentum space range of the weak decay and γ_0 is its strength. The full decay amplitude M to first order in the weak decay strength γ_0 but to all orders in the final state interaction strength λ can be written

$$M(\vec{p}_1, \vec{p}_2, \vec{p}_3) = \Gamma(\vec{p}_1, \vec{p}_2, \vec{p}_3) + \sum_{i=1}^{3} f(\vec{p}_i)\Pi(E-\tfrac{3}{4}p_i^{\,2})v(q_i^{\,2})\,, \tag{3}$$

where Π is the Yamaguchi propagator

$$\lambda\Pi^{-1}(\epsilon) \equiv D(\epsilon) = 1 - \frac{\lambda}{(2\pi)^3}\frac{1}{2}\int \frac{d^3k\, v^2(k^2)}{\epsilon - k^2} \tag{4}$$

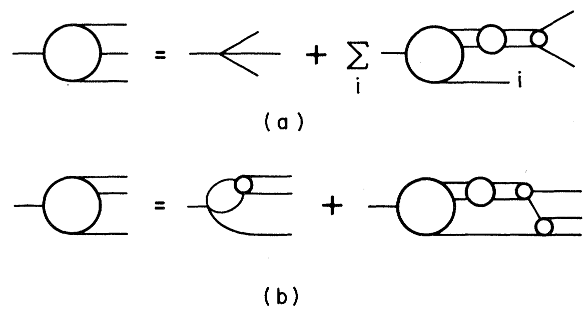

FIG. 1. Schematic representation of (a) Eq. (3) and (b) Eq. (5).

and $\vec{q}_i = \frac{1}{2}(\vec{p}_j - \vec{p}_k)$, $i \neq j \neq k$; $f(\vec{p})$ is defined by Eq. (3), which is represented diagrammatically in Fig. 1(a), but in fact it is the amplitude for decay into a particle and correlated pair or isobar. The factor $\Pi\, v$ represents the subsequent propagation and disassociation of that pair. $E-\frac{3}{4}p^2$ is just the energy of that pair in the two-body center of mass when the third particle has momentum p and we are in the over-all rest system. f satisfies the well-known integral equation[5]

$$f(\vec{p}) = \frac{\gamma_0}{(2\pi)^3}\int \frac{d^3q\, v(q^2)}{(\alpha^2+\frac{3}{4}p^2+q^2)(E-\frac{3}{4}p^2-q^2)} + \frac{1}{(2\pi)^3}\int \frac{d^3p'\lambda v((\vec{p}\,'+\frac{1}{2}\vec{p})^2)\, v((\vec{p}+\frac{1}{2}\vec{p}\,')^2) f(\vec{p}\,')}{D(E-\frac{3}{4}p'^2)(E-p^2-p'^2-\vec{p}\cdot\vec{p}\,')} \tag{5}$$

which is represented diagrammatically in Fig. 1(b). In fact only the s-wave projection of $f(\vec{p})$ matters in Eq. (5). It is the quasi-two-body amplitude f that we study by solving the s-wave projection of Eq. (5) numerically and comparing it with the solution of the unitarity equations.

The simplest of the unitarity equations for f assumes that the two-body t matrix has zero range. If we rewrite the two-body t matrix in the form

$$\tau(\epsilon) = N(\epsilon)/D(\epsilon) \tag{6}$$

with $D(\epsilon)$ as in Eq. (4), we find using the unitarity convention of Ref. 2 that

$$N(\epsilon) = 2\pi\lambda v^2(\epsilon)\,. \tag{7}$$

The zero-range convention corresponds to assuming $N(\epsilon)$ is independent of ϵ, which corresponds to taking $v^2(\epsilon)=1$. As is shown in Ref. 2 Eq. (40) this leads to the equation

$$f(\vec{p}) = R(\vec{p}) + \int \frac{d^3p'\lambda}{(2\pi)^3 D(E-\frac{3}{4}p'^2)(E-p^2-p'^2-\vec{p}\cdot\vec{p}\,')}\,. \tag{8}$$

(Note we are now using $m=1$ not $2m=1$ to correspond to Ref. 5.) In Eq. (8) we take the same inhomogeneous term as in Eq. (5). This is just Eq. (5) and the v's in the numerator of the integral kernel put equal to one. We do not also put them to one in the form for D [Eq. (4)] since then the integrals defining D will not converge. Hence, Eq. (8) is a mixed equation with very dubious mathematical credentials. Dropping the v's in the numerator, we lose convergence of the trace of the kernel at the upper limits, or of iterations of the kernel. It is for that reason that the Fredholm method is not guaranteed for this case and hence, we call its solution by Fredholm methods illegal. It is possible to include the left-hand singularities of τ, that is the ϵ dependence of N, in the unitarity and analyticity formalism. This leads to Eq. (48) of Ref. 2:

$$f(\vec{p}) = R(\vec{p}) + \int \frac{d^3p'}{(2\pi)^4}\frac{f(\vec{p}\,')}{D(E-\frac{3}{4}p'^2)}\left(\frac{N(E-\frac{3}{4}p^2)}{E-p^2-p'^2-\vec{p}\cdot\vec{p}\,'} + \frac{1}{\pi}\int \frac{dy\,\rho(y)}{(E-\frac{3}{4}p^2-y)[\frac{4}{3}y-p'^2-\frac{1}{3}E-\vec{p}\,'\cdot\hat{p}(\frac{4}{3}(E-y))^{1/2}]}\right), \tag{9}$$

where ρ is the discontinuity of N

$$N(\epsilon)=\frac{1}{\pi}\int\frac{dy\,\rho(y)}{y-\epsilon}\,. \tag{10}$$

For the Yamaguchi case we write this as

$$N(\epsilon)=2\pi\beta^4\lambda\,\frac{\partial}{\partial\beta^2}\int\frac{dy\,\delta(y+\beta^2)}{y-\epsilon}\,. \tag{11}$$

Again in Eq. (9) we take the same inhomogeneous term R as in Eq. (5).

We now solve Eqs. (5), (8), and (9) and compare their solutions. To simplify the parametrization we make a number of variable choices. Firstly β^2, the range of the forces, simply sets the energy scale and hence we take $\beta^2=1$. Now short range (large β^2) translates into small E. We are interested in effects on the p dependence of f generated by the strong final state interaction, not by the form of the weak vertex; hence, we want to take that vertex to be point-like. We take α in Eq. (2) to be 10. f is linear in γ_0, the strength of the weak decay and hence, γ_0 is irrelevant to the form of f. We set $\gamma_0=1$. We also redefine the interaction strength according to

$$\lambda=-16\pi\nu\,. \tag{12}$$

The minus is introduced to make positive ν correspond to attractive interactions (the only case we study) and the normalization is such that $\nu=1$ corresponds to a zero energy bound state of the two-body system. We study $\nu=0.98$, $\nu=0.5$, and $E=1$, 0.1, and 0.01. $\nu=0.98$ would be a case of very strong low energy pairwise final state interaction because we nearly have a bound state and the scattering length is large. $\nu=0.5$ corresponds to more moderate final state interactions. $E=1$ is a very large energy (recall $\beta^2=1$) or equivalently is a very long-range force. $E=0.1$ is intermediate, while $E=0.01$ corresponds to a very low energy or a very short-range interaction. For orientation the deuteron binding energy, expressed in units of the range of the nuclear force, is about 0.03.

In Figs. 2(a), 2(b), and 2(c) we show the f's for the three cases: Schrödinger, illegal, and Eq. (9) [which we call 48 because it comes from Eq. (48) of Ref. 2] for $E=1$, 0.1, and 0.01 and for $\nu=0.98$. We see that they do not agree, even at the lowest energy, although the shapes are much more alike for $E=0.01$ than for $E=1$. For $\nu=0.5$ the same is true. Why is that? To answer that question we need only look at the Fredholm denominators for the various cases. To solve the equation we convert the integrals to sums (we use the standard contour rotation technique) and then the denominator is just the determinant of the resulting algebraic equation. It helps to set the scale of the over-all solution. The determinant D is shown in Table I for the various values of E for $\nu=0.98$ and $\nu=0.5$ and for the three cases, Schrödinger, illegal, and 48. We see that the different forms give different values of D and that in particular the illegal case, as we would have expected, gives drastically different values. Recall that in this case we are simply using the fact that in replacing integrals by sums we introduce a momentum cutoff so that all terms are finite. The fact that the equation is not Fredholm implies that in the limit as the cutoff becomes large, the equations do not

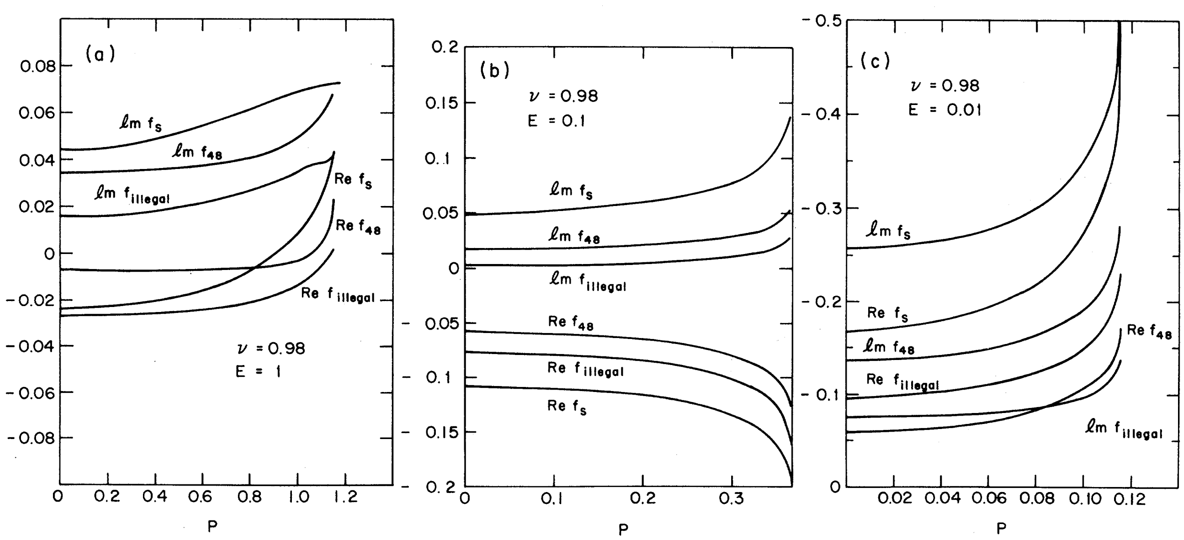

FIG. 2. Real and imaginary parts of the quasi-two-body amplitudes f for the three solution methods for coupling strength $\nu=0.98$ and energy (a) $E=1$, (b) $E=0.1$, and (c) $E=0.01$ as a function of the odd particle momentum.

TABLE I. The real and imaginary parts of the Fredholm denominator D of the three-body equations for the three solution methods, Schrödinger, 48, and illegal for various values of the coupling strength ν and energy E.

	Schrödinger		48		Illegal	
ν/E	Real D	Im D	Real D	Im D	Real D	Im D
0.98/1	0.822	−0.808	0.895	−0.928	38.7	56.2
0.98/0.1	−0.2933	−0.674	−0.545	−0.747	23.7	−11.9
0.98/0.01	−0.420	0.116	−0.537	0.266	8.01	−22.3
0.5/1	0.88	−0.397	0.890	−0.429	−3.41	4.0
0.5/0.1	0.420	−0.276	0.382	−0.279	−0.843	2.66
0.5/0.01	0.361	−0.070	0.343	−0.0687	−1.03	2.04

possess well defined solutions.

It is precisely because the various D's do not agree and because D sets the over-all scale that the various f's in Fig. 2 do not agree. The conclusion then is that the over-all (complex) scale of the amplitudes is a detailed dynamical quantity and approximate treatments of the dynamics can lead to very different values for it. But what of the shape of f? We can define a normalized f via

$$f_N(p) = f(p)/f(0). \tag{13}$$

This is just what one would do in trying to fit the shape of the final state particle distribution, but not its over-all magnitude. In Figs. 3(a), 3(b), and 3(c) we show the normalized amplitudes for $\nu = 0.98$, $E = 1.0$, 0.1, and 0.01. Now we see that the agreement is much better and in fact for the shortest-range case $E = 0.01$ is essentially perfect. Hence, even in this case of very important final state interaction ($\nu = 0.98$) the shape of f is determined nearly entirely by the nearby singularities which singularities are required by unitarity while only the magnitude of f reflects the details of the dynamics or, what is the same thing, the more distant singularities. For $\nu = 0.5$ we see similar results for the normalized amplitudes in Figs. 4(a), 4(b), and 4(c). Again for the longest-range case, the dynamics is important, as one would expect, and agreement among the cases is poor, but for $E = 0.1$ it is good and for $E = 0.01$ it is excellent. This is in spite of the fact that the unnormalized f's are very different in these three cases. This is shown in Fig. 5 where we show the $|f|^2$ for $\nu = 0.5$, $E = 0.01$. We see again the remarkable fact

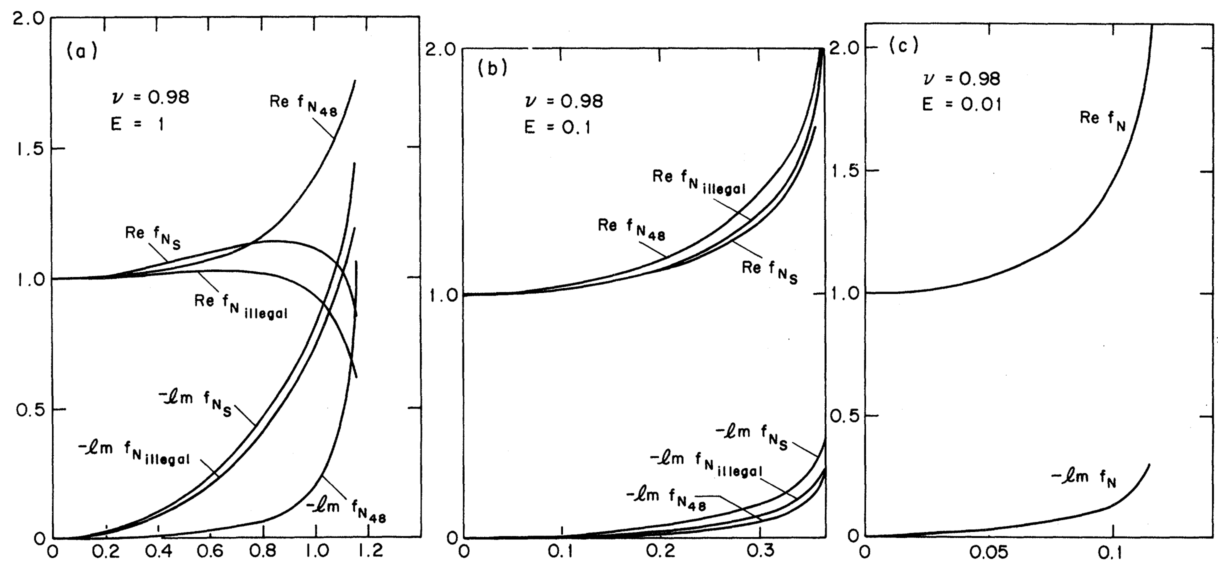

FIG. 3. Real and imaginary parts of the normalized quasi-two-body amplitudes f_n for the three solution methods for coupling strength $\nu = 0.98$ and energy (a) $E = 1$, (b) $E = 0.1$, and (c) $E = 0.01$ as a function of the odd particle momentum. For the case $E = 0.01$ three cases agree to within the numerical precision of our calculation and therefore only one line is shown.

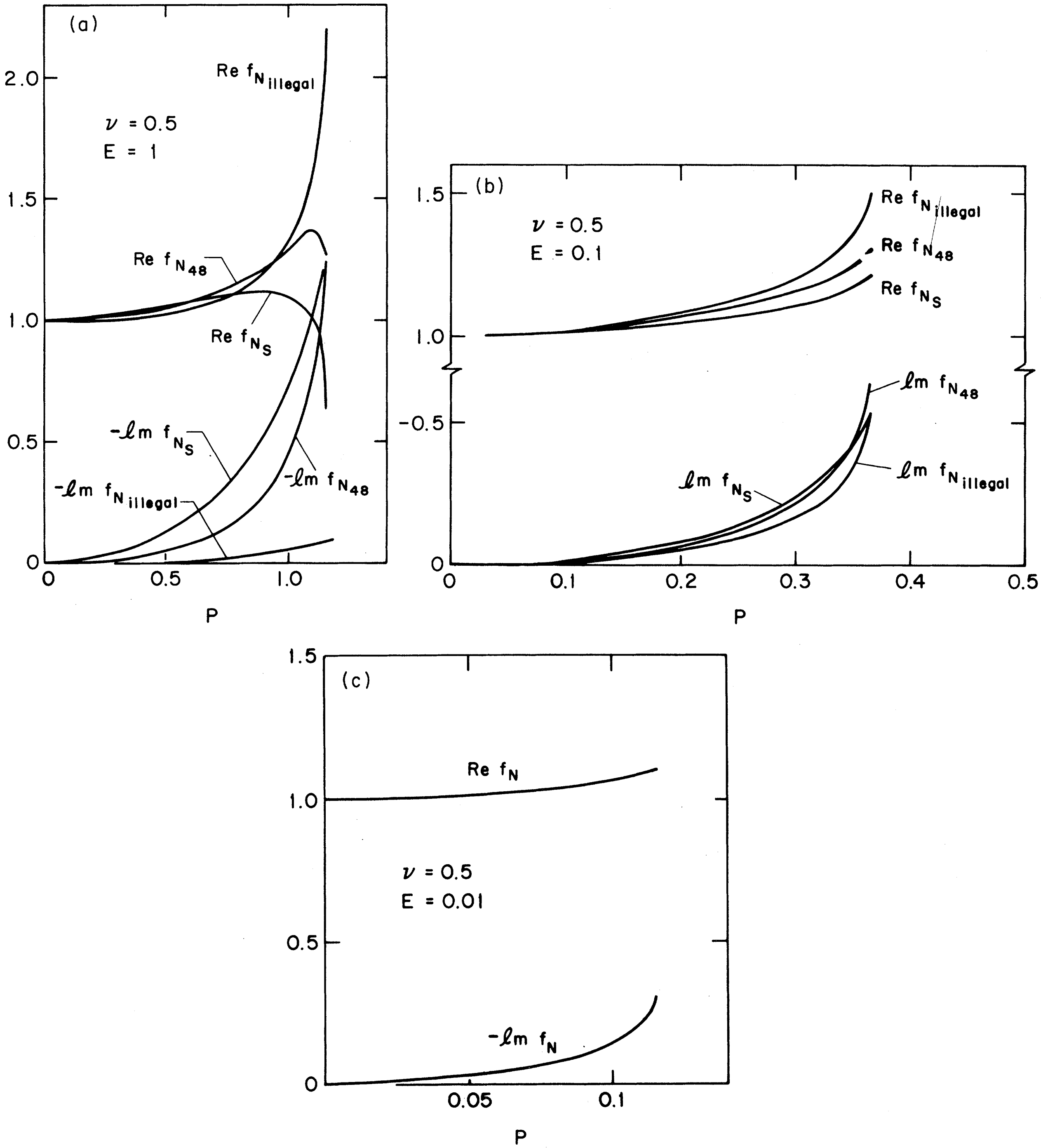

FIG. 4. Real and imaginary parts of the normalized quasi-two-body amplitudes f_n for the three solution methods for coupling strength $\nu = 0.5$ and energy (a) $E = 1$, (b) $E = 0.1$, and (c) $E = 0.01$ as a function of the odd particle momentum. For case $E = 0.01$ three cases agree to within the numerical precision of our calculation and therefore only one line is shown.

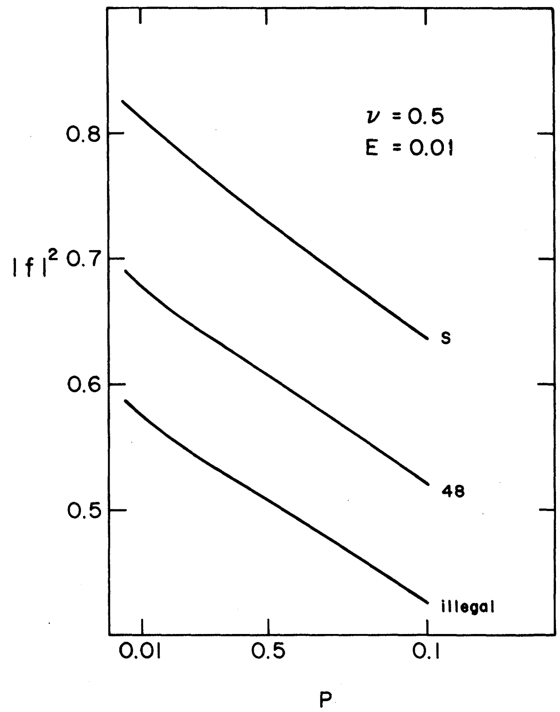

FIG. 5. Absolute square of the quasi-two-body amplitude f for the three solution methods for $\nu = 0.5$ and $E = 0.01$.

that the shape is well given while the magnitude is not. It should be noted that in getting the shape correctly the amplitudes are not simply getting the form of the singularity at $(E - \frac{3}{4}p^2)^{1/2} = 0$ correct. This can be done in Born approximations as we have shown,[5] but the shapes of $f(p)$ are agreeing throughout the physical range of p including parts of the p range that are not dominated by the singularity. As can be seen from Ref. 5, the various Born iterations that do get the shape of the singularity correctly do not get the shape for all p.

In summary, we have seen that at least in a simple s-wave model, the three-body equations derived from unitarity and analyticity do not reproduce the magnitude of the correct (Schrödinger) amplitudes, but they do reproduce the shape, particularly when the range of the force is short. We have seen that it is so even when we use a singular form of the equation which has no mathematical credential. We are presently investigating the question of why this works, as well as trying to determine which of the ingredients of the equations determines the shape. We are also exploring a numerical model involving resonant final state interactions and will report on these results in a subsequent note.

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[1]R. D. Amado, Phys. Rev. Lett. 33, 333 (1974).

[2]R. D. Amado, Phys. Rev. C 11, 719 (1975).

[3]R. D. Amado, Phys. Rev. 132, 485 (1963). See particularly Appendix II.

[4]We cannot use the method of Padé approximates since not even the integrals in the Neumann series exist.

[5]Cf., S. K. Adhikari and R. D. Amado, Phys. Rev. D 9, 1467 (1974).