Evolution equation for the energy-momentum moments of the nonequilibrium density function and regularized relativistic third-order hydrodynamics

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In this work, we first derive the evolution equation for the general energy-momentum moment of δf, where δf is the deviation from the local equilibrium phase-space density. We then introduce a relativistic extension of regularized hydrodynamics developed in the nonrelativistic case by Struchtrup and Torrilhon that judiciously mixes the method of moments and Chapman-Enskog expansion. Hydrodynamic equations up to the third-order in gradients are then systematically derived within the context of a single species system and the relaxation-time approximation. This is followed by a series of linear stability and causality analysis. For the massless particles without any charge conservation, the third-order hydrodynamics is shown to be linearly stable and causal.

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I. INTRODUCTION

The investigation of the hot and dense matter generated during ultrarelativistic heavy-ion collisions, commonly referred to as quark-gluon plasma (QGP), constitutes a prominent area of study within modern high-energy nuclear physics. One of the most challenging aspects of this study is the difficulty to obtain an analytic or numerical solution to a microscopic many-body QCD problem using first-principles calculations. What is accessible is the coarse-grained collective motion of the fluid-like system once approximate local thermal equilibrium is achieved [1]. Accordingly, relativistic viscous hydrodynamics is an indispensable theoretical tool for modeling the evolution of QGP in relativistic heavy-ion collisions.

The most intuitive and straightforward way of obtaining a relativistic viscous hydrodynamics theory is to extend the nonrelativistic Navier-Stokes theory to a relativistic one [2,3]. These theories are also commonly referred to as the "first-order theories," which only include terms up to first order in gradients. However, the Navier-Stokes theory is unstable and acausal when slightly perturbed around thermal equilibrium in linear regime [4–7], and it has been shown that this instability is in fact caused by the acausality of the theory [7–9]. For this reason, the original Navier-Stokes theory has been regarded as not suitable for relativistic hydrodynamics. However, recent work (usually referred to as the BDNK theory) [10–16] has shown that with some modification of the energy-momentum tensor, the first-order theory can be indeed made causal and stable. (See also Refs. [17,18] for relationship between BDNK and the second-order theories.)

The most well-known linearly stable and causal relativistic viscous hydrodynamics theory is the Müller-Isratel-Stewart (MIS) theory [19–22] that used the method of moments generalizing Grad's work on nonrelativistic hydrodynamics [23]. Unlike the first-order theories, the MIS theory contains terms that are up to second-order in gradients, thus it is also commonly referred to as the second-order theory. However, it has been shown that even the MIS theory is not always linearly stable and causal. Their transport coefficients must satisfy a set of constraints to be so [7–9,24,25]. Furthermore, the second-order theory is in fact, not unique. The original MIS paper derived the second-order theory by considering entropy production. More recent approaches start with the Boltzmann equation and derive hydrodynamic equations either using the Chapman-Enskog expansion [26–28], or the method of moments [29–31]. These approaches all give slightly different results depending on the truncation scheme. One goal of this work is to provide a framework where truncation scheme is dictated by the theory itself.

There have also been several recent works that derived the third-order hydrodynamics. One of the main motivation to obtain the third-order hydrodynamics is the fact that the third-order terms may significantly improve the agreement with the kinetic theory results when the value of the specific shear viscosity η/s is large [27,32,33]. In Refs. [32,34] positive entropy production rate argument was used to derive third-order hydrodynamic equations. A Chapman-Enskog approach to the third-order hydrodynamics was advocated in Refs. [33,35,36]. Naively, these approaches result in parabolic equations that may violate linear stability and causality as shown in Ref. [37] but causality may be restored by promoting gradients of viscous tensor to an independent variable [38] following the prescription from Ref. [39]. In contrast, the methods of moments was used to derive the third-order equations in Refs. [37,39] which were shown to be linearly stable and causal. In this work, we explore a method that combines a certain features of the method of moments and the Chapman-Enskog expansion. This will allow us to systematically derive relativistic viscous hydrodynamic equations up to the third order starting from the evolution equations of the energy-momentum moments.

This is accomplished by generalizing the nonrelativistic 13-moment regularized hydrodynamics (R13) developed by

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Struchtrup and Torrilhon [40–43], to the relativistic regularized hydrodynamics. In short, the regularization method combines both the method of moments and Chapman-Enskog expansion by applying a Chapman-Enskog-like expansion to the energy-momentum moments instead of the phase-space density function. Using this method, we derive the third-order hydrodynamic equations followed by a linear stability and causality analysis for the massless case with a similar procedure outlined in Ref. [37].

This paper is organized as follows: in Sec. II we introduce the conservation laws to mainly set the notations. In Sec. III, we present the derivation of the evolution equations for general energy-momentum moments of the phase-space density. The regularization method is also introduced in this section. In Sec. IV, we obtain the Chapman-Enskog-like expansion of the energy-momentum moments up to the fourth momentum rank to prepare for the derivation of the third-order hydrodynamics. In Sec. V we first briefly discuss the second-order equations obtained using regularization. Then, we proceed to the derivation of the third-order theory before discussing the special case of massless particles ($m = 0$) in Sec. VI. Section VII contains our linear analysis of the third-order hydrodynamics with $m = 0$. We demonstrate the linear stability and causality of the theory. Finally, we conclude this work in Sec. VIII. Appendixes A–E contains mathematical and computational details on the projectors, irreducible momentum polynomials, some derivative identities, details of the derivation of the general moment equation, and the integrals with the equilibrium density function.

Throughout this paper, we consider only one particle species. We use the natural units $c = \hbar = k_B = 1$, and adopt the mostly positive Minkowski metric $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. To convert tensorial quantities to the mostly negative metric, each subscripted (covariant) index is to be multiplied by -1 except the derivatives which work in the opposite way. In particular, for the Navier-Stokes tensor $\sigma_{\mu\nu}$ (which involves derivatives of the flow velocity), this means that $\sigma_{\mu\nu} \to -\sigma_{\mu\nu}$, $\sigma^{\mu\nu} \to -\sigma^{\mu\nu}$, but σ_μ^ν remains unchanged. The expansion rate defined as $\theta = \partial_\mu u^\mu$ (where u^μ is the local fluid velocity) and the local time derivative defined as $D = u^\mu \partial_\mu$ also remain the same.

II. CONSERVATION LAWS

The evolution equations of a hydrodynamics theory can be categorized into two parts: the conservation laws and the moment equations. The conservation laws are the continuity equations related to the energy-momentum conservation, and any other charge conservations. In this work, we only consider a single species system that does not possess any additional conserved charges (for instance, a real scalar $\lambda\phi^4$ theory) for the sake of simplicity. Hence, only the energy-momentum conservation is relevant:

$$\partial_\mu T^{\mu\nu} = 0, \tag{1}$$

where the energy-momentum tensor is further decomposed as

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}. \tag{2}$$

The fluid 4-velocity u^μ is defined by

$$T^{\mu\nu} u_\nu = -\varepsilon u^\mu, \tag{3}$$

where ε is the local energy density and the fluid 4-velocity u^μ is normalized to $u_\mu u^\mu = -1$. The thermal pressure at local equilibrium is subject to the equation of state, $P = P(\varepsilon)$, and Π is the bulk pressure. The local 3-metric, $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, is the projector that extracts the components of any 4-vector that is transverse to u^μ. The transverse, symmetric, and traceless rank-2 tensor $\pi^{\mu\nu}$ is the shear-stress tensor.

It is convenient to decompose Eq. (1) into the timelike and the spacelike components with respect to the fluid 4-velocity u^μ. Applying u_ν to $\partial_\mu T^{\mu\nu} = 0$ yields the timelike component

$$D\varepsilon + (\varepsilon + P + \Pi)\theta + \pi^{\alpha\beta}\sigma_{\alpha\beta} = 0. \tag{4}$$

Applying Δ^λ_ν to $\partial_\mu T^{\mu\nu} = 0$ yields the spacelike components

$$(\varepsilon + P + \Pi)Du^\lambda + \nabla^\lambda(P + \Pi) + \Delta^\lambda_\nu \partial_\mu \pi^{\mu\nu} = 0, \tag{5}$$

where we defined the relativistic substantial derivative (local time derivative) $D = u^\mu \partial_\mu$, the local spatial derivative $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$, the expansion rate $\theta = \partial_\mu u^\mu = \nabla_\mu u^\mu$, the Navier-Stokes tensor $\sigma^{\mu\nu} = \nabla^{\langle\mu} u^{\nu\rangle}$, and the fluid acceleration $Du^\lambda = a^\lambda$. The angular bracket around a set of indices represents the transverse (with respect to u^μ), symmetric, and traceless combination of the indices. In practice, this can be obtained by applying the projector:

$$A^{\langle\mu_1\cdots\mu_n\rangle} = \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} A^{\nu_1\cdots\nu_n}, \tag{6}$$

where $A^{\mu_1\cdots\mu_n}$ is an arbitrary rank-n tensor. Some useful facts about the projectors such as the explicit form for $n = 2, 3$, and recursive relationships can be found in Appendix A.

Equations (4) and (5) enforce the energy conservation and momentum conservation, respectively. Together, they constitute the evolution equations for ε and u^μ. However, at this point, the evolution equations for Π and $\pi^{\mu\nu}$ are not yet developed. In the following sections, we do so in the context of a single-species kinetic theory.

III. GENERAL METHODS

A. Energy-momentum moments

To obtain the evolution equations for the bulk pressure Π and the shear tensor $\pi^{\mu\nu}$, one can start with the kinetic theory equation

$$p^\mu \partial_\mu f = C[f], \tag{7}$$

where $f(x, p)$ is the phase-space density, and $C[f]$ is the collision integral. As stated, we consider a system with a single-particle species. This is also consistent with having no other conserved quantities. The energy-momentum tensor is defined as

$$T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 E_p} p^\mu p^\nu f, \tag{8}$$

with $E_p = p^0 = (\mathbf{p}^2 + m^2)^{1/2}$. This tensor satisfies the continuity equations $\partial_\mu T^{\mu\nu} = 0$ as long as the collisions conserve energy and momentum.

By further decomposing the phase-space density as the local equilibrium part and the correction

$$f(x,p)=f_0(x,p)+\delta f(x,p), \tag{9}$$

where $f_0(x,p)$ is the local equilibrium density, we can further define the ideal fluid part of the energy-momentum tensor

$$T_0^{\mu\nu}=\int\frac{d^3p}{(2\pi)^3E_p}p^\mu p^\nu f_0=\varepsilon u^\mu u^\nu+P\Delta^{\mu\nu}, \tag{10}$$

and the dissipative part

$$\delta T^{\mu\nu}=\int\frac{d^3p}{(2\pi)^3E_p}p^\mu p^\nu\delta f=\Pi\Delta^{\mu\nu}+\pi^{\mu\nu}. \tag{11}$$

The local energy density ε and the flow velocity u^μ are defined by the Landau matching condition

$$T^{\mu\nu}u_\nu=T_0^{\mu\nu}u_\nu=-\varepsilon u^\mu. \tag{12}$$

As one can see, various components of $T^{\mu\nu}$ are obtained as the energy-momentum moments of f_0 and δf. Accordingly, their evolution equations can be obtained from the kinetic theory equation Eq. (7). To obtain the evolution equations for Π and $\pi^{\mu\nu}$, it is convenient to define the energy-weighted rank-n tensor moment of δf as

$$\rho_r^{\mu_1\cdots\mu_n}=\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^r p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}, \tag{13}$$

where $\mathcal{E}_p=-u_\mu p^\mu$ is the energy of a particle in the rest frame of a fluid cell, and $p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}=\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}p^{\nu_1}p^{\nu_2}\cdots p^{\nu_n}$ is the symmetric and traceless combination of $p^{\langle\mu\rangle}=\Delta^\mu_\nu p^\mu$. Here, the integer n is the rank of the tensor, and $\mathcal{E}_p^r$ is the energy weight in which the integer exponent r indicates the energy order. In the fluid-cell rest frame, the local equilibrium density function f_0 takes the form of $f_0=\frac{1}{e^{\beta E_p}-\zeta}$, in which $\beta=1/T$ is the inverse temperature, and ζ could be 1 (Bose-Einstein statistics), 0 (Boltzmann statistics), or -1 (Fermi-Dirac statistics).

Using the decomposition $p^\mu=\mathcal{E}_pu^\mu+p^{\langle\mu\rangle}$, the Landau matching condition, Eq. (12), becomes the following two conditions on the moments

$$\rho_2=\rho_1^\mu=0. \tag{14}$$

In terms of the energy-momentum moments, the bulk pressure is given by

$$\Pi=-\frac{m^2}{3}\rho_0, \tag{15}$$

and the shear tensor is given by

$$\pi^{\mu\nu}=\rho_0^{\mu\nu}. \tag{16}$$

B. Derivation of the general moment equation

The evolution equation for $\rho_r^{\mu_1\cdots\mu_n}$ can be obtained by applying the local time derivative $D=u^\mu\partial_\mu$ to $\rho_r^{\mu_1\cdots\mu_n}$ and then using the kinetic equation Eq. (7) with $f=f_0+\delta f$. In this section, we outline the derivation of the evolution equation for the general energy-momentum moment $\rho_r^{\mu_1\cdots\mu_n}$. Full derivation can be found in Appendix D.

Applying the local time derivative to $\rho_r^{\mu_1\cdots\mu_n}$ in Eq. (13), and then projecting onto the transverse space, we get

$$\begin{aligned}\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}&D\rho_r^{\nu_1\cdots\nu_n}\\&=\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}(D\delta f)\mathcal{E}_p^r p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}\\&\quad-n\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r+1}p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle}-r\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}a_\sigma\\&\quad\times\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\sigma\rangle}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle},\end{aligned} \tag{17}$$

where we defined the fluid acceleration $a^\mu=Du^\mu$, and used the fact that $u_\mu Du^\mu=0$ so that $D\mathcal{E}_p=-a_\sigma p^\sigma=-a_\sigma p^{\langle\sigma\rangle}$, and also

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}Dp^{\langle\nu_1\cdots}p^{\nu_n\rangle}=-n\mathcal{E}_pp^{\langle\mu_1\cdots}p^{\mu_{n-1}}a^{\mu_n\rangle}, \tag{18}$$

which is derived in Appendix C. Using the identity

$$\begin{aligned}p^{\langle\lambda\rangle}p^{\langle\mu_1}\cdots p^{\mu_n\rangle}=p^{\langle\lambda}p^{\mu_1}\cdots p^{\mu_n\rangle}+\frac{n}{2n+1}\left(\mathcal{E}_p^2-m^2\right)\\\times p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_{n-1}}\Delta^{\mu_n\rangle\lambda}\end{aligned} \tag{19}$$

proven in Appendix B, we can expand the last term on the right-hand side to get

$$\begin{aligned}\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}&D\rho_r^{\nu_1\cdots\nu_n}\\&=\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}(D\delta f)\mathcal{E}_p^r p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}\\&\quad-n\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r+1}p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle}\\&\quad-ra_\sigma\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\sigma}p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}\\&\quad-r\frac{n}{2n+1}a_\sigma\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\left(\mathcal{E}_p^2-m^2\right)\\&\quad\times p^{\langle\mu_1}p^{\mu_2}\cdots\Delta^{\mu_n\rangle\sigma}.\end{aligned} \tag{20}$$

For $D\delta f$, we can use the following form of the Boltzmann equation

$$p^\mu\partial_\mu f_0+\mathcal{E}_pD\delta f+p^{\langle\mu\rangle}\nabla_\mu\delta f=C[f], \tag{21}$$

where $C[f]$ is the collision term of the relativistic Boltzmann equation, and we used

$$p^\mu\partial_\mu=\mathcal{E}_pD+p^{\langle\mu\rangle}\nabla_\mu. \tag{22}$$

This gives

$$\begin{aligned}
\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} D\rho_r^{\nu_1\cdots\nu_n} = &-n\int \frac{d^3p}{(2\pi)^3E_p}\delta f \mathcal{E}_p^{r+1} p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} - r a_\sigma \int \frac{d^3p}{(2\pi)^3E_p}\delta f \mathcal{E}_p^{r-1} p^{\langle\sigma}p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}\\
&- r\frac{n}{2n+1}\int \frac{d^3p}{(2\pi)^3E_p}\delta f \mathcal{E}_p^{r-1}\left(\mathcal{E}_p^2 - m^2\right) p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} + \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p} C[f]\mathcal{E}_p^{r-1} p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}\\
&- \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p}(\partial_\lambda f_0)\mathcal{E}_p^{r-1} p^\lambda p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} - \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p}(\nabla_\lambda \delta f)\mathcal{E}_p^{r-1} p^{\langle\lambda\rangle} p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}.
\end{aligned} \tag{23}$$

The first three lines of Eq. (23) can be expressed in terms of the energy-momentum moments. The term with the collision integral is in general a nonlinear functional of δf that will not admit a simple expression. In the rest of this work, we use the relaxation-time approximation so that this term *can* be expressed in terms of the energy-momentum moments. The line involving the equilibrium density f_0 will not result in the energy-momentum moments. Instead, it gives the constitutive relationships. The rest of the derivation is then to deal with the last line. Details of transferring ∇_λ from δf to the other factors can be found in Appendix D. The final result is

$$\begin{aligned}
\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} D\rho_r^{\nu_1\cdots\nu_n} = &\int \frac{d^3p}{(2\pi)^3E_p} C[f]\mathcal{E}_p^{r-1} p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} - \int \frac{d^3p}{(2\pi)^3E_p}(\partial_\lambda f_0)\mathcal{E}_p^{r-1} p^\lambda p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}\\
&- \frac{n(2n+r+1)}{2n+1}\rho_{r+1}^{\langle\mu_1\cdots\mu_{n-1}}a^{\mu_n\rangle} + rm^2\frac{n}{2n+1}\rho_{r-1}^{\langle\mu_1\cdots\mu_{n-1}}a^{\mu_n\rangle} - r a_\lambda \rho_{r-1}^{\lambda\mu_1\cdots\mu_n} - \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda \rho_{r-1}^{\lambda\nu_1\cdots\nu_n}\\
&- \frac{n}{2n+1}\nabla^{\langle\mu_1}\rho_{r+1}^{\mu_2\cdots\mu_n\rangle} + m^2\frac{n}{2n+1}\nabla^{\langle\mu_1}\rho_{r-1}^{\mu_2\cdots\mu_n\rangle} - \frac{n+r+2}{3}\theta\rho_r^{\mu_1\cdots\mu_n} - (r-1)\sigma_{\lambda\alpha}\rho_{r-2}^{\alpha\lambda\mu_1\cdots\mu_n}\\
&+ \frac{(r-1)m^2}{3}\theta\rho_{r-2}^{\mu_1\cdots\mu_n} - \frac{n(2n+2r+1)}{2n+3}\rho_r^{\lambda\langle\mu_1\cdots\mu_{n-1}}\sigma_\lambda^{\mu_n\rangle} - n\rho_r^{\lambda\langle\mu_1\cdots\mu_{n-1}}\omega_\lambda^{\mu_n\rangle}\\
&- \frac{(2n+r)(n-1)n}{(2n-1)(2n+1)}\rho_{r+2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle} + 2m^2\frac{(r-1)n}{2n+3}\rho_{r-2}^{\lambda\langle\mu_1\cdots\mu_{n-1}}\sigma_\lambda^{\mu_n\rangle}\\
&- m^4\frac{(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_{r-2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle} + m^2\frac{(2n+2r-1)(n-1)n}{(2n+1)(2n-1)}\rho_r^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}.
\end{aligned} \tag{24}$$

Here, $\omega^{\mu\nu} = \frac{1}{2}(\nabla^\mu u^\nu - \nabla^\nu u^\mu)$ is the antisymmetric vorticity tensor. For $n = 0, 1, 2, 3, 4$, Eq. (24) agrees with the results obtained by Denicol and others [31,39] as they should. This general evolution equation was first derived by one of the authors in Ref. [44]. As far as we know, this was the first time the evolution equation for a general energy-momentum moment was explicitly derived in literature. This equation also appeared in a recent paper [45]. Even though we eventually use Boltzmann statistics, Eq. (24) is valid for quantum statistics as well.

C. Regularization methods

As one can see in Eq. (24) the time evolution of $\rho_r^{\mu_1\cdots\mu_n}$ involves $\rho_r^{\mu_1\cdots\mu_n}$, $\rho_{r-2}^{\mu_1\cdots\mu_n}$, $\rho_{r\pm1}^{\mu_1\cdots\mu_{n-1}}$, $\rho_{r\pm2}^{\mu_1\cdots\mu_{n-2}}$, $\rho_r^{\mu_1\cdots\mu_{n-2}}$, $\rho_{r-1}^{\mu_1\cdots\mu_{n+1}}$, and $\rho_{r-2}^{\mu_1\cdots\mu_{n+2}}$. As such, Eq. (24) represents an infinite set of coupled partial differential equations. To get a closed set of equations for a finite number of moments, one must use a truncation scheme. The two well-known truncation schemes are the method of moments [20–23], and the Chapman-Enskog method [26]. In the method of moments, one assumes that δf is such that all nth rank moments are proportional to each other regardless of their energy weights [39]. On the other hand, the Chapman-Enskog method expands δf using the Boltzmann equation as the recursion equation to obtain δf as a derivative expansion.

In a series of papers [40–43], Struchtrup and Torrilhon developed a novel method they named the "regularized hydrodynamics" that combines both the method of moments and the Chapman-Enskog expansion. This technique applies a Chapman-Enskog-like expansion directly to the energy-momentum moments instead of δf, excluding the moments that serve as the dynamic hydrodynamic variables. This technique provides a more systematic way to produce a set of equations to any given order in the expansion parameter ϵ without introducing any additional assumptions.

In the usual Chapman-Enskog method, the collision term is scaled as $C[f] \to (1/\epsilon)C[f]$ and the nonequilibrium part of the phase-space density is expanded as

$$\delta f = \sum_{n=1}^{\infty} \epsilon^n \delta f_{|n}. \tag{25}$$

Here and here after, the vertical bar in the subscript indicates the relevant ϵ order. These are then plugged into the Boltzmann equation. Collecting terms having the same power of ϵ, the nth order piece $\delta f_{|n}$ can be found iteratively involving a maximum of n spatial derivatives of β and u^μ. The resulting equations are at best parabolic and hence potentially acausal.

This can lead to instability unless additional evolution equations for Π, $\pi^{\mu\nu}$ and other dissipative currents are postulated using the constitutive relationships [4–9,38].

In the method of Struchtrup and Torrilhon, instead of δf, the energy-momentum moments of δf are expanded in powers of ϵ

$$\rho_r^{\mu_1\cdots\mu_n} = \sum_{n=1}^{\infty} \epsilon^n \rho_{r|n}^{\mu_1\cdots\mu_n}. \tag{26}$$

Working out the order-by-order solution by putting Eq. (26) in Eq. (24) would be completely equivalent to the usual Chapman-Enskog method. What we would like to do differently, however, is *not* to expand the hydrodynamic variables, such as Π and $\pi^{\mu\nu}$, whenever they occur while expanding all other moments in terms of them. However, at higher orders of ϵ, there is no guarantee that Π and $\pi^{\mu\nu}$ [which are $O(\epsilon)$] are the only relevant dynamic variables. As we see below, we may need to promote some higher moments to be dynamic to get a closed set of equations.

IV. CHAPMAN-ENSKOG EXPANSION OF THE MOMENTS

In this section, we work out the ϵ expansion of the energy-momentum moments up to $n = 4$ within the relaxation-time approximation. The results from this section will be used in the later sections to build hydrodynamic equations.

To determine the ϵ order of each $\rho_r^{\mu_1\cdots\mu_n}$ explicitly, we consider the relaxation-time approximation for the collision term

$$C[f] = -\frac{\mathcal{E}_p}{\epsilon\tau_R}\delta f(x, p), \tag{27}$$

where we have explicitly indicated the expansion parameter ϵ. The relaxation time τ_R is assumed to be a constant. The parameter ϵ is set to one at the end of calculations. Putting Eqs. (26) and (27) into the general moment equation Eq. (24) and collecting the $O(\epsilon^0)$ terms, we get the first-order coefficient function

$$\rho_{r|1}^{\mu_1\cdots\mu_n} = -\tau_R F_{r-1|0}^{\mu_1\cdots\mu_n}, \tag{28}$$

where we defined the equilibrium density term to be

$$F_r^{\mu_1\cdots\mu_n} = \int \frac{d^3p}{(2\pi)^3 E_p}\mathcal{E}_p^r p^{\langle\mu_1}\cdots p^{\mu_{n-1}}p^{\mu_n\rangle}p^\lambda\partial_\lambda f_0. \tag{29}$$

Here, $\rho_{r|1}^{\mu_1\cdots\mu_n}$ is the $O(\epsilon)$ part of $\rho_r^{\mu_1\cdots\mu_n}$ and $F_{r-1|0}^{\mu_1\cdots\mu_n}$ is the $O(\epsilon^0)$ part of $F_{r-1}^{\mu_1\cdots\mu_n}$. Using Eq. (22), one can show that $p^\lambda\partial_\lambda f_0 = -f_0(1+\zeta f_0)p^\lambda\partial_\lambda(\mathcal{E}_p\beta)$ can contain only 1, $p^{\langle\mu_1\rangle}$, $p^{\langle\mu_1}p^{\mu_2\rangle}$. Hence the orthogonality of the irreducible polynomials $p^{\langle\mu_1}\cdots p^{\mu_n\rangle}$ [cf. Eq. (B6) in Appendix B and also Ref. [30]] demands that

$$F_r^{\mu_1\cdots\mu_n} = 0 \quad \text{for } n \geqslant 3. \tag{30}$$

For $n = 0, 1, 2$, we get

$$F_r = \phi_{r|0}\theta + \phi_{r|1}^{\pi\Pi}(\pi^{\gamma\rho}\sigma_{\gamma\rho} + \theta\Pi), \tag{31}$$

$$F_r^\mu = \psi_{r|1}\left(\Delta_\gamma^\mu\partial_\rho\pi^{\rho\gamma} + \nabla^\mu\Pi + a^\mu\Pi\right), \tag{32}$$

$$F_r^{\mu\nu} = \varphi_{r|0}\sigma^{\mu\nu}, \tag{33}$$

where the coefficient functions ϕ, ψ, and φ are functions of β only. Derivations can be found in Appendix E. Observe that F_r, F_r^μ, and $F_r^{\mu\nu}$ all involve gradients and time derivatives of the hydrodynamic variables. Consequently, they can be described as physical thermodynamic forces that are driving the evolution of the system. In deriving the above expressions, we have used Eq. (4) to express $D\beta$ in terms of spatial derivatives. The acceleration $a^\mu = Du^\mu$ can also be expressed in terms of spatial derivatives using Eq. (5) but we leave it as it is for brevity. Details can be found in Appendix E.

From Eqs. (28) and (30), it follows immediately that $\rho_{r|1}^{\mu_1\cdots\mu_n} = 0$ for $n \geqslant 3$. One should also note that $\rho_{r|1}^\mu = 0$ because there is no number (mass) conservation. Hence

$$\rho_r,\ \rho_r^{\mu_1\mu_2} = O(\epsilon), \tag{34}$$

$$\rho_r^{\mu_1\cdots\mu_n} = O(\epsilon^2) \ \text{ for } n = 1 \text{ and } n \geqslant 3. \tag{35}$$

In fact, only $n = 1, 3, 4$ moments are $O(\epsilon^2)$. To see this, note that in Eq. (24), the lowest momentum order on the right-hand side is $n-2$. Hence, for $n = 5, 6$, the lowest momentum order appearing on the right-hand side is $n = 3$ and $n = 4$, respectively. This implies that the right-hand sides for $n = 5, 6$ are at most $O(\epsilon^2)$, which further implies that $\rho_{r|2}^{\mu_1\cdots\mu_n}/(\epsilon\tau_R) = 0$ for $n = 5, 6$ since there are no $O(\epsilon)$ terms in the right-hand side of Eq. (24). Equivalently,

$$\rho_r^{\mu_1\cdots\mu_n} = O(\epsilon^3) \ \text{ for } \ n = 5, 6. \tag{36}$$

Continuing this way, it can be established that, in general,

$$\rho_r^{\mu_1\cdots\mu_n} = O(\epsilon^{\lceil n/2\rceil}) \ \text{ for } \ n \geqslant 3, \tag{37}$$

where $\lceil n/2\rceil$ is the closest integer that is larger than or equal to $n/2$.

The second-order hydrodynamics theory is based on energy density ε, fluid flow velocity u^μ, shear stress tensor $\pi^{\mu\nu}$, and bulk viscous pressure Π. From Eq. (34) one can see that Π and $\pi^{\mu\nu}$ are $O(\epsilon)$. Therefore, in this method, the second-order theory includes the $O(\epsilon^0)$ terms and the $O(\epsilon)$ terms. To obtain the third-order theory, we need to include the $O(\epsilon^2)$ terms.

Since we have now established the ϵ order of the energy-momentum moments, we do not have to carry ϵ around from here on although we keep referring to the ϵ order of specific terms. For the relaxation-time approximation, the ϵ order is the same as the number of τ_R factors.

As stated, the goal of this section is to work out the ϵ expansion of the energy-momentum moments up to $n = 4$. We start with the scalar moments. The general equation of motion for an arbitrary scalar moment ($n = 0$) is

$$D\rho_r = -\frac{\rho_r}{\tau_R} - F_{r-1} + \frac{1}{3}[(r-1)m^2\rho_{r-2} - (2+r)\rho_r]\theta - \nabla_\lambda\rho_{r-1}^\lambda - ra_\lambda\rho_{r-1}^\lambda - (r-1)\sigma_{\lambda\alpha}\rho_{r-2}^{\alpha\lambda}. \tag{38}$$

Collecting the $O(\epsilon^0)$ terms, we get

$$\rho_{r|1} = -\tau_R F_{r-1|0} = -\tau_R\phi_{r-1|0}\theta. \tag{39}$$

The scalar moment up to and including $O(\epsilon^2)$ terms are then

$$\rho_r = \tau_R\left\{-F_{r-1} - D\rho_{r|1} + \frac{1}{3}\left[(r-1)m^2\rho_{r-2|1} - (2+r)\rho_{r|1}\right]\theta - (r-1)\sigma_{\lambda\alpha}\rho^{\alpha\lambda}_{r-2|1}\right\} + O(\epsilon^3), \tag{40}$$

where we used the facts that $\tau_R = O(\epsilon)$, $\rho^{\lambda}_{r-1} = O(\epsilon^2)$, and F_{r-1} contains both the $O(\epsilon^0)$ terms and $O(\epsilon)$ terms. The time derivative term is

$$\begin{aligned} D\rho_{r|1} &= D(\tau_R\phi_{r-1|0}\theta) \\ &= \tau_R\left(\frac{\partial\phi_{r-1|0}}{\partial\beta}\right)\chi_{\beta|0}\theta^2 + \tau_R\phi_{r-1|0}D\theta + O(\epsilon^2), \end{aligned} \tag{41}$$

where $\chi_{\beta|0}$ is defined in Appendix E. To keep the theory from becoming parabolic, the right-hand side of Eq. (40) should not contain any derivatives of thermodynamic variables upon using suitable constitutive relationships. To deal with $D\theta = D\partial_\mu u^\mu$, which contains second derivatives, we can use

$$\begin{aligned} \rho_0 &= -\frac{3}{m^2}\Pi \\ &= \tau_R\left[-F_{-1} - \tau_R\left(\frac{\partial\phi_{-1|0}}{\partial\beta}\right)\chi_{\beta|0}\theta^2 - \tau_R\phi_{-1|0}D\theta - \frac{1}{3}(m^2\rho_{-2|1} + 2\theta\rho_{0|1})\theta + \sigma_{\lambda\alpha}\rho^{\alpha\lambda}_{-2|1}\right] + O(\epsilon^3), \end{aligned} \tag{42}$$

which will be used only in the context of obtaining the ϵ expansion of other moments. Replacing $D\theta$ in Eq. (41) with $D\theta$ in Eq. (42), we get

$$\begin{aligned} \rho_r = {}& -\frac{3}{m^2}\Phi_r\Pi + \tau_R\left[-(F_{r-1|1} - \Phi_r F_{-1|1}) - \tau_R\left(\frac{\partial\phi_{r-1|0}}{\partial\beta} - \Phi_r\frac{\partial\phi_{-1}}{\partial\beta}\right)\chi_{\beta|0}\theta^2\right. \\ & \left. -\left(\frac{2+r}{3}\rho_{r|1} - \frac{2\Phi_r}{3}\rho_{0|1}\right)\theta + \frac{m^2}{3}[(r-1)\rho_{r-2|1} + \Phi_r\rho_{-2|1}]\theta - \sigma_{\lambda\alpha}\left[(r-1)\rho^{\alpha\lambda}_{r-2|1} + \Phi_r\rho^{\alpha\lambda}_{-2|1}\right]\right] + O(\epsilon^3), \end{aligned} \tag{43}$$

where $\Phi_r = \phi_{r-1|0}/\phi_{-1|0}$. Using the first-order constitutive relationships

$$\Pi = \frac{m^2}{3}\tau_R\phi_{-1|0}\theta + O(\epsilon^2), \tag{44}$$

$$\pi^{\mu\nu} = -\tau_R\varphi_{-1|0}\sigma^{\mu\nu} + O(\epsilon^2), \tag{45}$$

$$\rho_{r|1} = -\frac{3}{m^2}\Phi_r\Pi, \tag{46}$$

ρ_r can then be expressed solely in terms of Π and $\pi^{\mu\nu}$ without involving any derivatives or an explicit factor of τ_R.

From Eq. (24), the evolution equation for the general rank-2 moment can obtained as

$$\begin{aligned} \Delta^{\mu_1\mu_2}_{\nu_1\nu_2}D\rho^{\nu_1\nu_2}_r = {}& -\frac{\rho^{\mu_1\mu_2}_r}{\tau_R} - F^{\mu_1\mu_2}_{r-1} + \frac{2}{15}[-(4+r)\rho_{r+2} + m^2(2r+3)\rho_r - m^4(r-1)\rho_{r-2}]\sigma^{\mu_1\mu_2} - ra_\alpha\rho^{\alpha\mu_1\mu_2}_{r-1} \\ & + \frac{2}{5}\left(rm^2\rho^{\langle\mu_1}_{r-1}a^{\mu_2\rangle} - (r+5)\rho^{\langle\mu_1}_{r+1}a^{\mu_2\rangle}\right) - \frac{2}{5}\left(\nabla^{\langle\mu_1}\rho^{\mu_2\rangle}_{r+1} - m^2\nabla^{\langle\mu_1}\rho^{\mu_2\rangle}_{r-1}\right) - 2\omega^{\langle\mu_1}_{\lambda}\rho^{\mu_2\rangle\lambda}_r - (r-1)\sigma_{\lambda\alpha}\rho^{\alpha\lambda\mu_1\mu_2}_{r-2} \\ & + \frac{2}{7}\left[-(2r+5)\sigma^{\langle\mu_1}_{\lambda}\rho^{\mu_2\rangle\lambda}_r + 2(r-1)m^2\sigma^{\langle\mu_1}_{\lambda}\rho^{\mu_2\rangle\lambda}_{r-2}\right] + \frac{1}{3}\left[m^2(r-1)\rho^{\mu_1\mu_2}_{r-2} - (4+r)\rho^{\mu_1\mu_2}_r\right]\theta - \Delta^{\mu_1\mu_2}_{\nu_1\nu_2}\nabla_\lambda\rho^{\lambda\nu_1\nu_2}_{r-1}, \end{aligned} \tag{47}$$

where $F^{\mu\nu}_r = F^{\mu\nu}_{r|0} = \varphi_{r|0}\sigma^{\mu\nu}$. Following the similar procedure as in the scalar case, we obtain

$$\begin{aligned} \rho^{\mu_1\mu_2}_r = {}& \Sigma_r\rho^{\mu_1\mu_2}_0 + \tau_R\left[-\frac{\theta}{3}\left(r\rho^{\mu_1\mu_2}_{r|1} - (r-1)m^2\rho^{\mu_1\mu_2}_{r-2|1} - \Sigma_r m^2\rho^{\mu_1\mu_2}_{-2|1}\right)\right. \\ & + \frac{2}{7}\left(-2r\sigma^{\langle\mu_2}_{\lambda}\rho^{\mu_1\rangle\lambda}_{r|1} + (2r-2)m^2\sigma^{\langle\mu_2}_{\lambda}\rho^{\mu_1\rangle\lambda}_{r-2|1} + 2m^2\Sigma_r\sigma^{\langle\mu_2}_{\lambda}\rho^{\mu_1\rangle\lambda}_{-2|1}\right) \\ & + \frac{2}{15}\sigma^{\mu_1\mu_2}\left[-(4+r)\rho_{r+2|1} + (2r+3)m^2\rho_{r|1} - (r-1)m^4\rho_{r-2|1}\right] \\ & \left. -\Sigma_r\frac{2}{15}\sigma^{\mu_1\mu_2}(-4\rho_{2|1} + 3m^2\rho_{0|1} + m^4\rho_{-2|1}) + \tau_R\left(\frac{\partial(\varphi_{r-1|0})}{\partial\beta} - \Sigma_r\frac{\partial(\varphi_{-1|0})}{\partial\beta}\right)\chi_{\beta|0}\theta\sigma^{\mu_1\mu_2}\right] + O(\epsilon^3), \end{aligned} \tag{48}$$

where $\Sigma_r = \varphi_{r-1|0}/\varphi_{-1|0}$ and we used the ϵ expansion of $\pi^{\mu\nu} = \rho_0^{\mu\nu}$ to replace $\Delta^{\mu_1\mu_2}_{\nu_1\nu_2} D\sigma^{\nu_1\nu_2}$. Upon using Eqs. (44), (45), and (46), $\rho_r^{\mu_1\mu_2}$ can be reexpressed solely in terms of $\pi^{\mu\nu}$ and Π without their derivatives or an explicit factor of τ_R.

For the $O(\epsilon^2)$ moments, we start with the vector moments whose evolution equation is given by

$$\begin{aligned}\Delta^{\mu_1}_{\nu_1} D\rho_r^{\nu_1} =& -\frac{\rho_r^{\mu_1}}{\tau_R} - F_{r-1}^{\mu_1} + \frac{1}{3}\left[(r-1)m^2\rho_{r-2}^{\mu_1} - (3+r)\rho_r^{\mu_1}\right]\theta - r a_\alpha \rho_{r-1}^{\alpha\mu_1} - \Delta^{\mu_1}_{\nu_1}\nabla_\lambda \rho_{r-1}^{\lambda\nu_1} - \omega^{\mu_1}_{\lambda}\rho_r^{\lambda}\\ &- (r-1)\sigma_{\lambda\alpha}\rho_{r-2}^{\alpha\lambda\mu_1} + \frac{1}{3}(rm^2\rho_{r-1} - (r+3)\rho_{r+1})a^{\mu_1} - \frac{1}{3}(\nabla^{\mu_1}\rho_{r+1} - m^2\nabla^{\mu_1}\rho_{r-1})\\ &+ \frac{1}{5}\left[-(2r+3)\rho_r^{\lambda} + 2(r-1)m^2\rho_{r-2}^{\lambda}\right]\sigma^{\mu_1}_{\lambda}. \end{aligned} \tag{49}$$

Since $\rho_r^{\mu} = O(\epsilon^2)$, the $O(\epsilon)$ terms on the right-hand-side must add up to zero, yielding

$$\begin{aligned}\rho_r^{\mu_1} =& -\tau_R \psi_{r-1|1}\left(\Delta^{\mu}_{\gamma}\partial_\rho \pi^{\rho\gamma} + \nabla^{\mu_1}\Pi + a^{\mu_1}\Pi\right) + \tau_R\Bigg[-\Delta^{\mu_1}_{\nu_1}\nabla_\lambda\rho_{r-1|1}^{\lambda\nu_1} - r a_{|0\alpha}\rho_{r-1|1}^{\alpha\mu_1}\\ &- \frac{1}{3}(\nabla^{\mu_1}\rho_{r+1|1} - m^2\nabla^{\mu_1}\rho_{r-1|1}) + \frac{1}{3}[rm^2\rho_{r-1|1} - (r+3)\rho_{r+1|1}]a^{\mu_1}_{|0}\Bigg] + O(\epsilon^3). \end{aligned} \tag{50}$$

Further details can be found in Appendix E. Unlike the $O(\epsilon)$ moments, this cannot be expressed solely in terms of Π and $\pi^{\mu\nu}$ without involving derivatives.

For the rank-3 moments, we have

$$\begin{aligned}\Delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3} D\rho_r^{\nu_1\nu_2\nu_3} =& -\frac{\rho_r^{\mu_1\mu_2\mu_3}}{\tau_R} + \frac{1}{3}\left[-(5+r)\rho_r^{\mu_1\mu_2\mu_3} + (r-1)m^2\rho_{r-2}^{\mu_1\mu_2\mu_3}\right]\theta\\ &+ \frac{6}{35}\left[-(6+r)\rho_{r+2}^{\langle\mu_1}\sigma^{\mu_2\mu_3\rangle} + (2r+5)m^2\rho_r^{\langle\mu_1}\sigma^{\mu_2\mu_3\rangle} - (r-1)m^4\rho_{r-2}^{\langle\mu_1}\sigma^{\mu_2\mu_3\rangle}\right] - 3\omega^{\langle\mu_1}_{\lambda}\rho_r^{\mu_2\mu_3\rangle\lambda}\\ &+ \frac{1}{3}\left[-(2r+7)\sigma^{\langle\mu_1}_{\lambda}\rho_r^{\mu_2\mu_3\rangle\lambda} + 2(r-1)m^2\sigma^{\langle\mu_1}_{\lambda}\rho_{r-2}^{\mu_2\mu_3\rangle\lambda}\right] - r a_\alpha\rho_{r-1}^{\alpha\mu_1\mu_2\mu_3}\\ &- \frac{3}{7}\left(\nabla^{\langle\mu_1}\rho_{r+1}^{\mu_2\mu_3\rangle} - m^2\nabla^{\langle\mu_1}\rho_{r-1}^{\mu_2\mu_3\rangle}\right) + \frac{3}{7}\left[rm^2\rho_{r-1}^{\langle\mu_1\mu_2}a^{\mu_3\rangle} - (r+7)\rho_{r+1}^{\langle\mu_1\mu_2}a^{\mu_3\rangle}\right]\\ &- \Delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\nabla_\lambda\rho_{r-1}^{\lambda\nu_1\nu_2\nu_3} - (r-1)\sigma_{\lambda\alpha}\rho_{r-2}^{\alpha\lambda\mu_1\mu_2\mu_3}. \end{aligned} \tag{51}$$

As before, the $O(\epsilon)$ terms on the right-hand side must add up to zero, yielding

$$\rho_r^{\mu_1\mu_2\mu_3} = -\frac{3\tau_R}{7}\left[\nabla^{\langle\mu_1}\rho_{r+1|1}^{\mu_2\mu_3\rangle} + (r+7)\rho_{r+1|1}^{\langle\mu_1\mu_2}a^{\mu_3\rangle} - m^2\nabla^{\langle\mu_1}\rho_{r-1|1}^{\mu_2\mu_3\rangle} - rm^2\rho_{r-1|1}^{\langle\mu_1\mu_2}a^{\mu_3\rangle}\right] + O(\epsilon^3). \tag{52}$$

Again, this cannot be expressed solely in terms of Π and $\pi^{\mu\nu}$ without any derivatives. One may take this as the first sign that the rank-1 and rank-3 moments need to be promoted to dynamic variables, as we do below.

For the rank-4 moments, we have

$$\begin{aligned}&\Delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4} D\rho_r^{\nu_1\nu_2\nu_3\nu_4}\\ &= -\frac{\rho_r^{\mu_1\mu_2\mu_3\mu_4}}{\tau_R} - r a_\alpha\rho_{r-1}^{\alpha\mu_1\mu_2\mu_3\mu_4} - \frac{4}{9}\left[(r+9)\rho_{r+1}^{\langle\mu_1\mu_2\mu_3}a^{\mu_4\rangle} - rm^2\rho_{r-1}^{\langle\mu_1\mu_2\mu_3}a^{\mu_4\rangle}\right] - \frac{4}{9}\left(\nabla^{\langle\mu_1}\rho_{r+1}^{\mu_2\mu_3\mu_4\rangle} - m^2\nabla^{\langle\mu_1}\rho_{r-1}^{\mu_2\mu_3\mu_4\rangle}\right)\\ &\quad - \Delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4}\nabla_\lambda\rho_{r-1}^{\lambda\nu_1\nu_2\nu_3\nu_4} + \frac{4}{21}\left[-(8+r)\rho_{r+2}^{\langle\mu_1\mu_2}\sigma^{\mu_3\mu_4\rangle} + (2r+7)m^2\rho_r^{\langle\mu_1\mu_2}\sigma^{\mu_3\mu_4\rangle} - (r-1)m^4\rho_{r-2}^{\langle\mu_1\mu_2}\sigma^{\mu_3\mu_4\rangle}\right]\\ &\quad - 4\omega^{\langle\mu_1}_{\lambda}\rho_r^{\mu_2\mu_3\mu_4\rangle\lambda} - (r-1)\sigma_{\lambda\alpha}\rho_{r-2}^{\alpha\lambda\mu_1\mu_2\mu_3\mu_4} + \frac{4}{11}\left[-(2r+9)\sigma^{\langle\mu_1}_{\lambda}\rho_r^{\mu_2\mu_3\mu_4\rangle\lambda} + 2(r-1)m^2\sigma^{\langle\mu_1}_{\lambda}\rho_{r-2}^{\mu_2\mu_3\mu_4\rangle\lambda}\right]\\ &\quad + \frac{1}{3}\left[(r-1)m^2\rho_{r-2}^{\mu_1\mu_2\mu_3\mu_4} - (6+r)\rho_r^{\mu_1\mu_2\mu_3\mu_4}\right]\theta. \end{aligned} \tag{53}$$

Collecting all $O(\epsilon)$ terms on the right-hand side, the rank-4 moments up to $O(\epsilon^2)$ are given by

$$\rho_r^{\mu_1\mu_2\mu_3\mu_4} = \tau_R\left[-(8+r)\tfrac{4}{21}\rho_{r+2|1}^{\langle\mu_1\mu_2}\sigma^{\mu_3\mu_4\rangle} + (7+2r)\tfrac{4}{21}m^2\rho_{r|1}^{\langle\mu_1\mu_2}\sigma^{\mu_3\mu_4\rangle} - (r-1)\tfrac{4}{21}m^4\rho_{r-2|1}^{\langle\mu_1\mu_2}\sigma^{\mu_3\mu_4\rangle}\right] + O(\epsilon^3), \tag{54}$$

which can be expressed using only $\pi^{\langle\mu_1\mu_2}\pi^{\mu_3\mu_4\rangle}$ and without an explicit factor of τ_R.

V. RELATIVISTIC REGULARIZED HYDRODYNAMICS UP TO $O(\epsilon^2)$

Within the relaxation-time approximation, the full evolution equation for the bulk pressure $\Pi = -(m^2/3)\rho_0$ can be obtained by setting $r = 0$ in Eq. (38):

$$D\Pi = -\frac{\Pi}{\tau_R} + \frac{m^2}{3}[\phi_{-1|0}\theta + \phi^{\pi\Pi}_{-1|1}(\theta\Pi + \pi^{\gamma\rho}\sigma_{\gamma\rho})] + \frac{m^2}{3}\nabla_\lambda\rho^\lambda_{-1} - \frac{2}{3}\theta\Pi - \frac{m^2}{3}\sigma_{\lambda\alpha}\rho^{\lambda\alpha}_{-2} + \frac{m^4}{9}\theta\rho_{-2}. \quad (55)$$

From this, one can identify the bulk viscosity as $\zeta = \tau_R m^2\phi_{-1|0}/3$. Similarly, the full evolution equation for $\pi^{\mu\nu} = \rho_0^{\mu\nu}$ is obtained from Eq. (47) by setting $r = 0$:

$$\Delta^{\mu\nu}_{\alpha\beta}D\pi^{\alpha\beta} = -\frac{\pi^{\mu\nu}}{\tau_R} - (\varphi_{-1|0}\sigma^{\mu\nu}) - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\rho^{\lambda\alpha\beta}_{-1} + \frac{2m^2}{5}\nabla^{\langle\mu}\rho^{\nu\rangle}_{-1} - \frac{4}{3}\theta\pi^{\mu\nu} + \sigma_{\lambda\alpha}\rho^{\alpha\lambda\mu\nu}_{-2} - \frac{m^2}{3}\theta\rho^{\mu\nu}_{-2} - \frac{10}{7}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda - 2\pi^{\lambda\langle\mu}\omega^{\nu\rangle}_\lambda - \frac{4m^2}{7}\rho^{\lambda\langle\mu}_{-2}\sigma^{\nu\rangle}_\lambda + \frac{2m^4}{15}\rho_{-2}\sigma^{\mu\nu} - \frac{6}{5}\Pi\sigma^{\mu\nu}. \quad (56)$$

The shear viscosity can be identified as $\eta = \tau_R\varphi_{-1|0}/2$. In obtaining Eqs. (55) and (56), we used the Landau condition $\rho_2 = \rho_1^\mu = 0$. These equations are not closed because the following moments appearing in the above two equations:

$$\rho_{-2}, \quad \rho^\mu_{-1}, \quad \rho^{\mu\nu}_{-2}, \quad \rho^{\lambda\alpha\beta}_{-1}, \quad \rho^{\alpha\lambda\mu\nu}_{-2}, \quad (57)$$

are not Π nor $\pi^{\mu\nu}$. The goal is to use the ϵ expansion of these moments to reexpress Eqs. (55) and (56) so that the equations are closed, adding extra dynamic degrees of freedom when necessary.

Before we carry out the $O(\epsilon^2)$ analysis, we can first check the $O(\epsilon)$ results. Using the $O(\epsilon)$ terms from the ϵ expansions of ρ_{-2} and $\rho^{\mu\nu}_{-2}$ [Eqs. (43) and (48)], the evolution equation for Π can be expressed as

$$D\Pi = -\frac{\Pi}{\tau_R} + \frac{m^2}{3}\phi_{-1|0}\theta - \frac{2}{3}\theta\Pi + \frac{m^2}{3}\phi^{\pi\Pi}_{-1|1}(\theta\Pi + \pi^{\gamma\rho}\sigma_{\gamma\rho}) - \frac{m^2}{3}\left(\frac{\varphi_{-3|0}}{\varphi_{-1|0}}\right)\sigma_{\lambda\alpha}\pi^{\lambda\alpha} - \frac{m^2}{3}\left(\frac{\phi_{-3|0}}{\phi_{-1|0}}\right)\theta\Pi + O(\epsilon^2). \quad (58)$$

Similarly, for $\pi^{\mu\nu}$, the second-order evolution equation is

$$\Delta^{\mu\nu}_{\alpha\beta}D\pi^{\alpha\beta} = -\frac{\pi^{\mu\nu}}{\tau_R} - \varphi_{-1|0}\sigma^{\mu\nu} - \frac{4}{3}\theta\pi^{\mu\nu} - \frac{m^2}{3}\theta\left(\frac{\varphi_{-3|0}}{\varphi_{-1|0}}\right)\pi^{\mu\nu} - \frac{10}{7}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda - \frac{4m^2}{7}\left(\frac{\varphi_{-3|0}}{\varphi_{-1|0}}\right)\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda - \frac{6}{5}\Pi\sigma^{\mu\nu} - \frac{2m^2}{5}\left(\frac{\phi_{-3|0}}{\phi_{-1|0}}\right)\Pi\sigma^{\mu\nu} - 2\pi^{\lambda\langle\mu}\omega^{\nu\rangle}_\lambda + O(\epsilon^2). \quad (59)$$

Note that these equations are hyperbolic, namely, involves the same number of temporal and spatial derivatives. This fact does not automatically guarantee that the theory is stable, but as long as $\tau_R > \eta/(\varepsilon + P)$, it is at least causal.

To go to the $O(\epsilon^2)$ order, one needs to examine ρ^μ_{-1} and $\rho^{\mu_1\mu_2\mu_3}_{-1}$ more closely. There is no need to consider $\rho^{\mu_1\mu_2\mu_3\mu_4}_{-2}$ any further since it can be expressed using $\pi^{\langle\mu_1\mu_2}\pi^{\mu_3\mu_4\rangle} = O(\epsilon^2)$. But the first moment and the third moment cannot be expressed solely in terms of Π and $\pi^{\mu\nu}$ without involving their derivatives. As such, if the ϵ expansion from Sec. IV is used, parabolic equations will result. One way to remedy this problem is to promote the first moment ρ^μ_{-1} and the third moment $\rho^{\mu_1\mu_2\mu_3}_{-1}$ to be dynamic variables. Denoting $W^\mu = m^2\rho^\mu_{-1}$, its evolution equation can be obtained from Eq. (49):

$$\Delta^{\mu_1}_{\nu_1}DW^{\nu_1} = -\frac{W^{\mu_1}}{\tau_R} - m^2F^{\mu_1}_{-2} - \frac{2}{3}\theta W^{\mu_1} - \frac{1}{5}\sigma^{\mu_1}_\lambda W^\lambda - \omega^{\mu_1}_\lambda W^\lambda + \nabla^{\mu_1}\Pi + 2\Pi a^{\mu_1} - \frac{2}{3}m^4\theta\rho^{\mu_1}_{-3} - m^4\frac{4}{5}\sigma^{\mu_1}_\lambda\rho^\lambda_{-3} - m^2\Delta^{\mu_1}_{\nu_1}\nabla_\lambda\rho^{\lambda\nu_1}_{-2} + m^2a_\alpha\rho^{\alpha\mu_1}_{-2} + \frac{m^4}{3}\nabla^{\mu_1}\rho_{-2} - \frac{m^4}{3}a^{\mu_1}\rho_{-2} + 2m^2\sigma_{\lambda\alpha}\rho^{\alpha\lambda\mu_1}_{-3}. \quad (60)$$

Denoting $\xi^{\mu_1\mu_2\mu_3} = \rho^{\mu_1\mu_2\mu_3}_{-1}$, its evolution equation can be obtained from Eq. (51):

$$\Delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}D\xi^{\nu_1\nu_2\nu_3} = -\frac{\xi^{\mu_1\mu_2\mu_3}}{\tau_R} - \frac{18}{7}\pi^{\langle\mu_1\mu_2}a^{\mu_3\rangle} - \frac{3}{7}\nabla^{\langle\mu_1}\pi^{\mu_2\mu_3\rangle} + \frac{3}{7}m^2\nabla^{\langle\mu_1}\rho^{\mu_2\mu_3\rangle}_{-2} - \frac{3}{7}m^2\rho^{\langle\mu_1\mu_2}_{-2}a^{\mu_3\rangle} - \frac{4}{3}\theta\xi^{\mu_1\mu_2\mu_3} - \frac{15}{9}\sigma^{\langle\mu_1}_\lambda\xi^{\mu_2\mu_3\rangle\lambda} - 3\omega^{\langle\mu_1}_\lambda\xi^{\mu_2\mu_3\rangle\lambda} + \frac{18}{35}W^{\langle\mu_1}\sigma^{\mu_2\mu_3\rangle} + m^4\frac{12}{35}\rho^{\langle\mu_1}_{-3}\sigma^{\mu_2\mu_3\rangle} - m^2\frac{2\theta}{3}\rho^{\mu_1\mu_2\mu_3}_{-3} - m^2\frac{4}{3}\sigma^{\langle\mu_1}_\lambda\rho^{\mu_2\mu_3\rangle\lambda}_{-3} - \Delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\nabla_\lambda\rho^{\lambda\nu_1\nu_2\nu_3}_{-2} + a_\alpha\rho^{\alpha\mu_1\mu_2\mu_3}_{-2} + O(\epsilon^3). \quad (61)$$

We can use the ϵ expansions, Eqs. (43), (48), and (54), in place of ρ_{-2}, $\rho^{\mu\nu}_{-2}$, and $\rho^{\mu_1\mu_2\mu_3\mu_4}_{-2}$, respectively, on the right-hand-sides of Eqs. (60) and (61). These terms do not contain any derivatives. We can also use the ϵ expansions, Eqs. (50) and (52), for $\rho^{\mu_1}_{-3|2}$ and $\rho^{\mu_1\mu_2\mu_3}_{-3|2}$, respectively. This replacement does involve derivatives, and since $\rho^\mu_{-3|2}$ and $\rho^{\mu_1\mu_2\mu_3}_{-3|2}$ above are accompanied by either θ or $\sigma^{\mu\nu}$, which results in terms with two derivatives. Fortunately, we can avoid having two derivatives by associating the explicit factor of τ_R from Eqs. (50) and (52) to the factors θ and $\sigma^{\mu\nu}$ to turn them into Π and $\pi^{\mu\nu}$. In this way, we have a closed set of equations for Π, $\pi^{\mu\nu}$, W^μ, and $\xi^{\mu_1\mu_2\mu_3}$ that involve no more than the first derivatives. Furthermore, the relaxation time τ_R does not appears explicitly except for the collision integral term (the $1/\tau_R$ term).

VI. THIRD-ORDER EQUATIONS FOR $m = 0$

The third-order hydrodynamic equations obtained in the previous sections are nonlinear coupled differential equation of 20 degrees of freedom, making them hard to analyze. For the sake of simplicity, from now on, we take the massless limit. In this limit, the bulk pressure does not exist, $\Pi = 0$, and it is consistent to set $W^\mu = 0$ as well. As such, the dynamic degrees of freedom reduce to the energy density ε, the flow vector $\mathbf{u}$, the shear-stress tensor $\pi^{\mu\nu}$ and the third moment $\xi^{\mu_1\mu_2\mu_3}$. In this limit, Eq. (56) reduces to

$$\Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} = -\frac{\pi^{\mu\nu}}{\tau_R} - \varphi_{-1|0}\sigma^{\mu\nu} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda \xi^{\lambda\alpha\beta} - \frac{4}{3}\theta\pi^{\mu\nu} + \sigma_{\lambda\alpha}\varsigma^{\alpha\lambda\mu\nu} - \frac{10}{7}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_{\lambda} - 2\pi^{\lambda\langle\mu}\omega^{\nu\rangle}_{\lambda}, \tag{62}$$

where

$$\varsigma^{\alpha\beta\mu\nu} = \rho^{\alpha\beta\mu\nu}_{-2} = -\frac{8}{7\varphi_{-1|0}}\pi^{\langle\alpha\beta}\pi^{\mu\nu\rangle} + O(\epsilon^3). \tag{63}$$

In the $m = 0$ limit, Eq. (61) reduces to

$$\Delta^{\lambda\mu\nu}_{\rho\alpha\beta} D\xi^{\rho\alpha\beta} = -\frac{1}{\tau_R}\xi^{\lambda\mu\nu} - \frac{4}{3}\theta\xi^{\lambda\mu\nu} - \frac{5}{3}\xi^{\alpha\langle\lambda\mu}\sigma^{\nu\rangle}_{\alpha} - 3\xi^{\alpha\langle\lambda\mu}\omega^{\nu\rangle}_{\alpha} - \frac{18}{7}\pi^{\langle\lambda\mu}a^{\nu\rangle} - \frac{3}{7}\nabla^{\langle\lambda}\pi^{\mu\nu\rangle} + a_\rho\varsigma^{\rho\lambda\mu\nu} - \Delta^{\lambda\mu\nu}_{\rho\alpha\beta}\nabla_\omega\varsigma^{\omega\rho\alpha\beta} + O(\epsilon^3). \tag{64}$$

The dynamics variables are ε, u^μ, $\pi^{\mu\nu}$, $\xi^{\lambda\mu\nu}$. The number of independent degrees of freedom is thus 16.

Equations (62) and (64) provide us with the third-order dissipative equations for massless particles without conservation of the net particle number. As far as terms linear in $\pi^{\mu\nu}$, $\xi^{\lambda\mu\nu}$, and u^μ are concerned, these equations are equivalent to the stable third-order theory postulated in Ref. [37] with $\tau_\rho = \eta_\rho = \tau_\pi$ in their notation. Consequently, our 16 moment formulation is also linearly stable and causal.

What we would like to do further here is to analyze an alternative third-order theory where $\varsigma^{\alpha\beta\mu\nu}$ is also promoted to be a dynamic variable. Setting $r = -2$ and $m = 0$, Eq. (53) becomes

$$\Delta^{\alpha\beta\mu\nu}_{\rho\lambda\omega\gamma} D\varsigma^{\rho\lambda\omega\gamma} = -\frac{1}{\tau_R}\varsigma^{\alpha\beta\mu\nu} - \frac{4}{3}\theta\varsigma^{\alpha\beta\mu\nu} - \frac{8}{7}\pi^{\langle\alpha\beta}\sigma^{\mu\nu\rangle} - \frac{28}{9}\xi^{\langle\alpha\beta\mu}a^{\nu\rangle} - \frac{4}{9}\nabla^{\langle\alpha}\xi^{\beta\mu\nu\rangle} - \frac{20}{11}\varsigma^{\lambda\langle\alpha\beta\mu}\sigma^{\nu\rangle}_{\lambda} - 4\varsigma^{\lambda\langle\alpha\beta\mu}\omega^{\nu\rangle}_{\lambda} + O(\epsilon^3). \tag{65}$$

Equations (64) and Eq. (65) are similar to, but not identical to, the equations for the third and the fourth moments in Ref. [39]. This is because the third and the fourth moments used in Ref. [39] are $\rho_0^{\mu_1\mu_2\mu_3}$ and $\rho_0^{\mu_1\mu_2\mu_3\mu_4}$ while ours are $\rho_{-1}^{\mu_1\mu_2\mu_3}$ and $\rho_{-2}^{\mu_1\mu_2\mu_3\mu_4}$ that naturally appear in the evolution equation of $\pi^{\mu\nu}$.

One way of justifying the promotion of $\varsigma^{\mu_1\mu_2\mu_3\mu_4}$ to a dynamic variable is to note that both are $O(\epsilon^2)$ and in Eq. (65), $\Delta^{\alpha\beta\mu\nu}_{\rho\lambda\omega\gamma}D\varsigma^{\rho\lambda\omega\gamma}$ is linearly coupled to $\nabla^{\langle\alpha}\xi^{\beta\mu\nu\rangle}$ while in Eq. (64), $\Delta^{\lambda\mu\nu}_{\rho\alpha\beta}D\xi^{\rho\alpha\beta}$ is linearly coupled to $\Delta^{\lambda\mu\nu}_{\rho\alpha\beta}\nabla_\omega\varsigma^{\omega\rho\alpha\beta}$. Hence, a consistent linear analysis can be carried out that includes both $\xi^{\mu_1\mu_2\mu_3}$ and $\varsigma^{\mu_1\mu_2\mu_3\mu_4}$. This way of including $\varsigma^{\mu_1\mu_2\mu_3\mu_4}$ to close the equations without incurring two derivatives, however, is possible only when $m = 0$. If $m \neq 0$, the right-hand side of Eq. (65) will contain $\nabla^{\langle\mu_1}\rho_{-3}^{\mu_2\mu_3\mu_4\rangle}$ and $a^{\langle\mu_1}\rho_{-3}^{\mu_2\mu_3\mu_4\rangle}$ resulting in two derivatives. Even though we can argue that promoting $\varsigma^{\mu_1\mu_2\mu_3\mu_4}$ to a dynamic variable is not strictly necessary, we feel that it is still beneficial to carry out a linear analysis as these types of equations do appear elsewhere in literature (for instance Ref. [39]) without the full linear analysis.

In the next section, we carry out linear analysis of this extended 25-moment theory. Before we do so, let us consider the physical meaning of the third moment $\xi^{\mu_1\mu_2\mu_3}$. We will not regard $\varsigma^{\mu_1\mu_2\mu_3\mu_4}$ as a dynamic variable for this consideration. Applying the thermodynamic identities $Ts = \varepsilon + P$ and $T\,ds = d\varepsilon$ to the local equilibrium part, the energy conservation law, Eq. (4), in the massless limit can be reexpressed as

$$\partial_\mu(su^\mu) = -\beta\pi^{\mu\nu}\sigma_{\mu\nu}, \tag{66}$$

where s is the local equilibrium entropy density. Within the first-order approach, the right-hand side becomes nonnegative upon using the first-order constitutive equation, Eq. (45), affirming the second law of thermodynamics in this limit. In our case, upon using the full evolution equation for $\pi^{\mu\nu}$ [Eq. (62)] to replace $\sigma_{\mu\nu}$, Eq. (66) can be re-arranged as

$$\partial_\mu s^\mu_{\text{hyd}} = \frac{\beta}{\varphi_{-1|0}}\bigg[\frac{1}{\tau_R}\pi_{\mu_1\mu_2}\pi^{\mu_1\mu_2} + \frac{8}{7}\tau_R\pi_{\langle\mu_1\mu_2}\sigma_{\lambda\alpha\rangle}\pi^{\langle\alpha\lambda}\sigma^{\mu_1\mu_2\rangle} - \frac{5}{2I_{3,0}}\pi^{\mu_1\mu_2}\pi_{\mu_1\mu_2}\pi^{\mu_3\mu_4}\sigma_{\mu_3\mu_4} + \frac{10}{7}\sigma^{\langle\mu_2}_{\lambda}\pi^{\mu_1\rangle\lambda}\pi_{\mu_1\mu_2} - \xi^{\mu_1\mu_2\mu_3}\big(\nabla_{\langle\mu_1}\pi_{\mu_2\mu_3\rangle} + 6a_{\langle\mu_1}\pi_{\mu_2\mu_3\rangle}\big)\bigg] + O(\epsilon^4), \tag{67}$$

where

$$s^\mu_{\text{hyd}} = \left(s - \frac{\beta}{2\varphi_{-1|0}}\pi_{\mu_1\mu_2}\pi^{\mu_1\mu_2}\right)u^\mu - \frac{\beta}{\varphi_{-1|0}}\pi_{\nu_1\nu_2}\xi^{\mu\nu_1\nu_2} \tag{68}$$

can be interpreted as the hydrodynamic nonequilibrium entropy current. In deriving Eq. (67), we used Eqs. (E9), (E19), and (E27) from Appendix E, and the constitutive relationship for the fourth moment, Eq. (54). The term in Eq. (62) involving the vorticity tensor $\omega_{\mu_1\mu_2}$ does not contribute because of its antisymmetric property. Expressed this way, the meaning of $\xi^{\mu_1\mu_2\mu_3}$ is clear: It is a part of the dissipative entropy current.

In Eq. (68), the first term in the parentheses indicates that the nonequilibrium entropy density is *lower* than the equilibrium one, as it should be. This $\pi_{\mu_1\mu_2}\pi^{\mu_1\mu_2}$ term appears in the original Israel-Stewart paper [22] and all subsequent second-order and third-order analyses. The dissipative term is transverse to u^μ because of $\xi^{\mu\nu_1\nu_2}$. Hence, the fact that one cannot assign definite sign to this term does not disturb the requirement that the nonequilibrium entropy to be lower than the equilibrium one.

The second law of thermodynamics dictates that the entropy of a system must increase when out of equilibrium. This is guaranteed if the right-hand side of Eq. (67) is non-negative. On the right-hand side of Eq. (67), the first line is non-negative. The second line is not guaranteed to be non-negative, but as $\pi^{\mu_3\mu_4}$ relaxes towards $-\tau_R\varphi_{-1|0}\sigma^{\mu_3\mu_4}$, it will become non-negative. A similar argument applies to the last line which is the third-order contribution. As $\xi^{\mu_1\mu_2\mu_3}$ relaxes towards $-\tau_R\frac{3}{7}(\nabla^{\langle\mu_1}\pi^{\mu_2\mu_3\rangle}+6a^{\langle\mu_1}\pi^{\mu_2\mu_3\rangle})$ [e.g., Eq. (61)], the last line in Eq. (67) will become non-negative. The third line cannot be manipulated into a total derivative and/or a square even as $\pi^{\mu_1\mu_2}$ relaxes towards $-\tau_R\varphi_{-1|0}\sigma^{\mu_1\mu_2}$. However, this may be an artifact of the way we defined the nonequilibrium entropy [18,46,47].

In Ref. [33], the entropy current was derived from the Chapman-Enskog expansion of δf. Comparing the two expressions one can see that they are almost the same except that their entropy current contains the third-order contribution proportional to $(\pi^{\gamma}_{\alpha}\pi_{\gamma\beta}\pi^{\alpha\beta})u^{\mu}$. The entropy density found in Refs. [32,34] also have a similar term although their entropy currents do not have our dissipative part. The difference between our expression and those from Refs. [32–34] mainly comes from the fact that they are using the Boltzmann's H-function definition of the entropy current whereas we are combining the energy conservation equation with thermodynamic identities to define the entropy current following Israel and Stewart's work on the second-order hydrodynamics. Unfortunately, it is not at all straightforward to make an exact correspondence because expressing the H-function definition of entropy (which involves $f\ln f$) as a linear combination of the energy-momentum moments of δf is highly nontrivial.

VII. LINEAR STABILITY AND CAUSALITY ANALYSIS OF THE 25 MOMENTS

A. Linearized moment equations

The previous section provided us with the third-order moment equations for massless particles without conservation of net particle number. The next step is to ensure that these equations lead to stable and causal solutions. In general, analyzing the stability and causality of nonlinear partial differential equations is a challenging task. In principle, one should carry out a full nonlinear analysis as advocated in Ref. [48]. However, in this study we only perform the linear analysis of the 25-moment equations as a first step towards establishing the stability and causality of our third-order hydrodynamics.

Consider small fluctuations in the energy density ε, fluid 4-velocity u^μ, and shear-stress tensor $\pi^{\mu\nu}$:

$$\varepsilon=\varepsilon_0+\delta\varepsilon,\quad u^\mu=u_0^\mu+\delta u^\mu,\quad \pi^{\mu\nu}=\delta\pi^{\mu\nu}, \tag{69}$$

where ε_0, u_0^μ are constants. Since $m=0$, the equation of state is simply $P=\varepsilon/3$. Consider the energy and momentum conservation laws Eqs. (4) and (5). The linearized conservation laws are straightforward to get:

$$\begin{gathered}D_0\delta\varepsilon+\tfrac{4}{3}\varepsilon_0\nabla_{\mu,0}\delta u^\mu=0,\\ D_0(\varepsilon_0\delta u^\mu)+\tfrac{1}{4}\nabla_0^\mu\delta\varepsilon+\tfrac{3}{4}\nabla_{\lambda,0}\delta\pi^{\lambda\mu}=0,\end{gathered} \tag{70}$$

where we defined $\Delta_0^{\mu\nu}=g^{\mu\nu}+u_0^\mu u_0^\nu$ and $\nabla_0^\mu=\Delta_0^{\mu\nu}\partial_\nu$. It is convenient to express the above equations in Fourier space. We use the following format of Fourier transform:

$$\begin{aligned}\tilde{f}(k)&=\int_{-\infty}^{\infty}d^4x\, e^{-ik_\mu x^\mu}f(x),\\ f(x)&=\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}e^{ik_\mu x^\mu}\tilde{f}(k).\end{aligned} \tag{71}$$

Here, $k^\mu=(\omega,\mathbf{k})$ is the wave 4-vector. Therefore, we can express each Fourier component of the variables in the linearized equations as a plane wave multiplied by a complex amplitude $\tilde{\phi}$:

$$\phi=\tilde{\phi}e^{ik_\mu x^\mu}=\tilde{\phi}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{72}$$

Note that since $g^{\mu\nu}=\mathrm{diag}(-1,1,1,1)$, we have $k_\mu x^\mu=\mathbf{k}\cdot\mathbf{x}-\omega t$. Furthermore, we rewrite the linearized equations in terms of the Lorentz-covariant variables defined below:

$$\begin{aligned}\Omega&\equiv u_0^\mu k_\mu,\\ \kappa^\mu&\equiv\Delta_0^{\mu\nu}k_\nu,\end{aligned} \tag{73}$$

which correspond to $-\omega$ and $\mathbf{k}$ in the local rest frame of the background system. We also define the covariant wave number κ as

$$\kappa\equiv\sqrt{\kappa^\mu\kappa_\mu}. \tag{74}$$

In terms of the covariant variables, the linearized conservation laws (70) can now be rewritten as

$$\begin{gathered}\Omega\delta\tilde{\varepsilon}+\tfrac{4}{3}\varepsilon_0\kappa_\mu\delta\tilde{u}^\mu=0,\\ \Omega\varepsilon_0\delta\tilde{u}^\mu+\tfrac{1}{4}\kappa^\mu\delta\tilde{\varepsilon}+\tfrac{3}{4}\kappa_\alpha\delta\tilde{\pi}^{\alpha\mu}=0.\end{gathered} \tag{75}$$

From now on, we omit the tilde above the Fourier space variables. All hydrodynamic variables below are expressed in Fourier space. Furthermore, we scale Ω and κ with the timescale $\tau_\eta=\eta/(\varepsilon_0+P_0)$ so that they become dimensionless quantities following Refs. [30,37]. Here, $\eta=\tau_R\varphi_{-1|0}/2$ is the shear viscosity.

The next step is to linearize the $\pi^{\mu\nu}$ equation. To do this, we drop all the higher-order terms in Eq. (62) and keep only the terms that are linear in $\delta\varepsilon$, δu^μ, $\delta\pi^{\mu\nu}$, $\xi^{\mu_1\mu_2\mu_3}$, and $\varsigma^{\mu_1\mu_2\mu_3\mu_4}$ to obtain the linearized $\pi^{\mu\nu}$ equation:

$$\Delta^{\mu\nu}_{\alpha\beta,0}D_0\delta\pi^{\alpha\beta}+\frac{1}{\tau_R}\delta\pi^{\mu\nu}+\varphi_{-1|0}\delta\sigma^{\mu\nu}+\Delta^{\mu\nu}_{\alpha\beta,0}\nabla_{\lambda,0}\xi^{\lambda\alpha\beta}=0, \tag{76}$$

where $\delta\sigma^{\mu\nu}=\nabla^{\langle\mu}\delta u^{\nu\rangle}$. Using (E28) to express the coefficient $\varphi_{-1|0}$ in terms of ε_0 leads us to the following linearized $\pi^{\mu\nu}$ equation:

$$\begin{aligned}&\left(i\Omega+\frac{1}{\tau_R}\right)\delta\pi^{\mu\nu}+\frac{4i\varepsilon_0}{15}\left(\kappa^\mu\delta u^\nu+\kappa^\nu\delta u^\mu-\frac{2}{3}\kappa_\alpha\delta u^\alpha\Delta_0^{\mu\nu}\right)\\ &\quad+i\kappa_\lambda\xi^{\lambda\mu\nu}=0.\end{aligned} \tag{77}$$

Similarly, the linearized equation for $\xi^{\lambda\mu\nu}$ is

$$\begin{aligned}&\Delta^{\lambda\mu\nu}_{\alpha\beta\gamma,0}D_0\xi^{\alpha\beta\gamma}+\frac{1}{\tau_R}\xi^{\lambda\mu\nu}+\frac{3}{7}\Delta^{\lambda\mu\nu}_{\alpha\beta\gamma,0}\nabla_0^\alpha\delta\pi^{\beta\gamma}\\ &\quad+\Delta^{\lambda\mu\nu}_{\alpha\beta\gamma,0}\nabla_{\omega,0}\varsigma^{\omega\alpha\beta\gamma}=0,\end{aligned} \tag{78}$$

which becomes

$$\left(i\Omega + \frac{1}{\tau_R}\right)\xi^{\lambda\mu\nu} + \frac{i}{7}(\kappa^\lambda \delta\pi^{\mu\nu} + \kappa^\mu \delta\pi^{\nu\lambda} + \kappa^\nu \delta\pi^{\mu\lambda}) - \frac{2i}{35}\left(\Delta_0^{\lambda\mu}\kappa^\omega \delta\pi_\omega^\nu + \Delta_0^{\lambda\nu}\kappa^\omega \delta\pi_\omega^\mu + \Delta_0^{\mu\nu}\kappa^\omega \delta\pi_\omega^\lambda\right) + i\kappa_\omega \varsigma^{\omega\lambda\mu\nu} = 0 \tag{79}$$

in the Fourier space after taking the derivatives D_0 and $\nabla_{\lambda,0}$. To derive the above expression, we have used Eq. (A6) from Appendix A for $n = 3$ to express $\kappa^{\langle\lambda}\pi^{\mu\nu\rangle}$. The linearized equation for $\varsigma^{\alpha\beta\mu\nu}$ is also straightforward to obtain

$$\Delta^{\alpha\beta\mu\nu}_{\lambda\gamma\rho\theta,0} D_0 \varsigma^{\lambda\gamma\rho\theta} + \frac{1}{\tau_R}\varsigma^{\alpha\beta\mu\nu} + \frac{4}{9}\Delta^{\alpha\beta\mu\nu}_{\lambda\gamma\rho\theta,0}\nabla_0^\lambda \xi^{\gamma\rho\theta} = 0, \tag{80}$$

which becomes

$$\left(i\Omega + \frac{1}{\tau_R}\right)\varsigma^{\alpha\beta\mu\nu} + \frac{4i}{9}\Delta^{\alpha\beta\mu\nu}_{\lambda\gamma\rho\theta,0}\kappa^\lambda \xi^{\gamma\rho\theta} = 0 \tag{81}$$

in the Fourier space after taking the derivatives. Using Eq. (A6) for $n = 4$ from Appendix A, one can show that

$$\begin{aligned}\Delta^{\alpha\beta\mu\nu}_{\lambda\gamma\rho\theta,0}\kappa^\lambda \xi^{\gamma\rho\theta} &= \frac{1}{4}(\kappa^\alpha \xi^{\beta\mu\nu} + \kappa^\beta \xi^{\alpha\mu\nu} + \kappa^\mu \xi^{\alpha\beta\nu} + \kappa^\nu \xi^{\alpha\mu\beta}) \\ &\quad - \frac{1}{14}\left(\Delta_0^{\beta\mu}\kappa_\lambda \xi^{\alpha\nu\lambda} + \Delta_0^{\beta\nu}\kappa_\lambda \xi^{\alpha\mu\lambda} + \Delta_0^{\mu\nu}\kappa_\lambda \xi^{\alpha\beta\lambda} \right. \\ &\quad \left. + \Delta_0^{\alpha\mu}\kappa_\lambda \xi^{\beta\nu\lambda} + \Delta_0^{\alpha\nu}\kappa_\lambda \xi^{\beta\mu\lambda} + \Delta_0^{\alpha\beta}\kappa_\lambda \xi^{\mu\nu\lambda}\right).\end{aligned} \tag{82}$$

Plugging this back into Eq. (81) gives the complete linearized evolution equation for $\varsigma^{\alpha\beta\mu\nu}$.

B. Transverse modes

The linear stability and causality analysis presented in this work adheres to the procedure outlined in de Brito & Denicol's work [25,37]. This involves decomposing the linearized equations in Fourier space into longitudinal (parallel to κ^μ) and transverse (orthogonal to κ^μ) components. This method offers the advantage of decoupling the equations in the linear regime, allowing them to be solved and analyzed independently and greatly simplifying the calculations [37]. Due to the superposition principle of solutions to linear PDEs, this procedure is equivalent to analyzing the complete three-dimensional linearized equations without decomposition.

It is beneficial to introduce a projector that is analogous to $\Delta^{\mu\nu}$ but with respect to κ^μ:

$$\Delta_\kappa^{\mu\nu} = g^{\mu\nu} - \frac{\kappa^\mu\kappa^\nu}{\kappa^2}, \tag{83}$$

where κ^2 is introduced to ensure normalization. Then, any 4-vector A^μ can be decomposed into a linear combination of the longitudinal and transverse parts:

$$A^\mu = A_{||}\frac{\kappa^\mu}{\kappa} + A_\perp^\mu, \tag{84}$$

where $A_{||} = \kappa_\mu A^\mu/\kappa$ and $A_\perp^\mu = \Delta_\kappa^{\mu\nu}A_\nu$. Similarly, a rank-2 tensor $A^{\mu\nu}$ can also be decomposed as

$$A^{\mu\nu} = A_{||}\frac{\kappa^\mu\kappa^\nu}{\kappa^2} + \frac{1}{3}A_\perp \Delta_\kappa^{\mu\nu} + A_\perp^\mu \frac{\kappa^\nu}{\kappa} + +A_\perp^\nu \frac{\kappa^\mu}{\kappa} + A_\perp^{\mu\nu}, \tag{85}$$

where $A_{||} = \kappa_\mu\kappa_\nu A^{\mu\nu}/\kappa^2, A_\perp = \Delta_\kappa^{\mu\nu}A_{\mu\nu}, A_\perp^\mu = \kappa^\lambda \Delta_\kappa^{\mu\nu}A_{\lambda\nu}/\kappa$, and $A_\perp^{\mu\nu} = \Delta_\kappa^{\mu\nu,\alpha\beta}A_{\alpha\beta}$. Here, we defined the rank-2κ projector to be

$$\Delta_\kappa^{\mu\nu,\alpha\beta} = \tfrac{1}{2}\left(\Delta_\kappa^{\mu\alpha}\Delta_\kappa^{\nu\beta} + \Delta_\kappa^{\mu\beta}\Delta_\kappa^{\nu\alpha} - \tfrac{2}{3}\Delta_\kappa^{\mu\nu}\Delta_\kappa^{\alpha\beta}\right). \tag{86}$$

In this section, we analyze the linear stability and causality of the transverse components of third-order regularized hydrodynamics for $m = 0$. We discuss two cases: in the first, the wave vector $\mathbf{k}$ is parallel to the background fluid velocity $\mathbf{v}$, while in the second, the wave vector is orthogonal to $\mathbf{v}$.

1. Case 1: k is parallel to v

For simplicity and without loss of generality, we assume that $\mathbf{k}$ and $\mathbf{v}$ are both in the x axis:

$$\begin{aligned} u_0^\mu &= \gamma(1, v, 0, 0), \\ k^\mu &= (\omega, k, 0, 0). \end{aligned} \tag{87}$$

It immediately follows that

$$\begin{aligned} \Omega &= \gamma(vk - \omega), \\ \kappa^2 &= \gamma^2(k - v\omega)^2. \end{aligned} \tag{88}$$

Note that the first equation in Eq. (75), which corresponds to the energy-conservation law, is a scalar equation. Thus it is purely longitudinal and does not contribute to the transverse analysis. The transverse component of the momentum conservation law and the moment equations can be easily obtained by applying the projector $\Delta_\kappa^{\mu\nu}$ and κ^μ. Doing so gives us

$$\begin{aligned} \Omega\varepsilon_0\delta u_\perp^\mu + \frac{3}{4}\kappa\delta\pi_\perp^\mu &= 0, \\ \left(i\Omega + \frac{1}{\tau_R}\right)\delta\pi_\perp^\mu + \frac{4}{15}i\kappa\varepsilon_0\delta u_\perp^\mu + i\kappa\xi_\perp^\mu &= 0, \\ \left(i\Omega + \frac{1}{\tau_R}\right)\xi_\perp^\mu + \frac{8}{35}i\kappa\delta\pi_\perp^\mu + i\kappa\varsigma_\perp^\mu &= 0, \\ \left(i\Omega + \frac{1}{\tau_R}\right)\varsigma_\perp^\mu + \frac{5}{21}i\kappa\xi_\perp^\mu &= 0, \end{aligned} \tag{89}$$

where we defined $\xi_\perp^\mu = \kappa_\alpha\kappa_\lambda\Delta^\mu_{\nu,\kappa}\xi^{\alpha\lambda\nu}/\kappa^2$ and $\varsigma_\perp^\mu = \kappa_\alpha\kappa_\beta\kappa_\lambda\Delta^\mu_{\nu,\kappa}\varsigma^{\alpha\beta\lambda\nu}/\kappa^3$. This can be written in the following matrix form:

$$\begin{pmatrix} \Omega & \frac{3}{4}\kappa & 0 & 0 \\ \frac{4}{15}i\kappa & i\Omega + \frac{1}{\tau_R} & i\kappa & 0 \\ 0 & \frac{8}{35}i\kappa & i\Omega + \frac{1}{\tau_R} & i\kappa \\ 0 & 0 & \frac{5}{21}i\kappa & i\Omega + \frac{1}{\tau_R} \end{pmatrix} \begin{pmatrix} \varepsilon_0\delta u_\perp^\mu \\ \delta\pi_\perp^\mu \\ \xi_\perp^\mu \\ \varsigma_\perp^\mu \end{pmatrix} = 0. \tag{90}$$

We require that the determinant of the 4×4 matrix be zero to obtain nontrivial solutions, the resulting equation is the

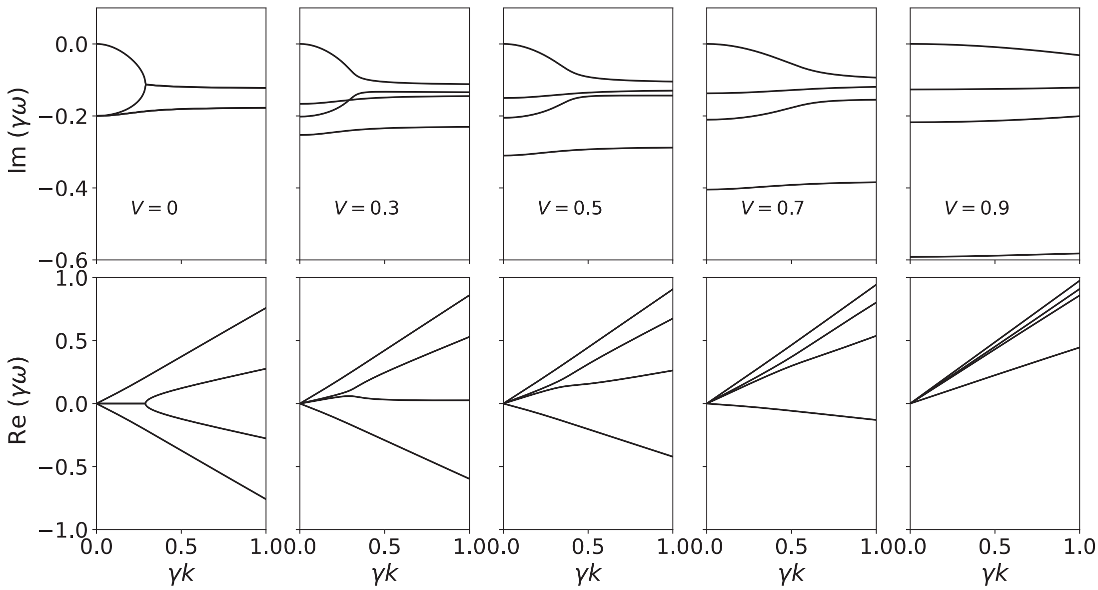

FIG. 1. Real and imaginary parts of the transverse modes of the massless third-order hydrodynamics without conservation of net particle number, in the case of fluid velocity vector being parallel to the wave vector. The relaxation time is chosen to be $\tau_R = 5$.

dispersion relation. However, we should note that the dispersion relation is extremely complicated, even displaying the leading-order terms is not feasible. Therefore, we only present the numerical solutions to the dispersion relation shown in Fig. 1, assuming $\tau_R = 5$ in units of τ_η [30,37]. This particular value for the shear relaxation time τ_R is calculated from the Boltzmann equation in the ultrarelativistic limit, using the 14 moments approximation. However, since the matrix is linear in $1/\tau_R$, Ω, and κ, the value of τ_R does not really matter in the current and the subsequent linear analysis. We chosen value for τ_R is just to facilitate the comparison between our results and those in Refs. [30,37] by having a common scale.

To determine whether these solutions are linearly stable, we first take a look at the plane waves formula [Eq. (72)]:

$$\phi \sim e^{i(kx-\omega t)} = e^{ikx}e^{-i\omega_r t}e^{\omega_i t}, \tag{91}$$

where $\omega = \omega_r + i\omega_i$ is complex. Note that the first two exponential terms are simply oscillating waves, therefore only the third term contributes to the damping, and thus, stability. To ensure exponential suppression of Eq. (91) for $t \geqslant 0$, it is necessary that ω_i be less than or equal to zero. Thus, in general, stability requires

$$\omega_i \leqslant 0 \tag{92}$$

for all $t \geqslant 0$. The determinant of the matrix in Eq. (90) results in a fourth order polynomial in ω, thus we should expect to obtain four modes. Indeed, Fig. 1 shows four distinct curves, two of which have the same imaginary parts for static fluids, i.e., $v = 0$. As one can easily see, all the modes have nonpositive imaginary parts for small k. We have also ascertained that the imaginary parts of all four modes become nonpositive constants for large k.

For the causality analysis, we plot the asymptotic behavior of the group velocity of the four modes in Fig. 2. In the large-k limit, the magnitude of the group velocity remains subluminal for all values of the fluid velocity v. Thus, the linear theory is causal.

2. *Case 2:* k *is orthogonal to* v

We now discuss the second case in which the wave vector is orthogonal to the fluid velocity vector. Without loss of generality, we assume that **V** is still in the x axis, but **k** is now in the y axis:

$$\begin{aligned} u_0^\mu &= \gamma(1, v, 0, 0), \\ k^\mu &= (\omega, 0, k, 0). \end{aligned} \tag{93}$$

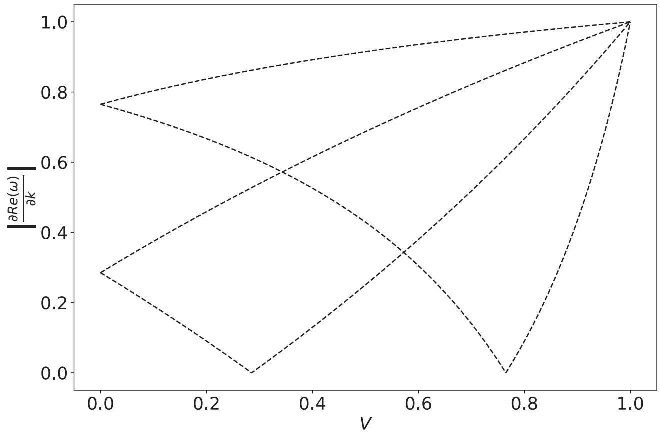

FIG. 2. Magnitude of the group velocity for the transverse modes of the massless third-order hydrodynamics without conservation of net particle number, as a function of the fluid velocity v in the large-k limit and with $\tau_R = 5$, in the case of the fluid velocity vector is parallel to the wave vector.

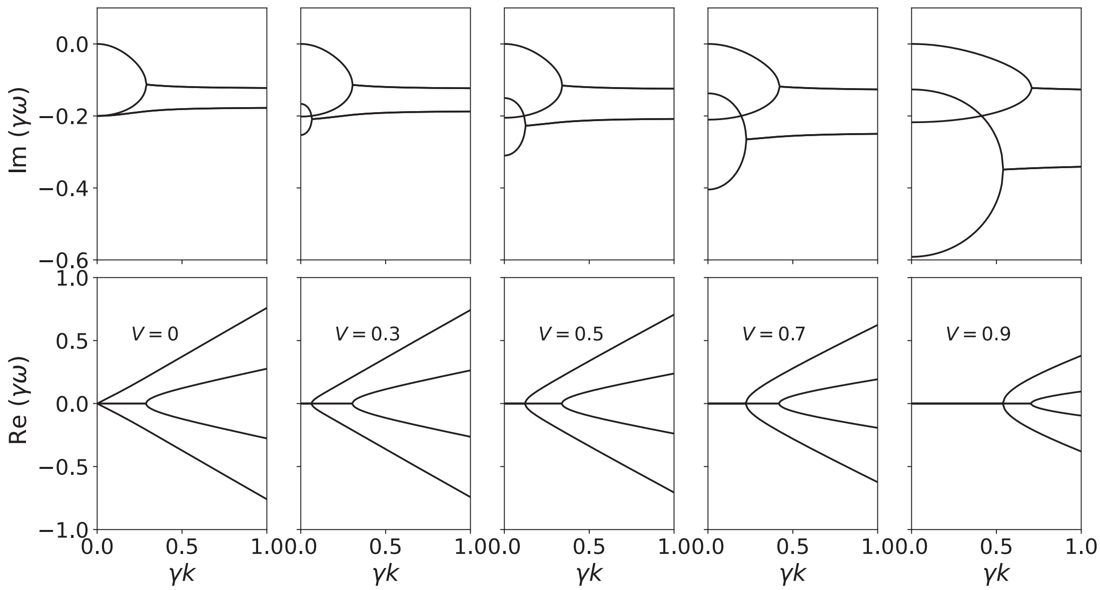

FIG. 3. Real and Imaginary parts of the transverse modes of the massless third-order hydrodynamics without conservation of net particle number, in the case of fluid velocity vector being orthogonal to the wave vector and with $\tau_R = 5$.

It follows that

$$\Omega = -\gamma\omega,$$
$$\kappa^2 = \gamma^2 v^2 \omega^2 + k^2. \tag{94}$$

It is then straightforward to obtain the solutions for this case by substituting Eq. (94) into the dispersion relation and then solving it numerically. The results are shown in Fig. 3. From the figure, we can again see that all the modes are linearly stable as their imaginary parts are always nonpositive for small k, regardless of the background fluid velocity. As before, we can further extend the linear stability of the modes to all $k \geqslant 0$ from the asymptotic behavior of the modes which asymptote to constant negative values.

Figure 4 shows asymptotic group velocity as a function of v. Note that there are only two curves for four solutions. This is because the group velocities for each pair of solutions are only off by a sign. Since the y axis is the absolute value of the group velocity, both solutions coincide in this case. Also, note that both curves approach zero when the fluid velocity reaches the speed of light. This is expected since the plane wave propagates in the orthogonal direction with respect to the fluid flow. As the fluid moves faster and faster, the wave is eventually "dragged" by the fluid flow under the effect of shear viscosity and moves in the fluid flow direction eventually, resulting in zero group velocity in the orthogonal direction.

C. Longitudinal modes

1. Case 1: k *is parallel to* v

Similar to the second-order case, the first step is to obtain the longitudinal components of the conservation laws and the equations for $\pi^{\mu\nu}$, $\xi^{\lambda\mu\nu}$, and $\varsigma^{\alpha\beta\mu\nu}$. Applying $\kappa^\mu\kappa^\nu$ and κ^μ to the corresponding equations, we get

$$\begin{aligned}
\Omega\delta\varepsilon + \frac{4}{3}\varepsilon_0\kappa\delta u_{||} &= 0,\\
\Omega\varepsilon_0\delta u_{||} + \frac{1}{4}\kappa\delta\varepsilon + \frac{3}{4}\kappa\delta\pi_{||} &= 0,\\
\left(i\Omega + \frac{1}{\tau_R}\right)\delta\pi_{||} + \frac{16}{45}i\varepsilon_0\kappa\delta u_{||} + i\kappa\xi_{||} &= 0,\\
\left(i\Omega + \frac{1}{\tau_R}\right)\xi_{||} + \frac{9}{35}i\kappa\delta\pi_{||} + i\kappa\varsigma_{||} &= 0,\\
\left(i\Omega + \frac{1}{\tau_R}\right)\varsigma_{||} + \frac{16}{63}i\kappa\xi_{||} &= 0,
\end{aligned} \tag{95}$$

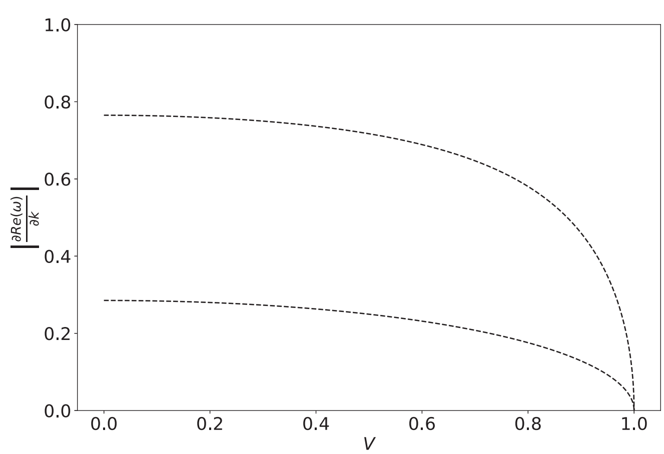

FIG. 4. Magnitude of the group velocity for the transverse modes of the massless third-order hydrodynamics without conservation of net particle number, as a function of the fluid velocity v in the large-k limit and with $\tau_R = 5$, in the case of fluid velocity vector being orthogonal to the wave vector.

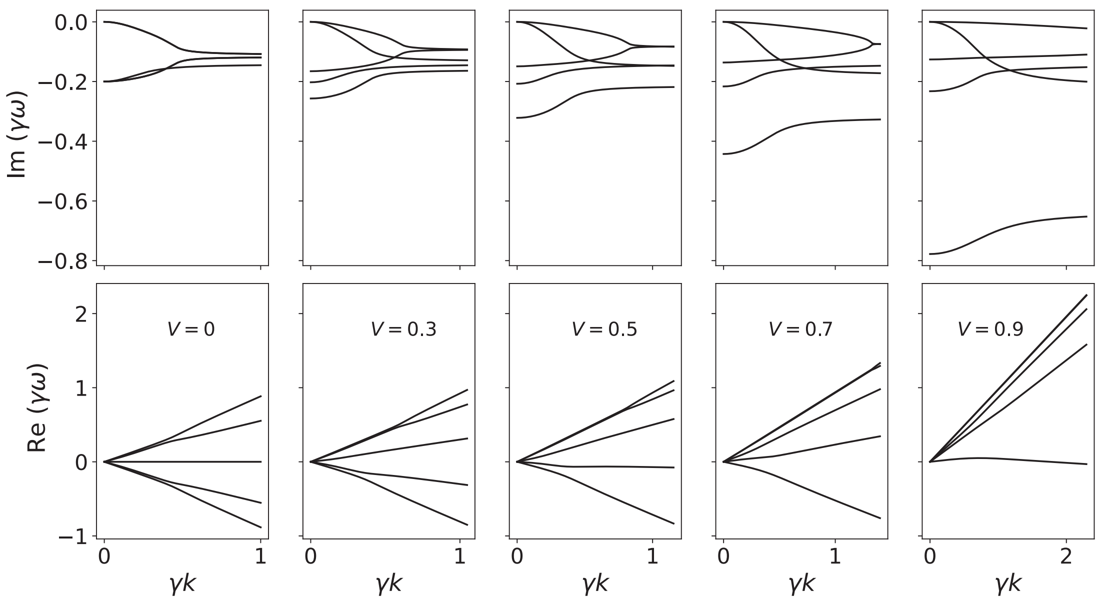

FIG. 5. Real and imaginary parts of the longitudinal modes of the massless third-order hydrodynamics without conservation of net particle number, in the case of fluid velocity vector being parallel to the wave vector and with $\tau_R = 5$.

where we defined $\xi_{||} = \kappa_\alpha\kappa_\beta\kappa_\lambda\xi^{\alpha\beta\lambda}/\kappa^3$ and $\varsigma_{||} = \kappa_\alpha\kappa_\beta\kappa_\mu\kappa_\nu\varsigma^{\alpha\beta\mu\nu}/\kappa^4$. Note that we have included the purely longitudinal energy conservation law in this system of equations. Written in the matrix form, this is equivalent to

$$\begin{pmatrix} \Omega & \frac{4}{3}\kappa & 0 & 0 & 0 \\ \frac{\kappa}{4} & \Omega & \frac{3}{4}\kappa & 0 & 0 \\ 0 & \frac{16}{45}i\kappa & i\Omega + \frac{1}{\tau_R} & i\kappa & 0 \\ 0 & 0 & \frac{9}{35}i\kappa & i\Omega + \frac{1}{\tau_R} & i\kappa \\ 0 & 0 & 0 & \frac{16}{63}i\kappa & i\Omega + \frac{1}{\tau_R} \end{pmatrix} \begin{pmatrix} \delta\varepsilon \\ \varepsilon_0\delta u_{||} \\ \delta\pi_{||} \\ \xi_{||} \\ \varsigma_{||} \end{pmatrix} = 0. \tag{96}$$

Since Ω is of fifth-order in the determinant, we should expect to obtain five modes. Indeed, Fig. 5 shows that all five solutions are linearly stable since for small k, their imaginary parts are all nonpositive for various background fluid velocities. Again, one can numerically show that all five modes asymptote to nonpositive constants.

In Fig. 6, we show asymptotic group velocities of 5 modes as a function of v. One can see that all solutions are linearly causal since the magnitude of the group velocity is less than 1 for all of them, in the large-k limit. Also, note that the straight diagonal line in the figure corresponds to a stationary mode in the fluid rest frame since its group velocity is simply the fluid flow velocity.

2. *Case 2:* k *is orthogonal to* v

As before, we insert Eq. (94) into the dispersion relation and solve numerically for the solutions. Figure 7 shows the result. Note that two out of the five solutions have the same imaginary parts, and we can see that all solutions are linearly stable since they all have nonpositive imaginary parts for small k. Again, we checked that all modes asymptote to nonpositive constants in the large-k limit.

To verify the causality of these solutions, we repeat the process from the previous sections. The group velocities of the solutions are shown in Fig. 8 as a function of the fluid flow velocity. Note that there are three curves in this figure, one of them lies on the x axis and corresponds to the stationary mode with zero group velocity.

VIII. DISCUSSIONS AND CONCLUSIONS

The main results of this work are the derivation of the evolution equation for the general energy-momentum moment of

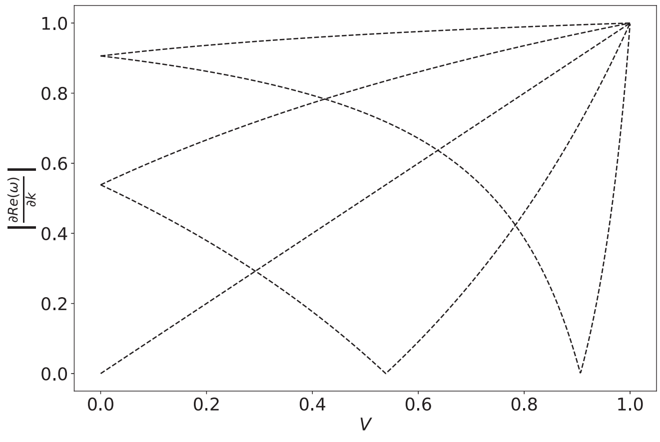

FIG. 6. Magnitude of the group velocity for the longitudinal modes of the massless third-order hydrodynamics without conservation of net particle number, as a function of the fluid velocity v in the large-k limit and with $\tau_R = 5$, in the case of fluid velocity vector being parallel to the wave vector.

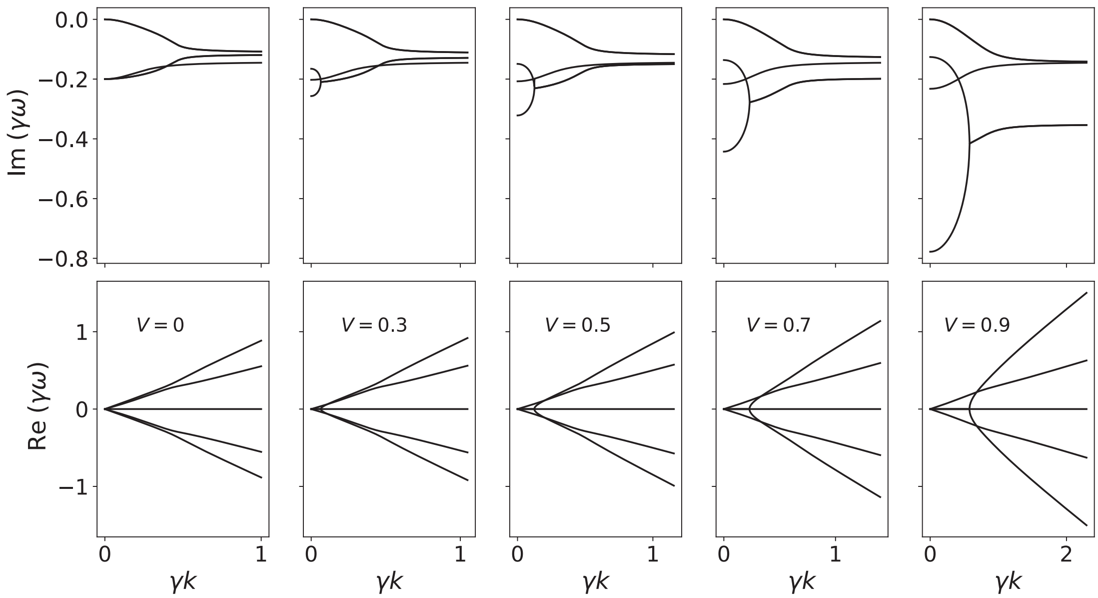

FIG. 7. Real and imaginary parts of the longitudinal modes of the massless third-order hydrodynamics without conservation of net particle number, in the case of fluid velocity vector being orthogonal to the wave vector and with $\tau_R = 5$.

the phase-space density function, introduction of the regularized hydrodynamics, and the derivation and the analysis of the third-order hydrodynamics. As far as we can find out, this is the first time that the derivation of the evolution equation for a general energy-momentum moment has appeared in literature.

Our derivation of hydrodynamic equations from the general moment equations follows closely the derivation of the regularized hydrodynamics by Struchtrup and Torrilhon in which the Chapman-Enskog-like expansion is applied to the moments, not to the density function, except for hydrodynamic variables. In this way, we avoided the inherent ambiguity in the method of moments [31] as well as possible acausality in the Chapman-Enskog method [35,37]. The third-order hydrodynamics unambiguously derived this way includes additional rank-1 moment and rank-3 moment as dynamic variable.

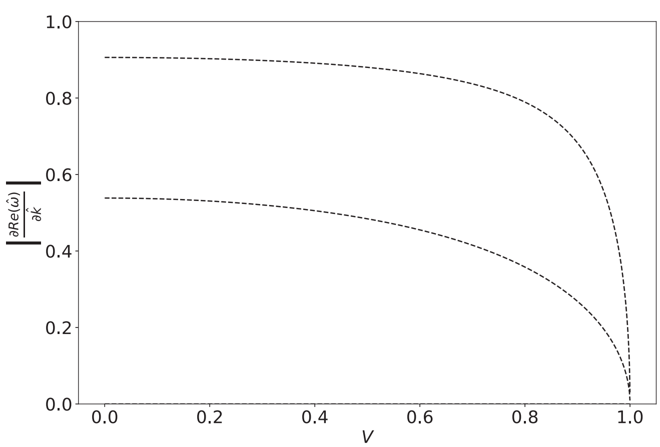

FIG. 8. Magnitude of the group velocity for the longitudinal modes of the massless third-order hydrodynamics without conservation of net particle number, as a function of the fluid velocity v in the large-k limit and with $\tau_R = 5$, in the case of fluid velocity vector being orthogonal to the wave vector. Notice that there is a stationary mode with zero group velocity along the direction of wave's propagation.

In recent literature, other versions of third-order theories appeared. The versions most closely related to ours are those from Refs. [37,39]. The authors of Ref. [37] proposed a third-order theory based on Ref. [35] in which they promoted the gradient of $\pi^{\mu\nu}$ to a new hydrodynamic variable

$$\nabla^{\langle\alpha}\pi^{\mu\nu\rangle} \to \rho^{\alpha\mu\nu} \tag{97}$$

to eliminate the second-order gradients in the evolution equation of $\pi^{\mu\nu}$. This is analogous to $\xi^{\lambda\mu\nu}$ we defined in Eqs. (63) and (64), but it was done in a heuristic way. This situation was remedied by the same authors in Ref. [39] where they derived the equations for the third and the fourth moments using $\rho_0^{\mu_1\mu_2\mu_3}$ and $\rho_0^{\mu_1\mu_2\mu_3\mu_4}$, while we use $\rho_{-1}^{\mu_1\mu_2\mu_3}$ and $\rho_{-2}^{\mu_1\mu_2\mu_3\mu_4}$. In their approach, all $\rho_r^{\mu_1\cdots\mu_n}$ up to $n = 4$ are proportional to $\rho_0^{\mu_1\cdots\mu_n}$ while ours clearly differ. Nevertheless, linear analysis should yield similar results.

To further analyze the properties of this theory and for simplicity, we assume the particles to be massless. A series of linear stability and causality analysis was then performed, and we showed that all the modes of the massless third-order theory are linearly stable and causal.

The hybrid method advocated in this work may be extended to higher orders. However, given that the Chapman-Enskog expansion is asymptotic in nature [49], and the fact that we need to promote higher and higher-order moments to be dynamic, this path may not be a profitable one to study the effect of higher-order moments. Instead, one may consider resummation approaches such as the generalized hydrodynamics formulated by Eu [50,51]. Other ways to extend our method include applying it to systems with multiple species and multiple conserved charges, to spin hydrodynamics [52–60], and to the general-frame theories with off-shell transport parameters [11,47]. We leave these as possible venues for further investigations.

ACKNOWLEDGMENTS

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APPENDIX A: ON PROJECTORS

The definition of the rank n projector is a tensor of rank (n, n) that selects the symmetric and traceless part of a tensor or rank (m, n) or rank (n, m). The basic building block is the spatial metric tensor for a fluid cell moving with the flow velocity u^μ:

$$\Delta^\mu_\nu = g^\mu_\nu + u^\mu u_\nu, \tag{A1}$$

which is the rank-1 projector. When applied to a 4-momentum, it gives

$$\begin{aligned} p^{\langle\mu\rangle} &= \Delta^\mu_\nu p^\nu \\ &= p^\mu - (\mathcal{E}_p)u^\mu, \end{aligned} \tag{A2}$$

where $\mathcal{E}_p = -u_\nu p^\nu$ is the time-component of the 4-vector in the fluid-cell rest frame. From here on the angular bracket around indices indicate the symmetric and traceless part of the tensor. For $n = 2$

$$\Delta^{\mu_1\mu_2}_{\nu_1\nu_2} = \tfrac{1}{2}\big(\Delta^{\mu_1}_{\nu_1}\Delta^{\mu_2}_{\nu_2} + \Delta^{\mu_1}_{\nu_2}\Delta^{\mu_2}_{\nu_1} - \tfrac{2}{3}\Delta_{\nu_1\nu_2}\Delta^{\mu_1\mu_2}\big), \tag{A3}$$

and for $n = 3$

$$\begin{aligned} \Delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3} = {}& \tfrac{1}{6}\big[\Delta^{\mu_1}_{\nu_1}\Delta^{\mu_2}_{\nu_2}\Delta^{\mu_3}_{\nu_3} + \Delta^{\mu_1}_{\nu_1}\Delta^{\mu_2}_{\nu_3}\Delta^{\mu_3}_{\nu_2} + \Delta^{\mu_1}_{\nu_2}\Delta^{\mu_2}_{\nu_1}\Delta^{\mu_3}_{\nu_3} \\ &+ \Delta^{\mu_1}_{\nu_2}\Delta^{\mu_2}_{\nu_3}\Delta^{\mu_3}_{\nu_1} + \Delta^{\mu_1}_{\nu_3}\Delta^{\mu_2}_{\nu_1}\Delta^{\mu_3}_{\nu_2} + \Delta^{\mu_1}_{\nu_3}\Delta^{\mu_2}_{\nu_2}\Delta^{\mu_3}_{\nu_1}\big] \\ &- \tfrac{1}{15}\big[\Delta^{\mu_1\mu_2}\big(\Delta_{\nu_1\nu_2}\Delta^{\mu_3}_{\nu_3} + \Delta_{\nu_2\nu_3}\Delta^{\mu_3}_{\nu_1} + \Delta_{\nu_3\nu_1}\Delta^{\mu_3}_{\nu_2}\big) \\ &+ \Delta^{\mu_2\mu_3}\big(\Delta_{\nu_1\nu_2}\Delta^{\mu_1}_{\nu_3} + \Delta_{\nu_2\nu_3}\Delta^{\mu_1}_{\nu_1} + \Delta_{\nu_3\nu_1}\Delta^{\mu_1}_{\nu_2}\big) \\ &+ \Delta^{\mu_3\mu_1}\big(\Delta_{\nu_1\nu_2}\Delta^{\mu_2}_{\nu_3} + \Delta_{\nu_2\nu_3}\Delta^{\mu_2}_{\nu_1} + \Delta_{\nu_3\nu_1}\Delta^{\mu_2}_{\nu_2}\big)\big]. \end{aligned} \tag{A4}$$

The above projectors are constructed in such a way that they are symmetric and traceless in both $(\mu_1, \ldots, \mu_n)$ and $(\nu_1, \ldots, \nu_n)$. For the sake of projecting $T^{\nu_1\cdots\nu_n}$ to $T^{\langle\mu_1\cdots\mu_n\rangle}$, this is actually not necessary. It turned out that we just need to make sure that the superscripted indices are symmetric and traceless. In that case, the following recursive construction works just as well as a projector [61],

$$\begin{aligned} \tilde{\Delta}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} = {}& \frac{1}{n}\sum_{i=1}^{n}\Delta^{\mu_i}_{\nu_1}\tilde{\Delta}^{\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n}_{\nu_2\cdots\nu_n} - \frac{2}{n(2n-1)} \\ &\times \sum_{i=1}^{n}\sum_{j=i+1}^{n}\Delta^{\mu_i\mu_j}\Delta_{\nu_1\alpha}\tilde{\Delta}^{\alpha\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_n}_{\nu_2\cdots\nu_n}. \end{aligned} \tag{A5}$$

This is explicitly constructed so that it is symmetric and traceless in $(\mu_1, \ldots, \mu_n)$, but not necessarily in $(\nu_1, \ldots, \nu_n)$. We do have $\Delta^{\mu_1\mu_2}_{\nu_1\nu_2} = \tilde{\Delta}^{\mu_1\mu_2}_{\nu_1\nu_2}$, but for $n > 2$, $\tilde{\Delta}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}$ is neither symmetric nor traceless in $(\nu_1, \ldots, \nu_n)$. As an example, applying this projector to $q^{\nu_1}P^{\nu_2\cdots\nu_n}$ yields

$$\begin{aligned} q^{\langle\mu_1}P^{\mu_2\cdots\mu_n\rangle} &= \tilde{\Delta}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}q^{\nu_1}P^{\nu_2\cdots\nu_n} \\ &= \frac{1}{n}\sum_{i=1}^{n}q^{\langle\mu_i\rangle}P^{\langle\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n\rangle} \\ &\quad - \frac{2}{n(2n-1)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\Delta^{\mu_i\mu_j}q_{\langle\alpha\rangle} \\ &\quad \times P^{\langle\alpha\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_n\rangle}, \end{aligned} \tag{A6}$$

where q^{ν_1} is an arbitrary 4-vector and $P^{\nu_2\cdots\nu_n}$ is an arbitrary rank-$(n-1)$ tensor. Equation (B3) is a particular example of this identity.

The full rank-n projector that are symmetric and traceless in both sets of indices can be recursively built by averaging Eq. (A5) over n different choices of ν_k that can be isolated

$$\begin{aligned} \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} = {}& \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\Delta^{\mu_i}_{\nu_k}\Delta^{\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n}_{\nu_1\cdots\nu_{k-1}\nu_{k+1}\cdots\nu_n} \\ &- \frac{2}{n^2(2n-1)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\Delta^{\mu_i\mu_j} \\ &\times \sum_{k=1}^{n}\Delta_{\nu_k\alpha}\Delta^{\alpha\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_n}_{\nu_1\cdots\nu_{k-1}\nu_{k+1}\cdots\nu_n}. \end{aligned} \tag{A7}$$

The right-hand side is explicitly constructed in such a way that it is symmetric and traceless in $(\mu_1, \ldots, \mu_n)$. It looks only symmetric in $(\nu_1, \ldots, \nu_n)$, but it would be also traceless provided that the following identities holds:

$$\Delta^{\alpha\mu_2\cdots\mu_{n-1}}_{\alpha\nu_2\cdots\nu_{n-1}} = \frac{(2n-1)}{(2n-3)}\Delta^{\mu_2\cdots\mu_{n-1}}_{\nu_2\cdots\nu_{n-1}}, \tag{A8}$$

$$\begin{aligned} \sum_{i=1}^{n}\Delta^{\mu_i\alpha}\Delta^{\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n}_{\alpha\nu_3\cdots\nu_n} = {}& \frac{2}{(2n-3)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\Delta^{\mu_i\mu_j} \\ &\times \Delta^{\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_n}_{\nu_3\cdots\nu_n}, \end{aligned} \tag{A9}$$

and

$$\Delta^{\mu_2\cdots\mu_n}_{\nu_2\cdots\nu_n} = \frac{(2n-1)}{n(n-1)}\left(\frac{1}{(2n-3)}\sum_{i=2}^{n}\sum_{k=2}^{n}\Delta^{\mu_i}_{\nu_k}\Delta^{\mu_2\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n}_{\nu_2\cdots\nu_{k-1}\nu_{k+1}\cdots\nu_n} - \frac{1}{(2n-1)}\sum_{j=2}^{n}\sum_{k=2}^{n}\Delta^{\mu_j\beta}\Delta_{\nu_k\alpha}\Delta^{\alpha\mu_2\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_n}_{\beta\nu_2\cdots\nu_{k-1}\nu_{k+1}\cdots\nu_n}\right). \tag{A10}$$

These identities can be proven by using the following mathematical induction strategy:

(1) Show that Eqs. (A7), (A8), (A9), and (A10) are valid for $n = 2$.
(2) Assume that Eqs. (A8), (A9), and (A10) are valid for an arbitrary n.
(3) Show that the projector recursion relationship, Eq. (A7), is valid for this n.
(4) Using Eqs. (A7)–(A10) for n, show that Eqs. (A8)–(A10) are valid for $n + 1$.

Due to the symmetry between μ and ν, the following is equivalent to Eq. (A9):

$$\sum_{i=1}^{n}\Delta_{\nu_i\alpha}\Delta^{\alpha\mu_3\cdots\nu_n}_{\nu_1\cdots\nu_{i-1}\nu_{i+1}\cdots\nu_n} = \frac{2}{(2n-3)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\Delta_{\nu_i\nu_j}\Delta^{\mu_3\cdots\mu_n}_{\nu_1\cdots\nu_{i-1}\nu_{i+1}\cdots\nu_{j-1}\nu_{j+1}\cdots\nu_n}. \tag{A11}$$

By combining Eq. (A7) and Eq. (A11), we can have a recursion relationship which is explicitly symmetric under $\mu_i \leftrightarrow \nu_i$ swapping:

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\Delta^{\mu_i}_{\nu_k}\Delta^{\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n}_{\nu_1\cdots\nu_{k-1}\nu_{k+1}\cdots\nu_n} - \frac{4}{n^2(2n-1)(2n-3)}\sum_{l=1}^{n}\sum_{m=l+1}^{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\Delta_{\nu_l\nu_m} \times \Delta^{\mu_i\mu_j}\Delta^{\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_n}_{\nu_1\cdots\nu_{l-1}\nu_{l+1}\cdots\nu_{m-1}\nu_{m+1}\cdots\nu_n}. \tag{A12}$$

APPENDIX B: IRREDUCIBLE POLYNOMIALS

In the rest frame of the fluid cell, the irreducible tensors of rank n is defined as the symmetric and traceless combinations of the n factors of p^m, where $m = 1, 2, 3$ is the spatial index. For instance, the rank-1 tensor is just p^m and the rank-2 tensor is

$$p^{\langle m_1}p^{m_2\rangle} = p^{m_1}p^{m_2} - \frac{\Delta^{m_1m_2}}{3}p^2, \tag{B1}$$

where $\Delta^{m_1m_2} = \delta^{m_1m_2}$ is the spatial metric tensor in the rest frame and $p^2 = p_{m_1}p_{m_2}\Delta^{m_1m_2}$. Here the angular bracket over indices indicate the symmetric and traceless part. For $n = 3$,

$$p^{\langle m_1}p^{m_2}p^{m_3\rangle} = p^{m_1}p^{m_2}p^{m_3} - \frac{p^2}{5}(\Delta^{m_1m_2}p^{m_3} + \Delta^{m_1m_3}p^{m_2} + \Delta^{m_2m_3}p^{m_1}). \tag{B2}$$

Higher rank irreducible tensors can be built using lower rank ones by using the following recursion relationship:

$$p^{\langle m_1}p^{m_2}\cdots p^{m_n\rangle} = \frac{1}{n}\sum_{i=1}^{n}p^{m_i}p^{\langle m_1}\cdots p^{m_{i-1}}p^{m_{i+1}}\cdots p^{m_n\rangle} - \frac{2}{n(2n-1)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\Delta^{m_im_j}p_a p^{\langle a} \times p^{m_1}\cdots p^{m_{i-1}}p^{m_{i+1}}\cdots p^{m_{j-1}}p^{m_{j+1}}\cdots p^{m_n\rangle}, \tag{B3}$$

which comes from applying Eq. (A6) to $p^{k_1}p^{\langle k_2}\cdots p^{k_n\rangle}$.

When the fluid-cell has a nonzero flow velocity u^μ, then the spatial metric tensor is

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \tag{B4}$$

and the spatial part of a 4-momentum is

$$p^{\langle\mu\rangle} = \Delta^\mu_\nu p^\nu = p^\mu - u^\mu\mathcal{E}_p, \tag{B5}$$

where $\mathcal{E}_p = -p_\mu u^\mu$ is the time component of the 4-momentum in the fluid-cell rest frame. All results in this sections can be generalized to the nonzero fluid velocity case by changing $m_i \to \langle\mu_i\rangle$ and $p^2 \to (\mathcal{E}_p^2 - m^2)$ where $m^2 = -p^\mu p_\mu$.

The orthogonality condition for the momentum polynomial is [30,62]

$$\int\frac{d^3p}{(2\pi)^3p^0}F(\mathcal{E}_p)p^{\langle\mu_1}\cdots p^{\mu_n\rangle}p_{\langle\nu_1}\cdots p_{\nu_m\rangle} = \frac{n!}{(2n+1)!!}\delta_{mn}\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3p^0}F(\mathcal{E}_p)\big(\mathcal{E}_p^2 - m^2\big)^n. \tag{B6}$$

In deriving the evolution equation for a general energy-momentum moment, the following identity is frequently needed:

$$p^{\langle\lambda\rangle}p^{\langle\mu_1}\cdots p^{\mu_n\rangle} = p^{\langle\lambda}p^{\mu_1}\cdots p^{\mu_n\rangle} + \frac{n}{2n+1}\big(\mathcal{E}_p^2 - m^2\big) \times p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_{n-1}}\Delta^{\mu_n\rangle\lambda}. \tag{B7}$$

To prove this, first we go to the rest frame where $u^\mu = (1, 0, 0, 0)$. In that case,

$$p^{\langle\mu\rangle} \to p^m, \tag{B8}$$

where $m = 1, 2, 3$ are the spatial component of a momentum and

$$\mathcal{E}_p^2 - m^2 \to p^2, \tag{B9}$$

where $p^2 = p_ip^i$.

The identity to prove is then

$$p^{\langle l}p^{m_1}\cdots p^{m_n\rangle} = p^lp^{\langle m_1}\cdots p^{m_n\rangle} - \frac{n}{2n+1}p^2p^{\langle m_1}p^{m_2}\cdots p^{m_{n-1}}\Delta^{m_n\rangle l}. \tag{B10}$$

Our starting point is the fact that these polynomials can be obtained from

$$\partial_{m_n}\cdots\partial_{m_2}\partial_{m_1}\frac{1}{p} = (-1)^n(2n-1)!!\frac{p^{\langle m_1}p^{m_2}\cdots p^{m_n\rangle}}{p^{2n+1}}, \tag{B11}$$

where $\partial_m = \partial/\partial p^m$. This expression is explicitly symmetric since derivatives commute. It is also traceless since

$$\nabla_p^2\frac{1}{p} \propto \delta(p). \tag{B12}$$

The normalization constant is chosen so that the coefficient of $p^{m_1}\cdots p^{m_n}$ in $p^{\langle m_1}\cdots p^{m_n\rangle}$ is one.

We can get the following recursion relation by considering the product rule of taking one more derivative of Eq. (B11):

$$\begin{aligned}(-1)^{n+1}(2n+1)!!\frac{p^{\langle m_1}p^{m_2}\cdots p^{m_n}p^{m_{n+1}\rangle}}{p^{2n+3}} &= \partial_{m_{n+1}}\partial_{m_n}\cdots\partial_{m_2}\partial_{m_1}\frac{1}{p}\\ &= (-1)^n(2n-1)!!\left((-1)(2n+1)\frac{p^{m_{n+1}}p^{\langle m_1}p^{m_2}\cdots p^{m_n\rangle}}{p^{2n+3}} + \frac{\partial_{m_{n+1}}(p^{\langle m_1}p^{m_2}\cdots p^{m_n\rangle})}{p^{2n+1}}\right),\end{aligned} \tag{B13}$$

which yields

$$p^{\langle m_1}p^{m_2}\cdots p^{m_n}p^{m_{n+1}\rangle} = p^{m_{n+1}}p^{\langle m_1}p^{m_2}\cdots p^{m_n\rangle} - \frac{p^2}{2n+1}\partial_{m_{n+1}}p^{\langle m_1}p^{m_2}\cdots p^{m_n\rangle}. \tag{B14}$$

The identity (B10) is proven if we can show

$$\partial_{m_{n+1}}p^{\langle m_1}p^{m_2}\cdots p^{m_n\rangle} = n\,p^{\langle m_1}p^{m_2}\cdots\Delta^{m_n\rangle m_{n+1}}. \tag{B15}$$

To start mathematical induction, consider $n = 2$:

$$\begin{aligned}\partial_{m_3}p^{\langle m_1}p^{m_2\rangle} &= \partial_{m_3}\left(p^{m_1}p^{m_2} - \frac{\Delta^{m_1m_2}}{3}p^2\right)\\ &= \Delta^{m_1m_3}p^{m_2} + \Delta^{m_2m_3}p^{m_1} - 2\frac{\Delta^{m_1m_2}}{3}p^{m_3}\\ &= 2\left(\frac{1}{2}(\Delta^{m_1m_3}p^{m_2} + \Delta^{m_2m_3}p^{m_1}) - \frac{\Delta^{m_1m_2}}{3}p^{m_3}\right)\\ &= 2p^{\langle m_1}\Delta^{m_2\rangle m_3},\end{aligned} \tag{B16}$$

which gives the correct expression.

To prove Eq. (B15) for general n, we need some identities first. The right-hand side of the following expression:

$$\begin{aligned}p^{\langle m_1}p^{m_2}\cdots p^{m_{n-1}}\Delta^{m_n\rangle m_{n+1}} &= \frac{1}{n}\sum_{i=1}^{n}p^{m_i}p^{\langle m_1}p^{m_2}\cdots p^{m_{i-1}}p^{m_{i+1}}\cdots\Delta^{m_n\rangle m_{n+1}}\\ &\quad - \frac{2}{n(2n-1)}\sum_{i=1}^{n}\sum_{j+1}^{n}\Delta^{m_im_j}p_a p^{\langle a}p^{m_1}p^{m_2}\cdots p^{m_{i-1}}p^{m_{i+1}}\cdots\Delta^{m_n\rangle m_{n+1}}\end{aligned} \tag{B17}$$

is explicitly constructed in such a way that it is symmetric and traceless in $(m_1,\ldots,m_n)$. The tensor $p^{\langle m_1}p^{m_2}\cdots p^{m_{n-1}}\Delta^{m_n\rangle m_{n+1}}$ can be also expressed as

$$\begin{aligned}p^{\langle m_1}p^{m_2}\cdots p^{m_{n-1}}\Delta^{m_n\rangle m_{n+1}} &= \frac{1}{n}\sum_{i=1}^{n}\Delta^{m_{n+1}m_i}p^{\langle m_1}\cdots p^{m_{i-1}}p^{m_{i+1}}\cdots p^{m_n\rangle}\\ &\quad - \frac{2}{n(2n-1)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\Delta^{m_im_j}p^{\langle m_1}\cdots p^{m_{i-1}}p^{m_{i+1}}\cdots p^{m_{j-1}}p^{m_{j+1}}\cdots p^{m_n}p^{m_{n+1}\rangle}.\end{aligned} \tag{B18}$$

Again the right-hand side (RHS) is explicitly constructed so that it is symmetric and traceless in $(m_1,\ldots,m_n)$.

To prove Eq. (B15), assume that it works for $n-1$. We then take another derivative of Eq. (B3):

$$\frac{\partial}{\partial p^{m_{n+1}}} p^{\langle m_1} p^{m_2} \cdots p^{m_n \rangle} = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial p^{m_{n+1}}} (p^{m_i} p^{\langle m_1} \cdots p^{m_{i-1}} p^{m_{i+1}} \cdots p^{m_n \rangle}) - \frac{2}{n(2n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \Delta^{m_i m_j} \frac{\partial}{\partial p^{m_{n+1}}} (p_a p^{\langle a} p^{m_1} \cdots p^{m_{i-1}} p^{m_{i+1}} \cdots p^{m_{j-1}} p^{m_{j+1}} \cdots p^{m_n \rangle}). \tag{B19}$$

One can then show Eq. (B15) can be reproduced with for $n+1$ using the identities (B17) and (B18).

This proves

$$p^{\langle l} p^{m_1} \cdots p^{m_n \rangle} = p^l p^{\langle m_1} \cdots p^{m_n \rangle} - \frac{n}{2n+1} p^2 p^{\langle m_1} p^{m_2} \cdots p^{m_{n-1}} \Delta^{m_n \rangle l}, \tag{B20}$$

which can be found in Ref. [63]. In a moving frame, this becomes

$$p^{\langle \lambda \rangle} p^{\langle \mu_1} \cdots p^{\mu_n \rangle} = p^{\langle \lambda} p^{\mu_1} \cdots p^{\mu_n \rangle} + \frac{n}{2n+1} (\mathcal{E}_p^2 - m^2) p^{\langle \mu_1} p^{\mu_2} \cdots p^{\mu_{n-1}} \Delta^{\mu_n \rangle \lambda}. \tag{B21}$$

One can also show

$$\begin{aligned} p^{\langle \alpha \rangle} p^{\langle \lambda \rangle} p^{\langle \mu_1} \cdots p^{\mu_{n-1}} p^{\mu_n \rangle} = {} & p^{\langle \alpha} p^{\lambda} p^{\mu_1} \cdots p^{\mu_{n-1}} p^{\mu_n \rangle} + \frac{1}{(2n+3)} (\mathcal{E}_p^2 - m^2) \sum_{i=1}^{n} \Delta^{\mu_i \alpha} p^{\langle \lambda} p^{\mu_1} \cdots p^{\mu_{i-1}} p^{\mu_{i+1}} \cdots p^{\mu_n \rangle} \\ & + \frac{1}{(2n+3)} (\mathcal{E}_p^2 - m^2) \sum_{i=1}^{n} \Delta^{\mu_i \lambda} p^{\langle \alpha} p^{\mu_1} \cdots p^{\mu_{i-1}} p^{\mu_{i+1}} \cdots p^{\mu_n \rangle} \\ & - \frac{4}{(2n+3)(2n-1)} (\mathcal{E}_p^2 - m^2) \sum_{i<j}^{n} \Delta^{\mu_i \mu_j} p^{\langle \alpha} p^{\lambda} p^{\mu_1} \cdots p^{\mu_{i-1}} p^{\mu_{i+1}} \cdots p^{\mu_{j-1}} p^{\mu_{j+1}} \cdots p^{\mu_n \rangle} \\ & + \frac{1}{(2n+3)} (\mathcal{E}_p^2 - m^2) (\Delta^{\lambda \alpha} p^{\langle \mu_1} \cdots p^{\mu_n \rangle}) \\ & + \frac{n(n-1)}{(2n+1)(2n-1)} (\mathcal{E}_p^2 - m^2)^2 \big(p^{\langle \mu_1} \cdots p^{\mu_{i-1}} p^{\mu_{i+1}} \cdots \Delta^{\mu_{n-1}}_{\alpha'} \Delta^{\mu_n \rangle}_{\lambda'} \big) \Delta^{\alpha \alpha'} \Delta^{\lambda \lambda'} \end{aligned} \tag{B22}$$

by using

$$\begin{aligned} p^{\langle \lambda \rangle} p^{\langle \mu_1} \cdots p^{\mu_{n-1}} \Delta^{\mu_n \rangle \alpha} = {} & \frac{1}{n} \sum_{i=1}^{n} \Delta^{\mu_i \alpha} p^{\langle \lambda} p^{\mu_1} \cdots p^{\mu_{i-1}} p^{\mu_{i+1}} \cdots p^{\mu_n \rangle} \\ & - \frac{2}{n(2n-1)} \sum_{i<j}^{n} \Delta^{\mu_i \mu_j} p^{\langle \lambda} p^{\alpha} p^{\mu_1} \cdots p^{\mu_{i-1}} p^{\mu_{i+1}} \cdots p^{\mu_{j-1}} p^{\mu_{j+1}} \cdots p^{\mu_n \rangle} \\ & + (\mathcal{E}_p^2 - m^2) \frac{n-1}{(2n-1)} p^{\langle \mu_1} \cdots p^{\mu_{n-2}} \Delta^{\mu_{n-1}}_{\lambda'} \Delta^{\mu_n \rangle}_{\alpha'} \Delta^{\lambda \lambda'} \Delta^{\alpha \alpha'}. \end{aligned} \tag{B23}$$

APPENDIX C: A USEFUL MATHEMATICAL IDENTITY

Consider the following rank-n tensor:

$$A^{\mu_1 \cdots \mu_n} = \Delta^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} D p^{\langle \nu_1 \cdots} p^{\nu_n \rangle}. \tag{C1}$$

Following Eqs. (C.8) and (C.9) in Ref. [63], for any symmetric tensor Π we have

$$\begin{aligned} \Pi_{\langle i_1 \cdots i_n \rangle} = {} & \Pi_{i_1 \cdots i_n} + \alpha_{n1} (\Delta_{i_1 i_2} \Pi_{i_3 \cdots i_n kk} + \text{permutation}) \\ & + \alpha_{n2} (\Delta_{i_1 i_2} \Delta_{i_3 i_4} \Pi_{i_5 \cdots i_n kkll} + \text{permutation}) + \cdots, \end{aligned} \tag{C2}$$

where

$$\alpha_{nk} = \frac{(-1)^k}{\Pi_{j=0}^{k-1} (2n - 2j - 1)}. \tag{C3}$$

Now, if we let

$$\Pi_{i_1 \cdots i_n} = p_{\langle i_1 \rangle} \cdots p_{\langle i_n \rangle}, \tag{C4}$$

then all terms in Eq. (C2) except the first one vanish under $\Delta^{i_1 \cdots i_n}_{j_1 \cdots j_n} D$ since

$$\begin{aligned} D(\Delta_{i_k i_l} F) &= (D \Delta_{i_k i_l}) F + \Delta_{i_k i_l} DF \\ &= \big(a_{i_k} u_{i_l} + u_{i_k} a_{i_l} \big) F + \Delta_{i_k i_l} DF. \end{aligned} \tag{C5}$$

This expression vanishes when the projector is applied due to the presence of u_{i_k}, u_{i_l}, or $\Delta_{i_l i_k}$. Consequently, we arrive at the following useful identity:

$$\Delta^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} D p^{\langle \nu_1 \cdots} p^{\nu_n \rangle} = \Delta^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} D p^{\langle \nu_1 \rangle} \cdots p^{\langle \nu_n \rangle}. \tag{C6}$$

Note that

$$\begin{aligned}
Dp^{\langle\mu\rangle} &= D\Delta^{\mu\nu}p_\nu \\
&= D(g^{\mu\nu}+u^\mu u^\nu)p_\nu \\
&= (u^\mu Du^\nu + u^\nu Du^\mu)p_\nu \\
&= u^\mu p_\nu a^\nu - \mathcal{E}_p a^\mu \\
&= u^\mu(p^{\langle\nu\rangle}+\mathcal{E}_p u^\nu)a_\nu - \mathcal{E}_p a^\mu \\
&= u^\mu p^{\langle\nu\rangle}a_\nu - \mathcal{E}_p a^\mu,
\end{aligned} \tag{C7}$$

where the term with u^μ vanishes when being projected. With some simple algebraic manipulations, we get

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} Dp^{\langle\nu_1\cdots}p^{\nu_n\rangle} = -n\mathcal{E}_p p^{\langle\mu_1\cdots}p^{\mu_{n-1}}a^{\mu_n\rangle}. \tag{C8}$$

Similarly, one can also argue for the same reasons

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda(p^{\langle\nu_1\cdots}p^{\nu_n\rangle}) = \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda(p^{\langle\nu_1\rangle}\cdots p^{\langle\nu_n\rangle}) \tag{C9}$$

and

$$\begin{aligned}
\nabla_\lambda p^{\langle\nu\rangle} &= \nabla_\lambda(p^\nu - \mathcal{E}_p u^\nu) \\
&= -u^\nu\nabla_\lambda\mathcal{E}_p - \mathcal{E}_p(\nabla_\lambda u^\nu).
\end{aligned} \tag{C10}$$

Once again, the first term vanishes when being projected. After some manipulations, we get:

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda(p^{\langle\nu_1\cdots}p^{\nu_n\rangle}) = -n\mathcal{E}_p p^{\langle\mu_1}\cdots p^{\mu_{n-1}}\nabla_\lambda u^{\nu\rangle}. \tag{C11}$$

APPENDIX D: DERIVATION OF THE GENERAL MOMENT EQUATION

The starting point is the general rank-n energy-momentum moments of δf:

$$\rho_r^{\mu_1\cdots\mu_n} = \int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^r p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}. \tag{D1}$$

Taking the comoving derivative $D=u^\mu\partial_\mu$, which corresponds to the time derivative in the fluid rest frame, and then projecting onto the transverse space, we get

$$\begin{aligned}
&\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}D\rho_r^{\nu_1\cdots\nu_n} \\
&= \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}(D\delta f)\mathcal{E}_p^r p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&\quad + \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^r Dp^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&\quad + \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\big(D\mathcal{E}_p^r\big)p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&= \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}(D\delta f)\mathcal{E}_p^r p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&\quad - n\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r+1}p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} \\
&\quad - r\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}a_\sigma\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\sigma\rangle}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle},
\end{aligned} \tag{D2}$$

where we defined the fluid acceleration by $a^\mu = Du^\mu$, and used the fact that $D\mathcal{E}_p = -a_\mu p^\mu = -a_\mu p^{\langle\mu\rangle}$, along with Eq. (C8). Using Eq. (B21), we can expand the last term on the right-hand side:

$$\begin{aligned}
&\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}D\rho_r^{\nu_1\cdots\nu_n} \\
&= \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}(D\delta f)\mathcal{E}_p^r p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&\quad - n\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r+1}p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} \\
&\quad - ra_\sigma\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\sigma}p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} \\
&\quad - r\frac{n}{2n+1}a_\sigma\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\big(\mathcal{E}_p^2-m^2\big) \\
&\quad \times p^{\langle\mu_1}p^{\mu_2}\cdots\Delta^{\mu_n\rangle\sigma}.
\end{aligned} \tag{D3}$$

To express $D\delta f$ in terms of δf, we can use the following form of the Boltzmann equation:

$$p^\mu\partial_\mu f_0 + \mathcal{E}_p D\delta f + p^{\langle\mu\rangle}\nabla_\mu\delta f = C[f], \tag{D4}$$

in Eq. (D3) where we used the decomposition

$$\partial_\mu = g_\mu^\alpha\partial_\alpha = \big(-u_\mu u^\alpha + \Delta_\mu^\alpha\big)\partial_\alpha = -u_\mu D + \nabla_\mu. \tag{D5}$$

This gives

$$\begin{aligned}
&\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}D\rho_r^{\nu_1\cdots\nu_n} \\
&= -n\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r+1}p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} \\
&\quad - ra_\sigma\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\sigma}p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} \\
&\quad - r\frac{n}{2n+1}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\big(\mathcal{E}_p^2-m^2\big)p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} \\
&\quad + \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}C[f]\mathcal{E}_p^{r-1}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&\quad - \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}(\partial_\lambda f_0)\mathcal{E}_p^{r-1}p^\lambda p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&\quad - \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}(\nabla_\lambda\delta f)\mathcal{E}_p^{r-1}p^{\langle\lambda\rangle}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}.
\end{aligned} \tag{D6}$$

Here, we define $\nabla_\mu = \Delta_\mu^\nu\partial_\nu$ as the projected derivative, corresponding to the spatial gradient in the fluid rest frame. Using the chain rule, we can pull ∇_λ in the last term on the right-hand side of Eq. (D6) out of the integral:

$$\begin{aligned}
\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} D\rho_r^{\nu_1\cdots\nu_n} = &-n\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r+1}p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} - ra_\sigma\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\sigma}p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} \\
&- r\frac{n}{2n+1}\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\big(\mathcal{E}_p^2-m^2\big)p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} + \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p}C[f]\mathcal{E}_p^{r-1}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&- \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p}(\partial_\lambda f_0)\mathcal{E}_p^{r-1}p^\lambda p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} - \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda\rangle}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&+ \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p}\delta f\big(\nabla_\lambda\mathcal{E}_p^{r-1}\big)p^{\langle\lambda\rangle}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} + \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}(\nabla_\lambda p^{\langle\lambda\rangle})p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&+ \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda\rangle}(\nabla_\lambda p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}).
\end{aligned} \tag{D7}$$

Now, note that the second-last term on the right-hand side can be simplified as

$$\begin{aligned}
\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\big(\nabla_\lambda p^{\langle\lambda\rangle}\big)p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} &= \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\nabla_\lambda\big(p^\lambda-\mathcal{E}_pu^\lambda\big)p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&= -\theta\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle},
\end{aligned} \tag{D8}$$

since $\nabla_\lambda p^\lambda = 0$ and $u^\lambda\nabla_\lambda\mathcal{E}_p = u^\lambda\Delta^\alpha_\lambda\partial_\alpha\mathcal{E}_p = 0$. Here, we define $\theta = \partial_\mu u^\mu = \nabla_\mu u^\mu$, which represents the expansion rate of the fluid.

To briefly summarize, so far we have

$$\begin{aligned}
\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} D\rho_r^{\nu_1\cdots\nu_n} = &-n\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r+1}p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} - ra_\sigma\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\sigma}p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} \\
&- r\frac{n}{2n+1}\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\big(\mathcal{E}_p^2-m^2\big)p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} + \int \frac{d^3p}{(2\pi)^3E_p}C[f]\mathcal{E}_p^{r-1}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} \\
&- \int \frac{d^3p}{(2\pi)^3E_p}(\partial_\lambda f_0)\mathcal{E}_p^{r-1}p^\lambda p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} - \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda\rangle}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle} \\
&+ \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda\rangle}(\nabla_\lambda p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}) - \theta\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^{r}\delta f p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} \\
&+ \int \frac{d^3p}{(2\pi)^3E_p}\delta f\big(\nabla_\lambda\mathcal{E}_p^{r-1}\big)p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}.
\end{aligned} \tag{D9}$$

We continue to simplify the last three terms by calculating the gradients. Observe that

$$\begin{aligned}
\nabla_\lambda\mathcal{E}_p^{r-1} &= (r-1)\mathcal{E}_p^{r-2}(\nabla_\lambda\mathcal{E}_p) \\
&= -(r-1)\mathcal{E}_p^{r-2}\nabla_\lambda(u_\alpha p^\alpha) \\
&= -(r-1)\mathcal{E}_p^{r-2}p^\alpha\nabla_\lambda u_\alpha \\
&= -(r-1)\mathcal{E}_p^{r-2}(p^{\langle\alpha\rangle}+\mathcal{E}_pu^\alpha)\nabla_\lambda u_\alpha \\
&= -(r-1)\mathcal{E}_p^{r-2}p^{\langle\alpha\rangle}\nabla_\lambda u_\alpha,
\end{aligned} \tag{D10}$$

using the normalization condition $u_\alpha u^\alpha = -1$. Plugging this into Eq. (D9) gives

$$\begin{aligned}
\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} D\rho_r^{\nu_1\cdots\nu_n} = &-n\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r+1}p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} - ra_\sigma\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\sigma}p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} \\
&- r\frac{n}{2n+1}\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\big(\mathcal{E}_p^2-m^2\big)p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle} + \int \frac{d^3p}{(2\pi)^3E_p}C[f]\mathcal{E}_p^{r-1}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} \\
&- \int \frac{d^3p}{(2\pi)^3E_p}(\partial_\lambda f_0)\mathcal{E}_p^{r-1}p^\lambda p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} - \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\int \frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda\rangle}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}
\end{aligned}$$

$$+\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda\rangle}(\nabla_\lambda p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle})-\theta\int\frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r\delta f p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}$$
$$-(r-1)\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}(\nabla_\lambda u_\alpha)p^{\langle\alpha\rangle}p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}. \tag{D11}$$

Now, using Eq. (C11) proven in Appendix C, the third-to-last term on the right-hand side can be written as

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda\rangle}(\nabla_\lambda p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle})=-n\int\frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r\delta f p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots\nabla_\lambda u^{\mu_n\rangle}. \tag{D12}$$

Equation (D11) now becomes

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}D\rho_r^{\nu_1\cdots\nu_n}=-n\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r+1}p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle}-ra_\sigma\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\sigma}p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}$$
$$-r\frac{n}{2n+1}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\left(\mathcal{E}_p^2-m^2\right)p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle}+\int\frac{d^3p}{(2\pi)^3E_p}C[f]\mathcal{E}_p^{r-1}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}$$
$$-\int\frac{d^3p}{(2\pi)^3E_p}(\partial_\lambda f_0)\mathcal{E}_p^{r-1}p^\lambda p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}-\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda\rangle}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}$$
$$-n\int\frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r\delta f p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots\nabla_\lambda u^{\mu_n\rangle}-\theta\int\frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r\delta f p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}$$
$$-(r-1)\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}(\nabla_\lambda u_\alpha)p^{\langle\alpha\rangle}p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}. \tag{D13}$$

Applying Eq. (B21) again to the sixth term on the right-hand side, we get

$$-\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda\rangle}p^{\langle\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}=-\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda}p^{\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}$$
$$-\frac{n}{2n+1}\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\left(\mathcal{E}_p^2-m^2\right)p^{\langle\nu_1}p^{\nu_2}\cdots\Delta^{\nu_n\rangle\lambda}. \tag{D14}$$

Plugging this back into Eq. (D13) gives us

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}D\rho_r^{\nu_1\cdots\nu_n}=\int\frac{d^3p}{(2\pi)^3E_p}C[f]\mathcal{E}_p^{r-1}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}-\int\frac{d^3p}{(2\pi)^3E_p}(\partial_\lambda f_0)\mathcal{E}_p^{r-1}p^\lambda p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}$$
$$-n\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r+1}p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle}-r\frac{n}{2n+1}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\left(\mathcal{E}_p^2-m^2\right)p^{\langle\mu_1}p^{\mu_2}\cdots a^{\mu_n\rangle}$$
$$-ra_\sigma\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\sigma}p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}-\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}p^{\langle\lambda}p^{\nu_1}p^{\nu_2}\cdots p^{\nu_n\rangle}$$
$$-\frac{n}{2n+1}\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-1}\left(\mathcal{E}_p^2-m^2\right)p^{\langle\nu_1}p^{\nu_2}\cdots\Delta^{\nu_n\rangle\lambda}-n\int\frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r\delta f p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots\nabla_\lambda u^{\mu_n\rangle}$$
$$-\theta\int\frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r\delta f p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}-(r-1)\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}(\nabla_\lambda u_\alpha)p^{\langle\alpha\rangle}p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}. \tag{D15}$$

Using the definition of the moments, we get

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}D\rho_r^{\nu_1\cdots\nu_n}=\int\frac{d^3p}{(2\pi)^3E_p}C[f]\mathcal{E}_p^{r-1}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}-\int\frac{d^3p}{(2\pi)^3E_p}(\partial_\lambda f_0)\mathcal{E}_p^{r-1}p^\lambda p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}$$
$$-\theta\rho_r^{\mu_1\cdots\mu_n}-\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\rho_{r-1}^{\lambda\nu_1\cdots\nu_n}-\frac{n}{2n+1}\left(\nabla^{\langle\mu_1}\rho_{r+1}^{\mu_2\cdots\mu_n\rangle}-m^2\nabla^{\langle\mu_1}\rho_{r-1}^{\mu_2\cdots\mu_n\rangle}\right)$$

$$- r a_\alpha \rho_{r-1}^{\alpha\mu_1\cdots\mu_n} + r\frac{n}{2n+1} m^2 \rho_{r-1}^{\langle\mu_1\cdots\mu_{n-1}} a^{\mu_n\rangle} - \frac{n(r+2n+1)}{2n+1}\rho_{r+1}^{\langle\mu_1\cdots\mu_{n-1}} a^{\mu_n\rangle}$$
$$- n\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f p^{\langle\lambda\rangle} p^{\langle\mu_1} p^{\mu_2}\cdots \nabla_\lambda u^{\mu_n\rangle} - (r-1)\int \frac{d^3p}{(2\pi)^3E_p}\delta f \mathcal{E}_p^{r-2}(\nabla_\lambda u_\alpha) p^{\langle\alpha\rangle} p^{\langle\lambda\rangle} p^{\langle\mu_1} p^{\mu_2}\cdots p^{\mu_n\rangle}. \tag{D16}$$

Now, we can further expand the term $-n\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f p^{\langle\lambda\rangle} p^{\langle\mu_1} p^{\mu_2}\cdots \nabla_\lambda u^{\mu_n\rangle}$ as the follows:

$$\begin{aligned}
&- n\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f p^{\langle\lambda\rangle} p^{\langle\mu_1} p^{\mu_2}\cdots \nabla_\lambda u^{\mu_n\rangle}\\
&= -\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f \left(\sum_{i=1}^n (\nabla_\lambda u^{\mu_i}) p^{\langle\lambda\rangle} p^{\langle\mu_1\cdots} p^{\mu_{i-1}} p^{\mu_{i+1}\cdots} p^{\mu_n\rangle}\right)\\
&\quad + \frac{2}{2n-1}\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f \left(\sum_{i\neq j}^n \Delta^{\mu_i\mu_j}(\nabla_\lambda u_\alpha) p^{\langle\lambda\rangle} p^{\langle\alpha} p^{\mu_1\cdots} p^{\mu_{i-1}} p^{\mu_{i+1}\cdots} p^{\mu_{j-1}} p^{\mu_{j+1}\cdots} p^{\mu_n\rangle}\right),
\end{aligned} \tag{D17}$$

where we used

$$p^{\langle\mu_1\cdots} p^{\mu_{n-1}} a^{\mu_n\rangle} = \frac{1}{n}\sum_{i=1}^n a^{\langle\mu_i\rangle} p^{\langle\mu_1\cdots} p^{\mu_{i-1}} p^{\mu_{i+1}\cdots} p^{\mu_n\rangle} - \frac{2}{n(2n-1)}\sum_{i\neq j}^n \Delta^{\mu_i\mu_j} a_{\langle\lambda\rangle} p^{\langle\lambda} p^{\mu_1\cdots} p^{\mu_{i-1}} p^{\mu_{i+1}\cdots} p^{\mu_{j-1}} p^{\mu_{j+1}\cdots} p^{\mu_n\rangle}, \tag{D18}$$

in which $a^{\langle\mu\rangle}$ is an arbitrary transverse vector. This identity comes frmo Eq. (A6). Using Eq. (B21) to combine the angular brackets, we can further expand Eq. (D17) as

$$\begin{aligned}
&- n\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f p^{\langle\lambda\rangle} p^{\langle\mu_1} p^{\mu_2}\cdots \nabla_\lambda u^{\mu_n\rangle}\\
&= -\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f \sum_{i=1}^n (\nabla_\lambda u^{\mu_i}) p^{\langle\lambda} p^{\mu_1\cdots} p^{\mu_{i-1}} p^{\mu_{i+1}\cdots} p^{\mu_n\rangle}\\
&\quad - \frac{n-1}{2n-1}\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f \sum_{i=1}^n (\nabla_\lambda u^{\mu_i})\big(\mathcal{E}_p^2 - m^2\big) p^{\langle\mu_1\cdots} p^{\mu_{i-1}} p^{\mu_{i+1}\cdots} \Delta^{\mu_n\rangle\lambda}\\
&\quad + \frac{2}{2n-1}\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f \sum_{i\neq j}^n \Delta^{\mu_i\mu_j}(\nabla_\lambda u_\alpha) p^{\langle\lambda} p^{\alpha} p^{\mu_1\cdots} p^{\mu_{i-1}} p^{\mu_{i+1}\cdots} p^{\mu_{j-1}} p^{\mu_{j+1}\cdots} p^{\mu_n\rangle}\\
&\quad + \frac{2(n-1)}{(2n-1)^2}\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f \sum_{i\neq j}^n \Delta^{\mu_i\mu_j}(\nabla_\lambda u_\alpha)\big(\mathcal{E}_p^2 - m^2\big) p^{\langle\alpha} p^{\mu_1\cdots} p^{\mu_{i-1}} p^{\mu_{i+1}\cdots} p^{\mu_{j-1}} p^{\mu_{j+1}\cdots} \Delta^{\mu_n\rangle\lambda},
\end{aligned} \tag{D19}$$

which can be written in terms of the moments:

$$\begin{aligned}
&- n\int \frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r \delta f p^{\langle\lambda\rangle} p^{\langle\mu_1} p^{\mu_2}\cdots \nabla_\lambda u^{\mu_n\rangle}\\
&= -\sum_{i=1}^n (\nabla_\lambda u^{\mu_i}) \rho_r^{\lambda\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n} + \frac{2}{2n-1}\sum_{i\neq j}^n \Delta^{\mu_i\mu_j}(\nabla_\lambda u_\alpha)\rho_r^{\lambda\alpha\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_n}\\
&\quad - \frac{n-1}{2n-1}\sum_{i=1}^n \rho_{r+2}^{\langle\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{n-1}}\nabla^{\mu_n\rangle} u^{\mu_i} + \frac{2(n-1)}{(2n-1)^2}\sum_{i\neq j}^n \Delta^{\mu_i\mu_j}\rho_{r+2}^{\langle\alpha\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_{n-1}}\nabla^{\mu_n\rangle} u_\alpha\\
&\quad + \frac{m^2(n-1)}{2n-1}\sum_{i=1}^n \rho_r^{\langle\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{n-1}}\nabla^{\mu_n\rangle} u^{\mu_i} - \frac{2m^2(n-1)}{(2n-1)^2}\sum_{i\neq j}^n \Delta^{\mu_i\mu_j}\rho_r^{\langle\alpha\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_{n-1}}\nabla^{\mu_n\rangle} u_\alpha\\
&= -\sum_{i=1}^n \nabla_\lambda u^{\langle\mu_i}\rho_r^{\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n\rangle\lambda} - \frac{n-1}{2n-1}\sum_{i=1}^n \rho_{r+2}^{\langle\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{n-1}}\sigma^{\mu_n\mu_i\rangle} + \frac{m^2(n-1)}{2n-1}\sum_{i=1}^n \rho_r^{\langle\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{n-1}}\sigma^{\mu_n\mu_i\rangle},
\end{aligned} \tag{D20}$$

where

$$\sigma^{\mu\nu} = \nabla^{\langle\mu}u^{\nu\rangle} \tag{D21}$$

is the symmetric Navier-Stokes shear tensor. Since the angular bracket represents the traceless and symmetric combination of the Lorentz indices, all permutations of the Lorentz indices inside the bracket give the same term. Thus,

$$-n\int\frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r\delta f p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots\nabla_\lambda u^{\mu_n\rangle} = -n\rho_r^{\lambda\langle\mu_1\cdots\mu_{n-1}}\nabla_\lambda u^{\mu_n\rangle} - \frac{n(n-1)}{2n-1}\rho_{r+2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle} + \frac{m^2(n-1)n}{2n-1}\rho_r^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}. \tag{D22}$$

Here, we can replace $\nabla_\lambda u^{\mu_n}$ using

$$\nabla^\lambda u^\mu = \sigma^{\mu\nu} + \omega^{\mu\nu} + \frac{\theta}{3}\Delta^{\mu\nu}, \tag{D23}$$

where

$$\omega^{\mu\nu} = \tfrac{1}{2}(\nabla^\mu u^\nu - \nabla^\nu u^\mu) \tag{D24}$$

is the antisymmetric vorticity tensor. Doing so gives us

$$\begin{aligned}-n\int\frac{d^3p}{(2\pi)^3E_p}\mathcal{E}_p^r\delta f p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots\nabla_\lambda u^{\mu_n\rangle} &= -n\rho_r^{\lambda\langle\mu_1\cdots\mu_{n-1}}\sigma_\lambda^{\mu_n\rangle} - n\rho_r^{\lambda\langle\mu_1\cdots\mu_{n-1}}\omega_\lambda^{\mu_n\rangle} - \frac{n}{3}\theta\rho_r^{\mu_1\cdots\mu_n}\\ &\quad - \frac{n(n-1)}{2n-1}\rho_{r+2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle} + \frac{m^2(n-1)n}{2n-1}\rho_r^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}.\end{aligned} \tag{D25}$$

Now let us go back to the general moment equation Eq. (D16) and take a look at the term $-(r-1)\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}(\nabla_\lambda u_\alpha)p^{\langle\alpha\rangle}p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}$. Using Eq. (D23), this term can be written as

$$\begin{aligned}&-(r-1)\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}(\nabla_\lambda u_\alpha)p^{\langle\alpha\rangle}p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}\\ &= -(r-1)\sigma_{\lambda\alpha}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}p^{\langle\alpha\rangle}p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} - \frac{(r-1)}{3}\theta\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}\big(\mathcal{E}_p^2-m^2\big)p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}\\ &= -(r-1)\sigma_{\lambda\alpha}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}p^{\langle\alpha\rangle}p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle} - \frac{(r-1)}{3}\theta\rho_r^{\mu_1\cdots\mu_n} + \frac{(r-1)m^2}{3}\theta\rho_{r-2}^{\mu_1\cdots\mu_n}.\end{aligned} \tag{D26}$$

Note that the term with $\omega_{\lambda\alpha}$ vanishes due to its antisymmetric property. We then proceed to expand the first term on the right-hand side using Eq. (B22)

$$\begin{aligned}&-(r-1)\sigma_{\lambda\alpha}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}p^{\langle\alpha\rangle}p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}\\ &= -(r-1)\sigma_{\lambda\alpha}\rho_{r-2}^{\alpha\lambda\mu_1\cdots\mu_n} - \frac{2(r-1)}{2n+3}\sum_{i=1}^{n}\sigma_\alpha^{\mu_i}\rho_r^{\alpha\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n} + \frac{4(r-1)}{(2n+3)(2n-1)}\sum_{i\neq j}^{n}\Delta^{\mu_i\mu_j}\sigma_{\lambda\alpha}\rho_r^{\alpha\lambda\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_n}\\ &\quad + \frac{2m^2(r-1)}{2n+3}\sum_{i=1}^{n}\sigma_\alpha^{\mu_i}\rho_{r-2}^{\alpha\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n} - \frac{4m^2(r-1)}{(2n+3)(2n-1)}\sum_{i\neq j}^{n}\Delta^{\mu_i\mu_j}\sigma_{\lambda\alpha}\rho_{r-2}^{\alpha\lambda\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_n}\\ &\quad - \frac{(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_{r+2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle} + \frac{2m^2(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_r^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle} - \frac{m^4(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_{r-2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}.\end{aligned} \tag{D27}$$

Note that each pair of summations give the traceless and symmetric combination of $\sigma_\alpha^{\mu_i}$ and $\rho_r^{\alpha\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n}$. Thus this reduces to

$$\begin{aligned}&-(r-1)\sigma_{\lambda\alpha}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}p^{\langle\alpha\rangle}p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}\\ &= -(r-1)\sigma_{\lambda\alpha}\rho_{r-2}^{\alpha\lambda\mu_1\cdots\mu_n} - \frac{2(r-1)}{2n+3}\sum_{i=1}^{n}\sigma_\alpha^{\langle\mu_i}\rho_r^{\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n\rangle\alpha} + \frac{2m^2(r-1)}{2n+3}\sum_{i=1}^{n}\sigma_\alpha^{\langle\mu_i}\rho_{r-2}^{\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n\rangle\alpha}\\ &\quad - \frac{(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_{r+2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle} + \frac{2m^2(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_r^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle} - \frac{m^4(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_{r-2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}.\end{aligned} \tag{D28}$$

Since all permutations of the Lorentz indices inside the angular brackets give the same term, this can be simplified to

$$
\begin{aligned}
&-(r-1)\sigma_{\lambda\alpha}\int\frac{d^3p}{(2\pi)^3E_p}\delta f\mathcal{E}_p^{r-2}p^{\langle\alpha\rangle}p^{\langle\lambda\rangle}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}\\
&=-(r-1)\sigma_{\lambda\alpha}\rho_{r-2}^{\alpha\lambda\mu_1\cdots\mu_n}-\frac{2(r-1)n}{2n+3}\rho_r^{\alpha\langle\mu_1\cdots\mu_{n-1}}\sigma_\alpha^{\mu_n\rangle}+\frac{2m^2(r-1)n}{2n+3}\rho_{r-2}^{\alpha\langle\mu_1\cdots\mu_{n-1}}\sigma_\alpha^{\mu_n\rangle}\\
&\quad-\frac{(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_{r+2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}+\frac{2m^2(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_r^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}-\frac{m^4(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_{r-2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}.
\end{aligned}
\tag{D29}
$$

Plugging all the above results back into Eq. (D16) and expressing everything in terms of the moments, we arrive at the final form of the general moment equation:

$$
\begin{aligned}
\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}D\rho_r^{\nu_1\cdots\nu_n}&=\int\frac{d^3p}{(2\pi)^3E_p}C[f]\mathcal{E}_p^{r-1}p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}-\int\frac{d^3p}{(2\pi)^3E_p}(\partial_\lambda f_0)\mathcal{E}_p^{r-1}p^\lambda p^{\langle\mu_1}p^{\mu_2}\cdots p^{\mu_n\rangle}\\
&\quad-\frac{n(2n+r+1)}{2n+1}\rho_{r+1}^{\langle\mu_1\cdots\mu_{n-1}}a^{\mu_n\rangle}+rm^2\frac{n}{2n+1}\rho_{r-1}^{\langle\mu_1\cdots\mu_{n-1}}a^{\mu_n\rangle}-ra_\lambda\rho_{r-1}^{\lambda\mu_1\cdots\mu_n}-\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\nabla_\lambda\rho_{r-1}^{\lambda\nu_1\cdots\nu_n}\\
&\quad-\frac{n}{2n+1}\nabla^{\langle\mu_1}\rho_{r+1}^{\mu_2\cdots\mu_n\rangle}+m^2\frac{n}{2n+1}\nabla^{\langle\mu_1}\rho_{r-1}^{\mu_2\cdots\mu_n\rangle}-\frac{n+r+2}{3}\theta\rho_r^{\mu_1\cdots\mu_n}\\
&\quad-(r-1)\sigma_{\lambda\alpha}\rho_{r-2}^{\alpha\lambda\mu_1\cdots\mu_n}+\frac{(r-1)m^2}{3}\theta\rho_{r-2}^{\mu_1\cdots\mu_n}-\frac{n(2n+2r+1)}{2n+3}\rho_r^{\lambda\langle\mu_1\cdots\mu_{n-1}}\sigma_\lambda^{\mu_n\rangle}\\
&\quad-n\rho_r^{\lambda\langle\mu_1\cdots\mu_{n-1}}\omega_\lambda^{\mu_n\rangle}-\frac{(2n+r)(n-1)n}{(2n-1)(2n+1)}\rho_{r+2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}+2m^2\frac{(r-1)n}{2n+3}\rho_{r-2}^{\lambda\langle\mu_1\cdots\mu_{n-1}}\sigma_\lambda^{\mu_n\rangle}\\
&\quad-m^4\frac{(r-1)(n-1)n}{(2n+1)(2n-1)}\rho_{r-2}^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}+m^2\frac{(2n+2r-1)(n-1)n}{(2n+1)(2n-1)}\rho_r^{\langle\mu_1\cdots\mu_{n-2}}\sigma^{\mu_{n-1}\mu_n\rangle}.
\end{aligned}
\tag{D30}
$$

APPENDIX E: F INTEGRALS AND ϕ, φ, ψ COEFFICIENTS

To evaluate the F integrals, we first need to know the conservation laws. The stress-energy tensor is

$$T^{\mu\nu}=\int\frac{d^3p}{(2\pi)^3E_p}f_0p^\mu p^\nu+\pi^{\mu\nu}+\Pi\Delta^{\mu\nu}. \tag{E1}$$

The energy-momentum conservation law is

$$
\begin{aligned}
0&=\partial_\mu T^{\mu\nu}\\
&=\int\frac{d^3p}{(2\pi)^3E_p}(\partial_\mu f_0)p^\mu p^\nu+\partial_\mu\pi^{\mu\nu}+(\nabla^\nu\Pi)\\
&\quad+\Pi(u^\nu\theta+a^\nu),
\end{aligned}
\tag{E2}
$$

where we used

$$
\begin{aligned}
\partial_\mu\Delta^{\mu\nu}&=\partial_\mu(u^\mu u^\nu)\\
&=u^\nu\theta+a^\nu,
\end{aligned}
\tag{E3}
$$

and we specify $f_0=e^{-\beta\mathcal{E}_p}$.

In the time direction $u_\nu\partial_\mu T^{\mu\nu}=0$ yields

$$0=-\pi^{\mu\nu}\sigma_{\mu\nu}-\theta\Pi+I_{3,0}D\beta-\frac{\beta}{3}\theta I_{3,1}, \tag{E4}$$

where we defined

$$I_{n,k}=\int\frac{d^3p}{(2\pi)^3E_p}f_0\mathcal{E}_p^{n-2k}\left(\mathcal{E}_p^2-m^2\right)^k, \tag{E5}$$

which can be evaluated in the local rest frame where $\mathcal{E}_p\to E_p$ and $(\mathcal{E}_p^2-m^2)\to p^2$. This integral is always finite when $m\neq 0$ and $k\geqslant 0$. In the $m\to 0$ limit, the integral behaves as $\log(m\beta)$ for $n=-2$, and $I_{n,k}\sim T^{n+2}$.

Using integration by part, it can be shown that

$$
\begin{aligned}
\beta I_{n,k}&=\beta\int\frac{d^3p}{(2\pi)^3E_p}E_p^{n-2k}p^{2k}e^{-\beta E_p}\\
&=-\int\frac{d^3p}{(2\pi)^3E_p}E_p^{n-2k+1}p^{2k-1}\partial_p e^{-\beta E_p}\\
&=(2k+1)I_{n-1,k-1}+(n-2k)I_{n-1,k},
\end{aligned}
\tag{E6}
$$

as long as all integrals are finite. In particular

$$
\begin{aligned}
\beta I_{3,1}&=3I_{2,0}+I_{2,1}\\
&=3(\varepsilon+P).
\end{aligned}
\tag{E7}
$$

In the spatial direction $\Delta^\rho_\nu\partial_\mu T^{\mu\nu}=0$ yields

$$0=\Delta^\rho_\nu\partial_\mu\pi^{\mu\nu}+(\nabla^\rho\Pi)+a^\rho\Pi-I_{3,1}\frac{\nabla^\rho\beta}{3}+\frac{\beta a^\rho}{3}I_{3,1}. \tag{E8}$$

Solving for the time derivatives $D\beta$ and $a^\rho=Du^\rho$, we obtain

$$D\beta=\chi_{\beta|0}\theta+\chi^{\pi\Pi}_{\beta|1}(\pi^{\gamma\rho}\sigma_{\gamma\rho}+\Pi\theta), \tag{E9}$$

where

$$\chi_{\beta|0}=\frac{\beta}{3}\frac{I_{3,1}}{I_{3,0}}\quad\text{and}\quad\chi^{\pi\Pi}_{\beta|1}=\frac{1}{I_{3,0}}. \tag{E10}$$

From Eq. (5), we get

$$a^\rho = \frac{1}{\Pi + (\varepsilon + P)}\left[-\nabla^\rho P - (\nabla^\rho \Pi) - \Delta^\rho_\nu \partial_\mu \pi^{\mu\nu}\right]$$
$$\approx \frac{1}{\varepsilon + P}\left[-\nabla^\rho P - (\nabla^\rho \Pi) - \Delta^\rho_\nu \partial_\mu \pi^{\mu\nu}\right] - \frac{\Pi}{(\varepsilon + P)^2}(-\nabla^\rho P), \tag{E11}$$

where we used

$$\frac{\nabla \beta}{3} I_{3,1} = -\nabla P. \tag{E12}$$

The zeroth-order acceleration is

$$a^\rho_{|0} = -\frac{\nabla^\rho P}{\varepsilon + P} = \nabla^\rho \beta \frac{I_{3,1}}{3(\varepsilon + P)} = \frac{\nabla^\rho \beta}{\beta}, \tag{E13}$$

and the first-order one satisfies

$$Q^\rho_\sigma a^\sigma_{|1} = \frac{1}{\varepsilon + P}\left(-\nabla^\rho \Pi - \Delta^\rho_\nu \nabla_\mu \pi^{\mu\nu}\right) - \frac{1}{(\varepsilon + P)^2}\Pi(-\nabla^\rho P), \tag{E14}$$

where

$$Q^\rho_\sigma = g^\rho_\sigma + \frac{1}{\varepsilon + P}\pi^\rho_\sigma. \tag{E15}$$

Now, observe that the only nonzero F integrals are the spin 0, 1, and 2 integrals. The scalar integral is

$$F_r = \int \frac{d^3p}{(2\pi)^3 E_p}\mathcal{E}^r_p p^\lambda(\partial_\lambda f_0)$$
$$= \int \frac{d^3p}{(2\pi)^3 E_p}\mathcal{E}^r_p f_0\left(-\mathcal{E}^2_p D\beta + \beta\frac{\theta}{3}\left(\mathcal{E}^2_p - m^2\right)\right)$$
$$= -I_{r+2,0}D\beta + \frac{\beta}{3}\theta I_{r+2,1}$$
$$= \phi_{r|0}\theta + \phi^{\pi\Pi}_{r|1}(\pi^{\rho\gamma}\sigma_{\rho\gamma} + \theta\Pi), \tag{E16}$$

where

$$\phi_{r|0} = \frac{\beta}{3}\left(I_{r+2,1} - \frac{I_{r+2,0}I_{3,1}}{I_{3,0}}\right), \tag{E17}$$

and

$$\phi^{\pi\Pi}_{r|1} = -\frac{I_{r+2,0}}{I_{3,0}} \tag{E18}$$

using Eq. (E9). Note that $\phi_{1|0} = 0$ and $\phi^{\pi\Pi}_{1|1} = -1$. In the massless limit, we have

$$I_{r,k} = \int \frac{d^3p}{(2\pi)^3}p^{r-1}e^{-p/T} = \frac{T^{r+2}}{2\pi^2}(r+1)!. \tag{E19}$$

For the 14 moments, we need F_{-1} whose coefficients are

$$\phi_{-1|0} = -4\frac{T^2}{2\pi^2}, \tag{E20}$$

and

$$\phi^{\pi\Pi}_{-1|1} = -\tfrac{1}{6}\beta^2. \tag{E21}$$

The vector integral is

$$F^\sigma_r = \int \frac{d^3p}{(2\pi)^3 E_p}\mathcal{E}^r_p p^\lambda(\partial_\lambda f_0)p^{\langle\sigma\rangle}$$
$$= \int \frac{d^3p}{(2\pi)^3 E_p}\mathcal{E}^r_p f_0(-\mathcal{E}_p p^{\langle\lambda\rangle}\nabla_\lambda\beta + \mathcal{E}_p\beta p^{\langle\lambda\rangle}a_\lambda)p^{\langle\sigma\rangle}$$
$$= \psi_{r|1}\left(\Delta^\rho_\nu\partial_\mu\pi^{\mu\nu} + \nabla^\sigma\Pi + a^\sigma\Pi\right), \tag{E22}$$

where we used a slight different form of Eq. (E11):

$$\beta a^\rho - \nabla^\rho\beta = -\frac{3}{I_{3,1}}\left[\Delta^\rho_\nu\partial_\mu\pi^{\mu\nu} + (\nabla^\rho\Pi) + a^\rho\Pi\right]. \tag{E23}$$

The coefficient is

$$\psi_{r|1} = -\frac{I_{r+3,1}}{I_{3,1}}. \tag{E24}$$

Note that $\psi_{0|1} = -1$. Here, $I_{3,1} = 3(\varepsilon + P)T$ and $I_{2,1} = 3P$ can be used if needed. With $r = -1$ and $m = 0$,

$$\psi_{-1|1} = -\tfrac{1}{4}\beta. \tag{E25}$$

The spin-2 integral is relatively simple

$$F^{\sigma\gamma}_r = \int \frac{d^3p}{(2\pi)^3 E_p}\mathcal{E}^r_p p^\lambda(\partial_\lambda f_0)p^{\langle\sigma}p^{\gamma\rangle}$$
$$= \int \frac{d^3p}{(2\pi)^3 E_p}\mathcal{E}^r_p f_0\left(\beta p^{\langle\lambda}p^{\alpha\rangle}\nabla_\lambda u_\alpha\right)p^{\langle\sigma}p^{\gamma\rangle}$$
$$= \varphi_{r|0}\sigma^{\sigma\gamma}, \tag{E26}$$

where

$$\varphi_{r|0} = \tfrac{2}{15}\beta I_{r+4,2} = \tfrac{2}{15}(5I_{r+3,1} + rI_{r+3,2}) \tag{E27}$$

is obtained with the help of the normalization condition Eq. (B6) (see also Refs. [31,62]), and Eq. (E6). With $r = -1$ and $m = 0$,

$$\varphi_{-1|0} = \frac{16}{5}\frac{T^4}{2\pi^2} = \frac{8}{15}\varepsilon. \tag{E28}$$

Let us check whether Landau conditions $\rho_2 = 0$ and $\rho^\mu_1 = 0$ are consistent with the F integrals. Setting $r = 2$ in Eq. (38), we get

$$D\rho_2 = -\frac{\rho_2}{\tau_R} - F_1 + m^2\frac{\theta}{3}\rho_0 - \frac{4}{3}\theta\rho_2 - \nabla_\lambda\rho^\lambda_1 - ra_\lambda\rho^\lambda_1 - \sigma_{\lambda\alpha}\rho^{\alpha\lambda}_0. \tag{E29}$$

Setting $r = 1$ in Eq. (49), we get

$$\Delta^{\mu_1}_{\nu_1}D\rho^{\nu_1}_1 = -\frac{\rho^{\mu_1}_1}{\tau_R} - F^{\mu_1}_0 - \frac{4}{3}\theta\rho^{\mu_1}_1 - a_\alpha\rho^{\alpha\mu_1}_0 - \Delta^{\mu_1}_{\nu_1}\nabla_\lambda\rho^{\lambda\nu_1}_0 - \omega^{\mu_1}_\lambda\rho^\lambda_1 + \frac{1}{3}(m^2\rho_0 - 4\rho_2)a^{\mu_1} - \frac{1}{3}(\nabla^{\mu_1}\rho_2 - m^2\nabla^{\mu_1}\rho_0) - \rho^\lambda_1\sigma^{\mu_1}_\lambda. \tag{E30}$$

Using $\pi^{\mu\nu} = \rho^{\mu\nu}_0$, $\Pi = -m^2\rho_0/3$ as well as

$$F_1 = -(\pi^{\rho\sigma}\sigma_{\rho\sigma} + \theta\Pi) \tag{E31}$$

and

$$F_0^\mu = -\left(\Delta^\mu_\nu \nabla_\lambda \pi^{\lambda\nu} + a_\lambda \pi^{\lambda\mu} + \nabla^\rho \Pi + a^\rho \Pi\right), \tag{E32}$$

these evolution equations become

$$D\rho_2 = -\frac{\rho_2}{\tau_R} - \frac{4}{3}\theta\rho_2 - \nabla_\lambda \rho_1^\lambda - r a_\lambda \rho_1^\lambda, \tag{E33}$$

and

$$\Delta^{\mu_1}_{\nu_1} D\rho_1^{\nu_1} = -\frac{\rho_1^{\mu_1}}{\tau_R} - \frac{4}{3}\theta\rho_1^{\mu_1} - \omega^{\mu_1}_{\lambda}\rho_1^\lambda - \frac{4}{3}\rho_2 a^{\mu_1} - \frac{1}{3}\nabla^{\mu_1}\rho_2 - \rho_1^\lambda \sigma^{\mu_1}_\lambda. \tag{E34}$$

Hence as long as the initial values for ρ_2 and ρ_1^μ all vanish, ρ_2 and ρ_1^μ remain zero.

[1] S. Jeon and U. Heinz, Introduction to hydrodynamics, Int. J. Mod. Phys. E **24**, 1530010 (2015).
[2] C. Eckart, The thermodynamics of irreversible processes. III. Relativistic theory of the simple fluid, Phys. Rev. **58**, 919 (1940).
[3] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics - Course of Theoretical Physics: Volume 6*, 2nd ed. (Butterworth-Heinemann, 1987).
[4] W. A. Hiscock and L. Lindblom, Generic instabilities in first-order dissipative relativistic fluid theories, Phys. Rev. D **31**, 725 (1985).
[5] W. A. Hiscock and L. Lindblom, Linear plane waves in dissipative relativistic fluids, Phys. Rev. D **35**, 3723 (1987).
[6] W. A. Hiscock and L. Lindblom, Nonlinear pathologies in relativistic heat-conducting fluid theories, Phys. Lett. A **131**, 509 (1988).
[7] G. S. Denicol, T. Kodama, T. Koide, and P. Mota, Stability and causality in relativistic dissipative hydrodynamics, J. Phys. G: Nucl. Part. Phys. **35**, 115102 (2008).
[8] W. A. Hiscock and L. Lindblom, Stability and causality in dissipative relativistic fluids, Ann. Phys. (NY) **151**, 466 (1983).
[9] S. Pu, T. Koide, and D. H. Rischke, Does stability of relativistic dissipative fluid dynamics imply causality? Phys. Rev. D **81**, 114039 (2010).
[10] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Causality and existence of solutions of relativistic viscous fluid dynamics with gravity, Phys. Rev. D **98**, 104064 (2018).
[11] P. Kovtun, First-order relativistic hydrodynamics is stable, J. High Energy Phys. 10 (2019) 034.
[12] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Nonlinear causality of general first-order relativistic viscous hydrodynamics, Phys. Rev. D **100**, 104020 (2019); Nonlinear causality of general first-order relativistic viscous hydrodynamics [Phys. Rev. D 100, 104020 (2019)], **105**, 069902(E) (2022).
[13] R. E. Hoult and P. Kovtun, Stable and causal relativistic Navier-Stokes equations, J. High Energy Phys. 06 (2020) 067.
[14] F. S. Bemfica, M. M. Disconzi, and J. Noronha, First-order general-relativistic viscous fluid dynamics, Phys. Rev. X **12**, 021044 (2022).
[15] R. E. Hoult and P. Kovtun, Causal first-order hydrodynamics from kinetic theory and holography, Phys. Rev. D **106**, 066023 (2022).
[16] F. S. Bemfica, M. Martinez, and M. Shokri, Causality and stability in first-order conformal anisotropic hydrodynamics, Phys. Rev. D **108**, 056004 (2023).
[17] A. Das, W. Florkowski, J. Noronha, and R. Ryblewski, Equivalence between first-order causal and stable hydrodynamics and Israel-Stewart theory for boost-invariant systems with a constant relaxation time, Phys. Lett. B **806**, 135525 (2020).
[18] T. Dore, L. Gavassino, D. Montenegro, M. Shokri, and G. Torrieri, Fluctuating relativistic dissipative hydrodynamics as a gauge theory, Ann. Phys. (NY) **442**, 168902 (2022).
[19] I. Möller, Zum paradoxon der warmeleitungstheorie, Eur. Phys. J. A **198**, 329 (1967).
[20] W. Israel and J. M. Stewart, Thermodynamics of nonstationary and transient effects in a relativistic gas, Phys. Lett. A **58**, 213 (1976).
[21] W. Israel, Nonstationary irreversible thermodynamics: A causal relativistic theory, Ann. Phys. (NY) **100**, 310 (1976).
[22] W. Israel and J. M. Stewart, Transient relativistic thermodynamics and kinetic theory, Ann. Phys. (NY) **118**, 341 (1979).
[23] H. Grad, On the kinetic theory of rarefied gases, Commun. Pure Appl. Math. **2**, 331 (1949).
[24] T. S. Olson, Stability and causality in the Israel-Stewart energy frame theory, Ann. Phys. (NY) **199**, 18 (1990).
[25] C. V. Brito and G. S. Denicol, Linear stability of Israel-Stewart theory in the presence of net-charge diffusion, Phys. Rev. D **102**, 116009 (2020).
[26] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, 3rd ed. (Cambridge University Press, 1970).
[27] A. Jaiswal, Relativistic dissipative hydrodynamics from kinetic theory with relaxation time approximation, Phys. Rev. C **87**, 051901(R) (2013).
[28] D. Dash, S. Jaiswal, S. Bhadury, and A. Jaiswal, Relativistic second-order viscous hydrodynamics from kinetic theory with extended relaxation-time approximation, Phys. Rev. C **108**, 064913 (2023).
[29] G. S. Denicol, T. Koide, and D. H. Rischke, Dissipative relativistic fluid dynamics: A new way to derive the equations of motion from kinetic theory, Phys. Rev. Lett. **105**, 162501 (2010).
[30] G. S. Denicol, H. Niemi, E. Molnár, and D. H. Rischke, Derivation of transient relativistic fluid dynamics from the Boltzmann equation, Phys. Rev. D **85**, 114047 (2012); Derivation of transient relativistic fluid dynamics from the Boltzmann equation [Phys. Rev. D **85**, 114047 (2012)], **91**, 039902(E) (2015).
[31] G. S. Denicol, E. Molnár, H. Niemi, and D. H. Rischke, Derivation of fluid dynamics from kinetic theory with the 14-moment approximation, Eur. Phys. J. A **48**, 170 (2012).
[32] A. El, Z. Xu, and C. Greiner, Extension of relativistic dissipative hydrodynamics to third order, Phys. Rev. C **81**, 041901(R) (2010).
[33] C. Chattopadhyay, A. Jaiswal, S. Pal, and R. Ryblewski,

Relativistic third-order viscous corrections to the entropy four-current from kinetic theory, Phys. Rev. C **91**, 024917 (2015).
[34] M. Younus and A. Muronga, Third order viscous hydrodynamics from the entropy four current, Phys. Rev. C **102**, 034902 (2020).
[35] A. Jaiswal, Relativistic third-order dissipative fluid dynamics from kinetic theory, Phys. Rev. C **88**, 021903(R) (2013).
[36] A. Jaiswal, Relaxation-time approximation and relativistic third-order viscous hydrodynamics from kinetic theory, Nucl. Phys. A **931**, 1205 (2014).
[37] C. V. Brito and G. S. Denicol, Linear causality and stability of third-order relativistic dissipative fluid dynamics, Phys. Rev. D **105**, 096026 (2022).
[38] P. Panday, A. Jaiswal, and B. K. Patra, Causal third-order viscous hydrodynamics within relaxation-time approximation, Phys. Rev. D **109**, 096039 (2024).
[39] C. V. P. de Brito and G. S. Denicol, Third-order relativistic dissipative fluid dynamics from the method of moments, Phys. Rev. D **108**, 096020 (2023).
[40] H. Struchtrup and M. Torrilhon, Regularization of Grad's 13 moment equations: Derivation and linear analysis, Phys. Fluids **15**, 2668 (2003).
[41] H. Struchtrup, Stable transport equations for rarefied gases at high orders in the Knudsen number, Phys. Fluids **16**, 3921 (2004).
[42] H. Struchtrup and M. Torrilhon, Regularized 13-moment equations: Shock structure calculations and comparison to Burnett models, J. Fluid Mech. **513**, 171 (2004).
[43] H. Struchtrup and M. Torrilhon, *H* theorem, regularization, and boundary conditions for linearized 13 moment equations, Phys. Rev. Lett. **99**, 014502 (2007).
[44] D. S. Ye, On the linear stability and causality of relativistic regularized hydrodynamics, Master's thesis, McGill University, 2023.
[45] C. V. P. de Brito and G. S. Denicol, Method of moments for a relativistic single-component gas, arXiv:2401.10098.
[46] L. Gavassino, M. Antonelli, and B. Haskell, When the entropy has no maximum: A new perspective on the instability of the first-order theories of dissipation, Phys. Rev. D **102**, 043018 (2020).
[47] M. Shokri and F. Taghinavaz, Conformal Bjorken flow in the general frame and its attractor: Similarities and discrepancies with the Müller-Israel-Stewart formalism, Phys. Rev. D **102**, 036022 (2020).
[48] F. S. Bemfica, M. M. Disconzi, V. Hoang, J. Noronha, and M. Radosz, Nonlinear constraints on relativistic fluids far from equilibrium, Phys. Rev. Lett. **126**, 222301 (2021).
[49] A. Santos, J. J. Brey, and J. W. Dufty, Divergence of the Chapman-Enskog expansion, Phys. Rev. Lett. **56**, 1571 (1986).
[50] B. C. Eu, On the modified moment method and irreversible thermodynamics, J. Chem. Phys. **85**, 1592 (1986).
[51] B. C. Eu and K. Mao, Kinetic theory approach to irreversible thermodynamics of radiation and matter, Phys. A (Amsterdam, Neth.) **180**, 65 (1992).
[52] D. Vasak, M. Gyulassy, and H. T. Elze, Quantum transport theory for Abelian plasmas, Ann. Phys. (NY) **173**, 462 (1987).
[53] J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang, and X.-N. Wang, Chiral anomaly and local polarization effect from quantum kinetic approach, Phys. Rev. Lett. **109**, 232301 (2012).
[54] Y. Hidaka, S. Pu, and D.-L. Yang, Relativistic chiral kinetic theory from quantum field theories, Phys. Rev. D **95**, 091901(R) (2017).
[55] W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza, Spin-dependent distribution functions for relativistic hydrodynamics of spin-1/2 particles, Phys. Rev. D **97**, 116017 (2018).
[56] S. Shi, C. Gale, and S. Jeon, From chiral kinetic theory to relativistic viscous spin hydrodynamics, Phys. Rev. C **103**, 044906 (2021).
[57] S. Li, M. A. Stephanov, and H.-U. Yee, Nondissipative second-order transport, spin, and pseudogauge transformations in hydrodynamics, Phys. Rev. Lett. **127**, 082302 (2021).
[58] H.-H. Peng, J.-J. Zhang, X.-L. Sheng, and Q. Wang, Ideal spin hydrodynamics from the Wigner function approach, Chin. Phys. Lett. **38**, 116701 (2021).
[59] J. Hu, Relativistic first-order spin hydrodynamics via the Chapman-Enskog expansion, Phys. Rev. D **105**, 076009 (2022).
[60] G. Sarwar, M. Hasanujjaman, J. R. Bhatt, H. Mishra, and J. Alam, Causality and stability of relativistic spin hydrodynamics, Phys. Rev. D **107**, 054031 (2023).
[61] M. Kröger, *Models for Polymeric and Anisotropic Liquids* (Springer, 2005).
[62] S. R. de Groot, W. A. van Leeuwen, and C. G. van Weert, *Relativistic Kinetic Theory: Principles and Applications* (North-Holland Publishing Company, 1980).
[63] H. Struchtrup, Zur irreversiblen thermodynamikder strahlung, Ph.D. thesis, Technical University of Berlin, 1996.