Symmetric cumulant $sc_{2,4}\{4\}$ and asymmetric cumulant $ac_2\{3\}$ from transverse momentum conservation and flow

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Multiparticle cumulants method can be used to reveal long-range collectivity in small and large colliding systems. The four-particle symmetric cumulant $sc_{2,4}\{4\}$, three-particle asymmetric cumulant $ac_2\{3\}$, and the normalized cumulants $nsc_{2,4}\{4\}$ and $nac_2\{3\}$ from the transverse momentum conservation and flow are calculated. The interplay between the two effects is also investigated. Our results are in a good agreement with the recent ATLAS measurements of multiparticle azimuthal correlations with the subevent cumulant method, which provides insight into the origin of collective flow in small systems.

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I. INTRODUCTION

High-energy nucleus-nucleus ($A + A$) collisions at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) can create an extremely dense and hot environment in which confined quarks and gluons are released into a deconfined state of matter called the quark-gluon plasma (QGP) [1–5]. One of the most significant experimental signatures of the QGP properties is the collective flow due to its sensitivity to the dynamical evolution of the QGP, which can transfer the asymmetries in the initial geometry space into the anisotropies in the final momentum space [6–10]. The magnitude of the azimuthal anisotropy in the transverse plane of the final momentum space can be quantified in terms of the Fourier expansion coefficient, $\frac{dN}{d\phi} \propto 1 + \sum_n v_n \cos[n(\phi - \Psi_n)]$ [11–13], where the anisotropic flow coefficients v_1, v_2, v_3, and v_4 are directed, elliptic, triangular, and quadrangular flows, respectively. Studies of collective flow have shown that the QGP is a nearly perfect fluid with strong coupling, i.e., the ratio of the shear viscosity to the entropy density η/s is close to the minimum value of $1/4\pi$ [14–16]. In addition to the collective flow, anisotropic flow in measurements contains nonflow, which includes the short-range correlation such as jets, resonance decays, and Bose-Einstein correlation, and long-range correlation such as the transverse momentum conservation (TMC) [17–22].

There are several methods for experimentally extracting flow coefficients, such as the event plane method requiring the estimation of reaction plane [12,23], two-particle correlation associated with ridge structure [24,25], or multiparticle cumulants suppressing nonflow [26,27]. The appearance of ridge structure with small azimuthal separation extending far in the longitudinal direction is considered to be direct evidence for the presence of collective flow, which was first observed in the two-particle correlation between the pseudorapidity gap $\Delta\eta$ and the azimuthal angle gap $\Delta\phi$ of the particle pairs in $A + A$ collisions [28–31]. This has been verified as the signature of the collective flow of the final particles in $A + A$ collisions, which has been reproduced by hydrodynamic models [32–36]. However, similar ridge structures have also been observed in small systems (e.g., $p + p$, $p + A$), posing a major challenge to previous understanding [37–43], as the applicability of hydrodynamics to small systems is controversial due to their extremely small size and short lifetime [44–49]. Recently, many theoretical models have been employed to study anisotropic flows in small systems to understand their origins, including the final-state hydrodynamics in response to geometry asymmetries in the initial state [50–54], the parton escape mechanism with similar hydrodynamics [55–58], and the color glass condensate (CGC) as an initial state mechanism [59–65]. In addition, both hydrodynamic and transport models have been used to study $c_2\{2\}$ and $c_2\{4\}$, since the multiparticle cumulant method can suppress nonflow contributions [66–70].

Previous studies have found that there is a linear relationship between the flow v_n and the corresponding eccentricity ε_n, i.e., $v_n \propto \varepsilon_n$ [71,72]. However, it has been argued that the set of flow coefficients $\{v_n\}$ and the set of eccentricities $\{\varepsilon_n\}$ can be linked by a response matrix, which implies that

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there is a nonlinear correlation between v_m and v_n [73–75]. The symmetric cumulants $sc_{n,m}\{4\} = \langle v_n^2 v_m^2\rangle - \langle v_n^2\rangle\langle v_m^2\rangle$ can be inscribed to carry a correlation between v_n and v_m capable of responding to the geometrical shape eccentricities ε_n and ε_m of the initial state phase during the evolution of the QGP, in addition to information about the interactions of the final state [76–78]. Moreover, $ac_n\{3\} = \langle v_n^2 v_{2n} \cos 2n(\Psi_n - \Psi_{2n})\rangle$ involves not only the correlation between flow harmonics v_n and v_{2n} but also the correlation between event planes Ψ_n and Ψ_{2n}. The generalized symmetric cumulants $sc_{k,l,m}\{6\}$ have been proposed to explore the collectivity in large systems such as Pb+Pb collisions at LHC energies [79]. The ALICE experiment measured $sc_{4,2}\{4\}$ and $sc_{3,2}\{4\}$ and found that there is a positive correlation between v_2 and v_4, and a negative correlation between v_2 and v_3 [80], which has shown that $sc_{n,m}\{4\}$ is very sensitive to the temperature dependence of η/s in noncentral collisions [81]. To suppress the nonflow contribution to $sc_{n,m}\{4\}$ and $ac_2\{3\}$, a subevent method has been proposed, in which particles in different pseudorapidity intervals are divided into two or more subevents. The subevent method has been performed in $p+p$ and p+Pb collisions using PYTHIA and HIJING models, which shows that the subevent method can indeed suppress nonflow contributions [82,83]. The recent ATLAS experimental results have demonstrated that the signal of four-particle symmetric cumulant $sc_{2,4}\{4\}$ and three-particle asymmetric cumulant $ac_2\{3\}$ gradually decreases from the standard method to the subevent method, as a result of the effective suppression of nonflow contribution from jets [84].

In this paper, we calculate the four-particle symmetric cumulant $sc_{2,4}\{4\}$, three-particle asymmetric cumulant $ac_2\{3\}$, and their normalized cumulants $nsc_{2,4}\{4\}$ and $nac_2\{3\}$ based on transverse momentum conservation and collective flow. Compared to the recent ATLAS experimental measurements with the subevent method, we aim to understand and explore the origin of the collectivity in small systems.

II. $sc_{2,4}\{4\}$ AND $ac_2\{3\}$ FROM TRANSVERSE MOMENTUM CONSERVATION

First, we summarize the calculation method of the TMC, which is assumed to be the only effect of correlations between final particles. The k-particle probability distribution $f(\vec{p}_1,\ldots,\vec{p}_k)$ for the N-particle system with imposed transverse momentum conservation is given by [85–88]

$$f(\vec{p}_1,\ldots,\vec{p}_k) = f(\vec{p}_1)\cdots f(\vec{p}_k)\frac{N}{N-k} \times \exp\left(-\frac{(\vec{p}_1+\cdots+\vec{p}_k)^2}{(N-k)\langle p^2\rangle_F}\right), \quad (1)$$

where $\langle p^2\rangle_F$ denotes the mean value of p^2 over the full space F,

$$\langle p^2\rangle_F = \frac{\int_F p^2 f(\vec{p})d^2\vec{p}}{\int_F f(\vec{p})d^2\vec{p}}. \quad (2)$$

Our goal is to calculate the four-particle symmetric cumulant $sc_{2,4}\{4\}$ and three-particle asymmetric cumulant $ac_2\{3\}$. The four-particle symmetric cumulant and and three-particle asymmetric cumulant are defined as follows:

$$sc_{2,4}\{4\} = \langle e^{i2(\phi_1-\phi_2)+i4(\phi_3-\phi_4)}\rangle - \langle e^{i2(\phi_1-\phi_2)}\rangle\langle e^{i4(\phi_3-\phi_4)}\rangle, \quad (3)$$

$$ac_2\{3\} = \langle e^{i2(\phi_1+\phi_2-2\phi_3)}\rangle. \quad (4)$$

A. $sc_{2,4}\{4\}$

For four particles, we have

$$f(\vec{p}_1,\ldots,\vec{p}_4) = f(\vec{p}_1)\cdots f(\vec{p}_4)\frac{N}{N-4} \times \exp\left(-\frac{p_1^2+p_2^2+p_3^2+p_4^2}{(N-4)\langle p^2\rangle_F}\right)\exp(-\Phi), \quad (5)$$

where

$$\Phi = \frac{2}{(N-4)\langle p^2\rangle_F}\sum_{i,j=1;i<j}^{4} p_i p_j \cos(\phi_i-\phi_j), \quad (6)$$

and $p_i = |\vec{p}_i|$.

To calculate $\langle e^{i2(\phi_1-\phi_2)+i4(\phi_3-\phi_4)}\rangle$ at given transverse momenta p_1, p_2, p_3, and p_4,

$$\langle e^{i2(\phi_1-\phi_2)+i4(\phi_3-\phi_4)}\rangle|p_1,p_2,p_3,p_4 = \frac{\int_0^{2\pi} e^{i2(\phi_1-\phi_2)+i4(\phi_3-\phi_4)}\exp(-\Phi)d\phi_1\cdots d\phi_4}{\int_0^{2\pi}\exp(-\Phi)d\phi_1\cdots d\phi_4}, \quad (7)$$

we expand $\exp(-\Phi)$ in Φ. In the numerator, the first nonzero term is given by $\Phi^6/720$ and we neglect all higher terms. In the denominator, it is enough to take the first term, $\exp(-\Phi)\approx 1$, since the next terms are suppressed by the power of $1/N$. To simplify our calculation, we assume that all the transverse momentum p_i are equal. In this case, we obtain

$$\langle e^{i2(\phi_1-\phi_2)+i4(\phi_3-\phi_4)}\rangle|p \approx \frac{5p^{12}}{16(N-4)^6\langle p^2\rangle_F^6}. \quad (8)$$

Performing analogous calculations we obtain

$$\langle e^{i2(\phi_1-\phi_2)}\rangle|p \approx \frac{p^4}{2(N-2)^2\langle p^2\rangle_F^2} \quad (9)$$

and

$$\langle e^{i4(\phi_3-\phi_4)}\rangle|p \approx \frac{p^8}{24(N-2)^4\langle p^2\rangle_F^4}. \quad (10)$$

Using Eq. (3) we find

$$sc_{2,4}\{4\} \approx \frac{5p^{12}}{16(N-4)^6\langle p^2\rangle_F^6} - \frac{p^{12}}{48(N-2)^6\langle p^2\rangle_F^6}. \quad (11)$$

B. $ac_2\{3\}$

For three particles, we have

$$f(\vec{p}_1,\ldots,\vec{p}_3) = f(\vec{p}_1)\cdots f(\vec{p}_3)\frac{N}{N-3} \times \exp\left(-\frac{p_1^2+p_2^2+p_3^2}{(N-3)\langle p^2\rangle_F}\right)\exp(-\Phi), \quad (12)$$

where

$$\Phi = \frac{2}{(N-3)\langle p^2\rangle_F} \sum_{i,j=1;i<j}^{3} p_i p_j \cos(\phi_i - \phi_j). \qquad (13)$$

Using Eq. (4) we find

$$ac_2\{3\} \approx \frac{p^8}{4(N-3)^2\langle p^2\rangle_F^4}. \qquad (14)$$

III. $sc_{2,4}\{4\}$ AND $ac_2\{3\}$ FROM TRANSVERSE MOMENTUM CONSERVATION AND FLOW

Next, we calculate the contribution of the TMC and the collective flow to the four-particle symmetric cumulant $sc_{2,4}\{4\}$ and three-particle asymmetric cumulant $ac_2\{3\}$. The particle emission azimuthal angle distribution measured with respect to the reaction plane is characterized by a Fourier expansion,

$$f(p,\phi) = \frac{g(p)}{2\pi}\left(1 + \sum_n 2v_n(p)\cos\left[n(\phi - \Psi_n)\right]\right), \qquad (15)$$

where v_n and Ψ_n denote the nth-order flow coefficient and the reaction plane angle. In our calculations we consider v_2, v_3, and v_4 only.

A. $sc_{2,4}\{4\}$

The four-particle probability distribution with TMC can be written as [87]

$$\begin{aligned} f_4&(p_1,\phi_1,\ldots,p_4,\phi_4) \\ &= f(p_1,\phi_1)\cdots f(p_4,\phi_4)\frac{N}{N-4} \\ &\quad\times \exp\left(-\frac{(p_{1,x}+\cdots+p_{4,x})^2}{2(N-4)\langle p_x^2\rangle_F} - \frac{(p_{1,y}+\cdots+p_{4,y})^2}{2(N-4)\langle p_y^2\rangle_F}\right), \end{aligned} \qquad (16)$$

where

$$p_x = p\cos(\phi), \quad p_y = p\sin(\phi), \qquad (17)$$

$$\langle p_x^2\rangle_F = \tfrac{1}{2}\langle p^2\rangle_F(1+v_{2F}), \quad \langle p_y^2\rangle_F = \tfrac{1}{2}\langle p^2\rangle_F(1-v_{2F}), \qquad (18)$$

$$v_{2F} = \frac{\int_F v_2(p)g(p)p^2 d^2p}{\int_F g(p)p^2 d^2p}. \qquad (19)$$

Using

$$\langle e^{i2(\phi_1-\phi_2)+i4(\phi_3-\phi_4)}\rangle \mid p_1,p_2,p_3,p_4 = \frac{\int_0^{2\pi} e^{i2(\phi_1-\phi_2)+i4(\phi_3-\phi_4)} f_4(p_1,\phi_1,\ldots,p_4,\phi_4)d\phi_1\ldots d\phi_4}{\int_0^{2\pi} f_4(p_1,\phi_1,\ldots,p_4,\phi_4)d\phi_1\ldots d\phi_4}, \qquad (20)$$

$$p_1 = p_2 = p_3 = p_4 = p, \qquad (21)$$

and including all the terms up to the one containing the pure TMC effect, $e^X \approx 1 + X + \frac{X^2}{2} + \frac{X^3}{3!} + \frac{X^4}{4!} + \frac{X^5}{5!} + \frac{X^6}{6!}$, we obtain

$$\langle e^{i2(\phi_1-\phi_2)+i4(\phi_3-\phi_4)}\rangle \mid p \approx A_0 + A_1Y_A + \tfrac{1}{2}A_2Y_A^2 + \tfrac{1}{6}A_3Y_A^3 + \tfrac{1}{24}A_4Y_A^4 + \tfrac{1}{120}A_5Y_A^5 + \tfrac{1}{720}A_6Y_A^6, \qquad (22)$$

where

$$Y_A = -\frac{p^2}{(N-4)\langle p^2\rangle_F\left(1-v_{2F}^2\right)} \qquad (23)$$

and

$$A_0 = v_2^2v_4^2,$$

$$A_1 = v_2^2v_3^2 + 4v_2^2v_4^2 + v_3^2v_4^2 - v_2v_{2F}v_4^2\cos(2\Psi_2) + 2v_2v_3^2v_4\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4),$$

$$\begin{aligned} A_2 = {}& v_2^4 + 16v_2^2v_3^2 + 4v_3^4 + v_4^2 + 30v_2^2v_4^2 + \frac{v_{2F}^2v_4^2}{2} + 16v_3^2v_4^2 + v_4^4 - 6v_2v_{2F}v_3^2\cos(2\Psi_2) - 20v_2v_{2F}v_4^2\cos(2\Psi_2) \\ & + 2v_2^2v_4\cos(4\Psi_2 - 4\Psi_4) - 6v_{2F}v_3^2v_4\cos(6\Psi_3 - 4\Psi_4) + 24v_2v_3^2v_4\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4), \end{aligned}$$

$$\begin{aligned} A_3 = {}& 24v_2^4 + 9v_3^2 + 234v_2^2v_3^2 + \frac{27v_{2F}^2v_3^2}{2} + 72v_3^4 + 24v_4^2 + 304v_2^2v_4^2 + 36v_{2F}^2v_4^2 + 234v_3^2v_4^2 + 24v_4^4 - 21v_2^3v_{2F}\cos(2\Psi_2) \\ & - 210v_2v_{2F}v_3^2\cos(2\Psi_2) - 369v_2v_{2F}v_4^2\cos(2\Psi_2) - 21v_2v_{2F}v_4\cos(2\Psi_2 - 4\Psi_4) + 48v_2^2v_4\cos(4\Psi_2 - 4\Psi_4) \\ & - 168v_{2F}v_3^2v_4\cos(6\Psi_3 - 4\Psi_4) + 300v_2v_3^2v_4\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4), \end{aligned}$$

$$\begin{aligned} A_4 = {}& 49v_2^2 + 436v_2^4 + 147v_2^2v_{2F}^2 + 264v_3^2 + 3328v_2^2v_3^2 + 792v_{2F}^2v_3^2 + 1056v_3^4 + 448v_4^2 + 3625v_2^2v_4^2 + 1344v_{2F}^2v_4^2 \\ & + 3328v_3^2v_4^2 + 436v_4^4 - 872v_2^3v_{2F}\cos(2\Psi_2) - 5004v_2v_{2F}v_3^2\cos(2\Psi_2) - 6728v_2v_{2F}v_4^2\cos(2\Psi_2) \\ & + 45v_2^2v_{2F}^2\cos(4\Psi_2) - 872v_2v_{2F}v_4\cos(2\Psi_2 - 4\Psi_4) + 898v_2^2v_4\cos(4\Psi_2 - 4\Psi_4) - 3492v_{2F}v_3^2v_4\cos(6\Psi_3 - 4\Psi_4) \\ & + 45v_{2F}^2v_4\cos(4\Psi_4) + 3920v_2v_3^2v_4\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4), \end{aligned}$$

$$
\begin{aligned}
A_5 = {} & 1820v_2^2 + 7120v_2^4 + 9100v_2^2v_{2F}^2 + 5545v_3^2 + 47025v_2^2v_3^2 + 27725v_{2F}^2v_3^2 + 14800v_3^4 + 7680v_4^2 + 47004v_2^2v_4^2 \\
& + 38400v_{2F}^2v_4^2 + 47001v_3^2v_4^2 + 7120v_4^4 - 525v_2v_{2F}\cos(2\Psi_2) - 23855v_2^3v_{2F}\cos(2\Psi_2) - \frac{1575}{2}v_2v_{2F}^3\cos(2\Psi_2) \\
& - 102960v_2v_{2F}v_3^2\cos(2\Psi_2) - 120970v_2v_{2F}v_4^2\cos(2\Psi_2) + 2940v_2^2v_{2F}^2\cos(4\Psi_2) - 23895v_2v_{2F}v_4\cos(2\Psi_2 - 4\Psi_4) \\
& + 15480v_2^2v_4\cos(4\Psi_2 - 4\Psi_4) - 65430v_{2F}v_3^2v_4\cos(6\Psi_3 - 4\Psi_4) + 2940v_{2F}^2v_4\cos(4\Psi_4) \\
& + 52730v_2v_3^2v_4\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4), \\
A_6 = {} & 225 + 45396v_2^2 + 110941v_2^4 + \frac{3375v_{2F}^2}{2} + 340470v_2^2v_{2F}^2 + \frac{10125v_{2F}^4}{8} + 102780v_3^2 \\
& + 666216v_2^2v_3^2 + 770850v_{2F}^2v_3^2 + 205956v_3^4 + 126720v_4^2 + 637704v_2^2v_4^2 + 950400v_{2F}^2v_4^2 + 664800v_3^2v_4^2 \\
& + 110686v_4^4 - 28140v_2v_{2F}\cos(2\Psi_2) - 545892v_2^3v_{2F}\cos(2\Psi_2) - 70350v_2v_{2F}^3\cos(2\Psi_2) - 1970532v_2v_{2F}v_3^2\cos(2\Psi_2) \\
& - 2142936v_2v_{2F}v_4^2\cos(2\Psi_2) + 115605v_2^2v_{2F}^2\cos(4\Psi_2) - 548580v_2v_{2F}v_4\cos(2\Psi_2 - 4\Psi_4) \\
& + 257412v_2^2v_4\cos(4\Psi_2 - 4\Psi_4) - 1169892v_{2F}v_3^2v_4\cos(6\Psi_3 - 4\Psi_4) + 113610v_{2F}^2v_4\cos(4\Psi_4) \\
& + 724092v_2v_3^2v_4\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4).
\end{aligned} \tag{24}
$$

Note that the above items are approximate results, where we kept the terms up to v_n^4. The full results and the ratios relative to the full results are shown in Eq. (A1) and the left panel of Fig. 5 in the Appendix, respectively.

Similarly, we have

$$\langle e^{i2(\phi_1-\phi_2)}\rangle|p \approx B_0 + B_1Y_B + \tfrac{1}{2}B_2Y_B^2, \tag{25}$$

where

$$Y_B = -\frac{p^2}{(N-2)\langle p^2\rangle_F\left(1 - v_{2F}^2\right)} \tag{26}$$

and

$$
\begin{aligned}
B_0 &= v_2^2, \\
B_1 &= 2v_2^2 + v_3^2 - v_2v_{2F}\cos(2\Psi_2) - v_2v_{2F}v_4\cos(2\Psi_2 - 4\Psi_4), \\
B_2 &= 1 + 6v_2^2 + \frac{v_{2F}^2}{2} + 3v_2^2v_{2F}^2 + 4v_3^2 + 2v_{2F}^2v_3^2 + v_4^2 + \frac{v_{2F}^2v_4^2}{2} - 8v_2v_{2F}\cos(2\Psi_2) + \frac{1}{2}v_2^2v_{2F}^2\cos(4\Psi_2) \\
&\quad - 8v_2v_{2F}v_4\cos(2\Psi_2 - 4\Psi_4) + 3v_{2F}^2v_4\cos(4\Psi_4).
\end{aligned} \tag{27}
$$

Finally

$$\langle e^{i4(\phi_3-\phi_4)}\rangle|p \approx C_0 + C_1Y_C + \tfrac{1}{2}C_2Y_C^2 + \tfrac{1}{6}C_3Y_C^3 + \tfrac{1}{24}C_4Y_C^4, \tag{28}$$

where

$$Y_C = -\frac{p^2}{(N-2)\langle p^2\rangle_F\left(1 - v_{2F}^2\right)} \tag{29}$$

and

$$
\begin{aligned}
C_0 &= v_4^2, \\
C_1 &= v_3^2 + 2v_4^2 - v_2v_{2F}v_4\cos(2\Psi_2 - 4\Psi_4), \\
C_2 &= v_2^2 + \frac{v_2^2v_{2F}^2}{2} + 4v_3^2 + 2v_{2F}^2v_3^2 + 6v_4^2 + 3v_{2F}^2v_4^2 - 8v_2v_{2F}v_4\cos(2\Psi_2 - 4\Psi_4) + \frac{1}{2}v_{2F}^2v_4\cos(4\Psi_4), \\
C_3 &= 6v_2^2 + 9v_2^2v_{2F}^2 + 15v_3^2 + \frac{45v_{2F}^2v_3^2}{2} + 20v_4^2 + 30v_{2F}^2v_4^2 - 3v_2v_{2F}\cos(2\Psi_2) - \frac{3}{4}v_2v_{2F}^3\cos(2\Psi_2) \\
&\quad - 45v_2v_{2F}v_4\cos(2\Psi_2 - 4\Psi_4) - \frac{45}{4}v_2v_{2F}^3v_4\cos(2\Psi_2 - 4\Psi_4) + 9v_{2F}^2v_4\cos(4\Psi_4) - \frac{1}{4}v_2v_{2F}^3v_4\cos(2\Psi_2 + 4\Psi_4), \\
C_4 &= 1 + 28v_2^2 + 3v_{2F}^2 + 84v_2^2v_{2F}^2 + \frac{3v_{2F}^4}{8} + \frac{21v_2^2v_{2F}^4}{2} + 56v_3^2 + 168v_{2F}^2v_3^2 + 21v_{2F}^4v_3^2 + 70v_4^2 + 210v_{2F}^2v_4^2
\end{aligned}
$$

$$+\frac{105v_{2F}^4v_4^2}{4}-32v_2v_{2F}\cos(2\Psi_2)-24v_2v_{2F}^3\cos(2\Psi_2)+3v_2^2v_{2F}^2\cos(4\Psi_2)+\frac{1}{2}v_2^2v_{2F}^4\cos(4\Psi_2)-v_{2F}^3v_3^2\cos(6\Psi_3)$$
$$-224v_2v_{2F}v_4\cos(2\Psi_2-4\Psi_4)-168v_2v_{2F}^3v_4\cos(2\Psi_2-4\Psi_4)+84v_{2F}^2v_4\cos(4\Psi_4)+14v_{2F}^4v_4\cos(4\Psi_4)$$
$$+\frac{1}{8}v_{2F}^4v_4^2\cos(8\Psi_4)-8v_2v_{2F}^3v_4\cos(2\Psi_2+4\Psi_4). \tag{30}$$

B. $ac_2\{3\}$

The three-particle probability distribution with TMC can be written as

$$f_3(p_1,\phi_1,\dots,p_3,\phi_3)=f(p_1,\phi_1)\cdots f(p_3,\phi_3)\frac{N}{N-3}\exp\left(-\frac{(p_{1,x}+\dots+p_{3,x})^2}{2(N-3)\langle p_x^2\rangle_F}-\frac{(p_{1,y}+\dots+p_{3,y})^2}{2(N-3)\langle p_y^2\rangle_F}\right). \tag{31}$$

We have

$$\langle e^{i2(\phi_1+\phi_2-2\phi_3)}\rangle \mid p=\frac{\int_0^{2\pi} e^{i2(\phi_1+\phi_2-2\phi_3)}f_3(p_1,\phi_1,\dots,p_3,\phi_3)d\phi_1\dots d\phi_3}{\int_0^{2\pi} f_3(p_1,\phi_1,\dots,p_3,\phi_3)d\phi_1\dots d\phi_3}$$
$$\approx D_0+D_1Y_D+\frac{1}{2}D_2Y_D^2+\frac{1}{6}D_3Y_D^3+\frac{1}{24}D_4Y_D^4, \tag{32}$$

where

$$Y_D=-\frac{p^2}{(N-3)\langle p^2\rangle_F\left(1-v_{2F}^2\right)} \tag{33}$$

and

$$D_0=v_2^2v_4\cos(4\Psi_2-4\Psi_4),$$
$$D_1=-v_2v_{2F}v_4\cos(2\Psi_2-4\Psi_4)+3v_2^2v_4\cos(4\Psi_2-4\Psi_4),$$
$$D_2=2v_2^2+4v_3^2+2v_4^2-14v_2v_{2F}v_4\cos(2\Psi_2-4\Psi_4)+15v_2^2v_4\cos(4\Psi_2-4\Psi_4)+\tfrac{3}{2}v_{2F}^2v_4\cos(4\Psi_4),$$
$$D_3=30v_2^2+48v_3^2+30v_4^2-12v_2v_{2F}\cos(2\Psi_2)-177v_2v_{2F}v_4\cos(2\Psi_2-4\Psi_4)+93v_2^2v_4\cos(4\Psi_2-4\Psi_4)+\tfrac{75}{2}v_{2F}^2v_4\cos(4\Psi_4),$$
$$D_4=6+322v_2^2+456v_3^2+320v_4^2-272v_2v_{2F}\cos(2\Psi_2)-2004v_2v_{2F}v_4\cos(2\Psi_2-4\Psi_4)+651v_2^2v_4\cos(4\Psi_2-4\Psi_4)$$
$$+645v_{2F}^2v_4\cos(4\Psi_4). \tag{34}$$

Note that the above items are approximate results, where we kept the terms up to v_n^3. The full results and the ratios relative to the full results are shown in Eq. (A2) and the right panel of Fig. 5 in the Appendix, respectively.

C. $nsc_{2,4}\{4\}$ and $nac_2\{3\}$

The normalized cumulants are defined as follows:

$$nsc_{2,4}\{4\}=\frac{sc_{2,4}\{4\}}{v_2\{2\}^2v_4\{2\}^2}, \tag{35}$$

$$nac_2\{3\}=\frac{ac_2\{3\}}{v_2\{2\}^2\sqrt{v_4\{2\}^2}}, \tag{36}$$

where $v_2\{2\}$ and $v_4\{2\}$ are from Eqs. (25) and (28). The normalized cumulants only reflect the strength of the correlation between v_2 and v_4, whereas the unnormalized cumulants have contributions from both the correlations between the two different flow harmonics and the individual harmonics.

IV. RESULTS

Based on Eqs. (11) and (14), we present the four-particle symmetric cumulants $sc_{2,4}\{4\}$ and three-particle asymmetric cumulants $ac_2\{3\}$ from transverse momentum conservation only as a function of the number of particles N for various values of transverse momenta $p = 0.6, 0.9, 1.2$ GeV in Fig. 1. In our calculation, $\langle p^2\rangle_F = 0.25$ GeV^2. It can be seen that the values of $sc_{2,4}\{4\}$ and $ac_2\{3\}$ from transverse momentum conservation decrease and tend to zero as N increases, and that $sc_{2,4}\{4\}$ and $ac_2\{3\}$ also increase with transverse momenta p, which are consistent with the trends found in Refs. [82,83] using the PYTHIA model. This is a manifestation of the property of the TMC that it is more effective at smaller N and rather negligible at larger N.

According to Eqs. (22), (25), (28), and (32), Fig. 2 shows $sc_{2,4}\{4\}$ and $ac_2\{3\}$ from transverse momentum conservation and flow as a function of the number of particles N for various values of transverse momenta $p = 0.6, 0.9, 1.2$ GeV. In our calculation, we set $v_2 = 0.08$, $v_3 = 0.0175$, $v_4 = 0.08^2$, $\langle p^2\rangle_F = 0.5^2$, $v_{2F} = 0.025$, $\Psi_2 = 0$, $\cos[4(\Psi_4 - \Psi_2)] = 0.8$, and $\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) = -0.15$. The values of the

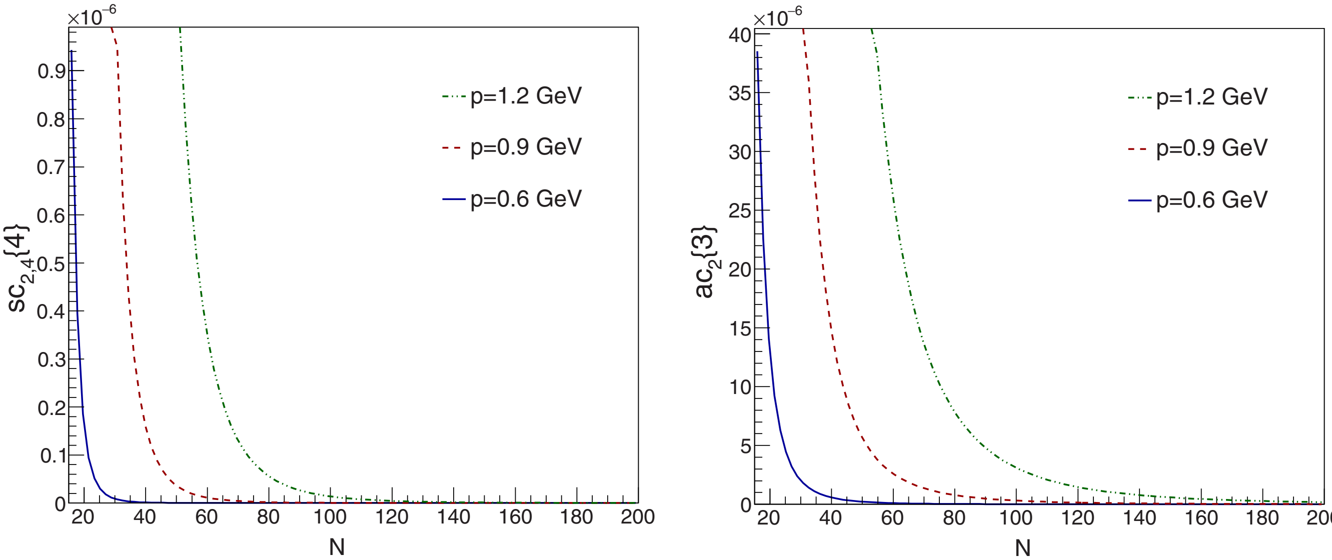

FIG. 1. The four-particle symmetric cumulants $sc_{2,4}\{4\}$ and three-particle asymmetric cumulants $ac_2\{3\}$ from transverse momentum conservation only as a function of the number of particles N for various values of transverse momenta p.

correlations among different combinations of event planes here are from Ref. [89]. We observe that both $sc_{2,4}\{4\}$ and $ac_2\{3\}$ decrease with the increase of multiplicity and their magnitudes are consistent with the data. Note that since the multiplicity N refers to the number of particles under the influence of the TMC rather than the number of experimentally detected charged particles, we multiply the experimental number of charged particles by 1.5 to obtain the total number of particles N in the experimental data in all figures. In comparison with Fig. 1, for larger N, $sc_{2,4}\{4\}$ and $ac_2\{3\}$ do not converge to zero, which is caused by the existence of flow due to hydrodynamics. When N is relatively small, $sc_{2,4}\{4\}$ and $ac_2\{3\}$ increase with increasing momentum p, whereas when N is large, $sc_{2,4}\{4\}$ and $ac_2\{3\}$ hardly change with momentum.

To expound how the TMC and collective flow affect $sc_{2,4}\{4\}$ and $ac_2\{3\}$, Fig. 3 presents the respective contributions from the TMC only (denoted as "TMC"), the TMC and collective flow (denoted as "TMC+flow"), and plus interplay (denoted as "'TMC+flow+interplay") for $p = 0.9$ GeV. Here "TMC" refers to the terms that depend only on N and p, "flow" refers to the terms that depend only on v_n and Ψ_n,

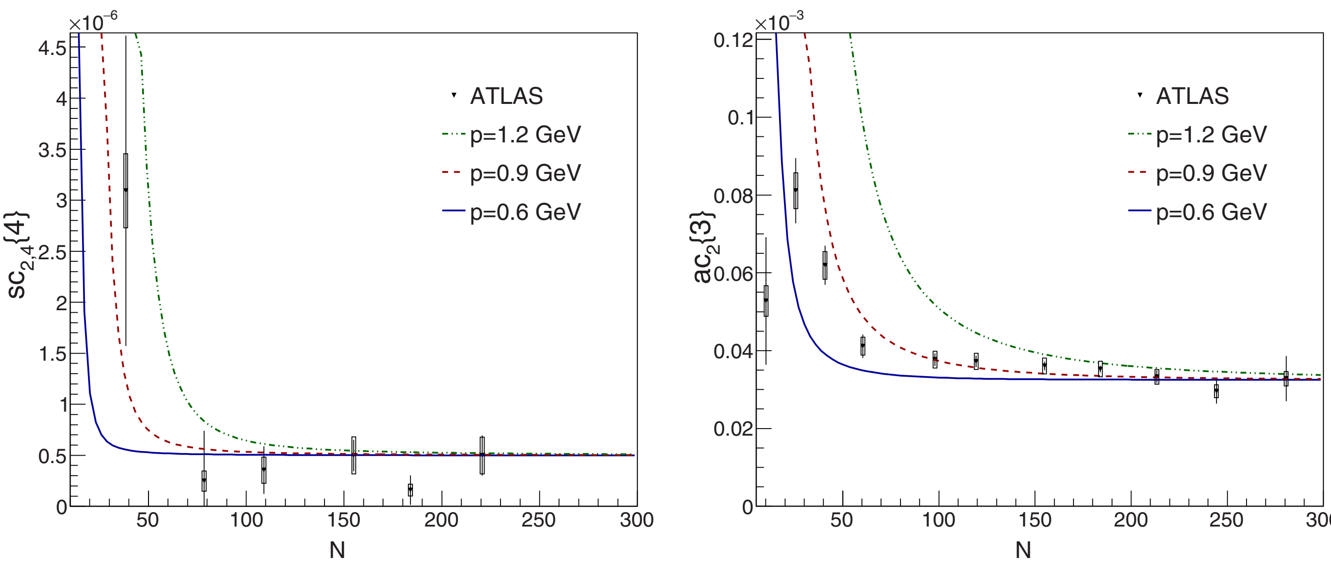

FIG. 2. $sc_{2,4}\{4\}$ and $ac_2\{3\}$ from transverse momentum conservation and flow as a function of the number of particles N for various values of transverse momenta p. The ATLAS data for $0.3 < p_T < 3$ GeV in $p + p$ collisions at 13 TeV using the four-subevent cumulant method or three-subevent cumulant method are shown for comparisons, where the error bars and boxes represent the statistical and systematic uncertainties, respectively [84].

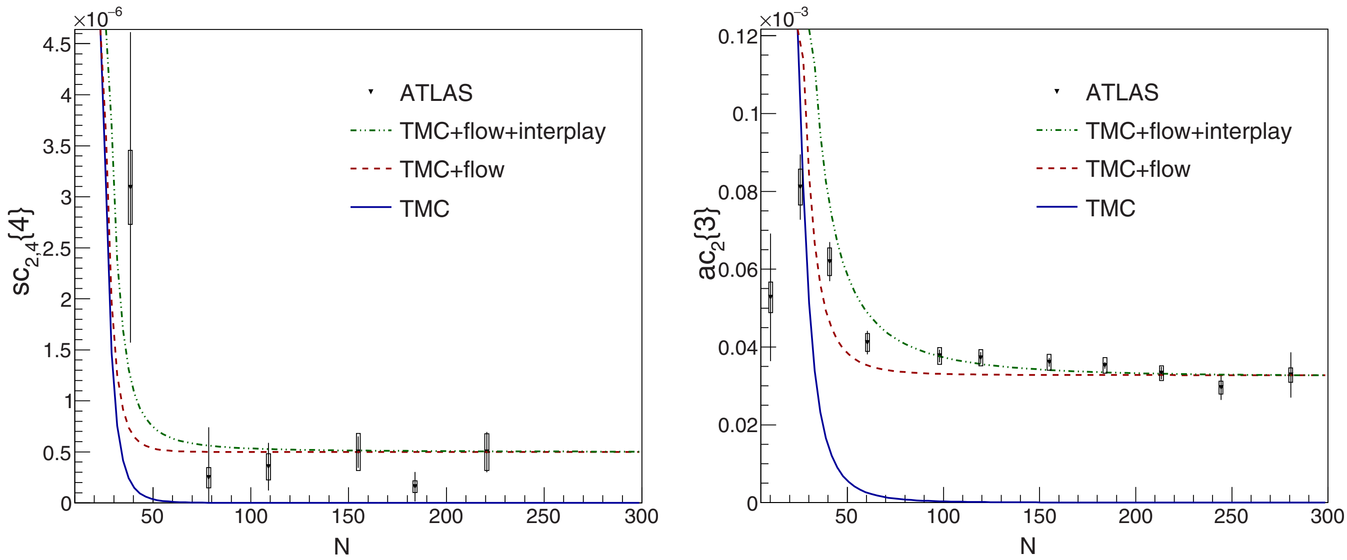

FIG. 3. $sc_{2,4}\{4\}$ and $ac_2\{3\}$ from the TMC, the TMC and collective flow, and plus interplay as a function of the number of particles N for momentum $p = 0.9$ GeV. The ATLAS data for $0.3 < p_T < 3$ GeV in $p + p$ collisions at 13 TeV using the four-subevent cumulant method or three-subevent cumulant method are shown for comparisons, where the error bars and boxes represent the statistical and systematic uncertainties, respectively [84].

and "interplay" refers to terms that depend on both N, p, v_n, and Ψ_n in Eqs. (22), (25), (28), and (32). In Fig. 3, "TMC+flow" means the sum of "TMC" and "flow", and "TMC+flow+interplay" means the combination of all three of the above. We see that collective flow makes the curve higher and the contribution from interplay is present when N is small, but almost negligible when N is large. It can be understood as when N is small, the TMC dominates, and when N is large, the contribution from collective flow becomes significant.

Figure 4 shows that the normalized cumulants $nsc_{2,4}\{4\}$ and $nac_2\{3\}$ from the TMC and flow decrease with the increase of multiplicity, which can basically describe the experimental data. Since Fig. 2 has shown that $sc_{2,4}\{4\}$ and $ac_2\{3\}$ can describe the ATLAS data, it indicates that our results on two-particle $v_2\{2\}$ and $v_4\{2\}$ should also be consistent with the experimental data. In the left plot of Fig. 4, because the TMC contribution is small when N is large, our results close to 2.0 for $nsc_{2,4}\{4\}$ at very large N reflect the correlation between $v_2\{2\}$ and $v_4\{2\}$ produced by hydrodynamics. In the right plot of Fig. 4, we see that $nac_2\{3\}$ is close to 1, suggesting that the event planes Ψ_2 and Ψ_4 gradually converge in the same direction at large N, consistent with the hydrodynamic expectation. The increase in $nsc_{2,4}\{4\}$ and

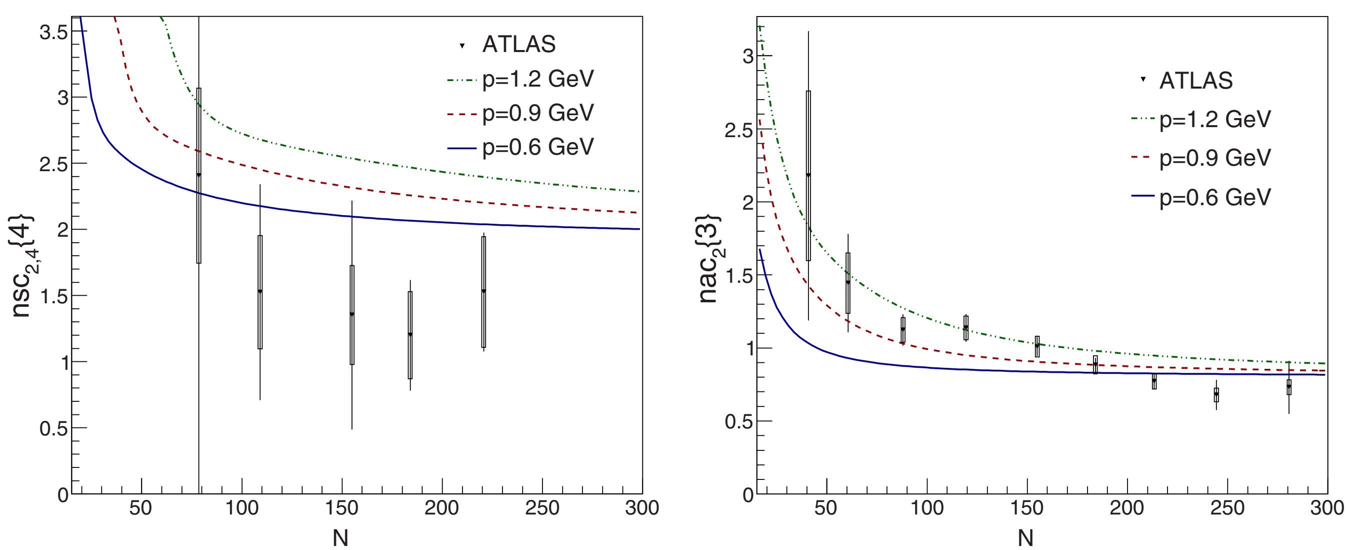

FIG. 4. $nsc_{2,4}\{4\}$ and $nac_2\{3\}$ from the TMC and flow as a function of the number of particles N for various values of transverse momenta p. The ATLAS data for $0.3 < p_T < 3$ GeV in $p + p$ collisions at 13 TeV using the three-subevent cumulant method are shown for comparisons, where the error bars and boxes represent the statistical and systematic uncertainties, respectively [84].

$nac_2\{3\}$ with decreasing N and the increase in $nsc_{2,4}\{4\}$ and $nac_2\{3\}$ with increasing p are both due to the TMC effect.

V. CONCLUSIONS

In this paper, we calculated the four-particle symmetric cumulants $sc_{2,4}\{4\}$, three-particle asymmetric cumulants $ac_2\{3\}$, and the normalized cumulants $nsc_{2,4}\{4\}$ and $nac_2\{3\}$, originating from the transverse momentum conservation and flow. As expected, when the number of particles is small, the correlation comes from the TMC, and when the number of particles is large, the collective flow is dominant. Our results are consistent with the ATLAS data using the subevent cumulant method and therefore allow for a better understanding of collectivity in small systems. In the future, we can calculate the higher order symmetric cumulants $sc_{k,l,m}\{6\}$ in the same way to understand how the TMC and collective flow affects the coupling between v_k, v_l, and v_m in small systems.

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APPENDIX

The full results of the Eq. (24) are as follows:

$$
\begin{aligned}
A_0 =\;& v_2^2 v_4^2,\\
A_1 =\;& v_2^2 v_3^2 + 4v_2^2 v_4^2 + v_3^2 v_4^2 - v_2 v_{2F} v_4^2\cos(2\Psi_2) + 2v_2 v_3^2 v_4\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) - v_2^3 v_{2F} v_4\cos(2\Psi_2 - 4\Psi_4)\\
& - 2v_2 v_{2F} v_3^2 v_4\cos(2\Psi_2 - 4\Psi_4) - v_2 v_{2F} v_4^3\cos(2\Psi_2 - 4\Psi_4),\\
A_2 =\;& v_2^4 + 16v_2^2 v_3^2 + 4v_3^4 + v_4^2 + 30v_2^2 v_4^2 + \frac{v_{2F}^2 v_4^2}{2} + 16v_3^2 v_4^2 + v_4^4 - 6v_2 v_{2F} v_3^2\cos(2\Psi_2) - 20v_2 v_{2F} v_4^2\cos(2\Psi_2)\\
& + 2v_2^2 v_4\cos(4\Psi_2 - 4\Psi_4) - 6v_{2F} v_3^2 v_4\cos(6\Psi_3 - 4\Psi_4) + 24v_2 v_3^2 v_4\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) + \frac{v_2^4 v_{2F}^2}{2}\\
& + 8v_2^2 v_{2F}^2 v_3^2 + 2v_{2F}^2 v_3^4 + \frac{v_{2F}^2 v_4^2}{2} + 15v_2^2 v_{2F}^2 v_4^2 + 8v_{2F}^2 v_3^2 v_4^2 + \frac{v_{2F}^2 v_4^4}{2} + \frac{1}{2}v_2^2 v_{2F}^2 v_4^2\cos(4\Psi_2)\\
& - 6v_2^2 v_{2F} v_3^2\cos(4\Psi_2 - 6\Psi_3) + 3v_2^2 v_{2F}^2 v_4^2\cos(4\Psi_2 - 8\Psi_4) - 6v_{2F} v_3^2 v_4^2\cos(6\Psi_3 - 8\Psi_4)\\
& - 20v_2^3 v_{2F} v_4\cos(2\Psi_2 - 4\Psi_4) - 48v_2 v_{2F} v_3^2 v_4\cos(2\Psi_2 - 4\Psi_4) - 20v_2 v_{2F} v_4^3\cos(2\Psi_2 - 4\Psi_4)\\
& + v_2^2 v_{2F}^2 v_4\cos(4\Psi_2 - 4\Psi_4) + \frac{7}{2}v_2^2 v_{2F}^2 v_4\cos(4\Psi_4) + 6v_{2F}^2 v_3^2 v_4\cos(4\Psi_4) + 3v_{2F}^2 v_4^3\cos(4\Psi_4)\\
& + 12v_2 v_{2F}^2 v_3^2 v_4\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4),\\
A_3 =\;& 24v_2^4 + 9v_3^2 + 234v_2^2 v_3^2 + \frac{27v_{2F}^2 v_3^2}{2} + 72v_3^4 + 24v_4^2 + 304v_2^2 v_4^2 + 36v_{2F}^2 v_4^2 + 234v_3^2 v_4^2 + 24v_4^4 - 21v_2^3 v_{2F}\cos(2\Psi_2)\\
& - 210v_2 v_{2F} v_3^2\cos(2\Psi_2) - 369v_2 v_{2F} v_4^2\cos(2\Psi_2) - 21v_2 v_{2F} v_4\cos(2\Psi_2 - 4\Psi_4) + 48v_2^2 v_4\cos(4\Psi_2 - 4\Psi_4)\\
& - 168v_{2F} v_3^2 v_4\cos(6\Psi_3 - 4\Psi_4) + 300v_2 v_3^2 v_4\cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) + 36v_2^4 v_{2F}^2 + 351v_2^2 v_{2F}^2 v_3^2 + 108v_{2F}^2 v_3^4\\
& + 36v_{2F}^2 v_4^2 + 456v_2^2 v_{2F}^2 v_4^2 + 351v_{2F}^2 v_3^2 v_4^2 + 36v_{2F}^2 v_4^4 - \frac{21}{4}v_2^3 v_{2F}^3\cos(2\Psi_2) - \frac{105}{2}v_2 v_{2F}^3 v_3^2\cos(2\Psi_2)\\
& - \frac{369}{4}v_2 v_{2F}^3 v_4^2\cos(2\Psi_2) + \frac{15}{2}v_2^2 v_{2F}^2 v_3^2\cos(4\Psi_2) + 24v_2^2 v_{2F}^2 v_4^2\cos(4\Psi_2) + \frac{105}{2}v_2 v_{2F}^2 v_3^2\cos(2\Psi_2 - 6\Psi_3)\\
& - 168v_2^2 v_{2F} v_3^2\cos(4\Psi_2 - 6\Psi_3) - 42v_2^2 v_{2F}^3 v_3^2\cos(4\Psi_2 - 6\Psi_3) - \frac{3}{2}v_{2F}^3 v_3^2 v_4^2\cos(6\Psi_3) - \frac{105}{4}v_2 v_{2F}^3 v_4^2\cos(2\Psi_2 - 8\Psi_4)\\
& + 120v_2^2 v_{2F}^2 v_4^2\cos(4\Psi_2 - 8\Psi_4) - 168v_{2F} v_3^2 v_4^2\cos(6\Psi_3 - 8\Psi_4) - 42v_{2F}^3 v_3^2 v_4^2\cos(6\Psi_3 - 8\Psi_4)\\
& - 363v_2^3 v_{2F} v_4\cos(2\Psi_2 - 4\Psi_4) - \frac{21}{4}v_2 v_{2F}^3 v_4\cos(2\Psi_2 - 4\Psi_4) - \frac{363}{4}v_2^3 v_{2F}^3 v_4\cos(2\Psi_2 - 4\Psi_4)
\end{aligned}
$$

$$
\begin{aligned}
&- 936v_2v_{2F}v_3^2v_4\cos(2\Psi_2-4\Psi_4) - 234v_2v_{2F}^3v_3^2v_4\cos(2\Psi_2-4\Psi_4) - 363v_2v_{2F}v_4^3\cos(2\Psi_2-4\Psi_4)\\
&- \frac{363}{4}v_2v_{2F}^3v_4^3\cos(2\Psi_2-4\Psi_4) + 72v_2^2v_{2F}^2v_4\cos(4\Psi_2-4\Psi_4) - 3v_2^3v_{2F}v_4\cos(6\Psi_2-4\Psi_4)\\
&- \frac{3}{4}v_2^3v_{2F}^3v_4\cos(6\Psi_2-4\Psi_4) - 42v_{2F}^3v_3^2v_4\cos(6\Psi_3-4\Psi_4) + 9v_2v_{2F}^2v_3^2v_4\cos(2\Psi_2+6\Psi_3-4\Psi_4)\\
&+ 144v_2^2v_{2F}^2v_4\cos(4\Psi_4) + \frac{525}{2}v_{2F}^2v_3^2v_4\cos(4\Psi_4) + 120v_{2F}^2v_4^3\cos(4\Psi_4) - \frac{15}{4}v_2v_{2F}^3v_4\cos(2\Psi_2+4\Psi_4)\\
&- 4v_2^3v_{2F}^3v_4\cos(2\Psi_2+4\Psi_4) - 9v_2v_{2F}^3v_3^2v_4\cos(2\Psi_2+4\Psi_4) - 4v_2v_{2F}^3v_4^3\cos(2\Psi_2+4\Psi_4)\\
&+ 450v_2v_{2F}^2v_3^2v_4\cos(2\Psi_2-6\Psi_3+4\Psi_4),
\end{aligned}
$$

$$
\begin{aligned}
A_4 =\;& 49v_2^2 + 436v_2^4 + 147v_2^2v_{2F}^2 + 264v_3^2 + 3328v_2^2v_3^2 + 792v_{2F}^2v_3^2 + 1056v_3^4 + 448v_4^2 + 3625v_2^2v_4^2 + 1344v_{2F}^2v_4^2\\
&+ 3328v_3^2v_4^2 + 436v_4^4 - 872v_2^3v_{2F}\cos(2\Psi_2) - 5004v_2v_{2F}v_3^2\cos(2\Psi_2) - 6728v_2v_{2F}v_4^2\cos(2\Psi_2)\\
&+ 45v_2^2v_{2F}^2\cos(4\Psi_2) - 872v_2v_{2F}v_4\cos(2\Psi_2-4\Psi_4) + 898v_2^2v_4\cos(4\Psi_2-4\Psi_4) - 3492v_{2F}v_3^2v_4\cos(6\Psi_3-4\Psi_4)\\
&+ 45v_{2F}^2v_4\cos(4\Psi_4) + 3920v_2v_3^2v_4\cos(2\Psi_2-6\Psi_3+4\Psi_4) + 1308v_2^4v_{2F}^2 + \frac{147v_2^2v_{2F}^4}{8} + \frac{327v_2^4v_{2F}^4}{2} + 9984v_2^2v_{2F}^2v_3^2\\
&+ 99v_{2F}^4v_3^2 + 1248v_2^2v_{2F}^4v_3^2 + 3168v_{2F}^2v_3^4 + 396v_{2F}^4v_3^4 + 10875v_2^2v_{2F}^2v_4^2 + 168v_{2F}^4v_4^2 + \frac{10875}{8}v_2^2v_{2F}^4v_4^2\\
&+ 9984v_{2F}^2v_3^2v_4^2 + 1248v_{2F}^4v_3^2v_4^2 + 1308v_{2F}^2v_4^4 + \frac{327v_{2F}^4v_4^4}{2} - 654v_2^3v_{2F}^3\cos(2\Psi_2) - 3753v_2v_{2F}^3v_3^2\cos(2\Psi_2)\\
&- 5046v_2v_{2F}^3v_4^2\cos(2\Psi_2) + 48v_2^4v_{2F}^2\cos(4\Psi_2) + \frac{15}{2}v_2^2v_{2F}^4\cos(4\Psi_2) + 8v_2^4v_{2F}^4\cos(4\Psi_2)\\
&+ 510v_2^2v_{2F}^2v_3^2\cos(4\Psi_2) + 85v_2^2v_{2F}^4v_3^2\cos(4\Psi_2) + 837v_2^2v_{2F}^2v_4^2\cos(4\Psi_2) + \frac{279}{2}v_2^2v_{2F}^4v_4^2\cos(4\Psi_2)\\
&+ 2196v_2v_{2F}^2v_3^2\cos(2\Psi_2-6\Psi_3) + 366v_2v_{2F}^4v_3^2\cos(2\Psi_2-6\Psi_3) - 3420v_2^2v_{2F}v_3^2\cos(4\Psi_2-6\Psi_3)\\
&- 2565v_2^2v_{2F}^3v_3^2\cos(4\Psi_2-6\Psi_3) - 105v_{2F}^3v_3^2\cos(6\Psi_3) - 134v_2^2v_{2F}^3v_3^2\cos(6\Psi_3) - 42v_{2F}^3v_3^4\cos(6\Psi_3)\\
&- 140v_{2F}^3v_3^2v_4^2\cos(6\Psi_3) - 1526v_2v_{2F}^3v_4^2\cos(2\Psi_2-8\Psi_4) + 3270v_2^2v_{2F}^2v_4^2\cos(4\Psi_2-8\Psi_4)\\
&+ 545v_2^2v_{2F}^4v_4^2\cos(4\Psi_2-8\Psi_4) - 3420v_{2F}v_3^2v_4^2\cos(6\Psi_3-8\Psi_4) - 2565v_{2F}^3v_3^2v_4^2\cos(6\Psi_3-8\Psi_4)\\
&- 6424v_2^3v_{2F}v_4\cos(2\Psi_2-4\Psi_4) - 654v_2v_{2F}^3v_4\cos(2\Psi_2-4\Psi_4) - 4818v_2^3v_{2F}^3v_4\cos(2\Psi_2-4\Psi_4)\\
&- 16640v_2v_{2F}v_3^2v_4\cos(2\Psi_2-4\Psi_4) - 12480v_2v_{2F}^3v_3^2v_4\cos(2\Psi_2-4\Psi_4) - 6424v_2v_{2F}v_4^3\cos(2\Psi_2-4\Psi_4)\\
&- 4818v_2v_{2F}^3v_4^3\cos(2\Psi_2-4\Psi_4) + 2694v_2^2v_{2F}^2v_4\cos(4\Psi_2-4\Psi_4) + \frac{1347}{4}v_2^2v_{2F}^4v_4\cos(4\Psi_2-4\Psi_4)\\
&- 152v_2^3v_{2F}v_4\cos(6\Psi_2-4\Psi_4) - 114v_2^3v_{2F}^3v_4\cos(6\Psi_2-4\Psi_4) + 22v_2v_{2F}^4v_3^2v_4\cos(2\Psi_2-6\Psi_3-4\Psi_4)\\
&- 2619v_{2F}^3v_3^2v_4\cos(6\Psi_3-4\Psi_4) + 510v_2v_{2F}^2v_3^2v_4\cos(2\Psi_2+6\Psi_3-4\Psi_4) + 85v_2v_{2F}^4v_3^2v_4\cos(2\Psi_2+6\Psi_3-4\Psi_4)\\
&+ 4419v_2^2v_{2F}^2v_4\cos(4\Psi_4) + \frac{15}{2}v_{2F}^4v_4\cos(4\Psi_4) + \frac{1473}{2}v_2^2v_{2F}^4v_4\cos(4\Psi_4) + 7434v_{2F}^2v_3^2v_4\cos(4\Psi_4)\\
&+ 1239v_{2F}^4v_3^2v_4\cos(4\Psi_4) + 3318v_{2F}^2v_4^3\cos(4\Psi_4) + 553v_{2F}^4v_4^3\cos(4\Psi_4) + \frac{105}{2}v_{2F}^4v_4^2\cos(8\Psi_4) + \frac{477}{8}v_2^2v_{2F}^4v_4^2\cos(8\Psi_4)\\
&+ 28v_{2F}^4v_3^2v_4^2\cos(8\Psi_4) + \frac{7}{2}v_{2F}^4v_4^4\cos(8\Psi_4) - 266v_2v_{2F}^3v_4\cos(2\Psi_2+4\Psi_4) - 288v_2^3v_{2F}^3v_4\cos(2\Psi_2+4\Psi_4)\\
&- 694v_2v_{2F}^3v_3^2v_4\cos(2\Psi_2+4\Psi_4) - 288v_2v_{2F}^3v_4^3\cos(2\Psi_2+4\Psi_4) + \frac{49}{4}v_2^2v_{2F}^4v_4\cos(4\Psi_2+4\Psi_4)\\
&+ 11760v_2v_{2F}^2v_3^2v_4\cos(2\Psi_2-6\Psi_3+4\Psi_4) + 1470v_2v_{2F}^4v_3^2v_4\cos(2\Psi_2-6\Psi_3+4\Psi_4),
\end{aligned}
$$

$$
\begin{aligned}
A_5 =\;& 1820v_2^2 + 7120v_2^4 + 9100v_2^2v_{2F}^2 + 5545v_3^2 + 47025v_2^2v_3^2 + 27725v_{2F}^2v_3^2 + 14800v_3^4 + 7680v_4^2 + 47004v_2^2v_4^2\\
&+ 38400v_{2F}^2v_4^2 + 47001v_3^2v_4^2 + 7120v_4^4 - 525v_2v_{2F}\cos(2\Psi_2) - 23855v_2^3v_{2F}\cos(2\Psi_2) - \frac{1575}{2}v_2v_{2F}^3\cos(2\Psi_2)
\end{aligned}
$$

$$
\begin{aligned}
&- 102960 v_2 v_{2F} v_3^2 \cos(2\Psi_2) - 120970 v_2 v_{2F} v_4^2 \cos(2\Psi_2) + 2940 v_2^2 v_{2F}^2 \cos(4\Psi_2) - 23895 v_2 v_{2F} v_4 \cos(2\Psi_2 - 4\Psi_4) \\
&+ 15480 v_2^2 v_4 \cos(4\Psi_2 - 4\Psi_4) - 65430 v_{2F} v_3^2 v_4 \cos(6\Psi_3 - 4\Psi_4) + 2940 v_{2F}^2 v_4 \cos(4\Psi_4) \\
&+ 52730 v_2 v_3^2 v_4 \cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) + 35600 v_2^4 v_{2F}^2 + \frac{6825 v_2^2 v_{2F}^4}{2} + 13350 v_2^4 v_{2F}^4 + 235125 v_2^2 v_{2F}^2 v_3^2 + \frac{83175 v_{2F}^4 v_3^2}{8} \\
&+ \frac{705375}{8} v_2^2 v_{2F}^4 v_3^2 + 74000 v_{2F}^2 v_3^4 + 27750 v_{2F}^4 v_3^4 + 235020 v_2^2 v_{2F}^2 v_4^2 + 14400 v_{2F}^4 v_4^2 + \frac{176265 v_2^2}{2} v_{2F}^4 v_4^2 \\
&+ 235005 v_{2F}^2 v_3^2 v_4^2 + \frac{705015}{8} v_{2F}^4 v_3^2 v_4^2 + 35600 v_{2F}^2 v_4^4 + 13350 v_{2F}^4 v_4^4 - \frac{71565}{2} v_2^3 v_{2F}^3 \cos(2\Psi_2) - \frac{525}{8} v_2 v_{2F}^5 \cos(2\Psi_2) \\
&- \frac{23855}{8} v_2^3 v_{2F}^5 \cos(2\Psi_2) - 154440 v_2 v_{2F}^3 v_3^2 \cos(2\Psi_2) - 12870 v_2 v_{2F}^5 v_3^2 \cos(2\Psi_2) - 181455 v_2 v_{2F}^3 v_4^2 \cos(2\Psi_2) \\
&- \frac{60485}{4} v_2 v_{2F}^5 v_4^2 \cos(2\Psi_2) + 3200 v_2^4 v_{2F}^2 \cos(4\Psi_2) + 1470 v_2^2 v_{2F}^4 \cos(4\Psi_2) + 1600 v_2^4 v_{2F}^4 \cos(4\Psi_2) \\
&+ 19785 v_2^2 v_{2F}^2 v_3^2 \cos(4\Psi_2) + \frac{19785}{2} v_2^2 v_{2F}^4 v_3^2 \cos(4\Psi_2) + 24780 v_2^2 v_{2F}^2 v_4^2 \cos(4\Psi_2) + 12390 v_2^2 v_{2F}^4 v_4^2 \cos(4\Psi_2) \\
&- 245 v_2^3 v_{2F}^3 \cos(6\Psi_2) - \frac{245}{8} v_2^3 v_{2F}^5 \cos(6\Psi_2) + 61755 v_2 v_{2F}^2 v_3^2 \cos(2\Psi_2 - 6\Psi_3) + \frac{61755}{2} v_2 v_{2F}^4 v_3^2 \cos(2\Psi_2 - 6\Psi_3) \\
&- 61920 v_2^2 v_{2F} v_3^2 \cos(4\Psi_2 - 6\Psi_3) - 92880 v_2^2 v_{2F}^3 v_3^2 \cos(4\Psi_2 - 6\Psi_3) - 7740 v_2^2 v_{2F}^5 v_3^2 \cos(4\Psi_2 - 6\Psi_3) \\
&- 6440 v_{2F}^3 v_3^2 \cos(6\Psi_3) - 9265 v_2^2 v_{2F}^3 v_3^2 \cos(6\Psi_3) - 805 v_{2F}^5 v_3^2 \cos(6\Psi_3) - \frac{9265}{8} v_2^2 v_{2F}^5 v_3^2 \cos(6\Psi_3) \\
&- 2840 v_{2F}^3 v_3^4 \cos(6\Psi_3) - 355 v_{2F}^5 v_3^4 \cos(6\Psi_3) - 7360 v_{2F}^3 v_3^2 v_4^2 \cos(6\Psi_3) - 920 v_{2F}^5 v_3^2 v_4^2 \cos(6\Psi_3) \\
&+ \frac{1305}{2} v_2 v_{2F}^4 v_3^2 \cos(2\Psi_2 + 6\Psi_3) - \frac{1305}{16} v_2 v_{2F}^5 v_4^3 \cos(2\Psi_2 - 12\Psi_4) - \frac{108855}{2} v_2 v_{2F}^3 v_4^2 \cos(2\Psi_2 - 8\Psi_4) \\
&- \frac{108855}{16} v_2 v_{2F}^5 v_4^2 \cos(2\Psi_2 - 8\Psi_4) + 74760 v_2^2 v_{2F}^2 v_4^2 \cos(4\Psi_2 - 8\Psi_4) + 37380 v_2^2 v_{2F}^4 v_4^2 \cos(4\Psi_2 - 8\Psi_4) \\
&- 61920 v_{2F} v_3^2 v_4^2 \cos(6\Psi_3 - 8\Psi_4) - 92880 v_{2F}^3 v_3^2 v_4^2 \cos(6\Psi_3 - 8\Psi_4) - 7740 v_{2F}^5 v_3^2 v_4^2 \cos(6\Psi_3 - 8\Psi_4) \\
&- 110770 v_2^3 v_{2F} v_4 \cos(2\Psi_2 - 4\Psi_4) - \frac{71685}{2} v_2 v_{2F}^3 v_4 \cos(2\Psi_2 - 4\Psi_4) - 166155 v_2^3 v_{2F}^3 v_4 \cos(2\Psi_2 - 4\Psi_4) \\
&- \frac{23895}{8} v_2 v_{2F}^5 v_4 \cos(2\Psi_2 - 4\Psi_4) - \frac{55385}{4} v_2^3 v_{2F}^5 v_4 \cos(2\Psi_2 - 4\Psi_4) - 282030 v_2 v_{2F} v_3^2 v_4 \cos(2\Psi_2 - 4\Psi_4) \\
&- 423045 v_2 v_{2F}^3 v_3^2 v_4 \cos(2\Psi_2 - 4\Psi_4) - \frac{141015}{4} v_2 v_{2F}^5 v_3^2 v_4 \cos(2\Psi_2 - 4\Psi_4) - 110695 v_2 v_{2F} v_4^3 \cos(2\Psi_2 - 4\Psi_4) \\
&- \frac{332085}{2} v_2 v_{2F}^3 v_4^3 \cos(2\Psi_2 - 4\Psi_4) - \frac{110695}{8} v_2 v_{2F}^5 v_4^3 \cos(2\Psi_2 - 4\Psi_4) + 77400 v_2^2 v_{2F}^2 v_4 \cos(4\Psi_2 - 4\Psi_4) \\
&+ 29025 v_2^2 v_{2F}^4 v_4 \cos(4\Psi_2 - 4\Psi_4) - 4875 v_2^3 v_{2F} v_4 \cos(6\Psi_2 - 4\Psi_4) - \frac{14625}{2} v_2^3 v_{2F}^3 v_4 \cos(6\Psi_2 - 4\Psi_4) \\
&- \frac{4875}{8} v_2^3 v_{2F}^5 v_4 \cos(6\Psi_2 - 4\Psi_4) + \frac{10665}{4} v_2 v_{2F}^4 v_3^2 v_4 \cos(2\Psi_2 - 6\Psi_3 - 4\Psi_4) - 98145 v_{2F}^3 v_3^2 v_4 \cos(6\Psi_3 - 4\Psi_4) \\
&- \frac{32715}{4} v_{2F}^5 v_3^2 v_4 \cos(6\Psi_3 - 4\Psi_4) + 18300 v_2 v_{2F}^2 v_3^2 v_4 \cos(2\Psi_2 + 6\Psi_3 - 4\Psi_4) + 9150 v_2 v_{2F}^4 v_3^2 v_4 \cos(2\Psi_2 + 6\Psi_3 - 4\Psi_4) \\
&+ 114820 v_2^2 v_{2F}^2 v_4 \cos(4\Psi_4) + 1470 v_{2F}^4 v_4 \cos(4\Psi_4) + 57410 v_2^2 v_{2F}^4 v_4 \cos(4\Psi_4) + 175995 v_{2F}^2 v_3^2 v_4 \cos(4\Psi_4) \\
&+ \frac{175995}{2} v_{2F}^4 v_3^2 v_4 \cos(4\Psi_4) + 77960 v_{2F}^2 v_4^3 \cos(4\Psi_4) + 38980 v_{2F}^4 v_4^3 \cos(4\Psi_4) + 4410 v_{2F}^4 v_4^2 \cos(8\Psi_4) \\
&+ \frac{10415}{2} v_2^2 v_{2F}^4 v_4^2 \cos(8\Psi_4) + \frac{22005}{8} v_{2F}^4 v_3^2 v_4^2 \cos(8\Psi_4) + 390 v_{2F}^4 v_4^4 \cos(8\Psi_4) - 12600 v_2 v_{2F}^3 v_4 \cos(2\Psi_2 + 4\Psi_4) \\
&- 13670 v_2^3 v_{2F}^3 v_4 \cos(2\Psi_2 + 4\Psi_4) - 1575 v_2 v_{2F}^5 v_4 \cos(2\Psi_2 + 4\Psi_4) - \frac{6835}{4} v_2^3 v_{2F}^5 v_4 \cos(2\Psi_2 + 4\Psi_4) \\
&- 31010 v_2 v_{2F}^3 v_3^2 v_4 \cos(2\Psi_2 + 4\Psi_4) - \frac{15505}{4} v_2 v_{2F}^5 v_3^2 v_4 \cos(2\Psi_2 + 4\Psi_4) - 12620 v_2 v_{2F}^3 v_4^3 \cos(2\Psi_2 + 4\Psi_4)
\end{aligned}
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\begin{aligned}
&- \frac{3155}{2} v_2 v_{2F}^5 v_4^3 \cos(2\Psi_2 + 4\Psi_4) + 1335 v_2^2 v_{2F}^4 v_4 \cos(4\Psi_2 + 4\Psi_4) - \frac{105}{8} v_2^3 v_{2F}^5 v_4 \cos(6\Psi_2 + 4\Psi_4) \\
&+ 263650 v_2 v_{2F}^2 v_3^2 v_4 \cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) + \frac{395475}{4} v_2 v_{2F}^4 v_3^2 v_4 \cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) \\
&- \frac{405}{4} v_{2F}^5 v_3^2 v_4 \cos(6\Psi_3 + 4\Psi_4) - \frac{4665}{16} v_2 v_{2F}^5 v_4^2 \cos(2\Psi_2 + 8\Psi_4),
\end{aligned}
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\begin{aligned}
A_6 = {}& 225 + 45396 v_2^2 + 110941 v_2^4 + \frac{3375 v_{2F}^2}{2} + 340470 v_2^2 v_{2F}^2 + \frac{10125 v_{2F}^4}{8} + 102780 v_3^2 + 666216 v_2^2 v_3^2 \\
&+ 770850 v_{2F}^2 v_3^2 + 205956 v_3^4 + 126720 v_4^2 + 637704 v_2^2 v_4^2 + 950400 v_{2F}^2 v_4^2 + 664800 v_3^2 v_4^2 + 110686 v_4^4 \\
&- 28140 v_2 v_{2F} \cos(2\Psi_2) - 545892 v_2^3 v_{2F} \cos(2\Psi_2) - 70350 v_2 v_{2F}^3 \cos(2\Psi_2) - 1970532 v_2 v_{2F} v_3^2 \cos(2\Psi_2) \\
&- 2142936 v_2 v_{2F} v_4^2 \cos(2\Psi_2) + 115605 v_2^2 v_{2F}^2 \cos(4\Psi_2) - 548580 v_2 v_{2F} v_4 \cos(2\Psi_2 - 4\Psi_4) \\
&+ 257412 v_2^2 v_4 \cos(4\Psi_2 - 4\Psi_4) - 1169892 v_{2F} v_3^2 v_4 \cos(6\Psi_3 - 4\Psi_4) + 113610 v_{2F}^2 v_4 \cos(4\Psi_4) \\
&+ 724092 v_2 v_3^2 v_4 \cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) + \frac{1664115 v_2^4 v_{2F}^2}{2} + \frac{510705 v_2^2 v_{2F}^4}{2} + \frac{4992345 v_2^4 v_{2F}^4}{8} \\
&+ \frac{1125 v_{2F}^6}{16} + \frac{56745 v_2^2 v_{2F}^6}{4} + \frac{554705 v_2^4 v_{2F}^6}{16} + 4996620 v_2^2 v_{2F}^2 v_3^2 + \frac{1156275 v_{2F}^4 v_3^2}{2} + 3747465 v_2^2 v_{2F}^4 v_3^2 \\
&+ \frac{128475 v_{2F}^6 v_3^2}{4} + \frac{416385}{2} v_2^2 v_{2F}^6 v_3^2 + 1544670 v_{2F}^2 v_3^4 + \frac{2317005 v_{2F}^4 v_3^4}{2} + \frac{257445 v_{2F}^6 v_3^4}{4} + 4782780 v_2^2 v_{2F}^2 v_4^2 \\
&+ 712800 v_{2F}^4 v_4^2 + 3587085 v_2^2 v_{2F}^4 v_4^2 + 39600 v_{2F}^6 v_4^2 + \frac{398565}{2} v_2^2 v_{2F}^6 v_4^2 + 4986000 v_{2F}^2 v_3^2 v_4^2 \\
&+ 3739500 v_{2F}^4 v_3^2 v_4^2 + 207750 v_{2F}^6 v_3^2 v_4^2 + 830145 v_{2F}^2 v_4^4 + \frac{2490435 v_{2F}^4 v_4^4}{4} + \frac{276715 v_{2F}^6 v_4^4}{8} - 1364730 v_2^3 v_{2F}^3 \cos(2\Psi_2) \\
&- \frac{35175}{2} v_2 v_{2F}^5 \cos(2\Psi_2) - \frac{682365}{2} v_2^3 v_{2F}^5 \cos(2\Psi_2) - 4926330 v_2 v_{2F}^3 v_3^2 \cos(2\Psi_2) - \frac{2463165}{2} v_2 v_{2F}^5 v_3^2 \cos(2\Psi_2) \\
&- 5357340 v_2 v_{2F}^3 v_4^2 \cos(2\Psi_2) - 1339335 v_2 v_{2F}^5 v_4^2 \cos(2\Psi_2) + 123315 v_2^4 v_{2F}^2 \cos(4\Psi_2) + 115605 v_2^2 v_{2F}^4 \cos(4\Psi_2) \\
&+ 123315 v_2^4 v_{2F}^4 \cos(4\Psi_2) + \frac{115605}{16} v_2^2 v_{2F}^6 \cos(4\Psi_2) + \frac{123315}{16} v_2^4 v_{2F}^6 \cos(4\Psi_2) + 592860 v_2^2 v_{2F}^2 v_3^2 \cos(4\Psi_2) \\
&+ 592860 v_2^2 v_{2F}^4 v_3^2 \cos(4\Psi_2) + \frac{148215}{4} v_2^2 v_{2F}^6 v_3^2 \cos(4\Psi_2) + \frac{1297635}{2} v_2^2 v_{2F}^2 v_4^2 \cos(4\Psi_2) + \frac{1297635}{2} v_2^2 v_{2F}^4 v_4^2 \cos(4\Psi_2) \\
&+ \frac{1297635}{32} v_2^2 v_{2F}^6 v_4^2 \cos(4\Psi_2) - 23220 v_2^3 v_{2F}^3 \cos(6\Psi_2) - \frac{17415}{2} v_2^3 v_{2F}^5 \cos(6\Psi_2) + \frac{1575}{4} v_2^4 v_{2F}^4 \cos(8\Psi_2) \\
&+ \frac{315}{8} v_2^4 v_{2F}^6 \cos(8\Psi_2) + 1467090 v_2 v_{2F}^2 v_3^2 \cos(2\Psi_2 - 6\Psi_3) + 1467090 v_2 v_{2F}^4 v_3^2 \cos(2\Psi_2 - 6\Psi_3) \\
&+ \frac{733545}{8} v_2 v_{2F}^6 v_3^2 \cos(2\Psi_2 - 6\Psi_3) - 1061160 v_2^2 v_{2F} v_3^2 \cos(4\Psi_2 - 6\Psi_3) - 2652900 v_2^2 v_{2F}^3 v_3^2 \cos(4\Psi_2 - 6\Psi_3) \\
&- 663225 v_2^2 v_{2F}^5 v_3^2 \cos(4\Psi_2 - 6\Psi_3) - 246240 v_{2F}^3 v_3^2 \cos(6\Psi_3) - 388860 v_2^2 v_{2F}^3 v_3^2 \cos(6\Psi_3) \\
&- 92340 v_{2F}^5 v_3^2 \cos(6\Psi_3) - \frac{291645}{2} v_2^2 v_{2F}^5 v_3^2 \cos(6\Psi_3) - 117990 v_{2F}^3 v_3^4 \cos(6\Psi_3) - \frac{176985}{4} v_{2F}^5 v_3^4 \cos(6\Psi_3) \\
&- 285210 v_{2F}^3 v_3^2 v_4^2 \cos(6\Psi_3) - \frac{427815}{4} v_{2F}^5 v_3^2 v_4^2 \cos(6\Psi_3) + \frac{246105}{4} v_2 v_{2F}^4 v_3^2 \cos(2\Psi_2 + 6\Psi_3) \\
&+ \frac{49221}{8} v_2 v_{2F}^6 v_3^2 \cos(2\Psi_2 + 6\Psi_3) - \frac{4653}{4} v_2^2 v_{2F}^5 v_3^2 \cos(4\Psi_2 + 6\Psi_3) - \frac{39435}{4} v_2 v_{2F}^5 v_4^3 \cos(2\Psi_2 - 12\Psi_4) \\
&- 1552110 v_2 v_{2F}^3 v_4^2 \cos(2\Psi_2 - 8\Psi_4) - \frac{2328165}{4} v_2 v_{2F}^5 v_4^2 \cos(2\Psi_2 - 8\Psi_4) + \frac{3099615}{2} v_2^2 v_{2F}^2 v_4^2 \cos(4\Psi_2 - 8\Psi_4) \\
&+ \frac{3099615}{2} v_2^2 v_{2F}^4 v_4^2 \cos(4\Psi_2 - 8\Psi_4) + \frac{3099615}{32} v_2^2 v_{2F}^6 v_4^2 \cos(4\Psi_2 - 8\Psi_4) - 1060542 v_{2F} v_3^2 v_4^2 \cos(6\Psi_3 - 8\Psi_4) \\
&- 2651355 v_{2F}^3 v_3^2 v_4^2 \cos(6\Psi_3 - 8\Psi_4) - \frac{2651355}{4} v_{2F}^5 v_3^2 v_4^2 \cos(6\Psi_3 - 8\Psi_4) - 1870296 v_2^3 v_{2F} v_4 \cos(2\Psi_2 - 4\Psi_4)
\end{aligned}
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\begin{aligned}
&- 1371450 v_2 v_{2F}^3 v_4 \cos(2\Psi_2 - 4\Psi_4) - 4675740 v_2^3 v_{2F}^3 v_4 \cos(2\Psi_2 - 4\Psi_4) - \frac{685725}{2} v_2 v_{2F}^5 v_4 \cos(2\Psi_2 - 4\Psi_4) \\
&- 1168935 v_2^3 v_{2F}^5 v_4 \cos(2\Psi_2 - 4\Psi_4) - 4655652 v_2 v_{2F} v_3^2 v_4 \cos(2\Psi_2 - 4\Psi_4) - 11639130 v_2 v_{2F}^3 v_3^2 v_4 \cos(2\Psi_2 - 4\Psi_4) \\
&- \frac{5819565}{2} v_2 v_{2F}^5 v_3^2 v_4 \cos(2\Psi_2 - 4\Psi_4) - 1864836 v_2 v_{2F} v_4^3 \cos(2\Psi_2 - 4\Psi_4) - 4662090 v_2 v_{2F}^3 v_4^3 \cos(2\Psi_2 - 4\Psi_4) \\
&- \frac{2331045}{2} v_2 v_{2F}^5 v_4^3 \cos(2\Psi_2 - 4\Psi_4) + 1930590 v_2^2 v_{2F}^2 v_4 \cos(4\Psi_2 - 4\Psi_4) + \frac{2895885}{2} v_2^2 v_{2F}^4 v_4 \cos(4\Psi_2 - 4\Psi_4) \\
&+ \frac{321765}{4} v_2^2 v_{2F}^6 v_4 \cos(4\Psi_2 - 4\Psi_4) - 124980 v_2^3 v_{2F} v_4 \cos(6\Psi_2 - 4\Psi_4) - 312450 v_2^3 v_{2F}^3 v_4 \cos(6\Psi_2 - 4\Psi_4) \\
&- \frac{156225}{2} v_2^3 v_{2F}^5 v_4 \cos(6\Psi_2 - 4\Psi_4) + \frac{635625}{4} v_2 v_{2F}^4 v_3^2 v_4 \cos(2\Psi_2 - 6\Psi_3 - 4\Psi_4) \\
&+ \frac{127125}{8} v_2 v_{2F}^6 v_3^2 v_4 \cos(2\Psi_2 - 6\Psi_3 - 4\Psi_4) - 2924730 v_{2F}^3 v_3^2 v_4 \cos(6\Psi_3 - 4\Psi_4) - \frac{1462365}{2} v_{2F}^5 v_3^2 v_4 \cos(6\Psi_3 - 4\Psi_4) \\
&+ 528240 v_2 v_{2F}^2 v_3^2 v_4 \cos(2\Psi_2 + 6\Psi_3 - 4\Psi_4) + 528240 v_2 v_{2F}^4 v_3^2 v_4 \cos(2\Psi_2 + 6\Psi_3 - 4\Psi_4) \\
&+ 33015 v_2 v_{2F}^6 v_3^2 v_4 \cos(2\Psi_2 + 6\Psi_3 - 4\Psi_4) + 2678325 v_2^2 v_{2F}^2 v_4 \cos(4\Psi_4) + 113610 v_{2F}^4 v_4 \cos(4\Psi_4) \\
&+ 2678325 v_2^2 v_{2F}^4 v_4 \cos(4\Psi_4) + \frac{56805}{8} v_{2F}^6 v_4 \cos(4\Psi_4) + \frac{2678325}{16} v_2^2 v_{2F}^6 v_4 \cos(4\Psi_4) + 3789540 v_{2F}^2 v_3^2 v_4 \cos(4\Psi_4) \\
&+ 3789540 v_{2F}^4 v_3^2 v_4 \cos(4\Psi_4) + \frac{947385}{4} v_{2F}^6 v_3^2 v_4 \cos(4\Psi_4) + 1673160 v_{2F}^2 v_4^3 \cos(4\Psi_4) + 1673160 v_{2F}^4 v_4^3 \cos(4\Psi_4) \\
&+ \frac{209145}{2} v_{2F}^6 v_4^3 \cos(4\Psi_4) + \frac{1660725}{8} v_{2F}^4 v_4^2 \cos(8\Psi_4) + \frac{2073165}{8} v_2^2 v_{2F}^4 v_4^2 \cos(8\Psi_4) + \frac{332145}{16} v_{2F}^6 v_4^2 \cos(8\Psi_4) \\
&+ \frac{414633}{16} v_2^2 v_{2F}^6 v_4^2 \cos(8\Psi_4) + \frac{602025}{4} v_{2F}^4 v_3^2 v_4^2 \cos(8\Psi_4) + \frac{120405}{8} v_{2F}^6 v_3^2 v_4^2 \cos(8\Psi_4) + \frac{48375}{2} v_{2F}^4 v_4^4 \cos(8\Psi_4) \\
&+ \frac{9675}{4} v_{2F}^6 v_4^4 \cos(8\Psi_4) + \frac{8415}{16} v_{2F}^6 v_4^3 \cos(12\Psi_4) - 459360 v_2 v_{2F}^3 v_4 \cos(2\Psi_2 + 4\Psi_4) \\
&- 497720 v_2^3 v_{2F}^3 v_4 \cos(2\Psi_2 + 4\Psi_4) - 172260 v_2 v_{2F}^5 v_4 \cos(2\Psi_2 + 4\Psi_4) - 186645 v_2^3 v_{2F}^5 v_4 \cos(2\Psi_2 + 4\Psi_4) \\
&- 1064970 v_2 v_{2F}^3 v_3^2 v_4 \cos(2\Psi_2 + 4\Psi_4) - \frac{1597455}{4} v_2 v_{2F}^5 v_3^2 v_4 \cos(2\Psi_2 + 4\Psi_4) - 429680 v_2 v_{2F}^3 v_4^3 \cos(2\Psi_2 + 4\Psi_4) \\
&- 161130 v_2 v_{2F}^5 v_4^3 \cos(2\Psi_2 + 4\Psi_4) + \frac{357255}{4} v_2^2 v_{2F}^4 v_4 \cos(4\Psi_2 + 4\Psi_4) + \frac{71451}{8} v_2^2 v_{2F}^6 v_4 \cos(4\Psi_2 + 4\Psi_4) \\
&- \frac{4059}{2} v_2^3 v_{2F}^5 v_4 \cos(6\Psi_2 + 4\Psi_4) + 5430690 v_2 v_{2F}^2 v_3^2 v_4 \cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) \\
&+ \frac{8146035}{2} v_2 v_{2F}^4 v_3^2 v_4 \cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) + \frac{905115}{4} v_2 v_{2F}^6 v_3^2 v_4 \cos(2\Psi_2 - 6\Psi_3 + 4\Psi_4) \\
&- \frac{68013}{4} v_{2F}^5 v_3^2 v_4 \cos(6\Psi_3 + 4\Psi_4) + \frac{1287}{4} v_2 v_{2F}^6 v_3^2 v_4 \cos(2\Psi_2 + 6\Psi_3 + 4\Psi_4) \\
&- \frac{137643}{4} v_2 v_{2F}^5 v_4^2 \cos(2\Psi_2 + 8\Psi_4) + \frac{15279}{32} v_2^2 v_{2F}^6 v_4^2 \cos(4\Psi_2 + 8\Psi_4).
\end{aligned} \tag{A1}
$$

The full results of the Eq. (32) are as follows:

$$
\begin{aligned}
D_0 &= v_2^2 v_4 \cos(4\Psi_2 - 4\Psi_4), \\
D_1 &= - v_2 v_{2F} v_4 \cos(2\Psi_2 - 4\Psi_4) + 3 v_2^2 v_4 \cos(4\Psi_2 - 4\Psi_4) - \tfrac{1}{2} v_2^3 v_{2F} \cos(2\Psi_2) - 2 v_2 v_{2F} v_3^2 \cos(2\Psi_2) \\
&\quad - v_2 v_{2F} v_4^2 \cos(2\Psi_2) v_{2F} v_3^2 v_4 \cos(6\Psi_3 - 4\Psi_4), \\
D_2 &= 2v_2^2 + 4v_3^2 + 2v_4^2 - 14 v_2 v_{2F} v_4 \cos(2\Psi_2 - 4\Psi_4) + 15 v_2^2 v_4 \cos(4\Psi_2 - 4\Psi_4) + \tfrac{3}{2} v_{2F}^2 v_4 \cos(4\Psi_4) + v_2^2 v_{2F}^2 + 2 v_{2F}^2 v_3^2 \\
&\quad + v_{2F}^2 v_4^2 - 7 v_2^3 v_{2F} \cos(2\Psi_2) - 24 v_2 v_{2F} v_3^2 \cos(2\Psi_2) - 14 v_2 v_{2F} v_4^2 \cos(2\Psi_2) + \tfrac{1}{4} v_2^2 v_{2F}^2 \cos(4\Psi_2) \\
&\quad + 3 v_2 v_{2F}^2 v_3^2 \cos(2\Psi_2 - 6\Psi_3) + \tfrac{15}{2} v_2^2 v_{2F}^2 v_4 \cos(4\Psi_2 - 4\Psi_4) - 12 v_{2F} v_3^2 v_4 \cos(6\Psi_3 - 4\Psi_4) \\
&\quad + \tfrac{7}{2} v_2^2 v_{2F}^2 v_4 \cos(4\Psi_4) + 6 v_{2F}^2 v_3^2 v_4 \cos(4\Psi_4) + \tfrac{3}{2} v_{2F}^2 v_4^3 \cos(4\Psi_4),
\end{aligned}
$$

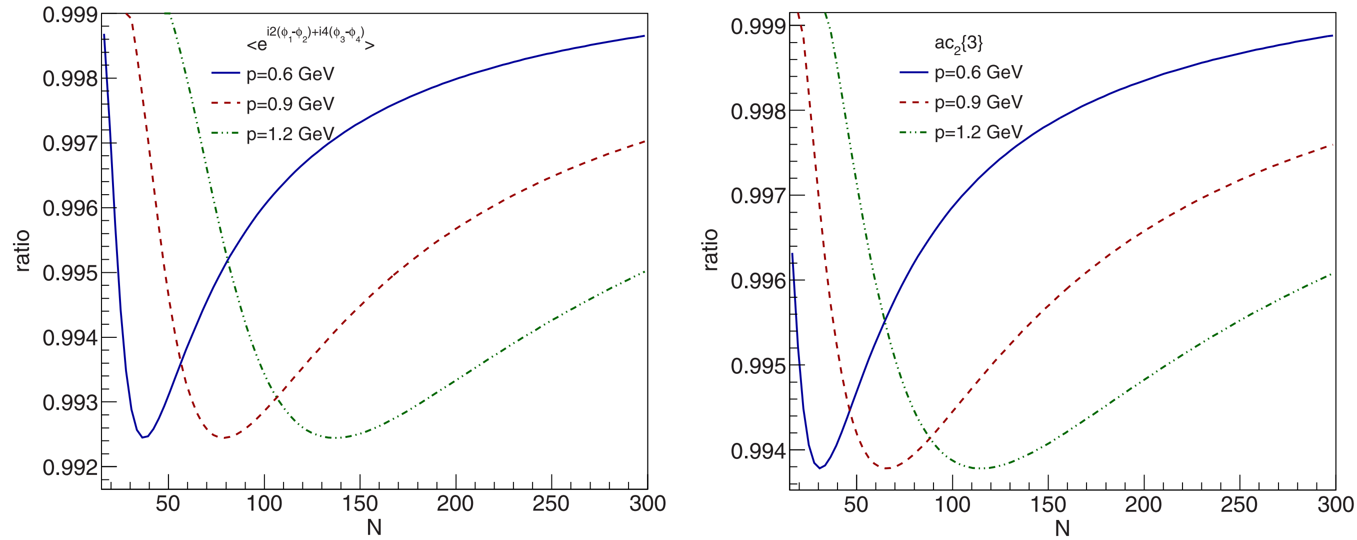

FIG. 5. The ratios of the approximate result to the full result, i.e., Eq. (24)/Eq. (A1) (left panel) and Eq. (34)/Eq. (A2) (right panel), as a function of the number of particles N, for different values of the transverse momenta p.

$$
\begin{aligned}
D_3 = {} & 30v_2^2 + 48v_3^2 + 30v_4^2 - 12v_2v_{2F}\cos(2\Psi_2) - 177v_2v_{2F}v_4\cos(2\Psi_2-4\Psi_4) + 93v_2^2v_4\cos(4\Psi_2-4\Psi_4) \\
& + \tfrac{75}{2}v_{2F}^2v_4\cos(4\Psi_4) + 45v_2^2v_{2F}^2 + 72v_{2F}^2v_3^2 + 45v_{2F}^2v_4^2 - \tfrac{165}{2}v_2^3v_{2F}\cos(2\Psi_2) - 3v_2v_{2F}^3\cos(2\Psi_2) \\
& - \tfrac{165}{8}v_2^3v_{2F}^3\cos(2\Psi_2) - 246v_2v_{2F}v_3^2\cos(2\Psi_2) - \tfrac{123}{2}v_2v_{2F}^3v_3^2\cos(2\Psi_2) - 162v_2v_{2F}v_4^2\cos(2\Psi_2) \\
& - \tfrac{81}{2}v_2v_{2F}^3v_4^2\cos(2\Psi_2) + \tfrac{33}{4}v_2^2v_{2F}^2\cos(4\Psi_2) - \tfrac{1}{8}v_2^3v_{2F}^3\cos(6\Psi_2) + \tfrac{123}{2}v_2v_{2F}^2v_3^2\cos(2\Psi_2-6\Psi_3) \\
& - \tfrac{15}{4}v_{2F}^3v_3^2\cos(6\Psi_3) - \tfrac{23}{2}v_2v_{2F}^3v_4^2\cos(2\Psi_2-8\Psi_4) - \tfrac{177}{4}v_2v_{2F}^3v_4\cos(2\Psi_2-4\Psi_4) \\
& + \tfrac{279}{2}v_2^2v_{2F}^2v_4\cos(4\Psi_2-4\Psi_4) - 120v_{2F}v_3^2v_4\cos(6\Psi_3-4\Psi_4) - 30v_{2F}^3v_3^2v_4\cos(6\Psi_3-4\Psi_4) \\
& + \tfrac{183}{2}v_2^2v_{2F}^2v_4\cos(4\Psi_4) + \tfrac{243}{2}v_{2F}^2v_3^2v_4\cos(4\Psi_4) + \tfrac{75}{2}v_{2F}^2v_4^3\cos(4\Psi_4) - \tfrac{15}{2}v_2v_{2F}^3v_4\cos(2\Psi_2+4\Psi_4), \\
D_4 = {} & 6 + 322v_2^2 + 456v_3^2 + 320v_4^2 - 272v_2v_{2F}\cos(2\Psi_2) - 2004v_2v_{2F}v_4\cos(2\Psi_2-4\Psi_4) + 651v_2^2v_4\cos(4\Psi_2-4\Psi_4) \\
& + 645v_{2F}^2v_4\cos(4\Psi_4) + 18v_{2F}^2 + 966v_2^2v_{2F}^2 + \tfrac{9}{4}v_{2F}^4 + \tfrac{483}{4}v_2^2v_{2F}^4 + 1368v_{2F}^2v_3^2 + 171v_{2F}^4v_3^2 + 960v_{2F}^2v_4^2 \\
& + 120v_{2F}^4v_4^2 - 914v_2^3v_{2F}\cos(2\Psi_2) - 204v_2v_{2F}^3\cos(2\Psi_2) - \tfrac{1371}{2}v_2^3v_{2F}^3\cos(2\Psi_2) \\
& - 2436v_2v_{2F}v_3^2\cos(2\Psi_2) - 1827v_2v_{2F}^3v_3^2\cos(2\Psi_2) - 1736v_2v_{2F}v_4^2\cos(2\Psi_2) - 1302v_2v_{2F}^3v_4^2\cos(2\Psi_2) \\
& + \tfrac{441}{2}v_2^2v_{2F}^2\cos(4\Psi_2) + \tfrac{147}{4}v_2^2v_{2F}^4\cos(4\Psi_2) - \tfrac{15}{2}v_2^3v_{2F}^3\cos(6\Psi_2) + 915v_2v_{2F}^2v_3^2\cos(2\Psi_2-6\Psi_3) \\
& + \tfrac{305}{2}v_2v_{2F}^4v_3^2\cos(2\Psi_2-6\Psi_3) - 147v_{2F}^3v_3^2\cos(6\Psi_3) + \tfrac{9}{2}v_2v_{2F}^4v_3^2\cos(2\Psi_2+6\Psi_3) - 330v_2v_{2F}^3v_4^2\cos(2\Psi_2-8\Psi_4) \\
& - 1503v_2v_{2F}^3v_4\cos(2\Psi_2-4\Psi_4) + 1953v_2^2v_{2F}^2v_4\cos(4\Psi_2-4\Psi_4) + \tfrac{1953}{8}v_2^2v_{2F}^4v_4\cos(4\Psi_2-4\Psi_4) \\
& - 1152v_{2F}v_3^2v_4\cos(6\Psi_3-4\Psi_4) - 864v_{2F}^3v_3^2v_4\cos(6\Psi_3-4\Psi_4) + 1548v_2^2v_{2F}^2v_4\cos(4\Psi_4) + \tfrac{215}{2}v_{2F}^4v_4\cos(4\Psi_4) \\
& + 258v_2^2v_{2F}^4v_4\cos(4\Psi_4) + 1776v_{2F}^2v_3^2v_4\cos(4\Psi_4) + 296v_{2F}^4v_3^2v_4\cos(4\Psi_4) + 603v_{2F}^2v_4^3\cos(4\Psi_4) \\
& + \tfrac{201}{2}v_{2F}^4v_4^3\cos(4\Psi_4) + \tfrac{119}{4}v_{2F}^4v_4^2\cos(8\Psi_4) - 322v_2v_{2F}^3v_4\cos(2\Psi_2+4\Psi_4) + \tfrac{127}{16}v_2^2v_{2F}^4v_4\cos(4\Psi_2+4\Psi_4).
\end{aligned}
\tag{A2}
$$

Based on Fig. 5, the ratios of the approximate results of Eqs. (24) and (34) to the full results of Eqs. (A1) and (A2) in this Appendix are both close to 1, with the worst approximations of 99.244% and 99.378% respectively, suggesting that Eqs. (24) and (34) can be good proxies for the results of Eqs. (A1) and (A2) in this Appendix.

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