Microscopic derivation of the generalized collective Hamiltonian

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The nucleus in the center-of-mass frame is described by $3A$-3 independent coordinates, including three Euler angles and three variables ρ, β, and γ to describe orientation, size, and shape of the nuclear inertia ellipsoid as well as $3A$-9 generalized Euler angles to describe internal motion of the nucleons. Based on the idea that the incompressible nucleus has constant density, we suggest a definition of the deformation parameter β. It allows us to derive the Hamiltonian, which describes large-amplitude collective motions. At $\beta \ll 1$ it reduces to a sum of the Bohr Hamiltonian and the Hamiltonian of monopole ρ vibrations. As a result of straightforward calculations we obtain explicit expressions for all the inertial parameters as functions of the variables ρ, β, and γ.

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I. INTRODUCTION

.The works of Rainwater, Bohr, and Mottelson [1–5] laid a foundation of the nuclear structure theory, where the Bohr Hamiltonian serves as one of the main cornerstones (see also [6]). It has been derived by Bohr [2], treating the nucleus as a liquid spherical drop having uniform density and sharp surface, which performs quadrupole vibrations with small amplitude. During such vibrations at any moment of time the nucleus attains an ellipsoidal shape, retaining its volume constant due to small compressibility of the nuclear matter. Using this fact Bohr defined the nuclear rotation as a rotation of the instantaneous principal axes of its shape ellipsoid, the orientation of which in the space is determined by the Euler angles. Two other coordinates were chosen as quadrupole deformation parameter β and triaxiality parameter γ describing the deviation of the quadrupole ellipsoid from an axially symmetric shape. Smallness of the vibrational amplitude meant that $\beta \ll 1$. By means of standard quantization procedure Bohr obtained the quantum kinetic energy $\hat{T}$ of the nucleus as a sum of kinetic energies of rotation and β and γ vibrations, which added with the potential energy $V(\beta, \gamma)$ gave the Bohr Hamiltonian $\hat{H}_{\mathrm{B}}$. Sometimes it is referred to as the Bohr-Mottelson Hamiltonian (see, e.g., [7]).

This Hamiltonian has been employed in a great many papers to study low-lying excitations of nuclei. In particular, Davydov and Filippov [8] calculated the rotational bands of a triaxial rigid rotor with fixed parameters β and γ. The β motion together with rotation were taken in consideration for the first time in the Davydov-Chaban model [9]. Such approach allowed us to explain significant deviation of experimental rotational bands in soft nuclei from the sequence $\sim I(I+1)$ typical for the rotational levels in rigid nuclei. A simple analytical solution for this task was obtained in the works [10,11] by applying the Davidson potential for the β motion. Bonatsos *et al.* [12] used the alternative Kratzer potential $V(\beta)$ and analyzed the role of the mass dependence on the deformation. Earlier the Schrödinger equation, containing the Bohr Hamiltonian with the same potentials, was solved by Skorobogatov [13].

Coexistence and mixing of the spherical and deformed shapes in a number of nuclei have been researched by calculating the eigenfunctions and eigenvalues of $\hat{H}_B$ with the sextic potential $V(\beta)$ in the articles [13–17]. In [18–22] tunneling through the barrier separating different nuclear shapes was analyzed using the WKB approximation.

The pioneering work of Bohr [2] has been followed by a number of papers (see [23] and references therein) regarding rotation of nuclei from the microscopic point of view as an ensemble of interacting nucleons rather than the semiempirical deformed liquid drop. It was shown that the rotation of the many-body system of A nucleons should be identified with the rotation of the body-fixed frame bound to principal axes of its inertia tensor. Unfortunately, all these papers used a redundant set of coordinates for description of the nuclei. Most correctly the expression for the collective energy

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operator was derived by Zickendraht and others for three-particle systems [24–29]. Later it was done for four-particle systems [30–32], which were described by three Euler angles φ, ϑ, and ψ defining the orientation of the inertia ellipsoid in space, three coordinates defining the magnitude of the principle axes of the inertia ellipsoid and the remaining three taken as internal Euler angles. Next, Zickendraht [33] expressed the kinetic energy of the nucleus having $A \geqslant 4$ nucleons in terms of the same collective coordinates except for the intrinsic ones, which were instead represented by radius vectors of the nucleons. In order to avoid spurious nuclear states these radius vectors have to satisfy some constraints, which lead to increased complexity when performing calculations [34].

In [35,36] the internal $3A$-9 coordinates of nucleons were taken as generalized Euler angles, which describe rotation in hyperspace of arbitrary dimensionality [37]. In the center-of-mass (c.m.) frame six collective coordinates were taken as three Euler angles and lengths a, b, and c of three orthogonal vectors in $(A-1)$-dimensional abstract space. These independent coordinates provided in straightforward manner decomposition of the kinetic energy into vibrational, rotational, and intrinsic parts. Later the same results were obtained by Buck *et al.* [38], using formalism of the dyadics instead of the inertia tensor approach. The papers [35,36] stimulated algebraic studies [39–44] of the collective dynamics as well as lead Filippov *et al.* [45,46] to creation of the method of generalized hyperspherical functions, unifying the shell and collective models. This method allowed researchers to obtain many physical results for light nuclei (see, e.g., [47–49]).

In the past years there has been a renewed interest in the problem of microscopic description of collective excitations, initiated by creation of nuclear structure models based on energy density functionals and other approaches to internal dynamics of nuclei [7,50].

The present paper is devoted to microscopic derivation of the quadrupole collective Hamiltonian, which describes the collective motions with arbitrary amplitude and reproduces the Bohr Hamiltonian at small deformations ($\beta \ll 1$). Our paper is tightly bound to the theory [35,36], the results of which for consistency are briefly repeated in the next two sections. At first we employ the same set of independent coordinates as in [35,36]. Then we transform the original coordinates a_1, a_2, and a_3 (semiaxes a, b, and c of the nuclear ellipsoid of inertia) to the shape coordinates β and γ and the variable ρ defining the volume conservation and characterizing absolute magnitude of the deformed ellipsoid axes. Next we express the kinetic energy of the nucleus T in terms of these coordinates, which is afterwards transformed to the corresponding operator $\hat{T}$. Simplifications of the obtained collective Hamiltonian at $\beta \approx 0$ or $\gamma \approx 0$ are analyzed.

Attempts to accomplish microscopic derivation of the collective Hamiltonian analogous to the Bohr Hamiltonian have been done in [38,46,51]. However, in all these works the variable β, which has to vary from zero to infinity, was treated as an angle, ranging in the interval from zero to $\pi/2$. Moreover, the main Bohr condition of the constant nuclear volume V was fulfilled there again only at $\beta \ll 1$.

Below, we propose a nonredundant set of variables, which describes large-amplitude collective motions of the nucleus keeping V constant, and express its kinetic-energy operator in terms of these variables.

II. COLLECTIVE COORDINATES

Let us start from the well-known expression for the kinetic energy of the nucleus, which consists of A nucleons with mass m, in a laboratory frame:

$$T_{\rm lab} = \frac{m}{2}\sum_{i=1}^{A}\dot{\mathbf{r}}_i^2. \tag{1}$$

Introducing the c.m. frame x', y', and z' allows us to transform (1) to

$$T_{\rm lab} = \frac{Am}{2}\dot{\mathbf{R}}^2 + T, \tag{2}$$

where

$$T = \frac{m}{2}\sum_{i=1}^{A}\dot{\mathbf{r}}_i'^2 \tag{3}$$

stands for the relative kinetic energy of the nucleus, and $\mathbf{r}_i' = \mathbf{r}_i - \mathbf{R}$ are coordinates of the nucleons with respect to the c.m. frame, the position of which in the laboratory frame is determined by the radius vector

$$\mathbf{R} = \frac{1}{A}\sum_{i=1}^{A}\mathbf{r}_i. \tag{4}$$

The coordinates of all the nucleons $\mathbf{r}_i' = \{x_i',\ y_i',\ z_i'\}$ obey the condition $\sum_i \mathbf{r}_i' = 0$. Usually such a constraint is abandoned with the aid of independent Jacobi vectors (see, e.g., [52])

$$\mathbf{q}_i = \sqrt{\frac{i}{i+1}}\left(\mathbf{r}_{i+1} - \frac{1}{i}\sum_{j=1}^{i}\mathbf{r}_j\right), \quad i = 1,\ 2, \ldots, n, \tag{5}$$

where $n = A-1$. Then the kinetic energy of relative motion takes the form

$$T = \frac{m}{2}\sum_{i=1}^{n}\dot{\mathbf{q}}_i^2. \tag{6}$$

In order to separate nuclear rotation we introduce the body-fixed frame with origin in c.m. and axes ξ, η, and ζ along the unit vectors $\boldsymbol{\epsilon}_v' = \{\boldsymbol{\epsilon}_1',\ \boldsymbol{\epsilon}_2',\ \boldsymbol{\epsilon}_3'\}$, parallel to the principal axes of the instantaneous inertia tensor of the nucleus. The inertia tensor for the nucleus with respect to c.m. is defined as [53]

$$I_{ss'} = m\sum_{i=1}^{A}\left(\delta_{ss'}r_i'^2 - r_{is}'r_{is'}'\right), \tag{7}$$

where $s, s' = 1, 2, 3$. Being rewritten in coordinates $\mathbf{q}_i$, it reads [46]

$$I_{ss'} = m\sum_{i=1}^{n}\left(\delta_{ss'}q_i^2 - q_{is}q_{is'}\right). \tag{8}$$

We denote the projections of the Jacobi vectors on the axes ξ, η, and ζ by $a_{iv} = \langle\boldsymbol{\epsilon}_v'|\mathbf{q}_i\rangle$. In the principal axes the

off-diagonal elements of the inertia tensor vanish, so that

$$\sum_{i=1}^{n} a_{i\xi} a_{i\eta} = \sum_{i=1}^{n} a_{i\xi} a_{i\zeta} = \sum_{i=1}^{n} a_{i\eta} a_{i\zeta} = 0, \tag{9}$$

while its diagonal elements are equal to the principal moments of inertia associated with a rigid rotation around the axes $\kappa \neq \mu, \nu$:

$$\mathfrak{J}_{\rm rig}^{(\kappa)} = m\big(a_\mu^2 + a_\nu^2\big), \quad \mu \neq \nu, \tag{10}$$

where

$$a_1 = \sqrt{\sum_{i=1}^{n} a_{i\xi}^2}, \quad a_2 = \sqrt{\sum_{i=1}^{n} a_{i\eta}^2}, \quad a_3 = \sqrt{\sum_{i=1}^{n} a_{i\zeta}^2}. \tag{11}$$

Rotation from the stationary c.m. frame to the body-fixed one is described by the Euler angles φ, ϑ, and ψ. It is realized by three successive rotations in the stationary frame: (1) by the angle ψ around the axis z, (2) by the angle ϑ around y, and (3) by φ around z. The same result is obtained doing the rotations (1) by φ around the initial axis z, (2) by ϑ around new axis y', and (3) by ψ around the axis z'' (see, e.g., [54]).

Next we introduce an abstract Euclidian space with basis orthonormal vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, which satisfy the condition $(\mathbf{e}_i \cdot \mathbf{e}_j) = \delta_{ij}$, and define there three vectors $\mathcal{A}_\nu$ [35,36]:

$$\mathcal{A}_\nu = \sum_{i=1}^{n} a_{i\nu} \mathbf{e}_i \quad (\nu = 1,\ 2,\ 3) \tag{12}$$

with lengths a_ν expressed by Eq. (11). The constraint (9) can be treated as an orthogonality condition $\mathcal{A}_\mu \cdot \mathcal{A}_\nu = \delta_{\mu,\nu}$ for these vectors, while their lengths a_1, a_2, and a_3 serve as three collective coordinates, specifying size and shape of the nucleus.

It is natural to determine the remaining $3n-6$ internal coordinates as any rotational variables, which describe orientation of three vectors $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ in the n-dimensional particle index space. The Eulerian angles are the most convenient parameters for the SO_3 group of rotations in the three-dimensional space. Therefore Vilenkin [37] generalized them on the SO_n group with $n \geqslant 3$. First he introduced the notation $g_{jk}(\theta)$ for the rotation through the angle θ in the plane $(j,\ k)$, calling positive the rotation from the vector $\mathbf{e}_j$ to $\mathbf{e}_k$. Next Vilenkin has shown that arbitrary rotation in the n-dimensional space may be represented in the form

$$g = g^{(n-1)} \cdots g^{(1)} \tag{13}$$

with

$$g^{(k)} = g_1\big(\theta_1^k\big) \cdots g_k\big(\theta_k^k\big), \tag{14}$$

where

$$g_k(\theta) \equiv g_{k+1,k}(\theta). \tag{15}$$

However, in the three-dimensional space the rotations by conventional Euler angles φ, ϑ, and ψ should obey the convention of the right-handed screw, so that they should be oriented in the opposite direction compared to (15). Therefore here we redefine Eq. (15) to read

$$g_k(\theta) \equiv g_{k,k+1}(\theta). \tag{16}$$

Then this operator $g_k(\theta)$ acts on the basis vectors in the following manner:

$$\begin{aligned} g_k(\theta)\ \mathbf{e}_k &= \cos\theta\ \mathbf{e}_k + \sin\theta\ \mathbf{e}_{k+1}, \\ g_k(\theta)\ \mathbf{e}_{k+1} &= -\sin\theta\ \mathbf{e}_k + \cos\theta\ \mathbf{e}_{k+1}. \end{aligned} \tag{17}$$

Such Euler angles vary in the limits

$$\begin{aligned} &0 \leqslant \theta_1^k < 2\pi, \\ &0 \leqslant \theta_j^k < \pi, \quad j \neq 1. \end{aligned} \tag{18}$$

Under the rotation g the basis vectors $\mathbf{e}_i$ of the stationary frame rotate to

$$\mathbf{e}_i' = g\mathbf{e}_i = \sum_{k=1}^{n} g_{ki} \mathbf{e}_k, \tag{19}$$

where the rotation matrix g_{ik} satisfies the orthogonality conditions

$$\sum_{i=1}^{n} g_{ki} g_{k'i} = \sum_{i=1}^{n} g_{ik} g_{ik'} = \delta_{kk'}, \tag{20}$$

providing that $\langle \mathbf{e}_i' | \mathbf{e}_j' \rangle = \delta_{ij}$.

Using Eq. (17) one can show that the operator $g^{(l)}(\theta_1^l, \ldots, \theta_l^l)$, acting on the vector $\mathbf{e}_{l+1}$, bears the vector [36]

$$\begin{aligned} g^{(l)}\mathbf{e}_{l+1} = {} & (-1)^l \sin\theta_l^l \sin\theta_{l-1}^l \cdots \sin\theta_1^l\ \mathbf{e}_1 \\ & + \sum_{k=2}^{l-1} (-1)^{l+1-k} \sin\theta_l^l \cdots \sin\theta_k^l \cos\theta_{k-1}^l\ \mathbf{e}_k \\ & + \cos\theta_l^l\ \mathbf{e}_{l+1}. \end{aligned} \tag{21}$$

From here we see that $\theta_1^l, \ldots, \theta_l^l$ can be interpreted as spherical angles of the vector $g^{(l)}\mathbf{e}_{l+1}$ in $(l+1)$-dimensional space.

Let us direct the basis vectors $\mathbf{e}'_{n-2}$, $\mathbf{e}'_{n-1}$, and $\mathbf{e}'_n$ along $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$, respectively, so that

$$\mathcal{A}_k = a_k \mathbf{e}'_{n-3+k}, \quad k = 1,\ 2,\ 3. \tag{22}$$

All possible orientations of the vectors $\mathcal{A}_k$ in the hyperspace are provided by the rotation

$$G = g^{(n-1)} g^{(n-2)} g^{(n-3)}, \tag{23}$$

depending on $3n-6$ angles $\theta_l^{(n-1)}$, $\theta_l^{(n-2)}$, and $\theta_l^{(n-3)}$, treated as independent coordinates to describe internal rotation of the nucleus.

With the operator $g = G$ the equality (19) takes the form

$$\mathbf{e}_i' = G\mathbf{e}_i = \sum_{k=1}^{n} G_{ki} \mathbf{e}_k. \tag{24}$$

III. KINETIC ENERGY

In this section we shall express the kinetic energy of the nucleus in terms of the collective variables introduced above. The task is reduced to calculation of the derivative of the

Jacobi vector over time:

$$\dot{\mathbf{q}}_i = \sum_{\nu=1}^{3} [\dot{a}_{i\nu}\boldsymbol{\epsilon}'_\nu + a_{i\nu}\dot{\boldsymbol{\epsilon}}'_\nu]. \tag{25}$$

In order to find the time derivative of the vector $\boldsymbol{\epsilon}'_\nu$ we use the relation of its change $d\boldsymbol{\epsilon}'_\nu$ with the infinitesimal rotations $d\phi_{\nu,\mu}$ in the planes, where lie the vectors $\boldsymbol{\epsilon}'_\nu$ and $\boldsymbol{\epsilon}'_\mu$:

$$d\boldsymbol{\epsilon}'_\nu = \boldsymbol{\epsilon}'_\nu(t+dt) - \boldsymbol{\epsilon}'_\nu(t) = \sum_{\mu\neq\nu} d\phi_{\nu,\mu}\boldsymbol{\epsilon}'_\mu. \tag{26}$$

This enables us to rewrite (25) as

$$\dot{\mathbf{q}}_i = \sum_{\nu} \left(\dot{a}_{i\nu}\boldsymbol{\epsilon}'_\nu + a_{i\nu} \sum_{\mu\neq\nu} \omega_{\nu,\mu}\boldsymbol{\epsilon}'_\mu \right), \tag{27}$$

where $\boldsymbol{\omega} = \{\omega_\xi \equiv \omega_{2,3},\ \omega_\eta \equiv \omega_{3,1},\ \omega_\zeta \equiv \omega_{1,2}\}$ denote the angular velocity of rotation in the ordinary space, $\omega_{\alpha,\beta} = \dot{\phi}_{\alpha,\beta}$. It is expressed in the Eulerian angles by

$$\begin{aligned} \omega_\xi &= -\cos\psi \sin\vartheta\ \dot{\varphi} + \sin\psi\ \dot{\vartheta}, \\ \omega_\eta &= \sin\psi \sin\vartheta\ \dot{\varphi} + \cos\psi\ \dot{\vartheta}, \\ \omega_\zeta &= \cos\vartheta\ \dot{\varphi} + \dot{\psi}. \end{aligned} \tag{28}$$

It remains to find the parameter $\dot{a}_{i\nu} = \langle \mathbf{e}_i | \dot{\mathcal{A}}_\nu \rangle$, which enters into Eq. (27). For this aim we introduce an infinitesimal rotation $d\alpha_{ik}$ from the vector $\mathbf{e}'_i$ to $\mathbf{e}'_k$ in the (i,k) plane of the frame $\mathbf{e}'_1,\ \mathbf{e}'_2, \ldots, \mathbf{e}'_n$, rotating in the n-dimensional hyperspace:

$$d\mathbf{e}'_i = \sum_{k\neq i} d\alpha_{i,k}\mathbf{e}'_k. \tag{29}$$

Then the differentials of the vectors $\mathcal{A}_\nu$ take the form

$$d\mathcal{A}_\nu = da_\nu \mathbf{e}'_{n-3+\nu} + a_\nu \sum_{l=1}^{n} d\alpha_{n-3+\nu,l}\ \mathbf{e}'_l. \tag{30}$$

By passing to time derivatives with the aid of (12) and (24) one finally has

$$\dot{a}_{i\nu} = \dot{a}_\nu G_{i,n-3+\nu} + a_\nu \sum_{l=1}^{n} \Omega_{n-3+\nu,l} G_{il}, \tag{31}$$

where $\Omega_{k,l} = \dot{\alpha}_{k,l}$ are the angular velocities of internal rotations in the (k,l) planes of the moving frame. In accordance with Eq. (29) in the sums (30) and (31) the term $l = n-3+\nu$ is absent, or in other words $\Omega_{n-3+\nu,n-3+\nu} = 0$.

With the aid of Eqs. (12), (19), and (22) one obtains

$$a_{i\nu} = a_\nu G_{i,n-3+\nu}. \tag{32}$$

By using Eqs. (31) and (32) and orthogonality of the G matrix, we obtain the following formulas:

$$\sum_{i,\nu} \dot{a}_{i\nu}^2 = \sum_{\nu=1}^{3} a_\nu^2 \sum_{l\neq n-3+\nu} \Omega_{l,n-3+\nu}^2 \tag{33}$$

and

$$\sum_{i=1}^{n} \dot{a}_{i\mu} a_{i\nu} = a_\mu a_\nu \Omega_{n-3+\mu,n-3+\nu}, \quad \mu \neq \nu, \tag{34}$$

which facilitate construction of the classical kinetic energy T.

Noting that the internal rotations in the planes of privileged unit vectors $\mathbf{e}'_{n-2}$, $\mathbf{e}'_{n-1}$, and $\mathbf{e}'_n$ with frequencies

$$\Omega_\xi \equiv \Omega_{n-1,n}, \quad \Omega_\eta \equiv \Omega_{n-2,n}, \quad \Omega_\zeta \equiv \Omega_{n-2,n-1} \tag{35}$$

are coupled by Coriolis interaction to the rotation in ordinary space with frequencies ω_ξ, ω_η, and ω_ζ, we include them in the rotational part $T_{\rm rot}$ of the kinetic energy T, so that [35,36]

$$T = T_{\rm vib} + T_{\rm rot} + T_{\rm cor} + T_{\rm int}, \tag{36}$$

where the vibrational kinetic energy is

$$T_{\rm vib} = \frac{m}{2}\left(\dot{a}_1^2 + \dot{a}_2^2 + \dot{a}_3^2\right), \tag{37}$$

while the kinetic energies of rotation and internal motion as well as the Coriolis interaction are

$$\begin{aligned} T_{\rm rot} &= \frac{1}{2}\sum_{\kappa=1}^{3} \mathfrak{J}_{\rm rig}^{(\kappa)}\left(\omega_\kappa^2 + \Omega_\kappa^2\right), \\ T_{\rm int} &= a_1^2 \sum_{l=1}^{n-3} \Omega_{l,n-2}^2 + a_2^2 \sum_{l=1}^{n-3} \Omega_{l,n-1}^2 + a_3^2 \sum_{l=1}^{n-3} \Omega_{l,n}^2, \\ T_{\rm cor} &= 4a_2a_3\Omega_\xi\omega_\xi + 4a_1a_3\Omega_\eta\omega_\eta + 4a_1a_2\Omega_\zeta\omega_\zeta. \end{aligned} \tag{38}$$

IV. BOHR HAMILTONIAN

In order to derive the Bohr Hamiltonian let us first express the parameters a_1, a_2, and a_3 in terms of the coordinates β and γ, which are usually used for description of the nuclei with quadrupole shape [5]. Also, we shall introduce the coordinate ρ, which in the method of hyperspherical functions is called the hyper-radius, defined by [52]

$$\rho^2 = \sum_{i=1}^{A} \mathbf{r}_i'^2 = \sum_{i=1}^{n} \mathbf{q}_i^2 \tag{39}$$

or

$$\rho^2 = a_1^2 + a_2^2 + a_3^2. \tag{40}$$

Its mean value is related to the mean-square radius of the nucleus by $\rho_0^2 = A\langle \mathbf{r}^2 \rangle$.

In the Bohr-Mottelson model a sharp surface of the nucleus is described by the expansion of the surface radius $R(\theta,\varphi)$ in spherical harmonics. For the nuclei with small quadrupole deformation the radii along the principal axes ξ, η, and ζ are determined by [5,6]

$$R_\kappa = R_0\left[1 + \tilde{\beta}\cos\left(\gamma - \frac{2\pi}{3}\kappa\right)\right], \tag{41}$$

where

$$\tilde{\beta} = (5/4\pi)^{1/2}\beta \tag{42}$$

and R_0 is the radius of the sphere with the same volume $V_0 = (4\pi/3)R_0^3$ as the deformed nucleus. The sharp surface, drawn by the vector $R(\theta,\varphi)$, is well approximated by the ellipsoid with semiaxes (41) [6]. Therefore we impose

$$R_0^3 = R_1R_2R_3. \tag{43}$$

The mean radius of any nucleus R_0 is defined by [5]

$$R_0^2 = \tfrac{5}{3}\langle \mathbf{r}^2\rangle + 0(\beta^2). \tag{44}$$

The radius vector $\mathbf{r}$ has components $\{\xi,\ \eta,\ \zeta\}$ along the axes $\xi,\ \eta,$ and ζ. Integrating such squared components over the volume of the nuclear ellipsoid with semiaxes $R_\kappa = \{R_1,\ R_2,\ R_3\}$ one obtains

$$R_1^2 = 5\langle\xi^2\rangle, \quad R_2^2 = 5\langle\eta^2\rangle, \quad R_3^2 = 5\langle\zeta^2\rangle. \tag{45}$$

Combining (43) and (45) we arrive at

$$R_0^2 = 5(\langle\xi^2\rangle\langle\eta^2\rangle\langle\zeta^2\rangle)^{1/3}. \tag{46}$$

Such definition of the nuclear radius R_0 in terms of the mean-square components of the radius $\mathbf{r}$ is valid at arbitrary deformations and transforms to Bohr's definition (44) as $\beta \to 0$.

The equalities (45) are equivalent to

$$R_\kappa^2 = 5a_\kappa^2/A, \quad \kappa = 1,\ 2,\ 3. \tag{47}$$

We see that three parameters a_κ are proportional to the radii R_κ with the same proportionality coefficient. Therefore the only choice for them is

$$a_\kappa = a_0\left[1 + \tilde{\beta}\cos\left(\gamma - \frac{2\pi}{3}\kappa\right)\right], \tag{48}$$

where the radius of the inertia ellipsoid is equal to $a_0 = (A/5)^{1/2}R_0$. At $\beta \ll 1$ it is related to the hyper-radius by $a_0 = \rho/\sqrt{3}$.

Having inserted time derivatives of a_κ in the equality (37), we obtain the kinetic energy of vibrations, expressed in new variables:

$$T_{\rm vib} = \frac{m}{2}\left[\frac{5}{8\pi}\rho^2(\dot\beta^2 + \beta^2\dot\gamma^2) + \dot\rho^2\right], \quad \beta \ll 1. \tag{49}$$

Time derivatives in all other parts (38) of the nuclear kinetic energy T are contained through angular velocities.

The quantization of T is done with the aid of the Pauli prescription [55], which is applied to any system, described by a set of curvilinear coordinates $x = \{x_1,\ x_2, \ldots,\ x_f\}$, the kinetic energy of which has the form

$$T = \frac{m}{2}\sum_{i,j} D_{ij}(x)\dot{x}_i\dot{x}_j, \tag{50}$$

where $D_{ij}(x) = D_{ji}(x)$ stands for the metric tensor. The corresponding operator is determined by

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{D(x)}}\sum_{i,j}\frac{\partial}{\partial x_i}\sqrt{D(x)}D^{ij}(x)\frac{\partial}{\partial x_j}, \tag{51}$$

where $D(x)$ denotes the determinant of the metric tensor, $D(x) = \det\{D_{ij}(x)\}$, and $D^{ij}(x) = D_{ij}^{-1}(x)$ is the inverse matrix to $D_{ij}(x)$.

The wave function, satisfying Schrödinger equation

$$\{\hat{T} + V(x)\}\psi(x) = E\psi(x), \tag{52}$$

is normalized by

$$\int |\psi(x)|^2 d\tau = 1, \tag{53}$$

where the volume element $d\tau = \sqrt{D(x)}dx$ with $dx = dx_1dx_2\cdots dx_f$.

When Eq. (49) is substituted into Eq. (36) instead of Eq. (37) and when the above quantization procedure is applied (for more details see the next section), one obtains the kinetic-energy operator $\hat{T}$.

Omitting in $\hat{T}$ all the terms related to the intrinsic motion, one gets the operator of collective kinetic energy $\hat{T}_{\rm col}(\rho,\beta,\gamma)$ as a sum of the quadrupole collective operator $\hat{T}^{(2)}_{\rm col}(\beta,\gamma)$ and the operator of monopole (breezing) vibrations $\hat{T}^{(0)}_{\rm vib}(\rho)$:

$$\hat{T}_{\rm col} = \hat{T}^{(2)}_{\rm col} + \hat{T}^{(0)}_{\rm vib}(\rho). \tag{54}$$

Then assuming that the potential energy splits into two terms, $V(\rho,\beta,\gamma) = V_0(\rho) + V_2(\beta,\gamma)$, we can write down the Hamiltonian $\hat{H}^{(2)}_{\rm col} = \hat{T}^{(2)}_{\rm col} + V_2(\beta,\gamma)$ of the collective five-dimensional motion as

$$\hat{H}^{(2)}_{\rm col} = \frac{\hbar^2}{2B(\rho)}\left\{-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\left(\beta^4\frac{\partial}{\partial\beta}\right) - \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4}\sum_{\kappa=1}^{3}\frac{\hat{L}_\kappa^2}{\beta^2\sin^2\left(\gamma - \frac{2\pi}{3}\kappa\right)}\right\} + V_2(\beta,\gamma), \tag{55}$$

where

$$B(\rho) = \frac{5}{8\pi}m\rho^2, \tag{56}$$

and $\hat{L}_\kappa$ are projections of the total orbital angular momentum operator $\hat{\mathbf{L}}$ (taken in units $\hbar$) on the body-fixed coordinate axes $\xi,\ \eta,$ and ζ. They are expressed in terms of the Euler angles [56]:

$$\begin{aligned}
\hat{L}_1 &= -i\left[\cos\psi\left(-\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} + \cot\vartheta\frac{\partial}{\partial\psi}\right) + \sin\psi\frac{\partial}{\partial\vartheta}\right],\\
\hat{L}_2 &= -i\left[\sin\psi\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \cot\vartheta\frac{\partial}{\partial\psi}\right) + \cos\psi\frac{\partial}{\partial\vartheta}\right],\\
\hat{L}_3 &= -i\frac{\partial}{\partial\psi}.
\end{aligned} \tag{57}$$

If the amplitude of the ρ vibrations is much less than the equilibrium hyper-radius ρ_0, then $B(\rho)$ may be approximated by its equilibrium value $B(\rho_0)$, and $\hat{H}^{(2)}_{\rm col}$ coincides with the Bohr Hamiltonian [5].

The kinetic-energy operator of the monopole (breezing) vibrations reads (see also [38])

$$\hat{T}^{(0)}_{\rm vib}(\rho) = -\frac{\hbar^2}{2m}\frac{1}{\rho^{3n-1}}\frac{\partial}{\partial\rho}\rho^{3n-1}\frac{\partial}{\partial\rho}. \tag{58}$$

It should be noted that here the variable ρ specifies the hyper-radius but not the nuclear density as it is usually done in the theories of giant monopole resonance (breezing vibrations) [57,58], the energy of which much exceeds the energies of β and γ vibrations (specifically, for ^{90}Zr it equals 17.87 MeV, for ^{116}Sn it equals 15.85 MeV, for ^{144}Sm it equals 15.40 MeV, and for ^{208}Pb it equals 13.96 MeV [59]).

Also, it is worth noting that in the general case the monopole vibrations are mixed with the internal motion, the kinetic-energy operator $\hat{T}_{\rm int}$ of which also depends on ρ.

However, the calculations of the monopole vibrations considerably simplify for double-magic nuclei such as ^{16}O and ^{40}Ca [46].

V. COLLECTIVE HAMILTONIAN FOR NUCLEI WITH ARBITRARY DEFORMATION

In this section we shall derive the collective Hamiltonian for quadrupole motion characterized by arbitrary value of the deformation parameter β. Following the Bohr-Mottelson model [5], we again require that vibrations and rotations change only the shape of the ellipsoidal nucleus keeping unchanged its radius R_0, related with the radii R_κ by (43). The radii, meeting this condition, can be chosen as

$$R_\kappa = R_0 \mathcal{E}_\kappa(\beta, \gamma), \tag{59}$$

where

$$\mathcal{E}_\kappa(\beta, \gamma) = \exp\left[\tilde{\beta}\cos\left(\gamma - \frac{2\pi}{3}\kappa\right)\right] \tag{60}$$

with $\tilde{\beta}$ defined in (42). The condition (43) is ensured by the fact that the product of the exponentials equals unity. In the limiting case of $\beta \ll 1$ Eq. (59) reduces to previous definition (41). Note also that such an exponential dependence of radii R_κ on β and γ has been introduced in [20,21]; however, the factor $(5/4\pi)^{1/2}$ was missed there, that led to incorrect estimations of the moments of inertia.

With varying triaxiality parameter γ the nucleus acquires the following shapes: for $\gamma = 0$ it is axially symmetric and prolate ($a_3 > a_1 = a_2$), for $\gamma = \pi/3$ it is axially symmetric and oblate ($a_2 < a_1 = a_3$), and the value $\pi/6$ corresponds to its maximal asymmetry. Beyond the interval $0 \leqslant \gamma \leqslant \pi/3$ the nuclear shapes are repeated. Therefore for single-valued correspondence of the shapes and coordinates one should impose standard constraints:

$$0 \leqslant \beta < \infty, \quad 0 \leqslant \gamma \leqslant \pi/3. \tag{61}$$

The shape of the inertia ellipsoid for the model discussed here must repeat the shape of the nucleus. Therefore we write parameters a_κ in the same form as R_κ:

$$a_\kappa = a_0 \mathcal{E}_\kappa(\beta, \gamma), \tag{62}$$

where $a_0 = \rho/\sqrt{3}$ represents a radius of the sphere with the same volume as the inertia ellipsoid, i.e.,

$$a_0^3 = a_1 a_2 a_3. \tag{63}$$

The radial coordinate ρ varies in the interval $0 \leqslant \rho < \infty$. It satisfies the condition $\rho^2 = a_1^2 + a_2^2 + a_3^2$ if $\beta \ll 1$ and the formula (62) may be replaced by (48). Only then the variable ρ takes the sense of the hyper-radius.

From Eq. (62) one can extract β^2:

$$\beta^2 = \frac{8\pi}{15}\sum_{\kappa=1}^{3}\ln^2\left(\frac{a_\kappa}{a_0}\right). \tag{64}$$

Here we see that our deformation coordinate β characterizes deviation of the inertia ellipsoid shape from the sphere. For nuclei with small deformation (64) simplifies:

$$\beta^2 = \frac{8\pi}{15}\sum_{\kappa=1}^{3}\left(\frac{a_\kappa - a_0}{a_0}\right)^2. \tag{65}$$

The kinetic energy of vibrations (37) can be represented as

$$T_{\rm vib} = \frac{m}{2}(D_{\beta\beta}\dot{\beta}^2 + 2D_{\beta\gamma}\dot{\beta}\dot{\gamma} + D_{\gamma\gamma}\dot{\gamma}^2 + D_{\rho\rho}\dot{\rho}^2). \tag{66}$$

For simplicity, we neglected here any coupling of the monopole ρ vibrations with β and γ vibrations. All other terms (38) of the classical kinetic energy conserve their form, but with the parameters a_κ being now the functions of time-dependent coordinates ρ, β, and γ.

Substituting the time derivatives of (62) into (37) one finds the metric associated with vibrations:

$$\begin{aligned} D_{\beta\beta} &= (5/4\pi)\rho^2 b_{\beta\beta}, \quad D_{\beta\gamma} = (5/4\pi)\rho^2\beta b_{\beta\gamma}, \\ D_{\gamma\gamma} &= (5/4\pi)\rho^2\beta^2 b_{\gamma\gamma}, \quad D_{\rho\rho} = b_{\rho\rho}, \end{aligned} \tag{67}$$

where $b_{ij} = b_{ij}(\beta, \gamma)$ denote the following expressions:

$$\begin{aligned} b_{\beta\beta} &= \sum_{\kappa=1}^{3}\cos^2(\gamma - 2\pi\kappa/3)\mathcal{E}_\kappa^2(\beta, \gamma), \\ b_{\gamma\gamma} &= \sum_{\kappa=1}^{3}\sin^2(\gamma - 2\pi\kappa/3)\mathcal{E}_\kappa^2(\beta, \gamma), \\ b_{\beta\gamma} &= -\sum_{\kappa=1}^{3}\sin(\gamma - 2\pi\kappa/3)\cos(\gamma - 2\pi\kappa/3)\mathcal{E}_\kappa^2(\beta, \gamma), \\ b_{\rho\rho} &= \frac{1}{3}\sum_{\kappa=1}^{3}\mathcal{E}_\kappa^2(\beta, \gamma). \end{aligned} \tag{68}$$

The metric tensor is block diagonal with a two-dimensional matrix for quadrupole vibrations

$$\begin{array}{c} \\ \dot{\beta} \\ \dot{\gamma} \end{array} \begin{array}{c} \begin{array}{cc} \dot{\beta} & \dot{\gamma} \end{array} \\ \begin{pmatrix} D_{\beta\beta} & D_{\beta\gamma} \\ D_{\gamma\beta} & D_{\gamma\gamma} \end{pmatrix} \end{array}, \tag{69}$$

and three rotational blocks

$$\begin{array}{c} \\ \omega_\kappa \\ \Omega_\kappa \end{array} \begin{array}{c} \begin{array}{cc} \omega_\kappa & \Omega_\kappa \end{array} \\ \begin{pmatrix} a_\mu^2 + a_\nu^2 & 2a_\mu a_\nu \\ 2a_\mu a_\nu & a_\mu^2 + a_\nu^2 \end{pmatrix} \end{array}, \tag{70}$$

enumerated by $\kappa = 1, 2, 3$, where $\kappa \neq \mu$, ν and $\mu \neq \nu$. They are followed by single matrix element $D_{\rho\rho}$ as well as by a_1^2, a_2^2, and a_3^2, each $n-3$ times repeated. All other matrix elements are zero.

The determinant of the metric tensor D equals a product of determinants of all the blocks, multiplied by $D_{\rho\rho}$ and by $(a_1 a_2 a_3)^{2(n-3)} \sim \rho^{2(3n-9)}$:

$$D = D_{\rho\rho} D_{\rm vib} D_{\rm rot} \rho^{2(3n-9)}, \tag{71}$$

where

$$D_{\rm vib} = D_{\beta\beta} D_{\gamma\gamma} - D_{\beta\gamma}^2 \tag{72}$$

and

$$D_{\rm rot} = \left(a_1^2 - a_3^2\right)^2 \left(a_2^2 - a_3^2\right)^2 \left(a_1^2 - a_2^2\right)^2 = \rho^{12}\Phi^2(\beta,\gamma) \tag{73}$$

with

$$\Phi^2(\beta,\gamma) = \left(\mathcal{E}_1^2 - \mathcal{E}_2^2\right)^2 \left(\mathcal{E}_1^2 - \mathcal{E}_3^2\right)^2 \left(\mathcal{E}_2^2 - \mathcal{E}_3^2\right)^2. \tag{74}$$

Everywhere in D we omit the unessential numerical factor.

The above structure of the metric tensor allows us to express the Jacobian of transformation $\sqrt{D}$ in the form

$$\sqrt{D} = \beta\sqrt{WR}\,\rho^{3n-1}, \tag{75}$$

where the functions W and R are defined by

$$\begin{aligned} W(\beta,\gamma) &= b_{\beta\beta}(\beta,\gamma) b_{\gamma\gamma}(\beta,\gamma) - b_{\beta\gamma}^2(\beta,\gamma), \\ R(\beta,\gamma) &= b_{\rho\rho}(\beta,\gamma)\Phi^2(\beta,\gamma). \end{aligned} \tag{76}$$

The volume element then takes the form

$$d\tau = d\tau_{\rm col} d\tau_\theta, \tag{77}$$

where the volume element associated with the six-dimensional collective motion and internal rotation are given, respectively, by

$$d\tau_{\rm col} = \sqrt{D}\, d\rho\, d\beta\, d\gamma\, d\varphi\, \sin\vartheta\, d\vartheta\, d\psi \tag{78}$$

and [37]

$$d\tau_\theta = \prod_{\kappa=n-2}^{n} \prod_{j=1}^{\kappa} \sin^{j-1}\theta_j^\kappa d\theta_j^\kappa. \tag{79}$$

As a result, the kinetic-energy operator of the collective motion becomes

$$\hat{T}_{\rm col} = \hat{T}_{\rm vib} + \hat{T}_{\rm rot} + \hat{T}_{\rm cor}. \tag{80}$$

In particular, the vibrational kinetic energy $\hat{T}_{\rm vib}$ is given by the sum of the operator for quadrupole β and γ vibrations

$$\begin{aligned} \hat{T}_{\rm vib}^{(2)} = &-\frac{\hbar^2}{2B(\rho)} \frac{3}{2\beta} \frac{1}{\sqrt{WR}} \\ &\times \left\{ \frac{\partial}{\partial\beta}\beta\sqrt{\frac{R}{W}} b_{\gamma\gamma} \frac{\partial}{\partial\beta} - \frac{\partial}{\partial\beta}\sqrt{\frac{R}{W}} b_{\beta\gamma} \frac{\partial}{\partial\gamma} \right. \\ &\left. - \frac{\partial}{\partial\gamma}\sqrt{\frac{R}{W}} b_{\beta\gamma} \frac{\partial}{\partial\beta} + \frac{1}{\beta}\frac{\partial}{\partial\gamma}\sqrt{\frac{R}{W}} b_{\beta\beta} \frac{\partial}{\partial\gamma} \right\} \end{aligned} \tag{81}$$

with $B(\rho)$ defined in Eq. (56) and the operator for the monopole ρ vibrations

$$\hat{T}_{\rm vib}^{(0)} = -\frac{\hbar^2}{2m b_{\rho\rho}(\beta,\gamma)} \frac{1}{\rho^{3n-1}} \frac{\partial}{\partial\rho} \rho^{3n-1} \frac{\partial}{\partial\rho}. \tag{82}$$

Let us rewrite the operator $\hat{T}_{\rm vib}^{(2)}$ in the form convenient for comparison with the Bohr-Mottelson model and the results of [50]. For this aim we shall introduce the functions

$$w(\beta,\gamma) = \frac{2}{3} W(\beta,\gamma), \quad \varrho(\beta,\gamma) = \frac{R(\beta,\gamma)}{\beta^6 \sin^2 3\gamma}. \tag{83}$$

Then expression (81) transforms to

$$\begin{aligned} \hat{T}_{\rm vib}^{(2)} = &-\frac{\hbar^2}{2B} \frac{1}{\sqrt{w\varrho}} \left\{ \frac{1}{\beta^4} \left[\frac{\partial}{\partial\beta}\beta^4 \sqrt{\frac{\varrho}{w}}\, b_{\gamma\gamma} \frac{\partial}{\partial\beta} \right. \right. \\ &\left. - \frac{\partial}{\partial\beta}\beta^3 \sqrt{\frac{\varrho}{w}}\, b_{\beta\gamma} \frac{\partial}{\partial\gamma} \right] + \frac{1}{\beta\sin 3\gamma} \left[-\frac{\partial}{\partial\gamma}\sqrt{\frac{\varrho}{w}} \right. \\ &\left. \left. \times \sin 3\gamma\, b_{\beta\gamma} \frac{\partial}{\partial\beta} + \frac{1}{\beta}\frac{\partial}{\partial\gamma}\sqrt{\frac{\varrho}{w}} \sin 3\gamma\, b_{\beta\beta} \frac{\partial}{\partial\gamma} \right] \right\} \end{aligned} \tag{84}$$

and the volume element for collective motion takes the form

$$d\tau_{\rm col} = \sqrt{w\varrho}\, \rho^{3n-1} d\rho\, \beta^4 d\beta\, |\sin 3\gamma| d\gamma\, d\varphi\, \sin\vartheta\, d\vartheta\, d\psi. \tag{85}$$

Before quantizing other parts of the kinetic energy it is worth noting that the rotation in the (k,l) plane of the particle label space is equivalent to rotation around the single axis through an angle $\alpha_{k,l}$. Therefore the angular velocities $\Omega_{k,l} = \dot{\alpha}_{k,l}$ are represented in the quantum picture by the angular momentum operators $\hat{\mathcal{L}}_{k,l} = -i\partial/\partial\alpha_{k,l}$. Keeping this in mind one has (see also [35,36])

$$\hat{T}_{\rm rot} = \frac{\hbar^2}{2m} \sum_{\kappa=1}^{3} \frac{a_\mu^2 + a_\nu^2}{\left(a_\mu^2 - a_\nu^2\right)^2} \left(\hat{L}_\kappa^2 + \hat{\mathcal{L}}_\kappa^2\right) \tag{86}$$

and

$$\hat{T}_{\rm cor} = -\frac{\hbar^2}{2m} \sum_{\kappa=1}^{3} \frac{4a_\mu a_\nu}{\left(a_\mu^2 - a_\nu^2\right)^2} \hat{L}_\kappa \hat{\mathcal{L}}_\kappa, \tag{87}$$

where the operators $\hat{\mathcal{L}}_{k,l}$, corresponding to rotations in the planes of the privileged unit vectors are designated as

$$\hat{\mathcal{L}}_\xi \equiv \hat{\mathcal{L}}_{n-1,n}, \quad \hat{\mathcal{L}}_\eta \equiv \hat{\mathcal{L}}_{n-2,n}, \quad \hat{\mathcal{L}}_\zeta \equiv \hat{\mathcal{L}}_{n-2,n-1}. \tag{88}$$

In [60] the operators $\hat{\mathcal{L}}_\kappa$ have been called vortex momentum components. According to Buck *et al.* [38], these operators coincide with the total orbital momenta for intrinsic rotation only in the case of the nuclei having axial symmetry.

By employing (62) we rewrite the operators (86) and (87) as

$$\hat{T}_{\rm rot} = \frac{\hbar^2}{2B} \sum_{\kappa=1}^{3} \frac{\hat{L}_\kappa^2 + \hat{\mathcal{L}}_\kappa^2}{\mathfrak{J}_{\rm hydr}^{(\kappa)}}, \tag{89}$$

and

$$\hat{T}_{\rm cor} = -\frac{\hbar^2}{2B} \sum_{\kappa=1}^{3} \left(\frac{4\mathcal{E}_\mu \mathcal{E}_\nu}{\mathcal{E}_\mu^2 + \mathcal{E}_\nu^2} \right) \frac{\hat{L}_\kappa \hat{\mathcal{L}}_\kappa}{\mathfrak{J}_{\rm hydr}^{(\kappa)}}, \tag{90}$$

where the hydrodynamic moments of inertia (in units B) are

$$\mathfrak{J}_{\rm hydr}^{(\kappa)} = \frac{8\pi}{15} \frac{\left(\mathcal{E}_\mu^2 - \mathcal{E}_\nu^2\right)^2}{\mathcal{E}_\mu^2 + \mathcal{E}_\nu^2}. \tag{91}$$

They may be expressed in terms of the rigid moments of inertia (also taken in units B) [38]:

$$\mathfrak{J}^{(\kappa)}_{\rm hydr} = \frac{\left(\mathfrak{J}^{(\mu)}_{\rm rig} - \mathfrak{J}^{(\nu)}_{\rm rig}\right)^2}{\mathfrak{J}^{(\kappa)}_{\rm rig}}, \quad \kappa \neq \mu,\ \nu \ \text{ and } \ \mu \neq \nu. \tag{92}$$

Note that appearance of $\mathfrak{J}^{(\kappa)}_{\rm hydr}$ in the quantized kinetic energy is a direct consequence of mixing of the external and internal rotations, reflected in the rotational metric blocks (70). When $\beta \ll 1$ they take the form $\mathfrak{J}^{(\kappa)}_{\rm hydr} = 4\beta^2 \sin^2(\gamma - 2\pi\kappa/3)$, predicted by the Bohr model [5]. Therefore following [50] we rewrite (91) in the form

$$\mathfrak{J}^{(\kappa)}_{\rm hydr} = 4b_\kappa \beta^2 \sin^2\left(\gamma - \frac{2\pi}{3}\kappa\right), \tag{93}$$

where our symbols b_κ designate the dimensionless functions

$$b_\kappa(\beta,\gamma) = \frac{2\pi}{15}\frac{1}{\beta^2\sin^2(\gamma - 2\pi\kappa/3)}\frac{\left(\mathcal{E}^2_\mu - \mathcal{E}^2_\nu\right)^2}{\mathcal{E}^2_\mu + \mathcal{E}^2_\nu}, \tag{94}$$

satisfying the boundary condition $b_\kappa(0,\gamma) = 1$.

At last, for the kinetic-energy operator of the intrinsic motion one finds

$$\hat{T}_{\rm in} = \frac{\hbar^2}{2B}\frac{15}{8\pi}\sum_{\kappa=1}^{3}\frac{1}{\mathcal{E}^2_\kappa(\beta,\gamma)}\sum_{l=1}^{n-3}\hat{\mathscr{L}}^2_{l,n-3+\kappa}. \tag{95}$$

In the case of small deformation the above formulas considerably simplify. In accordance with definition (68), when $\beta \to 0$, the function $b_{\beta\gamma} \to 0$ that is a coupling of the β and γ motions disappears. Also, $b_{\beta\beta}$, $b_{\gamma\gamma}$, and w approach 3/2; $b_{\rho\rho}$ and all rotational coefficients b_κ converge to unity; in addition, $\lim_{\beta\to 0}\varrho(\beta,\gamma) = (15/2\pi)^3/2$. Therefore dropping the internal kinetic energy described by (95), we get the total collective kinetic-energy operator $\hat{T}_{\rm col}$ again as a sum of $\hat{T}^{(0)}_{\rm vib}$ and $\hat{T}^{(2)}_{\rm col}$, where $\hat{T}^{(0)}_{\rm vib}$ retains the same form as in (82), while in $\hat{T}^{(2)}_{\rm col}$ we included now contributions from both the intrinsic rotation and the Coriolis interaction. In this case the collective Hamiltonian $\hat{H}^{(2)}_{\rm col}$ looks like the nuclear Hamiltonian in the unified model:

$$\hat{H}^{(2)}_{\rm col} = \frac{\hbar^2}{2B(\rho)}\left\{-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\left(\beta^4\frac{\partial}{\partial\beta}\right) - \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4}\sum_{\kappa=1}^{3}\frac{(\hat{I}_\kappa - \hat{J}_\kappa)^2}{\beta^2\sin^2\left(\gamma - \frac{2\pi}{3}\kappa\right)}\right\} + V_2(\beta,\gamma), \tag{96}$$

where $\hat{I}_\kappa$ and $\hat{\mathscr{L}}_\kappa$ are the components along the principal axes for the operators of the nuclear spin $\hat{\mathbf{I}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ and the internal angular momentum $\hat{\mathbf{J}} = \hat{\mathscr{L}} + \hat{\mathbf{S}}$, depending on the total spin $\hat{\mathbf{S}}$ of all the nucleons.

Let us consider simplifications arising in the case of axially symmetric prolate nuclei, for which $\gamma \approx 0$ while β may take arbitrary values. Then from (68) one has the rotational inertia coefficients

$$b_{\rm rot} = b_1 = b_2 = \frac{2}{9}\frac{1}{\tilde{\beta}^2}\frac{(e^{-\tilde{\beta}} - e^{2\tilde{\beta}})^2}{e^{-\tilde{\beta}} + e^{2\tilde{\beta}}}, \quad b_3 = e^{-\tilde{\beta}} \tag{97}$$

and the vibrational ones

$$b_{\rho\rho} = \frac{1}{3}(2e^{-\tilde{\beta}} + e^{2\tilde{\beta}}), \quad b_{\beta\beta} = \frac{1}{2}e^{-\tilde{\beta}} + e^{2\tilde{\beta}},$$
$$b_{\beta\gamma} = 0, \quad b_{\gamma\gamma} = \frac{3}{2}e^{-\tilde{\beta}}, \quad \tilde{\beta} = \sqrt{\frac{5}{4\pi}}\beta. \tag{98}$$

In the same case, the Hamiltonian of five-dimensional collective motion takes the form

$$\hat{H}^{(2)}_{\rm col} = \hat{T}^{(2)}_{\rm vib} + \frac{\hbar^2}{2B\beta^2}\left\{\frac{1}{3b_{\rm rot}}[(\hat{\mathbf{I}} - \hat{\mathbf{J}})^2 - (\hat{I}_3 - \hat{J}_3)^2] + \frac{1}{4b_3\gamma^2}(\hat{I}_3 - \hat{J}_3)^2\right\} + V_2(\beta,\gamma), \tag{99}$$

where the kinetic-energy term $\hat{T}^{(2)}_{\rm vib}$ is determined by Eq. (84) with parameters defined by Eq. (98).

Further simplification is achieved when we replace the functions $b_{ij}(\beta,\gamma)$ by $b^{(0)}_{ij} = b_{ij}(\beta_0,\gamma_0)$ with coordinates β_0 and $\gamma_0 = 0$, specifying the minimum in the potential-energy landscape:

$$\hat{T}^{(2)}_{\rm vib} = -\frac{3\hbar^2}{4B}\left[\frac{1}{b^{(0)}_{\beta\beta}}\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{b^{(0)}_{\gamma\gamma}}\frac{1}{\beta^2\gamma}\frac{\partial}{\partial\gamma}\gamma\frac{\partial}{\partial\gamma}\right]. \tag{100}$$

VI. CONCLUSION

Thus, in the framework of a microscopic theory, using 3A-3 independent coordinates in the c.m. frame, we derived the quadrupole collective Hamiltonian for the nuclei with arbitrary deformation, which coincides with the Bohr Hamiltonian at $\beta \ll 1$ and $\rho \approx \rho_0$.

It should be observed here that similarity of Eq. (84) for the kinetic-energy operator of β and γ vibrations with the corresponding expression, suggested by Nikšić *et al.* [50], is only formal. The quadrupole collective Hamiltonian, derived in [50], operates with the unknown inertia parameters $B_{\lambda\lambda'}(\beta,\gamma)$, to be found by means of additional calculations like the cranking model or by fitting to experimental data. In contrast, the collective model derived here provides the exactly determined inertia functions $b_{\lambda\lambda'}(\beta,\gamma)$, the dependence of which on β and γ is dictated by Eq. (68) or in the case of spheroidal nuclei by simplified Eqs. (97) and (98).

The theory presented above paves the way for more strict and simple calculations of the nuclear collective states, especially in nuclei at large deformation of $\beta \approx 0.5$. It may be very useful in addressing coexistence of nuclear shapes and transitions between states in coexisting minima in the nuclear potential-energy surface. As an example, let us consider careful calculations [18] of low-lying collective states with $\mathbf{J} = 0$ of even-even nucleus ^{96}Zr, spherical and deformed states coexisting. The authors of [18] analyzed β motion in the potential having two minima, one with $\beta_0 = 0$, corresponding to spherical shape, and another (deformed shape) with $\beta_0 = 0.24$ and $\gamma \approx 0$. The fitting procedure forced authors to accept $b_{\beta\gamma} = 0$, while the same result follows from our formulas without any calculations for both spherical shape with $\beta \ll 1$ and deformed shape with γ near zero.

Sections II and III are very general and the current formalism is not limited to nuclear systems. It can be applicable to any N-particle system, in particular to molecules. It has been done for the molecules like O_3 [61], CO_2 [62], and NH_3 [63] with the aim to determine the role of the Coriolis interaction. Usually, for description of the vibrational-rotational spectra of molecules, Eckart's coordinates [64] are used, which include three Euler angles and the normal vibrational coordinates. The angular variables define orientation of a rather obscure equilibrium configuration of molecules. In such an approach the kinetic-energy operator $\hat{T}_{\rm rot}$ already in zeroth approximation depends on the rigid moments of inertia. On the other hand, if the rotation is identified with rotation of the principal axes of the inertia tensor, the operator $\hat{T}_{\rm rot}$ contains the hydrodynamical moments of inertia (93), written in units B. In [61–63] it was shown that the rotational bands of molecules are obtained with rigid moments of inertia, when contribution of the Coriolis interaction is calculated in the second order of the perturbation theory.

Our paper presents strict mathematical derivation of the collective Hamiltonian, which can be used in future as a basis for the analysis of different physical problems. In particular, we are planning to study gamma decay of the superdeformed rotational band into the normal one, caused by mixing of states with different deformation due to tunneling through the barrier, separating these states.

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