Avoiding renormalization of the elastic transition amplitude in the proton-deuteron scattering calculations

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We discuss two approaches which, by applying the screening method, permit one to include the long range proton-proton (*pp*) Coulomb force in proton-deuteron (*pd*) momentum-space scattering calculations. In the first one, based on the Alt-Grassberger-Sandhas (AGS) equation, presented in Phys. Rev. C **71**, 054005 (2005) and **72**, 054004 (2005), one needs to renormalize elastic scattering amplitude before calculating observables. In the second treatment, proposed by us in Eur. Phys. J. A **41**, 369 (2009), **41**, 385 (2009), and arXiv:2310.03433 [nucl.th], this renormalization is avoided. For the proton induced deuteron breakup reaction both approaches require renormalization of the corresponding transition amplitudes. We derive the basic equations underlying both methods under the assumption that all contributing partial wave states are included and explain why in our approach renormalization of the elastic scattering amplitude is superfluous. We show that in order to take into account in the screening limit all partial waves it is required that four additional terms, based on the three-dimensional and partial-wave projected *pp* Coulomb *t* matrices, identical for both approaches, must appear in transition amplitudes. We investigate the importance of these terms for elastic *pd* scattering below the breakup threshold.

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I. INTRODUCTION

The need for the present investigation arose when two preprints [1] and [2], both dealing with the problem of how to include the long range proton-proton (*pp*) Coulomb force in momentum space *pd* scattering calculations through a screened Coulomb interaction, were posted. The arguments presented in [2] show that in the well established approach of Refs. [3,4] the interplay of the *pp* Coulomb potential and the deuteron bound state pole in the neutron-proton (*np*) *t* matrix makes renormalization of the elastic scattering transition amplitude necessary prior to calculating observables. Contrary to that, in our approach presented in [1,5,6], one avoids such renormalization. In the following we explain similarities and differences of both treatments and provide justification why the renormalization in our method for elastic scattering is unnecessary. We also discuss a very important problem, indispensable in any treatment of the long-range Coulomb force: how to take into account, in addition to partial waves utilised when solving corresponding three-nucleon (3*N*) scattering equations, all higher partial wave states.

II. FORMALISM

Let us start with the well-established approach of Refs. [3,4] based on the Alt-Grassberger-Sandhas (AGS) equation for the *pd* transition operator U [7,8]:

$$U|\Phi\rangle = PG_0^{-1}|\Phi\rangle + PtG_0U|\Phi\rangle, \tag{1}$$

where P is defined in terms of transposition operators, $P = P_{12}P_{23} + P_{13}P_{23}$, G_0 is the free 3*N* propagator, and $|\Phi >$ is the initial state composed of a deuteron and a momentum eigenstate of the proton. The t matrix t is a solution of the two-body Lippmann-Schwinger (LS) equation, with the interaction which contains in case of the *pp* system in addition to the nuclear part also the Coulomb *pp* force (assumed to be screened and parametrized by some parameter R). If the state $U|\Phi >$ is known, the elastic *pd* scattering amplitude $< \Phi'|U|\Phi >$ with $|\Phi' >$ being the final *pd* state, can be obtained by quadratures in the standard manner.

In our approach we use the breakup operator T defined as

$$T = tG_0U. \tag{2}$$

It fulfills the 3*N* Faddeev equation which, when nucleons interact with pairwise forces only, is given by [8,9]

$$T|\Phi > = tP|\Phi > + tPG_0T|\Phi > . \tag{3}$$

The above form of the Faddeev equation ensures that the T operator reflects directly the properties of the t matrix. Here, the elastic scattering amplitude is calculated from solutions of Eq. (3) by [8,9]

$$\langle\Phi'|U|\Phi\rangle = \langle\Phi'|PG_0^{-1}|\Phi\rangle + \langle\Phi'|PT|\Phi\rangle, \tag{4}$$

and the transition amplitude for breakup $< \Phi_0|U_0|\Phi >$ is expressed in terms of $T|\Phi\rangle$ by [8,9]

$$\langle\Phi_0|U_0|\Phi\rangle = \langle\Phi_0|(1+P)T|\Phi\rangle, \tag{5}$$

where $|\Phi_0 >= |\vec{p}\,\vec{q}\,m_1m_2m_3\nu_1\nu_2\nu_3 >$ is the state of three free outgoing nucleons. In the approach based on the AGS

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equation the transition amplitude for breakup is given also by Eq. (5) but with T replaced by U.

The AGS, Eq. (1), as well as the Faddeev, Eq. (3), equations are solved in the momentum-space partial-wave basis $|pq\bar{\alpha}>$:

$$|pq\bar{\alpha}>\equiv|pq(ls)j\left(\lambda\frac{1}{2}\right)I(jI)J\left(t\frac{1}{2}\right)T>, \qquad (6)$$

where one can differentiate between the partial wave states $|pq\alpha>$ with total $2N$ angular momentum j below some value $j_{\max}$: $j\leqslant j_{\max}$, in which the nuclear, V_N, as well as the pp screened Coulomb interaction, V_c^R (in isospin $t=1$ states only), act, and the states $|pq\beta>$ with $j>j_{\max}$, for which only the screened Coulomb force V_c^R is present in the pp subsystem. Incorporation of the $|pq\beta>$ states is indispensable due to the long-range nature of the pp Coulomb force and the necessity to perform finally the screening limit $R\to\infty$. In the following we derive for both approaches the equations in a subspace restricted to $|pq\alpha>$ states only, which, however, incorporate all contributions from the complementary subspace of $|pq\beta>$ states. The states $|pq\alpha>$ and $|pq\beta>$ form together a complete system of states (in the following we use shorthand notation $\sum_\alpha\int p^2dpq^2dq|pq\alpha\rangle\langle pq\alpha|\equiv|\alpha\rangle\langle\alpha|$):

$$\int p^2dpq^2dq\left(\sum_\alpha|pq\alpha\rangle\langle pq\alpha|+\sum_\beta|pq\beta\rangle\langle pq\beta|\right)$$
$$=|\alpha\rangle\langle\alpha|+|\beta\rangle\langle\beta|=\mathrm{I}, \qquad (7)$$

where I is the identity operator.

Let us start with our approach. Projecting Eq. (3) for $T|\Phi>$ on the $|pq\alpha>$ and $|pq\beta>$ states one gets the following system of coupled integral equations [1]:

$$\langle pq\alpha|T|\Phi\rangle=\langle pq\alpha|t_{N+c}^RP|\Phi\rangle+\langle pq\alpha|t_{N+c}^RPG_0|\alpha'\rangle\langle\alpha'|T|\Phi\rangle$$
$$+\langle pq\alpha|t_{N+c}^RPG_0|\beta'\rangle\langle\beta'|T|\Phi\rangle, \qquad (8)$$

$$\langle pq\beta|T|\Phi\rangle=\langle pq\beta|t_c^RP|\Phi\rangle+\langle pq\beta|t_c^RPG_0|\alpha'\rangle\langle\alpha'|T|\Phi\rangle, \qquad (9)$$

where t_{N+c}^R and t_c^R are t matrices generated by the interactions $V_N+V_c^R$ and V_c^R, respectively. The omitted term $\langle pq\beta|t_c^RPG_0|\beta'\rangle\langle\beta'|T|\Phi\rangle$ on the right-hand side of Eq. (9) is generated by $\langle pq\beta|t_c^RPG_0|\beta'\rangle\langle\beta'|t_c^R$. A direct calculation shows that it vanishes, independently of the value of the total isospin T.

Inserting $<pq\beta|T|\Phi>$ from Eq. (9) into Eq. (8) and using Eq. (7) one gets

$$\langle pq\alpha|T|\Phi\rangle=\langle pq\alpha|t_{N+c}^RP|\Phi\rangle+\langle pq\alpha|t_{N+c}^RPG_0t_c^{R3d}P|\Phi\rangle$$
$$-\langle pq\alpha|t_{N+c}^RPG_0|\alpha'\rangle\langle\alpha'|t_c^RP|\Phi\rangle$$
$$+\langle pq\alpha|t_{N+c}^RPG_0|\alpha'\rangle\langle\alpha'|T|\Phi\rangle$$
$$+\langle pq\alpha|t_{N+c}^RPG_0t_c^{R3d}PG_0|\alpha'\rangle\langle\alpha'|T|\Phi\rangle$$
$$-\langle pq\alpha|t_{N+c}^RPG_0|\alpha'\rangle\langle\alpha'|t_c^RPG_0|\alpha''\rangle\langle\alpha''|T|\Phi\rangle. \qquad (10)$$

This is a set of coupled integral equations in the space of the $|\alpha\rangle$ states, which exactly incorporates the contributions of the pp Coulomb interaction from all partial wave states up to infinity. It is clear that there is a price to pay for taking into account all states $|pq\beta>$: the necessity to work with the three-dimensional Coulomb t matrix, t_c^{R3d}, obtained by solving the three-dimensional LS equation [10].

Presently it is practically impossible to solve Eq. (10) in its completeness. The reasons are a drastic amount of computer resources and of computer time required to calculate the second and the fifth terms with the three-dimensional Coulomb t matrix. Luckily enough, one can rather easily eliminate them at the expense of increasing the basis of $|\alpha\rangle$ states. Namely, extending the set $|\alpha\rangle$ by adding channels with higher angular momenta, in which only the pp Coulomb interaction is present, permits one to completely neglect the four terms in Eq. (10) due to their mutual cancellation: the second with the third and the fifth with the sixth term. The set (10) is then reduced to

$$\langle pq\alpha|T|\Phi\rangle=\langle pq\alpha|t_{N+c}^RP|\Phi\rangle+\langle pq\alpha|t_{N+c}^RPG_0|\alpha'\rangle\langle\alpha'|T|\Phi\rangle, \qquad (11)$$

which is a basic equation in our approach (in [1] called a simplified one). It has identical structure as so frequently used $3N$ Faddeev equation for neutron-deuteron (nd) scattering [9].

To calculate in our approach the elastic scattering transition amplitude one needs in Eq. (4) the second term $\langle\vec{p}\vec{q}|T|\Phi\rangle$ composed of low (α) and high (β) partial wave contributions for $T|\Phi>$. Using the completeness relation (7) one gets

$$\langle\vec{p}\vec{q}\,|T|\Phi\rangle=\langle\vec{p}\vec{q}\,|\alpha'\rangle\langle\alpha'|T|\Phi\rangle+\langle\vec{p}\vec{q}\,|t_c^{R3d}P|\Phi\rangle$$
$$-\langle\vec{p}\vec{q}\,|\alpha'\rangle\langle\alpha'|t_c^RP|\Phi\rangle+\langle\vec{p}\vec{q}\,|t_c^{R3d}PG_0|\alpha'\rangle$$
$$\times\langle\alpha'|T|\Phi\rangle-\langle\vec{p}\vec{q}\,|\alpha'\rangle\langle\alpha'|t_c^RPG_0|\alpha''\rangle\langle\alpha''|T|\Phi\rangle. \qquad (12)$$

To account correctly for contributions from $|\beta\rangle$ states again four terms are required, two of which contain the three-dimensional Coulomb t matrix. The first one, $\langle\vec{p}\vec{q}|t_c^{R3d}P|\Phi\rangle$, corresponds to the amplitude of the Rutherford point-deuteron pd scattering and the second one, $\langle\vec{p}\vec{q}|t_c^{R3d}PG_0|\alpha'\rangle\langle\alpha'|T|\Phi\rangle$, is a modification of the first one by nucleon-nucleon (NN) interactions.

Now we derive analogous relations in the approach based on the AGS equation. Projecting Eq. (1) on the $|pq\alpha>$ and $|pq\beta>$ states and using shorthand notation

$$\sum_{\alpha,\tilde{\alpha}}\int p^2dpq^2dqp'^2dp'|pq\alpha\rangle t^{\alpha\tilde{\alpha}}\left(p,p';E-\frac{3}{4m}q^2\right)$$
$$\times G_0\langle pq\tilde{\alpha}|\equiv|\alpha\rangle t^\alpha G_0\langle\alpha|,$$

one gets the following system of coupled integral equations:

$$\langle pq\alpha|U|\Phi\rangle=\langle pq\alpha|PG_0^{-1}|\Phi\rangle+\langle pq\alpha|P|\alpha'\rangle t_{N+c}^{R\alpha'}G_0\langle\alpha'|U|\Phi\rangle$$
$$+\langle pq\alpha|P|\beta'\rangle t_c^{R\beta'}G_0\langle\beta'|U|\Phi\rangle, \qquad (13)$$

$$\langle pq\beta|U|\Phi\rangle=\langle pq\beta|PG_0^{-1}|\Phi\rangle+\langle pq\beta|P|\alpha'\rangle t_{N+c}^{R\alpha'}G_0\langle\alpha'|U|\Phi\rangle$$
$$+\langle pq\beta|P|\beta'\rangle t_c^{R\beta'}\langle\beta'|P|\Phi\rangle$$
$$+\langle pq\beta|P|\beta'\rangle t_c^{R\beta'}G_0\langle\beta'|P|\alpha'\rangle t_{N+c}^{R\alpha'}G_0\langle\alpha'|U|\Phi\rangle. \qquad (14)$$

Inserting $< pq\beta|U|\Phi >$ from Eq. (14) into Eq. (13) and using Eq. (7) one gets finally

$$\begin{aligned}\langle pq\alpha|U|\Phi\rangle &= \langle pq\alpha|PG_0^{-1}|\Phi\rangle + \langle pq\alpha|P|\alpha'\rangle t_{N+c}^{R\alpha'} G_0 \langle\alpha'|U|\Phi\rangle \\ &-\langle pq\alpha|P|\alpha'\rangle t_c^{R\alpha'}\langle\alpha'|P|\Phi\rangle + \langle pq\alpha|Pt_c^{R3d}P|\Phi\rangle \\ &-\langle pq\alpha|P|\alpha'\rangle t_c^{R\alpha'} G_0\langle\alpha'|P|\alpha''\rangle t_{N+c}^{R\alpha''}G_0\langle\alpha''|U|\Phi\rangle \\ &+\langle pq\alpha|Pt_c^{R3d}G_0P|\alpha''\rangle t_{N+c}^{R\alpha''}G_0\langle\alpha''|U|\Phi\rangle. \quad (15)\end{aligned}$$

This is a set of coupled integral equations in the space spanned by the $|\alpha\rangle$ states, analogous to Eq. (10) in our approach.

Again, extending the set $|\alpha\rangle$ by adding a finite number of channels with higher angular momenta leads to cancellations between last four terms and set (15) is reduced to the following basic equation for approach based on AGS equation [3,4]:

$$\langle pq\alpha|U|\Phi\rangle = \langle pq\alpha|PG_0^{-1}|\Phi\rangle + \langle pq\alpha|P|\alpha'\rangle t_{N+c}^{R\alpha'}G_0\langle\alpha'|U|\Phi\rangle. \quad (16)$$

To calculate the elastic scattering transition amplitude $\langle\Phi'|U|\Phi\rangle$ one needs $\langle\vec{p}\vec{q}|U|\Phi\rangle$ composed of low (α) and high (β) partial wave contributions for $U|\Phi >$. Employing the completeness relation (7) and Eq. (15) one gets

$$\begin{aligned}\langle\vec{p}\vec{q}\,|U|\Phi\rangle &= \langle\vec{p}\vec{q}\,|PG_0^{-1}|\Phi\rangle + \langle\vec{p}\vec{q}\,|P|\alpha'\rangle t_{N+c}^{R\alpha'}G_0\langle\alpha'|U|\Phi\rangle \\ &-\langle\vec{p}\vec{q}\,|P|\alpha'\rangle t_c^{R\alpha'}\langle\alpha'|P|\Phi\rangle + \langle\vec{p}\vec{q}\,|Pt_c^{R3d}P|\Phi\rangle \\ &-\langle\vec{p}\vec{q}\,|P|\alpha'\rangle t_c^{R\alpha'}G_0\langle\alpha'|P|\alpha''\rangle t_{N+c}^{R\alpha''}G_0\langle\alpha''|U|\Phi\rangle \\ &+\langle\vec{p}\vec{q}\,|Pt_c^{R3d}G_0P|\alpha''\rangle t_{N+c}^{R\alpha''}G_0\langle\alpha''|U|\Phi\rangle. \quad (17)\end{aligned}$$

Using relation (2) between U and T one finds that indeed amplitudes and thus also observables are the same in both treatments.

It should be emphasized that only by extending the set of $|\alpha\rangle$ states is it possible to neglect in Eqs. (10) and (15) the terms which contain the three-dimensional Coulomb t matrices, and to reduce the problem in both approaches to numerically well treatable Eqs. (11) and (16). The indication that cancellations takes place is given by convergence of predictions with respect to the total angular momentum in the two-nucleon ($2N$) subsystem $j_{\rm max}$, which defines the set of $|\alpha\rangle$ states. It will be denoted in the following by $j_s\, j_{\rm max}$ with j_s being the largest angular momentum in which the $2N$ interaction acts [1].

It is evident that a correct treatment of the Coulomb force in both approaches requires inclusion of four additional terms in the elastic (and also breakup) transition amplitudes [the last four terms in Eqs. (12) and (17)].

It was shown in [2] (see also [3,4] and references therein) that in the treatment based on AGS equation (16) the elastic scattering transition amplitude acquires, in the screening limit $R \to \infty$, an infinitely oscillating phase factor and must be renormalized before calculating observables. As a consequence, each term in Eq. (17) containing $U|\Phi\rangle$ has to be renormalized. The origin for that phase factor is the interplay of the pp Coulomb potential and the deuteron pole in the np t matrix leading to coinciding singularities in the AGS equation [2]. Analogous mechanism leads to the same diverging phase factor in the half- and on-shell pp t-matrix solutions of the LS equation when taking the screening limit [11]. In this case the oscillating phase originates from the interplay of the pp Coulomb potential and the pole of the free propagator in the LS equation. In the case of the off-shell pp t matrix this mechanism is deactivated and no phase factor emerges.

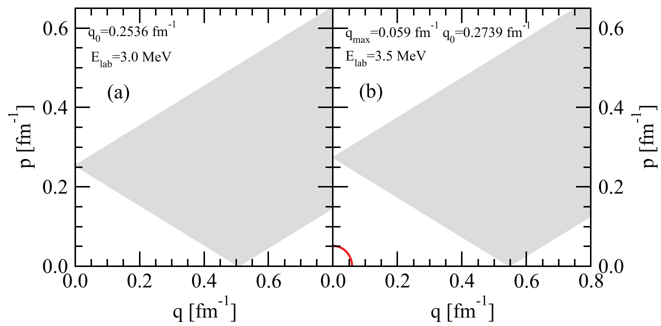

FIG. 1. Regions of the Jacobi momenta q and p values in $(q-p)$ plane which contribute to the breakup reaction [(red) solid line at $E = 3.5$ MeV, showing ellipse $\frac{p^2}{m} + \frac{3}{4m}q^2 = \frac{3}{4m}q_{\rm max}^2 = \frac{3}{4m}q_0^2 + E_d$) and elastic scattering ($< \Phi'|PT|\Phi >$ term] (gray highlighted region) at the incoming nucleon laboratory energy $E = 3.0$ and 3.5 MeV.

In our approach we solve instead of the AGS the $3N$ Faddeev equation (11) for the $\langle pq\alpha|T|\Phi\rangle$ with off-shell p-q values (see below), from which later elastic scattering transition amplitude is calculated using Eq. (4). The structure of the $3N$ Faddeev equation guarantees that their solutions inherit properties from the two-nucleon t matrices. On the one hand this permits us to rewrite them in a form where the deuteron pole is extracted from $\langle pq\alpha|T|\Phi\rangle$ amplitudes for all $\alpha = \alpha_d$ channels which contain ${}^3S_1$-${}^3D_1$ quantum numbers (see Eq. (187) in [9]). Being off-shell we avoid the source of the oscillating phase factor described in [2] and thus also the necessity of renormalization of the elastic scattering amplitude. On the other hand transfer of properties from t matrix to T amplitude provides an additional argument that such renormalization is redundant. Namely, the properties of t matrices generated by the screened Coulomb force alone (in the case of partial wave decomposed t matrices also those generated by a combination of Coulomb and nuclear parts) as well as their screening limits were studied theoretically in the past in numerous papers [11–19] and later some of these properties were confirmed numerically in [10]. The most important finding was that such off-shell t matrices have a well-defined screening limit while the half- and on-shell ones acquire in this limit an infinitely oscillating phase factor. At the same time, the elastic pd scattering amplitude gets contributions of $\langle pq\alpha|T|\Phi\rangle$ states only from the off-shell region of the Jacobi momenta magnitudes q and p in $(q-p)$ plane: $\frac{p^2}{m} + \frac{3}{4m}q^2 \neq \frac{3}{4m}q_{\rm max}^2 = \frac{3}{4m}q_0^2 + E_d$, where m is the nucleon mass, E_d is the (negative) deuteron binding energy, and q_0 is the magnitude of the relative pd momentum. That off-shell region of $q-p$ values does not overlap with the ellipse from which half-on-shell contributions to the breakup reaction come. In Fig. 1 we exemplify that off-shell part and the separation of the breakup and elastic scattering regions in the $(q-p)$ plane for the energy of a pd system $E = 3.5$ MeV, which is slightly above the breakup threshold and for which both reactions are possible, and at $E = 3.0$ MeV, which is below the breakup threshold and for which only elastic scattering is allowed. The fact that elastic *pd* scattering requires only off-shell solutions of the Faddeev equations and that the off-shell two-nucleon t matrices have a well-defined screening limit is the reason why in our method no renormalization of elastic scattering amplitudes is needed. Contrary to that, the breakup half-on-shell amplitudes acquire the oscillating phase factor originating from half-shell t matrices which are used in Eq. (11) together with the off-shell solutions $\langle pq\alpha|T|\Phi\rangle$ to calculate them [1].

III. RESULTS AND DISCUSSION

In order to compare results of two approaches and check that indeed our method does not need the renormalization, we applied our approach at a low proton energy below the breakup threshold, where effects of the *pp* Coulomb force as well as contributions of different terms to the elastic scattering amplitude are expected to be sizable and where also results of the Pisa group and of the AGS approach are available at $E = 3.0$ MeV [20–22]. In this energy region the Coulomb force problem in *pd* elastic scattering was for the first time exactly solved with realistic nuclear forces by the Pisa group who applied the pair correlated hyperspherical harmonic basis method [20,21]. Their results formed a solid base to cross-check the precision of the AGS method [22] and to establish it as a correct approach for including the *pp* Coulomb force in momentum space calculations. In Fig. 2 we show our predictions obtained with the AV18 *NN* potential [23], which was also used in [20–22], and $j_s3j_{\max}7$ $|\alpha\rangle$ basis at 3.0 MeV and compare them to existing elastic scattering data for the cross section and the analyzing powers. The red short dashed line shows results obtained with only the first three terms in the elastic scattering transition amplitude (12), which is the approximation used also in Ref. [3]. The red solid lines are predictions for neutron-deuteron scattering. It is clear that in this region of energies the Coulomb force effects indeed are large and dominant at all angles as evidenced by comparing the red solid and short dashed lines. It is astonishing how good the overall description of tensor analyzing power data is in spite of their small magnitudes of ≈1%. The vector analyzing powers A_y and iT_{11} are underestimated by theory as is very well known in the literature under the name "low energy analyzing power puzzle".

In Fig. 2 we show also by dotted blue lines results with the last term in Eq. (12) included. It is evident that the term $-\langle\vec{p}\vec{q}|\alpha'\rangle\langle\alpha'|t_c^R PG_0|\alpha''\rangle\langle\alpha''|T|\Phi\rangle$ is significant at low energies and that it deteriorates a good description of data obtained with the first three terms. In [1] it was shown that at energies above ≈10 MeV the contribution of that term to elastic scattering observables is negligible and at 10 MeV it starts to influence some spin observables. It is thus unavoidable below the breakup threshold to investigate how significant are effects of inclusion of the fourth term $\langle\vec{p}\vec{q}|t_c^{R3d}PG_0|\alpha'\rangle\langle\alpha'|T|\Phi\rangle$ in the elastic scattering transition amplitude. Since the fifth term has a negative sign and contains partial wave contributions to the Coulomb t matrix whose full three-dimensional form is contained in the fourth term, one would expect that they would

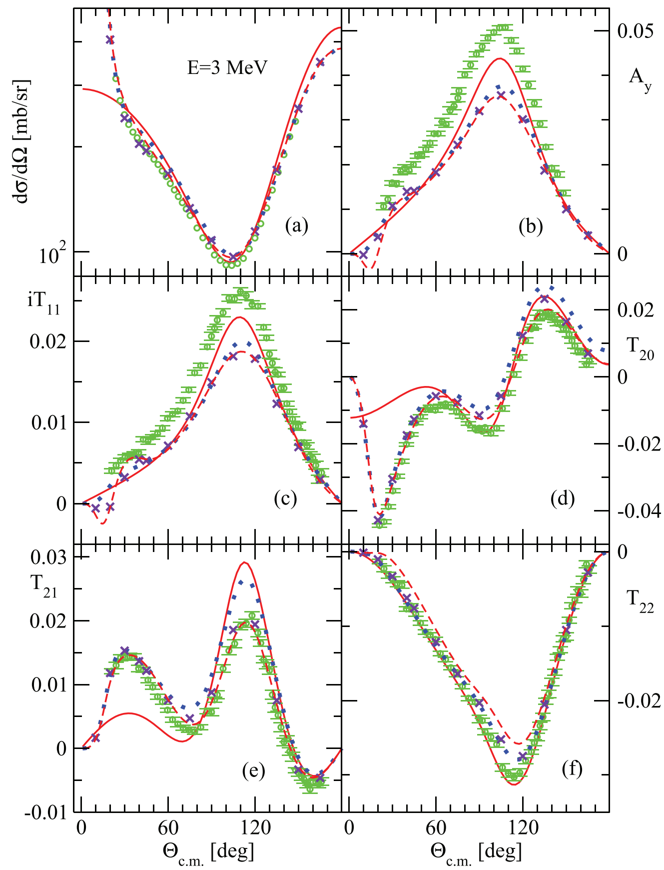

FIG. 2. Comparison of data and predictions for the *pd* scattering cross section $\frac{d\sigma}{d\Omega}$, proton vector A_y, deuteron vector iT_{11}, and deuteron tensor T_{20}, T_{21}, T_{22} analyzing powers. They are shown as functions of a c.m. proton scattering angle $\Theta_{\text{c.m.}}$ and were calculated at the incoming proton laboratory energy $E = 3.0$ MeV with the approach based on Faddeev equation (11) and transition amplitude (12). The exponentially screened Coulomb force ($R = 40$ fm, $n = 4$) and the AV18 potential [23] restricted to the $j \leqslant 3$ partial waves have been applied. To solve Faddeev equation the set $j_s3j_{\max}7$ of $|\alpha\rangle$ states was used. The red short dashed lines show the results when only the first three terms in Eq. (12) are taken into account. The blue dotted lines are predictions when also the fifth term in Eq. (12) $(-\langle\vec{p}\vec{q}|\alpha'\rangle\langle\alpha'|t_c^R PG_0|\alpha''\rangle\langle\alpha''|T|\Phi\rangle)$ is included. The pure Coulomb term $\langle\Phi'|Pt_cP|\Phi\rangle$ was determined using the screening limit expression for the off-shell three-dimensional Coulomb t matrix (Eq. (19) in Ref. [1]). The indigo crosses show the results with all terms in Eq. (12) included. The red solid lines are predictions for *nd* elastic scattering and green circles represent the *pd* data from Ref. [26].

at least partially cancel each other and the inclusion of the fourth term should restore at least partly the good description of data.

The computation of the fourth term with the three-dimensional Coulomb t matrix, t_c^{R3d}, can be done according to expressions (D.9), (D.6), and (D.8) of Ref. [5]. It requires integrations over components of two vectors: over vector $\vec{q}$ in (D.9), and over $\vec{p}\,'$ or $\vec{q}_4$ in (D.6) or (D.8), respectively. Below the breakup threshold only channels $\alpha \neq \alpha_d$ contribute to (D.6). Since below the breakup threshold the decomposition (D.7) is superfluous, (D.8) provides the full contribution from

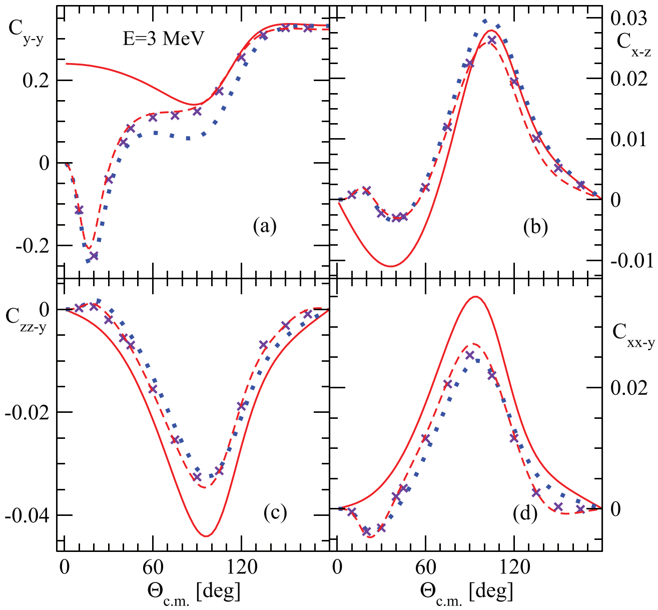

FIG. 3. The same as in Fig. 2 but for selected spin correlation coefficients. For description of lines see Fig. 2.

α_d channels, obtained by replacing the second part of splitting (D.7) with the left side of (D.7). The contributions from (D.6) and (D.8) must be determined numerically and this is the most time consuming part of the calculations.

In Fig. 2 the indigo crosses show the results obtained with all the terms in Eq. (12) included. As expected the fourth and fifth terms cancel each other to a large extent and a good description of data for the cross section and tensor analyzing powers is essentially regained.

Even more interesting than the overall good description of data found in Fig. 2 is the good agreement for practically all the shown observables, between our 3.0 MeV results and predictions of both the Pisa group and the AGS approach, as far as it can be judged from figures of Refs. [20,22]. This good agreement strongly supports the statement that our approach and the AGS one have to provide the same predictions for all observables and that in our approach renormalization of the elastic scattering amplitude is indeed superfluous.

The above comparison of predictions from different approaches was done on the level of observables. To get better insight into the accuracy of our method when applied below and above the deuteron breakup threshold a comparison on the level of phase shifts, for which abundant predictions are available in [20–22,24], or even a direct comparison of the K matrices [25], would be desirable. Since this requires additional programming we postpone it to a future study.

To get an idea about the magnitude of the Coulomb force effects for other elastic scattering observables we show in Figs. 3–6 analogous predictions as in Fig. 2 but for selected spin correlations (Fig. 3), proton to proton (Fig. 4), proton to deuteron (Fig. 5), and deuteron to proton (Fig. 6) spin transfer coefficients. The figures reveal a wide spectrum of importance and magnitude of the Coulomb force effects, dependent on the observable. For most of observables the effects are large in a wide range of angles, for example for spin correlations from Fig. 3 and some of spin transfers $[K_y^x(N-N), K_y^{xz}(N-D), K_{zz}^y(D-N)]$. For some large effects are restricted to forward region of angles below $\approx 90^\circ$ $[K_x^x(N-N), K_z^x(N-N), K_z^z(N-N), K_x^x(N-D), K_z^x(N-D), K_z^z(N-D), K_x^z(D-N), K_z^z(D-N)]$. There are some interesting cases of observables which for the neutron-deuteron scattering vanish and become nonzero for the proton-deuteron interaction, as for example the nucleon to nucleon spin transfer coefficient $K_y^x(N-N)$ shown in Fig. 4. These nonzero values are due to a large charge independence breaking of pp and np interactions in isospin $t=1$ states, caused by the

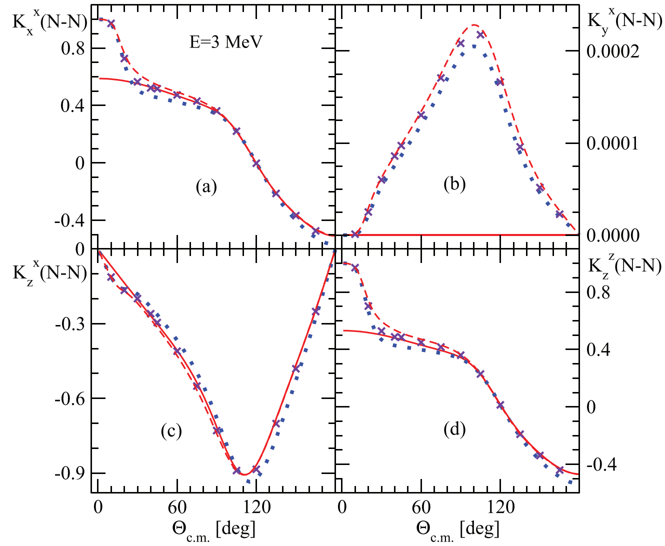

FIG. 4. The same as in Fig. 2 but for selected proton to proton spin transfer coefficients. For description of lines see Fig. 2.

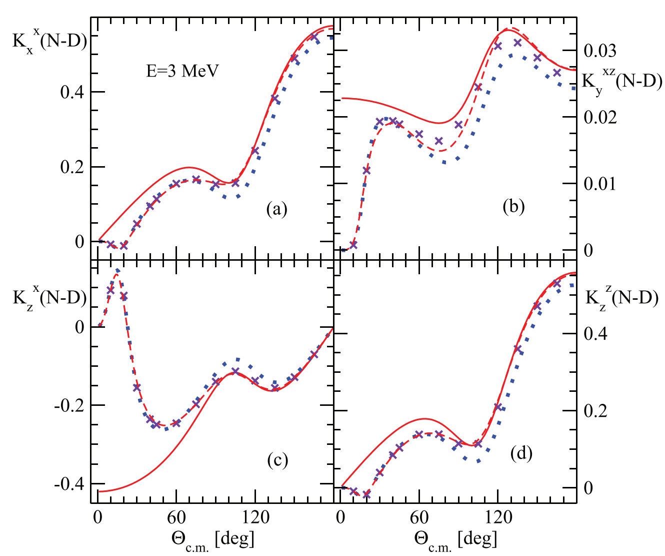

FIG. 5. The same as in Fig. 2 but for selected proton to deuteron spin transfer coefficients. For description of lines see Fig. 2.

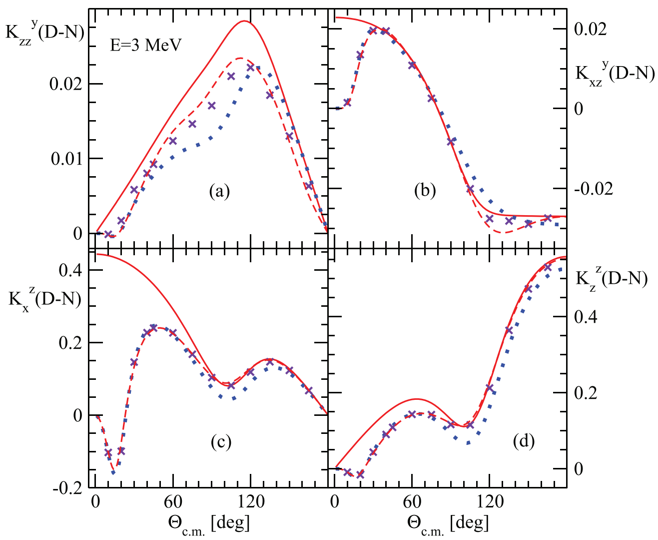

FIG. 6. The same as in Fig. 2 but for selected deuteron to proton spin transfer coefficients. For description of lines see Fig. 2.

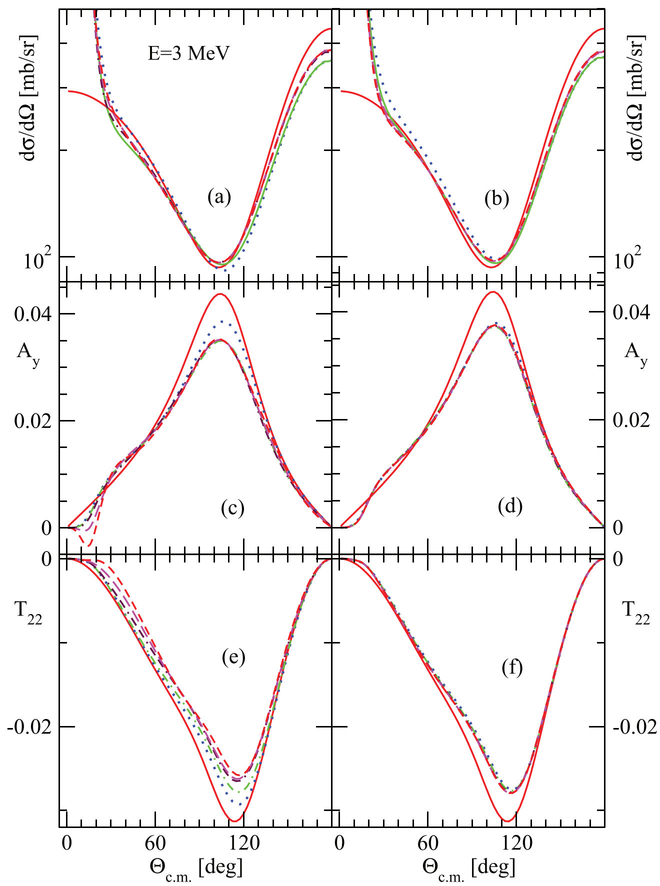

FIG. 7. The convergence in screening radius R of predictions for pd cross section and selected analyzing powers. In the left column, (a), (c), and (e) are the results obtained when only the first three terms in the elastic scattering amplitude of Eq. (12) were taken into account. In the right, (b), (d), and (f) are predictions obtained when in addition to the first three terms also the last fifth term was included. The differently colored lines correspond to different screening radii: blue dotted—$R = 5$ fm, green short dash-dotted—$R = 10$ fm, maroon double-short-dash-dotted—$R = 20$ fm, magenta long dashed—$R = 30$ fm, and red short dashed—$R = 40$ fm. The red solid lines are predictions for nd elastic scattering.

Coulomb pp force. In our calculations we used the charge dependent AV18 potentials, taking np and pp NN interactions of this model for the pd and nd systems. In all isospin $t = 1$ states both total isospins of the $3N$ system $T = \frac{1}{2}$ and $T = \frac{3}{2}$ were taken into account. Vanishing of the $K_y^x(n-n)$ for nd scattering shows that the difference between np and pp NN AV18 potentials is too weak to induce nonzero values for this observable.

The very interesting and most important effect seen in all the figures is that practically in all cases (large) effects caused by adding the fifth term to the elastic scattering transition amplitude are appreciably removed when including simultaneously the fourth term. In consequence, it is, for many observables, needless to account for these terms in elastic scattering amplitude which drastically simplifies and accelerates the determination of the Coulomb force effects.

When performing screening of the Coulomb potential, one is interested in the minimal value of the screening radius necessary to get converged predictions. The proper treatment of the screened Coulomb potential requires for any particular value of the screening radius R to accommodate all contributing partial waves, whose number grows with increasing R. That in consequence compels one to work with the three-dimensional Coulomb t matrix and in turn gives rise to four additional terms in the transition amplitude. Two of them contain the three-dimensional Coulomb t matrix and the other two its partial wave decomposed counterpart [see Eqs. (12) and (17)]. It is evident that these last two terms will contribute, together with the leading one $\langle\vec{p}\vec{q}|\alpha'\rangle\langle\alpha'|T|\Phi\rangle$, to the actual pattern of convergence. To study that pattern we calculated predictions for all (55) elastic scattering observables with five values of R: 5, 10, 20, 30, and 40 fm and including in the elastic scattering transition amplitude (12) in addition to first two terms either the third one, $-\langle\vec{p}\vec{q}|\alpha'\rangle\langle\alpha'|t_c^R P|\Phi\rangle$, or the third and the fifth, $-\langle\vec{p}\vec{q}|\alpha'\rangle\langle\alpha'|t_c^R P G_0|\alpha''\rangle\langle\alpha''|T|\Phi\rangle$, ones. In Figs. 7 and 8 we show for some selected observables the pattern of convergence in R for the first and second cases (left and right column, respectively). It turned out that among 55 observables about 30 revealed worse convergence when only the third term was included and there were observables for which the contribution from the third term vanishes (some such cases are shown in Figs. 7 and 8, with the exception of the differential cross section and T_{22}). In addition to deteriorated convergence some of these 30 observables exhibited angular oscillations which grow with the increasing screening radius R. They occur at forward angles and are exemplified for A_y in Figs. 2(b) and 7(c), for iT_{11} in Fig. 2(c), and for $K_{yy}^y(D-N)$ in Fig. 8(e). For T_{22} (see Fig. 7(e)), which gets contributions from the third term, such oscillations do not appear, nevertheless its convergence in R is poor and similar to other spin observables shown in the left columns of Figs. 7 and 8. Including the fifth term not only removes oscillations but significantly improves convergence for all the investigated observables as demonstrated in the right columns of Figs. 7

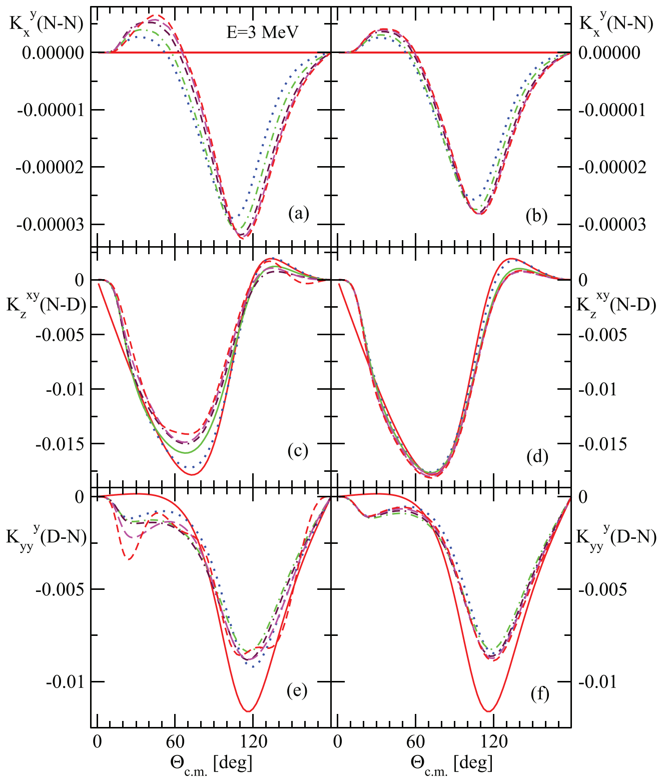

FIG. 8. The same as in Fig. 7 but for selected spin transfer coefficients.

and 8. We found that at the energy of $E = 3$ MeV considered in the present paper, the value of the screening radius $R = 20$ fm was actually sufficient to reach the convergence, what agrees with statements about convergence given in [22].

It is evident that the fifth term is mostly responsible for improving the convergence and for removing the angular oscillations appearing in some observables calculated with amplitude (12) restricted to the first three terms only. We would like to emphasize again that the appearance of the four additional terms in the transition amplitude (12) is compelled by the requirement to incorporate into Faddeev equations contributions from all $|\beta >$ states, which seems to be essential for the proper treatment of the long-ranged *pp* Coulomb force. The importance of this demand is brought clearly to light only at low energies, particularly below the breakup threshold, where large effects of the *pp* Coulomb force become emphasized. At these energies one needs, for some observables, both pairs of terms to get the final predictions, contrary to energies above ≈10 MeV, where the contribution of the second pair to elastic scattering becomes negligible [1].

One could wonder why in the approach of Ref. [3] whose results at 3 MeV are shown in [22], one does not encounter angular oscillations seen, e.g., for A_y in Fig. 2(b) or for iT_{11} in Fig. 2(c). Based on the results of the present study [see Eq. (17)], one would expect that both methods should provide the same final predictions as well as patterns of convergence with the only difference being that in our approach the renormalization of the elastic scattering amplitude is superfluous. Very probably, the reason for that seeming contradiction can be traced back to specific details of performance of both approaches. The most weighty difference seems to be an additional screened Coulomb potential between the spectator proton and the center of mass of the remaining neutron-proton pair introduced in [3]. That is probably the reason why in this approach the inclusion of all $|\beta >$ states was not considered at any stage and instead incorporation of many partial waves was compulsory when solving the AGS equation. Actually, in the AGS approach *pd* observables are calculated using the amplitude in the form $U = T^{\rm cm} + \lim_{R\to\infty}[U^R - T_R^{\rm cm}]$ with screening radius R and a screened Coulomb amplitude $T_R^{\rm cm}$ corresponding to the added Coulomb potential (renormalization factors are suppressed for simplicity). In Ref. [3] it is argued that the term $[U^R - T_R^{\rm cm}]$ is a short-range operator, which can be calculated within a restricted space of lower partial wave states $|\alpha\rangle$. The agreement between the results of the AGS approach and those of other methods suggests that it is reasonable to expect a successful cancellation between U^R and $T_R^{\rm cm}$ for the $|\beta\rangle$ states.

IV. SUMMARY

Summarizing, we have shown that the two discussed approaches which enable to include the long-range Coulomb force in momentum-space *pd* scattering calculations by applying a screening method have to provide the same results for all observables. In each method the cancellation between terms containing three-dimensional and partial wave decomposed Coulomb t matrices is decisive for establishing workable equations, whose structure is identical to the commonly used equations for neutron-deuteron scattering. Solutions of these equations together with four additional terms, two of which contain the three-dimensional Coulomb t matrices, permit one to get the elastic scattering (and breakup) transition amplitudes. At low energies, particularly below the breakup threshold, the complete inclusion of $|\beta\rangle$ states, as evidenced by two pairs of additional terms in the elastic scattering amplitude, is required for the calculation of some *pd* observables. In the approach based on the AGS equation it is unavoidable to perform renormalization of the elastic scattering amplitudes before calculating observables. In the approach based on the Faddeev equation such renormalization can be completely avoided.

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