Induced Coulomb corrections to the $0^- \leftrightarrow 0^+$ β decay

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We calculate the *induced* Coulomb correction as well as the usual Coulomb correction, to order αZRW_0, to the first forbidden $0^- \leftrightarrow 0^+$ β decay and demonstrate that the induced correction, through the pseudoscalar form factor term, is dominant among all the Coulomb corrections.

I. INTRODUCTION

In a recent paper[1] we discussed the importance, in precision analysis of β-decay spectra, of Coulomb corrections through the induced terms (hereafter referred to as the induced Coulomb correction) such as the weak magnetism and induced pseudoscalar terms. It was demonstrated using an example of ^{12}B β decay that there is a Coulomb correction through the weak magnetism term of order $(\alpha Z/m_p R)$ in addition to the usual correction of order $(\alpha Z)(RW_0)(W_0/2m_p)$ expected from simple consideration of the various factors involved, where R is the nuclear radius, W_0 the maximum energy available in β decay, and m_p is the proton mass. The usual Coulomb correction[2] for the finite size of the nuclei through dominant vector or axial-vector terms is of order $(\alpha Z)(RW_0)$. Since we have $RW_0 \sim (m_p R)^{-1}$, the induced Coulomb correction is of the same order of magnitude as the usual finite size correction. Recently Holstein[3] generalized the calculation to include arbitrary allowed transitions and the induced tensor term.

The induced Coulomb corrections are small, in general, as are the usual Coulomb corrections in allowed transitions. However, their effect can be significantly enhanced when the dominant allowed contributions are accidentally suppressed.

In order to emphasize the importance of this effect, we present in this paper an analytical expression for the shape factor for the $0^- \leftrightarrow 0^+$ β decay which is the first forbidden one. In this decay, when the transition hadron matrix element is expressed in terms of the nuclear weak form factors, as usually done in the elementary particle treatment of nuclei,[4] the contribution of the induced pseudoscalar term is significantly enhanced relative to that of the usual impulse approximation. In fact, the induced pseudoscalar term becomes comparable in magnitude to the leading axial-vector term, and hence, the induced Coulomb correction is dramatically enhanced. We demonstrate here that in the $0^- \leftrightarrow 0^+$ transition, the induced Coulomb correction is indeed a dominant Coulomb correction.

II. $0^- \leftrightarrow 0^+$ TRANSITION

The most general hadron matrix elements for the $0^- \leftrightarrow 0^+$ transition are, from general invariance principle,

$$\langle f(\vec{p}_f)|\, V_\alpha^{(\pm)}(0)\,|i(\vec{p}_i)\rangle = 0\,,$$

$$\langle f(\vec{p}_f)|A_\alpha^{(\pm)}(0)\,|i(\vec{p}_i)\rangle = f_+(q^2)Q_\alpha + \frac{(\Delta M)2M}{m_\pi^{\,2}} f_-(q^2)q_\alpha\,; \tag{1}$$

$$q_\alpha = (p_f - p_i)_\alpha\,, \quad Q_\alpha = (p_i + p_f)_\alpha\,,$$

where M is the nuclear mass $[M \cong \frac{1}{2}(M_i + M_f)]$, $\Delta M = M_i - M_f \cong W_0$, and $f_+(q^2)$ and $f_-(q^2)$ are, respectively, the nuclear axial-vector and the induced pseudoscalar form factors. In Eq. (1), $V_\alpha^{(\pm)}(x)$ and $A_\alpha^{(\pm)}(x)$ are, respectively, the vector and axial-vector hadron weak currents. The normalization of the $f_-(q^2)$ term is simply for convenience.

Since the $0^- \leftrightarrow 0^+$ transitions are first-forbidden ones, $f_+(0)$ cannot be of order unity. In fact, from the observed ft values for the $0^- \leftrightarrow 0^+$ transitions (e.g., ^{144}Ce, ^{144}Pr, ^{166}Ho, ^{206}Tl, ^{212}Pb), we find $|f_+(0)| \cong 10^{-1} \sim 10^{-2}$. From comparison of Eq. (1) with the corresponding impulse approximation expression, we find[5]

$$\begin{aligned} f_+(q^2) &\cong g_A\langle\gamma_5\rangle - \tfrac{1}{3} g_A W_0 \langle i\vec{\sigma}\cdot\vec{r}\rangle + \cdots\,, \\ f_-(q^2) &\cong -\tfrac{1}{3} g_A\left(\frac{m_\pi^{\,2}}{W_0}\right)\langle i\vec{\sigma}\cdot\vec{r}\rangle + \cdots\,, \end{aligned} \tag{2}$$

where $g_A = 1.24 \pm 0.01$ is the nucleon axial-vector coupling constant and we have used the definition

$$\langle O\rangle \equiv \langle \psi_f | \sum_{a=1}^{A} O^{(a)} \tau_{\pm}^{(a)} | \psi_i\rangle . \tag{3}$$

It is clear from Eq. (2) that $f_+(0)$ is of order $\langle\gamma_5\rangle \sim O(m_\pi/m_p)$, consistent with the above estimate. From Eq. (2) we have

$$\frac{f_-(q^2)}{f_+(q^2)} \cong \left(\frac{m_\pi}{W_0}\right)^2 \frac{1}{1+\Lambda},$$

where

$$\Lambda = -\frac{3}{W_0 R}\frac{\langle\gamma_5\rangle}{\langle i\vec{\sigma}\cdot\hat{r}\rangle}. \tag{4}$$

The ratio $\langle\gamma_5\rangle/\langle i\vec{\sigma}\cdot\hat{r}\rangle$ can be estimated using the well-known Ahrens-Feenberg approximation.[6] It is given by[7]

$$\frac{\langle\gamma_5\rangle}{\langle i\vec{\sigma}\cdot\hat{r}\rangle} \cong \mp\tfrac{1}{2}\lambda\alpha Z \quad \text{for } \beta^{\mp} \text{ decay}, \tag{5}$$

where λ takes the values of $1\sim 2$ depending on details on the nuclear models used. Since some evidence exists in favor of the value $\lambda \simeq 2$,[8] we shall use, in the following, $\lambda = 2$ for definiteness. Thus, the parameter Λ in Eq. (4) is given by

$$\Lambda \cong \pm\frac{3\alpha Z}{W_0 R}. \tag{6}$$

For example, $\Lambda \cong 1$, 48, 12, and 19, respectively, for $^{16}N^* \to {}^{16}O$, $^{144}Ce \to {}^{144}Pr$, $^{144}Pr \to {}^{144}Nd$, and $^{166}Ho \to {}^{166}Er$.[9]

From Eq. (6) it is clear that the ratio $[f_-(q^2)/f_+(q^2)]$ is considerably larger than unity since W_0 is of order of MeV for β decay. The ratio of the two contributions, $f_-(q^2)$ and $f_+(q^2)$, in the transition matrix element is then

$$\begin{aligned}\frac{(2MW_0/m_\pi^2)m_e f_-(0)}{2Mf_+(0)} &= \frac{W_0 m_e}{m_\pi^2}\left(\frac{f_-(0)}{f_+(0)}\right)\\ &= \left(\frac{m_e}{W_0}\right)\frac{1}{1+\Lambda}\\ &\equiv x .\end{aligned} \tag{7}$$

From the numerical values of Eq. (6) and of W_0 we have $x \cong 0.01 \sim 0.03$.[8] It appears, after all, that the contribution of the $f_-(q^2)$ term is only a few percent of the leading $f_+(q^2)$ term.[8] However, as will be seen later, the induced Coulomb correction through $f_-(q^2)$ becomes significant. In the following the shape factor will be given in terms of the parameter x defined in Eq. (7).

Before we proceed to calculate the shape factor, we give a rough estimate of the magnitude of the induced Coulomb correction.

From the minimal coupling replacement in the electromagnetic interaction

$$\partial_\alpha \to \partial_\alpha \pm ie\mathcal{A}_\alpha \tag{8}$$

we have

$$q_\alpha \to q_\alpha \mp e\mathcal{A}_\alpha , \tag{9}$$

where $\mathcal{A}_\alpha$ is the vector potential. This implies that whenever we have q_α in the matrix element, we replace it by $q_\alpha \mp e\mathcal{A}_\alpha$ which then gives the induced Coulomb correction. The vector potential produced by the nucleus of a uniformly charged sphere of the radius R is given by[10]

$$\mathcal{A}_\alpha \cong \left[i\frac{eZ}{r}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right), 0\right]. \tag{10}$$

Taking, for simplicity, $r = R$, we find, from Eqs. (9) and (10)

$$q_0 \to q_0 \mp \frac{\alpha Z}{R}, \tag{11}$$

which gives rise to, when combined with Eqs. (1) and (7), the induced Coulomb correction of order

$$\frac{\alpha Z}{R}\frac{W_0 2M}{m_\pi^2} f_-(0) = (2M)f_+(0)\left(\frac{\alpha Z x}{R m_e}\right). \tag{12}$$

Since the factor, $2Mf_+(0)$, is the leading contribution to the transition matrix element, as can be seen from Eq. (1), this correction is significant even when x is small.

Next we proceed to calculate the electron shape factor. The Coulomb-corrected amplitude for the process $i \to f + e^- + \bar{\nu}_e$ (calculated through a perturbative expansion to order αZ) is given by[11]

$$T = \frac{G\cos\theta_c}{\sqrt{2}}\,\bar{u}(\vec{p}_e)\left[\mathcal{J}_\alpha(q) + \frac{\alpha Z}{2\pi^2}\int d^3p\,\gamma_0 \frac{F_{\rm ch}(q_2^2)}{\vec{q}_2^{\,2}}\,\frac{p\cdot\gamma + m_e}{\vec{p}^{\,2} - \vec{p}_e^{\,2} - i\epsilon}\cdot \mathcal{J}_\alpha(q_1)\right]\gamma^\alpha(1-\gamma_5)v(\vec{p}_\nu). \tag{13}$$

In Eq. (13),

$$\mathcal{J}_\alpha(q) = \langle f(\vec{p}_f)|\, V_\alpha^{(\pm)}(0) + A_\alpha^{(\pm)}(0)\,|i(\vec{p}_i)\rangle ,$$

$$(q_1)_\alpha = -(p+p_\nu)_\alpha, \quad (q_2)_\alpha = (p - p_e)_\alpha , \tag{14}$$

where $(p_e)_\alpha$ and $(p_\nu)_\alpha$ are the lepton four-momenta, p_α the intermediate electron four-momentum, and $G\cos\theta_c = 10^{-5}/m_p^2$. Also, $F_{\rm ch}(q^2)$ is the elastic charge form factor of the final nucleus, normalized to unity.

The electron energy spectrum is then[12]

$$d\Gamma = \frac{p_e E_\nu}{2^6 \pi^3 M^2} \int d(\cos\theta) \frac{1}{2J_i + 1} \sum_{\text{spins}} |T|^2 dE_e\,, \tag{15}$$

where θ is the angle between $\vec{p}_e$ and $\vec{p}_\nu$, J_i is the spin of the initial nucleus, and E_e, E_ν are the electron and neutrino energies. From Eq. (13), we have

$$\sum_{\text{spins}} |T|^2 = \frac{G^2 \cos^2\theta_c}{2} \Big\{ \mathcal{J}_\alpha(q) \mathcal{J}_\beta^*(q) 2\,\mathrm{Tr}[(\gamma\cdot p_e + m_e)\gamma^\alpha(1-\gamma_5)\gamma\cdot p_\nu \gamma^\beta] + \frac{\alpha Z}{\pi^2}\,\mathrm{Re}\int d^3p \frac{F_{\mathrm{ch}}(q_2{}^2)}{\vec{q}_2{}^2(\vec{p}^2 - \vec{p}_e{}^2 - i\epsilon)}$$
$$\times \mathcal{J}_\alpha(q_1)\mathcal{J}_\beta^*(q) 2\,\mathrm{Tr}[(m_e + \gamma\cdot p_e)\gamma_0(m_e+\gamma\cdot p)\gamma^\alpha(1-\gamma_5)\gamma\cdot p_\nu\gamma^\beta]\Big\}. \tag{16}$$

Introducing the usual Fermi function $F_0(Z, E_e)$, the shape factor, $S'(Z, E_e)$ (un-normalized), is defined by

$$d\Gamma = \frac{G^2\cos^2\theta_c}{2\pi^3} S'(Z, E_e) p_e E_e (W_0 - E_e)^2 F_0(Z, E_e) dE_e\,. \tag{17}$$

The integrals which appear in Eq. (16) are evaluated in Appendix A. The form factors in Eqs. (1) and (16) are assumed to be of the form

$$f_\pm(q^2) = \frac{f_\pm(0)}{1 - b_\pm{}^2 q^2}\,,$$
$$F_{\mathrm{ch}}(q^2) = \frac{1}{1 + a^2\vec{q}^2}\,. \tag{18}$$

Using the integrals given in Appendix A and omitting terms of order (m_e/M) and (W_0/M) as well as of order $(p_l R)^2$, we obtain, from Eqs. (15) and (16)

$$\frac{d\Gamma}{dE_e} = \frac{G^2\cos^2\theta_c}{4\pi^3} p_e E_e (W_0 - E_e)^2 \int d(\cos\theta) f_+{}^2(q^2)$$
$$\times\Bigg(\Big[\Big(1 - \frac{m_e}{E_e}x(q^2)\Big)^2 + \frac{p_e{}^2}{E_e{}^2}x^2(q^2)\Big] + \frac{\alpha Z}{\pi^2}\Big(\frac{f_+(0)}{f_+(q^2)}\Big)\Big\{2E_e\Big[\Big(1 - \frac{m_e}{E_e}x(q^2)\Big)\Big(Y_{1+} - \frac{m_e}{E_e}x(0)Y_{1-}\Big) + \frac{\vec{p}_e{}^2}{E_e{}^2}x(0)x(q^2)Y_{1-}\Big]$$
$$+ \frac{1}{E_e}[Y_{2+} + x(0)x(q^2)Y_{2-}] + \frac{1}{W_0 - E_e}\Big[\Big(1 - \frac{m_e}{E_e}x(q^2)\Big)\Big(Y_{3+} + \frac{m_e}{E_e}x(0)Y_{3-}\Big) - \frac{\vec{p}_e{}^2}{E_e{}^2}x(0)x(q^2)Y_{3-}\Big]$$
$$+ \frac{1}{m_e}\Big(1 - \frac{m_e}{E_e}x(q^2)\Big)x(0)Y_{4-} + \frac{1}{m_e}\frac{\vec{p}_e\cdot\vec{p}_\nu}{E_eE_\nu}x(0)Y_{4-}\Big\}\Bigg)\,, \tag{19}$$

where $x(0) \equiv x$ which was previously defined in Eq. (7) and

$$x(q^2) = \frac{W_0 m_e}{m_\pi{}^2}\Big(\frac{f_-(q^2)}{f_+(q^2)}\Big)\,. \tag{20}$$

In Eq. (19), the quantities $Y_{1\pm}$, $Y_{2\pm}$, $Y_{3\pm}$, and $Y_{4\pm}$ are defined in Appendix B.

Next, integrating over $\cos\theta$ with the help of the integrals given in Appendix B and retaining terms up to order $p_l R$, we find

$$\frac{d\Gamma}{dE_e} = \frac{G^2\cos^2\theta_c}{2\pi^3} f_+{}^2(0) p_e E_e (W_0 - E_e)^2 \Bigg(\Big[\Big(1 - x\frac{m_e}{E_e}\Big)^2 + \frac{\vec{p}_e{}^2}{E_e{}^2}x^2\Big]\Big(1 + \alpha Z\pi\frac{E_e}{p_e}\Big) + \frac{\alpha Z}{\pi^2}$$
$$\times\Big\{2E_e\Big[\Big(1 - x\frac{m_e}{E_e}\Big)\Big((I_{1+} - I_p) - \frac{m_e}{E_e}x(I_{1-} - I_p)\Big) + \frac{\vec{p}_e{}^2}{E_e{}^2}x(I_{1-} - I_p)\Big]$$
$$+ \frac{1}{E_e}(I_{2+} + x^2 I_{2-}) + \frac{1}{W_0 - E_e}\Big[\Big(1 - x\frac{m_e}{E_e}\Big)\Big(I_{3+} + x\frac{m_e}{E_e}I_{3-}\Big) - \frac{\vec{p}_e{}^2}{E_e{}^2}x^2 I_{3-}\Big]$$
$$+ \frac{1}{m_e}x\Big(I_{4+} - x\frac{m_e}{E_e}I_{4-}\Big) + \frac{1}{m_e}xI_{5-}\Big\}\Bigg)\,, \tag{21}$$

where $I_{1\pm}$, $I_{2\pm}$, $I_{3\pm}$, $I_{4\pm}$, $I_{5\pm}$, and I_p are defined in Appendix B. The shape factor (normalized) is then given by, from Eqs. (17) and (21),[13]

$$S(Z,E_e)=1+x^2-2x\frac{m_e}{E_e}+\alpha Z\Big[-E_e\left(c_{1+}+x^2c_{1-}-x(c_{1+}+c_{1-})\frac{m_e}{E_e}\right)-(c_{2+}+x^2c_{2-})\frac{\vec{p}_e^{\,2}}{E_e}+3(W_0-E_e)$$
$$\times\left(c_{3+}-x^2c_{3-}-x(c_{3+}-c_{3-})\frac{m_e}{E_e}\right)-x\left(c_{5+}-x\frac{m_e}{E_e}c_{5-}\right)\frac{\vec{p}_e^{\,2}+(W_0-E_e)^2}{m_e}$$
$$+x\left(c_{6+}-x\frac{m_e}{E_e}c_{6-}\right)\frac{W_0^2}{m_e}-\tfrac{2}{3}xc_{5+}\frac{\vec{p}_e^{\,2}(W_0-E_e)}{m_eE_e}\Big]+\alpha Z\frac{x}{m_ec_{4-}}\left(1-x\frac{m_e}{E_e}\right), \tag{22}$$

where $c_{i\pm}$ $(i=1,2,\ldots,8)$ are given in Appendix B and are all of order R.

The first αZ term (given in brackets) is the usual finite-size Coulomb correction from both $f_+(q^2)$ and $f_-(q^2)$ and the last is the induced Coulomb correction due to the minimal coupling replacement mentioned earlier. It should also be pointed out that the above expression for the shape factor [Eq. (22)], being only linear in αZ, is valid, strictly speaking, only for light and medium nuclei such as the case of $^{16}N^*\to{}^{16}O$. However, even in the case of $\alpha Z\approx\frac{1}{2}$, the $(\alpha Z)^2$ terms are usually small; in particular the shape of the spectrum is not altered significantly by the $(\alpha Z)^2$ terms and Eq. (22) is perhaps adequate for a qualitative analysis of the shape factor for heavier nuclei.

We remind here again that in Eq. (22) terms of order (m_e/M), $(m_eR)^2$, ... are neglected. The result in Eq. (22) with $\alpha Z=0$ then agrees with the result given in Ref. 14.

In order to investigate the relative magnitude of the two Coulomb corrections, we assume for simplicity

$$a\cong b_+\cong b_-=\eta R, \quad \eta=\frac{1}{\sqrt{10}}. \tag{23}$$

The approximate equalities amount to saying that the charge radius of the final nucleus is roughly the same as the *transition* radii characterizing the form factors $f_\pm(q^2)$ and of course that the transition radii for $f_\pm(q^2)$ are roughly the same (in fact, the impulse approximation supports this). The factor $\eta=1/\sqrt{10}$ follows from the relation $R^2=\frac{5}{3}\langle r^2\rangle$.

For this simplified case, we have, from Eqs. (B7) and (B18) in Appendix B:

$$c_1=6\eta R, \quad c_2=-\tfrac{3}{2}\eta R, \quad c_3=-\tfrac{1}{6}\eta R, \quad c_4=\eta R$$
$$c_5=\tfrac{19}{12}\eta R, \quad c_6=\tfrac{7}{4}\eta R, \quad c_7=\tfrac{7}{12}\eta R, \quad c_8=\tfrac{3}{4}\eta R. \tag{24}$$

The shape factor Eq. (22) is then reduced to

$$S(Z,E_e)=1+x^2-2x\frac{m_e}{E_e}+\alpha Z(\mathrm{I})+\alpha Z(\mathrm{II}), \tag{25}$$

where

$$\alpha Z(\mathrm{I})=\alpha Z\eta R\left(A+BE_e+\frac{C}{E_e}+DE_e^2\right) \tag{26}$$

with

$$A=-\frac{W_0}{2}-\tfrac{8}{3}x^2W_0+\tfrac{1}{6}x\left(\frac{W_0^2}{m_e}\right)+\tfrac{451}{36}m_ex,$$
$$B=-4-\tfrac{11}{6}x^2+\tfrac{19}{9}x\left(\frac{W_0}{m_e}\right),$$
$$C=-\tfrac{3}{2}m_e^2-\tfrac{37}{12}m_e^2x^2-\tfrac{1}{6}x^2W_0^2+\tfrac{19}{18}m_eW_0x,$$
$$D=-\frac{19}{9}\frac{x}{m_e}, \tag{27}$$

and

$$\alpha Z(\mathrm{II})=\frac{\alpha Zx}{m_e\eta R}\left(1-x\frac{m_e}{E_e}\right). \tag{28}$$

It is clear that the usual Coulomb correction [αZ(I) term] is of order αZRW_0, whereas the induced Coulomb correction [αZ(II)] is of order $\alpha Zx/m_eR$. Thus, the ratio [$\alpha Z(\mathrm{II})/\alpha Z(\mathrm{I})$] is

$$\frac{\alpha Z(\mathrm{II})}{\alpha Z(\mathrm{I})}\sim\frac{x}{m_eR(W_0R)}\sim\frac{1}{A^{2/3}}\left(\frac{m_\pi^2}{m_eW_0}\right)x, \tag{29}$$

where we have used $R\simeq(1/m_\pi)A^{1/3}$. Since $x\simeq0.01\sim0.03$, the above ratio is considerably larger than unity. For example, the ratio is, respectively, 19, 17, 4, and 5 for $^{16}N^*\to{}^{16}O$, $^{144}Ce\to{}^{144}Pr$, $^{144}Pr\to{}^{144}Nd$, and $^{166}Ho\to{}^{166}Er$. Therefore, the induced Coulomb correction is dominant numerically. However, since $x\ll1$, it does not modify the energy dependence to any appreciable extent [see Eq. (28)], while the rate is significantly changed.

Finally, a comparison of the shape factor given by Eq. (22) or Eqs. (25)–(28) with the observed shape factors for heavy nuclei[9] shows a clear qualitative difference between theory and experiment. The theoretical shape factors are always monotonic functions of the energy E_e, whereas most of the observed ones show minima and/or maxima inside the energy range. This implies that either neglected $(\alpha Z)^2$ terms are unexpectedly large or the data need be reexamined.

APPENDIX A

Evaluation of all the integrals which appear in Eq. (15) is based on the following general formula:

$$B(\mu,\nu,\lambda) \equiv \int d^3p[(\mu^2+\vec{q}_1^{\,2})(\nu^2+\vec{q}_2^{\,2})(\vec{p}^{\,2}-\lambda^2-i\epsilon)]^{-1}$$

$$=\pi^2(\beta^2-\alpha)^{1/2}\ln\left[\frac{\beta+(\beta^2-\alpha)^{1/2}}{\beta-(\beta^2-\alpha)^{1/2}}\right], \tag{A1}$$

where

$$\alpha=[\vec{q}^{\,2}+(\mu+\nu)^2][\vec{p}_\nu^{\,2}+(\mu-i\lambda)^2][\vec{p}_e^{\,2}+(\nu-i\lambda)^2],$$

$$\beta=-i\lambda[\vec{q}^{\,2}+(\mu+\nu)^2]+\nu(\vec{p}_\nu^{\,2}+\mu^2-\lambda^2)+\mu(\vec{p}_e^{\,2}+\nu^2-\lambda^2). \tag{A2}$$

The integrals in Eq. (15) may be divided into the following three types of the integrals:

$$J\equiv\int d^3p f(\vec{p},\vec{p}_e,\vec{p}_\nu), \tag{A3}$$

$$J_r\equiv\int d^3p f(\vec{p},\vec{p}_e,\vec{p}_\nu)p_r, \quad r=1,2,3, \tag{A4}$$

$$J_{rs}\equiv\int d^3p f(\vec{p},\vec{p}_e,\vec{p}_\nu)p_r p_s, \quad r,s=1,2,3, \tag{A5}$$

with

$$f(\vec{p},\vec{p}_e,\vec{p}_\nu)$$
$$=[(1-b^2q_1^{\,2})(1+a^2\vec{q}_2^{\,2})\vec{q}_2^{\,2}(\vec{p}^{\,2}-\vec{p}_e^{\,2}-i\epsilon)]^{-1}$$
$$=(\mu^2+W_0^{\,2})\nu^2[(\mu^2+\vec{q}_1^{\,2})(\nu^2+\vec{q}_2^{\,2})\vec{q}_2^{\,2}(\vec{p}^{\,2}-\vec{p}_e^{\,2}-i\epsilon)]^{-1}, \tag{A6}$$

where $b^{-2}=\mu^2+W_0^{\,2}$ and $a^{-2}=\nu^2$. The integral J can then be easily expressed in terms of $B(\mu,\nu,p_e)$ in (A1) as follows:

$$J=(\mu^2+W_0^{\,2})[B(\mu,0,p_e)-B(\mu,\nu,p_e)]. \tag{A7}$$

On the other hand, J_r and J_{rs} can be obtained from J by successively applying to it the operator

$$\Omega_r\equiv-(\mu^2+W_0^{\,2})\int_\mu^\infty\frac{\mu'd\mu'}{\mu'^2+W_0^{\,2}}\frac{\partial}{\partial p_{\nu,r}}-p_{\nu,r}. \tag{A8}$$

In fact, one has the relations

$$J_r=\Omega_r J,$$
$$J_{rs}=\Omega_r\Omega_s J=\Omega_r J_s=\Omega_s J_r. \tag{A9}$$

In Eq. (15), we also have the following combination:

$$J_4\equiv\sum_{r=1}^{3}(J_{rr})-\vec{p}_e^{\,2}J=\int d^3p f(\vec{p},\vec{p}_e,\vec{p}_\nu)(\vec{p}^{\,2}-\vec{p}_e^{\,2}), \tag{A10}$$

which can also be obtained directly from the B's:

$$J_4=-\lim_{\lambda\to\infty}\lambda^2(\mu^2+W_0^{\,2})[B(\mu,0,\lambda)-B(\mu,\nu,\lambda)]$$

$$=i\pi^2\frac{\mu^2+W_0^{\,2}}{q}\ln\left[\frac{(\mu-iq)(\mu+\nu+iq)}{(\mu+iq)(\mu+\nu-iq)}\right]. \tag{A11}$$

We now give the low-energy expansion of the real part of the above integrals (p_l stands for either electron or neutrino momentum and R is of order $1/\mu, 1/\nu$):

$$\mathrm{Re}B(\mu,\nu,p_e)=\pi^2\frac{2}{\mu\nu(\mu+\nu)}\left\{1+\left[\frac{1}{\mu(\mu+\nu)}-\frac{(\mu+\nu)^2}{\mu^2\nu^2}-\frac{\mu^2}{3\nu^2(\mu+\nu)^2}\right]\vec{p}_e^{\,2}-\frac{(\mu+\nu)^3-\mu^3}{3\mu^2\nu(\mu+\nu)^2}\vec{p}_\nu^{\,2}-\frac{2}{3(\mu+\nu)^2}\vec{p}_e\cdot\vec{p}_\nu+O(p_l^{\,4}R^4)\right\}, \tag{A12}$$

$$\mathrm{Re}B(\mu,0,p_e)=\frac{\pi^2}{\mu^2 p_e}\left[\frac{\pi}{2}\left(1-\frac{\vec{q}^{\,2}}{\mu^2}+\frac{\vec{q}^{\,4}}{\mu^4}\right)-\frac{2p_e}{\mu}\left(1-\frac{\vec{q}^{\,2}+\vec{p}_\nu^{\,2}-\vec{p}_e^{\,2}}{\mu^2}\right)+\frac{8}{3}\frac{\vec{p}_e^{\,3}}{\mu^3}+O(p_l^{\,5}R^5)\right]. \tag{A13}$$

We wish to point out that Eq. (A13) cannot be derived from Eq. (A12) since the latter has meaning only for $\mu,\nu\neq0$. From Eq. (A7) we get

$$\mathrm{Re}J=\frac{\mu^2+W_0^{\,2}}{\mu^2}\frac{\pi^2}{p_e}\left(\frac{\pi}{2}\left(1-\frac{\vec{q}^{\,2}}{\mu^2}+\frac{\vec{q}^{\,4}}{\mu^4}\right)-\frac{2p_e}{\mu}\left(1-\frac{\vec{q}^{\,2}+\vec{p}_\nu^{\,2}-\vec{p}_e^{\,2}}{\mu^2}\right)+\frac{8}{3}\frac{\vec{p}_e^{\,3}}{\mu^3}-\frac{2\mu p_e}{\nu(\mu+\nu)}\right.$$
$$\left.\times\left\{1+\left[\frac{1}{\mu(\mu+\nu)}-\frac{(\mu+\nu)^2}{\mu^2\nu^2}-\frac{\mu^2}{3\nu^2(\mu+\nu)^2}\right]\vec{p}_e^{\,2}-\frac{(\mu+\nu)^3-\mu^3}{3\mu^2\nu(\mu+\nu)^2}\vec{p}_\nu^{\,2}-\frac{2}{3(\mu+\nu)^2}\vec{p}_e\cdot\vec{p}_\nu+O(p_l^{\,5}R^5)\right\}\right), \tag{A14}$$

and from Eq. (A9)

$$\mathrm{Re}J_r=\frac{\mu^2+W_0^{\,2}}{\mu^2}\pi^2\left(\left\{\frac{\pi}{2p_e}\left(1-\frac{\vec{q}^{\,2}}{\mu^2}\right)-\frac{4}{3}\left[\frac{1}{\mu}+\frac{\mu^2}{2\nu(\mu+\nu)^2}\right]\right\}p_{e,r}-\frac{2}{3}\frac{2\mu\nu+\nu^2}{\mu(\mu+\nu)^2}p_{\nu,r}+O(p_l^{\,3}R^3)\right), \tag{A15}$$

$$\mathrm{Re}J_{rs}=\frac{\mu^2+W_0^{\,2}}{\mu^2}\pi^2 p_e\left[\frac{2\mu\nu}{3(\mu+\nu)p_e}\delta_{rs}+\frac{\pi}{2}\frac{p_{e,r}p_{\nu,s}}{\vec{p}_e^{\,2}}+O(p_l R)\right]. \tag{A16}$$

From Eq. (A11) we have

$$\mathrm{Re}J_4 = \frac{\mu^2 + W_0^2}{\mu^2}\pi^2 p_e \left\{\frac{2\mu\nu}{(\mu+\nu)p_e} + \frac{2}{3}\left[\frac{\mu^2}{(\mu+\nu)^3} - \frac{1}{\mu}\right]\frac{\vec{q}^2}{p_e} + O(p_l^3R^3)\right\}. \tag{A17}$$

APPENDIX B

In this appendix we give the expansion of the integrals of the form

$$\frac{1}{2}\int d(\cos\theta)\,\mathrm{Re}g(q^2)\int d^3p\frac{g'(q_1^2)F_{\mathrm{ch}}(q_2^2)}{\vec{q}_2^2(\vec{p}^2-\vec{p}_e^2-i\epsilon)}h \tag{B1}$$

up to linear terms in (p_1R). In this equation g and g' represent any one of the form factors $f_\pm$ as defined in Eq. (19) for $0^- \to 0^+$ transitions, and h is one of the following terms:

$$1,\ \ \vec{p}_e\cdot\vec{p}_\nu,\ \ \vec{p}_e\cdot\vec{q}_2,\ \ \vec{p}_\nu\cdot\vec{p},\ \ \vec{p}^2-\vec{p}_e^2,\ \ (\vec{p}^2-\vec{p}_e^2)(\vec{p}_e\cdot\vec{p}_\nu)$$

which are the only relevant three-dimensional scalars that survive when summed over spin indices.

Consequently the integration over the intermediate electron momentum $\vec{p}$ generates four typical integrals

$$\begin{aligned}
&Y_1 = \mathrm{Re}J \qquad h=1,\ \ \vec{p}_e\cdot\vec{p}_\nu\,,\\
&Y_2 = \mathrm{Re}\vec{p}_e\cdot(\vec{J}-J\vec{p}_e) \qquad h=\vec{p}_e\cdot\vec{q}_2\,,\\
&Y_3 = \mathrm{Re}\vec{p}_\nu\cdot\vec{J} \qquad h=\vec{p}_\nu\cdot\vec{p}\,,\\
&Y_4 = \mathrm{Re}J_4 \qquad h=\vec{p}^2-\vec{p}_e^2,\ \ (\vec{p}^2-\vec{p}_e^2)(\vec{p}_e\cdot\vec{p}_\nu)\,.
\end{aligned} \tag{B2}$$

The low-energy expansions of Y_1, Y_2, Y_3, and Y_4 are given by

$$Y_1 = \frac{\pi^2}{p_e}\left[\frac{\pi}{2} - \frac{1}{2}c_1p_e + O(p_l^2R^2)\right], \tag{B3}$$

$$Y_2 = \pi^2\vec{p}_e\cdot\left[-c_2\vec{p}_e + 3c_3\vec{p}_\nu + O(p_l^2R^2)\right], \tag{B4}$$

$$Y_3 = \pi^2\vec{p}_\nu\cdot\left[\left(\frac{\pi}{2p_e} - \frac{1}{2}c_1 - c_2\right)\vec{p}_e + 3c_3\vec{p}_\nu + O(p_l^2R^2)\right], \tag{B5}$$

$$Y_4 = \pi^2p_e\left[\frac{1}{p_ec_4} - c_7\frac{\vec{q}^2}{p_e} + c_8\frac{W_0^2}{p_e} + O(p_l^2R^2)\right], \tag{B6}$$

where

$$\begin{aligned}
&c_1 = 4(a^2+ab+b^2)/(a+b)\,,\\
&c_2 = -\tfrac{2}{3}(2a^3+4a^2b+2ab^2+b^3)/(a+b)^2\,,\\
&c_3 = -\tfrac{2}{9}(2a+b)b^2/(a+b)^2\,,\\
&c_4 = \tfrac{1}{2}(a+b)\,,\\
&c_7 = \tfrac{2}{3}b^2(3a^2+3ab+b^2)/(a+b)^3\,,\\
&c_8 = b^2(2a+b)/(a+b)^2\,.
\end{aligned} \tag{B7}$$

The c's carry $\pm$ subscripts depending on which b (b_+ or b_-) enters their definition; for example

$$c_{1\pm} = 4(a^2+ab_\pm+b_\pm^2)/(a+b_\pm)\,.$$

Analogously for subscripts attached to Y's.

According to (B1) and (B2) we must now consider the integrals of the type

$$\begin{aligned}
&\frac{1}{2}\int d(\cos\theta)\frac{1}{1-b_\pm^2q^2}Y_{k\pm}\,, \quad k=1,2,3,4\,,\\
&\frac{1}{2}\int d(\cos\theta)\frac{\vec{p}_e\cdot\vec{p}_\nu}{E_eE_\nu}\frac{1}{1-b_\pm^2q^2}Y_{k\pm}\,, \quad k=1,4
\end{aligned} \tag{B8}$$

and expand them up to terms linear in (p_1R). For $k=1,2,3$ we have

$$\begin{aligned}
\frac{1}{2}\int d(\cos\theta)\frac{1}{1-b_\pm^2q^2}Y_{k\pm} &= \frac{1}{2}\int d(\cos\theta)Y_{k\pm} + O(p_l^2R^2)\\
&\equiv I_{k\pm} + O(p_l^2R^2)\,,
\end{aligned} \tag{B9}$$

$$\frac{1}{2}\int d(\cos\theta)\frac{\vec{p}_e\cdot\vec{p}_\nu}{E_eE_\nu}\frac{1}{1-b_\pm^2q^2}Y_{1\pm} = O(p_l^2R^2)\,. \tag{B10}$$

For $k=4$ we have only integrals with Y_{4-}; in fact these terms come from the space part of the axial-vector current

$$\frac{1}{2}\int d(\cos\theta)\frac{1}{1-b_\pm^2q^2}Y_{4-} \equiv I_{4\pm} + O(p_l^2R^2)\,, \tag{B11}$$

$$\frac{1}{2}\int d(\cos\theta)\frac{\vec{p}_e\cdot\vec{p}_\nu}{E_eE_\nu}\frac{1}{1-b_\pm^2q^2}Y_{4-} \equiv I_{5\pm} + O(p_l^2R^2)\,. \tag{B12}$$

The explicit expressions of I's are

$$I_1 = \frac{\pi^2}{p_e}\left[\frac{\pi}{2} - \frac{1}{2}c_1p_e\right], \tag{B13}$$

$$I_2 = -\pi^2p_e(c_2p_e)\,, \tag{B14}$$

$$I_3 = 3\pi^2p_\nu(c_3p_\nu)\,, \tag{B15}$$

$$I_{4\pm} = \pi^2p_e\left[\frac{1}{c_{4-}p_e} - c_{5\pm}\frac{\vec{p}_e^2+\vec{p}_\nu^2}{p_e} + c_{6\pm}\frac{W_0^2}{p_e}\right], \tag{B16}$$

$$I_{5\pm} = -\tfrac{2}{3}\pi^2p_e\left(c_{5\pm}\frac{p_ep_\nu}{E_e}\right) \tag{B17}$$

with

$$c_{5\pm} = c_{7-} + \frac{b_\pm^2}{c_{4-}}\,, \quad c_{6\pm} = c_{8-} + \frac{b_\pm^2}{c_{4-}}\,. \tag{B18}$$

We finally define

$$I_p,\ Y_p = \lim_{R\to 0} I_1\,, \qquad Y_1 = \frac{\pi^3}{2p_e}\,. \tag{B19}$$

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