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Analysis of three-body final states: Nonrelativistic

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Quantum mechanics forces rapid variation and coherent interrelation on amplitudes usually taken to be independent and constant in the Watson approximation to final state interactions. These effects are particularly important when more than one pair of final state interactions is simultaneously big. We develop these amplitude conditions using unitarity only. We show how these constraints combined with analyticity give a set of linear scattering integral equations. These equations are not only the minimal set compatible with the general constraints of quantum mechanics, but also turn out to be the simplest form of the separable interaction equations.

I. INTRODUCTION

Very little is understood about the structure of many particle final states and about how two-body interaction information is distributed over these states. Yet there is a great deal of data on such systems and the analysis of this data to obtain information on the interaction of unstable particles, like π-π or n-n, or to study how the pair information affects reaction mechanisms, is an important part of nuclear and particle physics. In this paper we take a step toward providing the theoretical basis for final state analysis, and show that the general principles of quantum mechanics constrain the final state in a way not normally included in phenomenological analyses. We develop these constraints, discuss their implementation, and describe where they can be important.

We will develop our discussion in terms of three-hadron final states, with occasional contact with the two-body final state problem as examples, but the methods we develop and many of the cautionary constraints clearly also apply to more than three final particles. It is perhaps surprising to find that after all the recent work in the three-body problem a correct theory of three-body final state interaction hardly exists, but that is the case. There are, of course, the many formally correct starting points, with that of Faddeev as the most familiar, but these are of no help in phenomenology. They are like writing down the Schrödinger equation for the problem; it is certainly correct, but much work is needed to extract consequences, even in a general way. At the other extreme there are the remarkably successful separable interaction calculations, particularly of $n+d$ breakup. These are not phenomenological either, but rather detailed dynamical solutions of the three-body equations, containing assumptions about the form of the two-body interaction on and off shell. There is no way to tell whether their agreement with experiment is due to these detailed assumptions or to the general features of the problem. In this paper we develop a theory of three-body final states containing only the general principles of quantum mechanics, in particular unitarity and its implications for analyticity.[1] We show how these principles can be used to establish the domain of applicability of the usual phenomenology—it turns out to be a rather slim domain—and how a better phenomenology can be developed that is generally applicable. This better phenomenology leads to a set of integral equations for the final state amplitudes, which set is nearly identical to the integral equations of the separable interaction theory.[2] Hence we have both established the "minimal" theory of three-body final state interactions compatible with the general constraints of unitarity and its implied analyticity, and also, by showing the great similarity of that theory to the separable interaction approach, we have shed light on the remarkable effectiveness of that approach.

As the discussion above implies, the usual phenomenology employed for the analysis of three-body final states violates unitarity. By quantifying this violation we are able to establish the range of validity of the usual approach as well as providing an alternative (albeit complicated) for use when the normal phenomenology fails. The usual approach to three-body final states is through a form of "Watson's theorem," a method borrowed from the two-body final state problem.[3] Even in the two-body case, what is usually called Watson's theorem is a special approximation of limited validity rather than a general theorem, but at least in the two-body case how to implement the full correct theorem on the basis of the general quantum mechanics is known. The Watson *approximation* in the two-body case consists of assuming that the pair's final state distribution in a reaction amplitude is proportional to that pair's free scattering amplitude times a slowly varying real factor that reflects the reaction mechanism itself. In the three-body case the corresponding assumption is that the reaction amplitude can be written as a sum of three terms, each one proportional to the free scattering amplitude of a given pair, and each multiplied by a factor that is essentially the amplitude for the creation of the spectator particle and correlated pair in the reaction. The assumption that is crucial to the approximation is that that last factor is slowly varying over the three-body phase space, and that turns out to violate unitarity. Many formalisms, for example the Faddeev equations, write the final state amplitude as a sum of three terms, each ending with one of the pair's t matrices depending on which pair interacts "last." In fact, the seductive simplicity of this form in the Faddeev case is sometimes mistakenly taken to justify the Watson approximation. Of course, the Faddeev theory makes no "slowly varying" approximation for the coefficients of the pair t matrices. The essence of the Watson approximation can be seen in the case where the final pair interactions are resonant. The approximation is then called the sequential decay model in nuclear physics and the isobar model in particle physics. The final state is then made in two steps. First the reaction forms a spectator and an unstable particle, or resonating pair, and then that state decays. The decay distribution is given by a Breit-Wigner form and hence by something proportional to the pair t matrix while the amplitude for forming the spectator and unstable states is a quasi-two-body amplitude. In fact the reaction amplitude is a sum of three such terms, one for each pair grouping. If the pair resonances are narrow they will live long, and escape the three-body interaction volume before decay. Each three-body event will then have a clear and unambiguous parentage. We will only need the quasi-two-body amplitudes at a pair subenergy corresponding to the resonance energy and there are no problems. But what if the resonances are not narrow? The particles begin to decay in the three-body interaction volume, or, equivalently, we have difficulty tracing a particular event to an unambiguous resonant pair plus spectator parentage. Clearly, interference effects become important and any coherence enforced by the general constraints of quantum mechanics becomes particularly important. Furthermore, if the pair resonances are broad, the quasi-two-body amplitudes leading to them can have important variation over the energy band in which pair interactions are important. These problems have been known for some time in both the isobar model and the sequential decay model, but since no solution was known, the models have continued to be used even in wide resonance cases where difficulties were almost certain to occur. It is precisely in these cases that we shall show that the simple isobar or sequential model violates unitarity. Our discussion is by no means restricted to the case of resonant pairwise final state interactions. Our corrections are just as important in cases involving strong but nonresonant overlapping final state interactions, as encountered for example in n-d breakup.

Our basic idea is to apply unitarity to an amplitude with a three-body final state parametrized as a sum of factors as in the Watson approximation. We then concentrate on those terms giving the strong or singular dependence of the spectator amplitudes on the pair subenergy for fixed total energy and total angular momentum. The pair subenergy and the spectator particle energy are then related by total energy conservation. We use the fact that each term in unitarity represents a singular term in the amplitude with the singularity at the threshold given by unitarity. We find the spectator term has a square-root branch point as a function of its pair's subenergy and that the strength of that singularity is proportional to the magnitude of the other pair terms. Hence unitarity requires that the spectator function, rather than being constant, have a rapid (in fact singular) dependence on the pair subenergy. Furthermore, because the strength of that square root is related to the other pair terms, there is an important coherence between the final state terms imposed by unitarity above. The presence of the square-root singularity in Schrödinger theories has been known for some time and its importance for controlling the dependence of the spectator function has been emphasized.[4] Also, the coherence between final state terms has been stressed before,[5] but the general nature of both these effects and their relationship has seldom been stressed.

The unitarity relations allow one to test the usual Watson approximation which neglects the square-root singularity and the coherence. One uses the approximate nonsingular amplitudes "on the right" in unitarity. If through unitarity they generate a very small singular part, the approximation of neglecting that part is valid. If they generate a large singular part, it is not. This is similar to using unitarity in elastic two-body scattering to test the validity of a purely real amplitude. One puts the purely real amplitude on the right; if it generates a small imaginary part, unitarity corrections are small, but if it generates a large imaginary part, unitarity is not just a correction, but an essential feature that must be included. In the two-body problem one knows how to use phase shifts to parametrize the amplitude so that unitarity is guaranteed. Unfortunately there seems to be no simple way to parametrize the spectator amplitudes in the three-body case so as to guarantee unitarity. However, by exploiting the fact that unitarity provides the discontinuity across the singularity associated with its threshold, we can implement the unitarity constraint in terms of a dispersion relation—that is, by writing Cauchy's theorem for a function with that branch cut and that discontinuity. If we assume the spectator function has only the branch cut required by unitarity, and recall that unitarity gives the discontinuity as a *linear* function of the other spectator functions, the dispersion relation gives a set of linear coupled integral equations for the spectator functions. This set turns out to be just the separable potential equations with neglect of vertex functions. The vertex functions give left hand cuts and they are neglected by taking only the unitarity cuts. Hence, in this way we derive a set of integral equations which are the minimal embodiment of the constraints of unitarity and analyticity and also shed light on the meaning of the separable interaction equations without recourse to the Schrödinger equation, etc. The approach is much like the procedure developed by Omnes for the implementation of unitarity in two-body final state interaction theory.[6] Unfortunately, the integral equation one now encounters is too difficult to solve in general, in contrast to the two-body case. Hence, though we have established the minimal structure required for the correct analysis of three-body final states by the general principles of quantum mechanics, we have also found that a full implementation of that structure requires the solution of a complex set of integral equations. These equations require much of the same technology for their solution as the separable interaction equation. Furthermore, they admit the possibility of making subtractions so that, for example, the breakup data in a reaction can be expressed in terms of the elastic scattering. Experience with n-d breakup indicates that very little latitude is left in the breakup if the elastic scattering is fitted.[7] Much experience will have to be developed with the formalism presented here to determine the general validity of such ideas, as well as the usefulness of the entire formalism.

As a technical note it should be stressed that in most phenomenological treatments of this problem as well as in our earlier discussions in letters of the unitarity constraint and its implementation,[1,2] the pair decay factor in the Watson approximation is taken to be the pair t matrix. Much better analytic structure and better convergence of the integral equations is obtained if the pair D function is used instead. This is precisely the factor that carries the pair final state interaction information in the two-body case as well.[6]

In Sec. II we develop the unitarity constraints on the Watson approximation form of the parametrization. This section is long and in places complicated since we develop the constraint for the general case, but it is also self-contained and begins with a discussion of our convention for unitarity. The reader uninterested in detail may wish to skip from Eq. (16) to the results, Eq. (20a), and the subsequent discussion. Section III discusses a simple three-boson example and hopefully clarifies the content of the general result. Section IV discusses implementation. In order to keep the algebra simple we only discuss implementation for the simple three-boson case. We first discuss implementation without the constraint of analyticity and, by comparing with two-body final state theory, show its shortcomings. We then present implementation with analyticity and show how one obtains an integral equation very much like that of the separable interaction theory. We also show here the superiority of using the pair D function rather than the t matrix to represent the pair correlations. Discussion of the results and an outline of future problems is presented in Sec. V. The unitarity constraint in the case of identical particles is derived in the Appendix.

II. UNITARITY CONSTRAINTS

A. Unitarity—two-body

Since our basic tool is unitarity, we begin with a discussion of our convention for it. In terms of the S matrix, we define the T matrix by[8]

$$\langle \alpha|S|\beta\rangle = \langle \alpha|1|\beta\rangle - 2\pi i\delta(E_\alpha - E_\beta)\langle \alpha|T|\beta\rangle . \quad (1)$$

Unitarity of the S matrix combined with symmetry of T (time reversal) allows us to write

$$\mathrm{Im}\langle \alpha|T|\beta\rangle = -\pi\sum_\gamma \langle \alpha|T|\gamma\rangle\langle \gamma|T|\beta\rangle^* \delta(E_\alpha - E_\gamma) \quad (2)$$

so long as $E_\alpha = E_\beta$. The sum over γ includes the usual integration over continuous variables and summation over discrete variables. Equation (2) applies to states of any number of particles so long as α, β, and γ can be connected by the quantum numbers and by energy conservation, but in order to clarify our procedures let us first study two-body scattering, assuming that the energy is low enough that only the elastic channel is open. Consider the scattering of two particles of mass m_1 and m_2 ($\hbar = 1$) with internal quantum numbers (spin, isospin, etc.) α_1 and α_2. We can write

$$\langle \vec{p}_1\alpha_1, \vec{p}_2\alpha_2|T|\vec{p}_1'\alpha_1', \vec{p}_2'\alpha_2'\rangle = (2\pi)^3\delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_1' - \vec{p}_2')\langle \vec{q}_{12}, \alpha_1\alpha_2|\tau|\vec{q}_{12}', \alpha_1'\alpha_2'\rangle , \quad (3)$$

where $(m_1+m_2)\vec{q}_{12}=m_2\vec{p}_1-\vec{p}_2 m_1$. Substituting (3) into (2) and changing variables to $\vec{q}''$ and $\vec{P}''$ $(\vec{P}=\vec{p}_1+\vec{p}_2)$, which transformation has unit Jacobian, we get

$$\mathrm{Im}\langle\vec{q}_{12}\alpha_1\alpha_2|\tau|\vec{q}\,'_{12}\alpha'_1\alpha'_2\rangle = -\pi\sum_{\alpha''}\int\frac{d^3q''_{12}}{(2\pi)^3}\,\delta\left(\frac{q''^2_{12}-q_{12}{}^2}{2\mu_{12}}\right) \times\langle\vec{q}_{12}\alpha_1\alpha_2|\tau|\vec{q}\,''_{12}\alpha''_1\alpha''_2\rangle \times\langle\vec{q}\,''_{12}\alpha''_1\alpha''_2|\tau|\vec{q}\,'_{12}\alpha'_1\alpha'_2\rangle^*, \tag{4}$$

where $(m_1+m_2)\mu_{12}=m_1m_2$, and $|\vec{q}_{12}|=|\vec{q}\,'_{12}|$ is energy conservation. If we assume, for simplicity, that there is no coupling between the internal quantum number and the orbital motion, we can write

$$\langle\vec{q}_{12}\alpha_1\alpha_2|\tau|\vec{q}\,'_{12}\alpha'_1\alpha'_2\rangle = \sum_{lmt}Y^*_{lm}(\hat{q}_{12})C^{\,t}_{\alpha_1\alpha_2}\tau_{lt}(q_{12}{}^2/2\mu_{12})C^{\,t}_{\alpha'_1\alpha'_2}Y_{lm}(\hat{q}'_{12}), \tag{5}$$

where $\hat{q}$ is a unit vector and $C^{\,t}_{\alpha\beta}$ is an element of a unitary transformation from the α representation to the t representation, in which representation the scattering amplitude is diagonal. The diagonal on-shell T matrix in this l, t state is then $\tau_{lt}(\epsilon)$. (The quantity t can be thought of as the channel spin, or isospin, or both.) Substituting (5) in (4) and using the orthonormality of the Y's and the C's gives

$$\mathrm{Im}\tau_{lt}(\epsilon)=-(\mu/8\pi^2)q|\tau_{lt}(\epsilon)|^2, \tag{6}$$

which is our convention for partial wave unitarity. Equation (6) can be satisfied by writing

$$\tau_{lt}(\epsilon)=-(8\pi^2/\mu q)e^{i\delta}\sin\delta \tag{7}$$

for any real δ.

It is often useful to write $\tau(\epsilon)$ (suppressing the lt labels) in the form

$$\tau(\epsilon)=N(\epsilon)/D(\epsilon), \tag{8}$$

where $D(\epsilon)$ carries only the unitarity cut and hence has the phase $-\delta$ while $N(\epsilon)$ is real for positive ϵ and has only left-hand cuts. In terms of (8), unitarity (6) gives

$$\mathrm{Im}D(\epsilon)=(\mu q/8\pi^2)N(\epsilon). \tag{9}$$

The analytic structure assumed for N and D means that in the simplest case, $D(\epsilon)$ can be written

$$D(\epsilon)=\exp\frac{-1}{\pi}\int_0^\infty\frac{d\epsilon'\delta(\epsilon')}{\epsilon'-\epsilon} \tag{10}$$

in terms of the phase shift.[9] There is considerable literature for generalizing (10) to include the effects of bound state zeros, Castillejo-Dalitz-Dyson poles, etc., hence (10) gives an expression for $D(\epsilon)$ at all ϵ in terms of the physical phase shift, then (8) or (9) can be used to construct $N(\epsilon)$ for real positive ϵ.

Let us now examine briefly the particular case of identical particles. No special care is needed with sums, etc. if the states are defined with the appropriate normalization. In particular, an n-body state of identical particles $|\alpha\beta\gamma\cdots\rangle$ is constructed according to

$$|\alpha\beta\gamma\cdots\rangle=\frac{1}{\sqrt{n}}\psi^\dagger_\alpha|\beta\gamma\cdots\rangle=\frac{1}{\sqrt{n!}}\psi^\dagger_\alpha\psi^\dagger_\beta\psi^\dagger_\gamma\cdots|0\rangle, \tag{11}$$

where the $\psi^\dagger_i$ are the creation operators and they obey the appropriate commutation or anticommutation relation. In that case the states $|\alpha\beta\gamma\cdots\rangle$ will have the correct symmetry as well as normalization. Using this, the unitarity relation will still be (6). The only restriction is that in the decomposition (5) we maintain the appropriate symmetry. Since interchanging the particles sends $\vec{q}$ to $-\vec{q}$ and since $Y_{lm}(\hat{q})=(-1)^lY_{lm}(-\hat{q})$, we need to take t's such that $C^{\,t}_{\alpha_1\alpha_2}=\pm C^{\,t}_{\alpha_2\alpha_1}$, the plus going with even l for bosons and odd l for fermions, and the minus with odd l for bosons and even l for fermions.

B. Unitarity—three-body

Let us now turn to the three particle case, and in particular the pair subenergy dependence of a three-body final state amplitude as required by unitarity. Consider an amplitude $T_{2,3}$ describing two stable particles going to three as in a breakup or particle production reaction. Assuming only the two- and three-body channels are open, unitarity for $T_{2,3}$ can be written

$$\mathrm{Im}T_{2,3}=-\pi\sum_{2'}T_{2,2'}\delta(E-E_{2'})T^\dagger_{2',3} -\pi\sum_{3'}T_{2,3'}\delta(E-E_{3'})T^\dagger_{3',3}, \tag{12}$$

where $T_{2,2'}$ and $T_{3',3}$ are the elastic two-body and three-body amplitudes, respectively. $T_{3',3}$ can be decomposed into a connected part $T_{3',3\ \mathrm{con}}$ and a sum of disconnected parts $T_{3',3\ \mathrm{discon}}$, which represent one particle going by while the other two scatter. These disconnected pieces of $T_{3',3}$ are a correct and necessary consequence of our definition of the S and T matrix (1). Hence (12) can be written

$$\mathrm{Im}T_{2,3} = -\pi \sum_{2'} T_{2,2'}\delta(E - E_{2'})T^{\dagger}_{2',3}$$
$$-\pi \sum_{3'} T_{2,3'}\delta(E - E_{3'})T_{3',3\,\mathrm{con}}$$
$$-\pi \sum_{3'} T_{2,3'}\delta(E - E_{3'})T^{\dagger}_{3',3\,\mathrm{discon}}\,. \quad (13)$$

This equation is represented schematically in Fig. 1. We are interested in exploiting (13) to obtain the dependence of $T_{2,3}$ on the pair subenergies for fixed total energy. We are particularly interested in its singularities, since a singularity represents rapid dependence. As is well known, each term in unitarity represents a singularity at the threshold of that term.[10] Strictly speaking, each term in unitarity contributes the discontinuity across the singularity beginning at the threshold and in the variable carrying that threshold. For a square-root singularity such as we encounter here for the subenergy, the discontinuity and the singular part are the same thing. We are interested in singularities in the pair subenergies, not in the total three-body energy, since that is kept fixed while the subenergy varies over the phase-space. Only terms in unitarity having pair subenergy thresholds will yield these subenergy singularities. Clearly, the $T_{2,2'}T^{\dagger}_{2',3}$ term has a threshold in E, the total energy, at $E = E_{2,0}$, the minimum two-body energy. Similarly, the $T_{2,3'}T_{3',3\,\mathrm{con}}$ term has a threshold in E at $E = E_{3,0}$ the minimum three-body energy. So, apparently, does the $T_{2,3}T_{3',3\,\mathrm{discon}}$ term from (13), but as is clear from Fig. 1 and as will become clear in our development, the δ function in $T_{3',3\,\mathrm{discon}}$ coming from the fly-by particle will give it a threshold in the interacting pair's subenergy. Hence this is the only term we need keep to study subenergy singularities. Keeping this term alone, we no longer have $\mathrm{Im}T_{2,3}$, but only the discontinuity of $T_{2,3}$ across the subenergy singularity. This could be called the absorptive part of $T_{2,3}$, $\mathrm{Abs}T_{2,3}$, (which is what we called it in the letter[1]) to stress the fact that we are no longer dividing the amplitude into real and imaginary parts but into absorptive and dispersive parts, each of which can be complex, while the absorptive part contains the appropriate physical threshold singularities. We could call it $\mathrm{Sing}T_{2,3}$

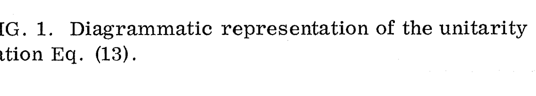

FIG. 1. Diagrammatic representation of the unitarity relation Eq. (13).

since it is the part of $T_{2,3}$ carrying the full subenergy singularity of $T_{2,3}$. This is a special feature of a square-root singularity; that the absorptive part or discontinuity is the same as the singular part. In fact, we choose to call it $\mathrm{Disc}T_{2,3}$ to stress that it is the discontinuity of $T_{2,3}$ across the subenergy cut, and that the other terms in unitarity do not contribute to that discontinuity. This name stresses the aspect we exploit in deriving the minimal separable interaction equations from unitarity and analyticity (Ref. 2 and Sec. IV), but equally important to phenomenology is the square-root singularity at the subenergy threshold, since this threshold is on the boundary of the physical region and the square root therefore represents a rapid variation not normally acknowledged in phenomenological analyses.

Keeping only the subenergy term from (13) we have

$$\mathrm{Disc}T_{2,3} = -\pi \sum_{3'} T_{2,3'}\delta(E - E_{3'})T^{\dagger}_{3',3\,\mathrm{discon}}\,. \quad (14)$$

We now flesh out (14) for three final particles of mass m_1, m_2, and m_3 ($\hbar = 1$), using the conventions of Sec. II A. For $T_{3,3'\,\mathrm{discon}}$ we have

$$\langle \vec{p}_1\alpha_1, \vec{p}_2\alpha_2, \vec{p}_3\alpha_3 | T_{3,3'\,\mathrm{discon}} | \vec{p}\,'_1\alpha'_1, \vec{p}\,'_2\alpha'_2, \vec{p}\,'_3\alpha'_3\rangle$$
$$= \sum_{\substack{ijk,\\ \mathrm{cyclic}}} (2\pi)^3\delta(\vec{p}_i - \vec{p}\,'_i)\delta_{\alpha_i,\alpha'_i}$$
$$\times \langle \vec{p}_j\alpha_j, \vec{p}_k\alpha_k | T | \vec{p}\,'_j\alpha'_j, \vec{p}\,'_k\alpha'_k\rangle\,. \quad (15)$$

For $T_{2,3}$ we take a particular form suggested by a number of approaches, sequential decay models, quasiparticle models, the Faddeev equations, and isobar models to name just a few. They are all equivalent for our purposes. In general one can always separate $T_{2,3}$ into a sum of three terms characterized by which pair interacts last. That final interaction will be through that pair's half-off-shell t matrix. If that t matrix is dominated by a particular partial wave (or by a sum of a few waves), the term ending in that pair's interaction can be written as a product of a factor depending on the initial state and on the spectator momentum times a factor depending only on the relative momentum of the interacting pair (or as a sum of such terms). The first term is the amplitude for going from the initial two-body state to a state of the spectator plus the correlated interacting pair, while the second factor represents the subsequent propagation and decay of that pair. This factor is proportional to the on-shell t matrix of the pair. In the Faddeev formalism, for example, the factor is the half-shell t matrix, but half-shell and on-shell t matrix differ only by a real factor that has no physical region singularities. Hence in the usual isobar or sequential decay phenomenology the propagation and decay factor is taken to be the on-shell pair t matrix (with certain kinematic corrections) and that is how we presented it in previous work.[1,2] In fact, there are a number of reasons for preferring to represent this factor by the two-body D function (again with kinematic correction) of the pair instead. This function can be constructed directly from the two-body scattering data, as we discussed above. It is the essential part of the pair propagator, and furthermore since $D \to 1$ for weak pair interaction while $t \to 0$, using $1/D$ rather than t allows one to include weakly interacting channels easily on the same footing as strong. The usefulness of $1/D$ to represent pairwise final state interaction in the two-body case is well known.[6] In the three-body case we shall find considerable technical advantage to its use when we come to implementing the unitarity constraint. However, since the t-matrix form is normally used in phenomenology we will give the unitarity condition in both forms. The two forms (essentially equivalent in the physical region) are

$$\langle \vec{k}, \rho | T_{2,3} | \vec{p}_1\alpha_1, \vec{p}_2\alpha_2, \vec{p}_3\alpha_3\rangle$$

$$= (2\pi)^3\delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \sum_{\substack{ijk \\ \text{cyclic} \\ l_{jk}, t_{jk}, m}} \frac{\langle \vec{k}\rho | f | \vec{p}_i\alpha_i, l_{jk}t_{jk}m\rangle}{|q_{jk}|^{l_{jk}}} \tau_{l_{jk}, t_{jk}}(q_{jk}^2/2\mu_{jk}) Y_{l_{jk}, m}(\hat{q}_{jk}) C^{t_{jk}}_{\alpha_j\alpha_k} \tag{16a}$$

$$= (2\pi)^3\delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \sum_{\substack{ijk \\ \text{cyclic} \\ l_{jk}, t_{jk}, m}} \frac{\langle \vec{k}\rho | F | \vec{p}_i\alpha_i, l_{jk}t_{jk}m\rangle |q_{jk}|^{l_{jk}}}{D_{l_{jk}, t_{jk}}(q_{jk}^2/2\mu_{jk})} Y_{l_{jk}, m}(\hat{q}_{jk}) C^{t_{jk}}_{\alpha_j\alpha_k}, \tag{16b}$$

where $\vec{k}$ is the relative momentum in the initial state and ρ represents the internal quantum numbers in that state. We work in the three-body center of mass. The $\tau_{l_{jk}, t_{jk}}$ or $D_{l_{jk}, t_{jk}}$ are the τ or D of (7) or (8). $l_{jk}t_{jk}$ are the states in which interaction of the j-k pair is important. Strictly F or f is defined by (16), but they are clearly the amplitude for going from the initial state to a final state of spectator particle i in the state $\vec{p}_i$, α_i and a correlated j-k pair in the state l_{jk}, t_{jk}. This correlated pair has total momentum $-\vec{p}_i$ and center of mass energy $q_{jk}^2/2\mu_{jk}$, which energy is related to p_i^2 by total energy conservation. The remaining factors in Eqs. 16 carry the propagation and subsequent decay of the correlated pair.

The factor of q^l in Eqs. 16 gives the decay amplitude the correct threshold behavior. It is needed in the denominator of (16a) because τ_l has a q^{2l} threshold. If the forms (16a) and (16b) are to be used far from threshold and/or if Coulomb forces are important, this factor should be replaced by the appropriate penetrability factor.

We now substitute the form (16) into the unitarity constraint (14) using the partial wave decomposition (5) in the expression for the disconnected amplitude (15). We give our discussion in terms of the usual phenomenological form, (16a), and will quote the corresponding result for (16b) at the end. Equating coefficients of $C^{t_{jk}}_{\alpha_j\alpha_k} Y_{l_{jk}, m}(\hat{q}_{jk})$ on both sides of the fleshed out unitarity expression, we obtain

$$(2\pi)^3\delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)\,\mathrm{Disc}\frac{\langle \vec{k}\rho | f | \vec{p}_i\alpha_i, l_{jk}t_{jk}m\rangle}{|q_{jk}|^{l_{jk}}} \tau_{l_{jk}, t_{jk}}\left(\frac{q_{jk}^2}{2\mu_{jk}}\right)$$

$$= -\pi \sum_{\substack{\beta' m' \\ qrs\ \text{cyclic} \\ l_{rs}t_{rs}}} \int \frac{d^3p_1' d^3p_2' d^3p_3'}{(2\pi)^9} (2\pi)^3\delta(\vec{p}_1' + \vec{p}_2' + \vec{p}_3')$$

$$\times \frac{\langle \vec{k}\rho | f | \vec{p}_q' \beta_q', l_{rs}t_{rs}m'\rangle}{|q_{rs}|^{l_{rs}}} \tau_{l_{rs}, t_{rs}}\left(\frac{q_{rs}'^2}{2\mu_{rs}}\right) Y_{l_{rs}, m'}(\hat{q}_{rs}) C^{t_{rs}}_{\beta_r'\beta_s'}$$

$$\times (2\pi)^3\delta(\vec{p}_i - \vec{p}_q')\delta_{\alpha_i, \beta_q'}(2\pi)^3\delta(\vec{p}_j + \vec{p}_k - \vec{p}_r' - \vec{p}_s') C^{t_{jk}}_{\beta_j'\beta_k'} Y^*_{l_{jk}, m}(\hat{q}_{jk})$$

$$\times \tau^*_{l_{jk}, t_{jk}}\left(\frac{q_{jk}^2}{2\mu_{jk}}\right) \delta\left(\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} - \frac{p_1'^2}{2m_1} - \frac{p_2'^2}{2m_2} - \frac{p_3'^2}{2m_3}\right), \tag{17}$$

where we have used the fact that the various δ functions force the condition $|q'_{jk}| = |q_{jk}|$. On the left hand side of (17) we use

$$\mathrm{Disc}(f\tau) = (\mathrm{Disc}f)\tau^* + f\,\mathrm{Disc}\tau = (\mathrm{Disc}f)\tau^* + f\,\mathrm{Im}\tau\ , \tag{18}$$

which follows from the original relation of Disc to Im, and from the definition of the imaginary part of a product

$$2i\,\mathrm{Im}AB = AB - A^*B^* = AB - A^*B + A^*B - A^*B^* = 2iB\,\mathrm{Im}A + 2iA^*\mathrm{Im}B\ . \tag{19}$$

In (18) we have also used $\mathrm{Disc}\tau \to \mathrm{Im}\tau$ because the only discontinuity of τ comes from its one threshold and that gives its imaginary part. There are two types of terms on the right hand side of (17). In the first $q=i$, $sr=jk$. In this term the Y_{lm} integrals are easily done by orthonormality since they are of the same argument and the $C^t_{\alpha\beta}$ sums are similarly done. One then finds that the $f\,\mathrm{Im}\tau$ term on the left cancels exactly with the $f|\tau|^2$ term on the right by two-body unitarity (6). This result that the $f\,\mathrm{Im}\tau$ term must cancel with a corresponding term on the right by two-body unitarity is general in all such calculations, identical particle or not, relativistic or not, (16a) or (16b) form, and serves as a useful check on these calculations. One is now left only with the $(\mathrm{Disc}f)\tau^*$ term on the left and the term on the right where $q \neq i$. Canceling the τ^* in both of these, one finally gets

$$\begin{aligned}\mathrm{Disc}\langle \vec{k}\rho|f|\vec{p}_i\,\alpha_i, l_{jk}t_{jk}m\rangle &= -\pi|q_{jk}|^{l_{jk}}\int\frac{d^3q'_{jk}}{(2\pi)^3}\,\delta\left(\frac{q_{jk}^2-q'^2_{jk}}{2\mu_{jk}}\right)\\ &\times\sum_{\substack{m',\beta_j\beta_k\\ l_{ik}t_{ik},l_{ij}t_{ij}}}\left[\frac{\langle\vec{k}\rho|f|\vec{p}'_j\beta_j, l_{ik}t_{ik}m'\rangle}{|q'_{ik}|^{l_{ik}}}\,\tau_{l_{ik},t_{ik}}\left(\frac{q'^2_{ik}}{2\mu_{ik}}\right)\right.\\ &\times Y_{l_{ik},m'}(\hat{q}'_{ik})C^{t_{ik}}_{\alpha_i\beta_k}+\frac{\langle\vec{k}\rho|f|\vec{p}'_k\beta_k, l_{ij}t_{ij}m'\rangle}{|q'_{ij}|^{l_{ij}}}\\ &\left.\times\tau_{l_{ij},t_{ij}}\left(\frac{q'^2_{ij}}{2\mu_{ij}}\right)Y_{l_{ij},m'}(\hat{q}_{ij})C^{t_{ij}}_{\alpha_i\beta_j}\right]C^{t_{jk}}_{\beta_j\beta_k}\,Y^*_{l_{jk},m}(\hat{q}'_{jk})\ ,\end{aligned} \tag{20a}$$

where in evaluating $\vec{p}'_j$ and $\vec{p}'_k$ it is realized that $\vec{p}_i$ is fixed and $\vec{p}'_j+\vec{p}'_k=\vec{p}_i$, while $\vec{q}'_{jk}$ is varied. Equation (20a) shows that $\langle\vec{k}\rho|f|\vec{p}_i\alpha_i, l_{jk}t_{jk}m\rangle$, the quasi-two-body amplitude for producing particle i and the correlated j-k pair, has a rapid dependence on the j-k pair subenergy $E_{jk}=q_{jk}^2/2\mu_{jk}$, which is related to p_i^2 by total energy conservation according to

$$E = p_i^2\left[\frac{1}{2m_i}+\frac{1}{2(m_j+m_k)}\right]+E_{jk}\ . \tag{21}$$

In fact, f has a $(E_{jk})^{1/2}$ singularity. The strength of that singularity depends on the "non-i" parts of the amplitude, that is, on the $f_j\tau_{ik}$ and $f_k\tau_{ij}$ terms. Hence it is important when a number of different final state pair interactions are strong in overlapping parts of the final state phase space.

If one begins with (16b), an exactly parallel argument can be made to give

$$\begin{aligned}\mathrm{Disc}\langle\vec{k}\rho|F|\vec{p}_i\,\alpha_i, l_{jk}t_{jk}m\rangle &= \frac{-\pi N_{l_{jk}t_{jk}}(q_{jk}^2/2\mu_{jk})}{|q_{jk}|^{l_{jk}}}\int\frac{d^3q'_{jk}}{(2\pi)^3}\,\delta\left(\frac{q_{jk}^2-q'^2_{jk}}{2\mu_{jk}}\right)\\ &\times\sum_{\substack{m'\beta_j\beta_k\\ l_{ik}t_{ik},k_{ij}t_{ij}}}\left[\frac{\langle\vec{k}\rho|F|\vec{p}'_j\beta_j, l_{ik}t_{ik}m'\rangle}{D_{l_{ik},t_{ik}}(q'^2_{ik}/2\mu_{ik})}|q'_{ik}|^{l_{ik}}\,Y_{l_{ik},m'}(\hat{q}'_{ik})C^{t_{ik}}_{\alpha_i\beta_k}\right.\\ &\left.+\frac{\langle\vec{k}\rho|F|\vec{p}'_k\beta_k, l_{ij}t_{ij}m'\rangle|q'_{ij}|^{l_{ij}}Y_{l_{ij},m'}(\hat{q}_{ij})}{D_{l_{ij},t_{ij}}(q'^2_{ij}/2\mu_{ij})}\,C^{t_{ij}}_{\alpha_i\beta_j}\right]C^{t_{jk}}_{\beta_j\beta_k}\,Y^*_{l_{kj},m}(\hat{q}'_{jk})\ ,\end{aligned} \tag{20b}$$

where all symbols and the importance of singularities are the same as given following (20a). Equation (20) represents a set of integral constraints on the singular parts of the f_i's or F_i's, the nonsingular parts being arbitrary; that is, determined by dynamics and analyticity but not fixed by the unitarity constraints. The constraints can be used to check the unitarity of any particular parametrization of a final state. Since most phenomenological studies treat the various f's or F's as independent and constant, while unitarity requires that they be neither, there must be a number of cases of overlapping interactions where the usual analysis is wrong. If one is given a theoretically generated (or guessed at) set of nonsingular f_i's and the two-body τ's (or F's and D's), one can use (20) as a set of integral equations to determine the additional singular parts of f_i or F_i required by unitarity. The importance of the unitarity correction will depend on the relative strength of the non-i final state interaction. Unfortunately, it is not simple to guess the nonsingular parts of f_i or $\boldsymbol{F}_i$. In particular, they are not just the Born term or equivalent. In order to fully implement the unitarity constraint (20) one needs to add analyticity. This is discussed in Sec. IV B. What goes wrong with using a simple guess for the nonsingular part directly in (20) and using it as an integral equation for the singular part is discussed in Sec. IV A.

We shall also need the result corresponding to (20) for bosons and for fermions. These are explicitly derived in an Appendix, but in fact they are nearly the same as (20) except for an important sign change in the fermion case.

III. SIMPLE EXAMPLE—THREE BOSONS

The unitarity constraint we have derived Eq. (20) is complicated in its general form and hence difficult to analyze. In order to see its content clearly, let us consider the special case of a final state of three spinless bosons. Let us further assume that their only important interaction is in S waves and that neither the bosons nor the initial state have any internal quantum numbers. For identical particles ($2m=1$) there is only one spectator function f or F and (20a) and (20b) become (see the appendix for an explicit treatment of identical particles):

$$\mathrm{Disc}\langle \vec{k}|f|\vec{p}\rangle = -\frac{1}{2}\int \frac{d^3p^1}{(2\pi)^3}\langle \vec{k}|f|\vec{p}'\rangle\,\tau(E-\tfrac{3}{2}p'^2)\delta(E-2p^2-2p'^2-2\vec{p}\cdot\vec{p}') \tag{21a}$$

and

$$\mathrm{Disc}\langle \vec{k}|F|\vec{p}\rangle = -\tfrac{1}{2}N(E-\tfrac{3}{2}p^2)\int \frac{d^3p'}{(2\pi)^3}\frac{\langle \vec{k}|F|\vec{p}'\rangle}{D(E-\tfrac{3}{2}p'^2)}\,\delta(E-2p^2-2p'^2-2\vec{p}\cdot\vec{p}')\,, \tag{21b}$$

where we have suppressed a number of unneeded labels. In particular τ, N, and D all refer to two-body S-wave scattering. The content of Eq. (21) is clarified if we make a partial wave decomposition according to (we take the a case)

$$\langle\vec{k}|f|\vec{p}\rangle = \sum_{lm} Y_{lm}(\hat{k})\langle k|f_k|p\rangle\, Y^*_{lm}(\hat{p})\,. \tag{22}$$

l corresponds to the total three-body angular momentum since the correlated pair has zero angular momentum. Equation (21a) then becomes

$$\mathrm{Disc}\langle k|f_l|p\rangle = -\pi\int\frac{p'^2dp'}{(2\pi)^3}\langle k|f_l|p'\rangle\tau(E-\tfrac{3}{2}p'^2)\times\int_{-1}^{1}dz\,P_l(z)\delta(E-2p^2-2p'^2-2pp'z) \tag{23}$$

in terms of the Legendre polynomial $P_l(z)$. Clearly the δ function can be used to do the z integral, but since $-1\le z\le 1$, there are restrictions on the p' range for fixed E and p. These are most easily expressed by changing variables to $p^2=x$ and $p'^2=y$. Let us call $\langle k|f_l|p\rangle = f_l(E,x)$. Equation (23) then becomes

$$\mathrm{Disc}f_l(E,x) = -\frac{1}{32\pi^2x^{1/2}}\int_{y_-}^{y_+}dy\,f_l(E,y)\tau(E-\tfrac{3}{2}y)\times P_l\left(\frac{E-2x-2y}{2(xy)^{1/2}}\right)\,, \tag{24}$$

where $y_\pm=\tfrac{1}{2}\{E-x\pm[2x(E-\tfrac{3}{2}x)]^{1/2}\}$. These limits on y are just those allowed by phase space. One sees clearly that (24) gives f a singular part at a given x in terms of an integral over f at different values. For example, for x near zero, y values near $\tfrac{1}{2}E$ contribute, while for x near $\tfrac{2}{3}E$, y near $\tfrac{1}{6}E$ contributes. Since as x approaches $\tfrac{2}{3}E$ the integration region shrinks to zero, $\mathrm{Disc}f_l(E,x)$ goes like $(E-\tfrac{3}{2}x)^{1/2}$ in this region. This is the square-root branch point we have stressed. $E-\tfrac{3}{2}x=0$ is precisely the two-body threshold of the pair associated with the "x" spectator. The integration region also shrinks to zero as $x\to 0$ like

$x^{1/2}$, but because of the $x^{-1/2}$ in front $f_l(E,x)$ is finite at $x=0$ and has no singularity there. In fact the branch cut associated with (24) runs from $E-\frac{3}{2}x=0$ to $E-\frac{3}{2}x=\infty$ or from $x=\frac{2}{3}E$ to $x=-\infty$, just as we would expect for a two-body scattering cut, rather than from $x=\frac{2}{3}E$ to $x=0$, as one might naively expect from phase-space considerations only. In the physical region $(0\leq x\leq\frac{2}{3}E)$, (23) or (24) may be used to calculate the discontinuity across the cut; for negative x these expressions do not apply and some form of analytic continuation in needed to get the discontinuity or singular part.

We see that (21) or (24) require the spectator function to have a singular or rapidly varying part not normally included in phenomenological analyses. The size of the singular part depends on the importance of the final pair scattering. Just as with (20), (21) can be used to check the importance of the unitarity-generated discontinuity or singular part by substituting the assumed nonsingular part on the right and seeing how much discontinuity it generates. This is just like using (4) or (6) in the two-body case with a purely real t matrix on the right. If that purely real t matrix is small, it will generate very little imaginary part and the assumption that is is purely real and has no $\epsilon^{1/2}$ branch cut is justified. If the purely real assumption on the right generates a large imaginary part, unitarity is violated. Of course, in the two-body case we know that the phase-shift parametrization (7) will satisfy unitarity for any δ, but we do not yet know how to parametrize the spectator function so that (20) or (21) is automatically satisfied. We must therefore take a more complex route to implement unitarity.

IV. IMPLEMENTATION

In this section we discuss ways of implementing the unitarity constraint (20) or (21). To keep the algebra simple we shall concentrate on the three-boson example of Sec. III. We shall first discuss implementation without analyticity and then with analyticity.

A. Without analyticity

Equation (24) implies that in the physical region we can write

$$f_l(E,x)=A_l(E,x)+i\,(E-\tfrac{3}{2}x)^{1/2}B_l(E,x)\ , \qquad (25)$$

with A and B analytic in x in the physical domain $0\leq x\leq\frac{2}{3}E$. In terms of (25), (24) becomes

$$32\pi^2B_l(E,x)[x(E-\tfrac{3}{2}x)]^{1/2}=-\int_{y-}^{y+}dy\,A_l(E,y)\tau(E-\tfrac{3}{2}y)P_l\left(\frac{E-2x-2y}{2(xy)^{1/2}}\right)$$
$$-i\int_{y-}^{y+}dy\,B_l(E,y)(E-\tfrac{3}{2}y)^{1/2}\tau(E-\tfrac{3}{2}y)P_l\left(\frac{E-2x-2y}{2(xy)^{1/2}}\right)\ . \qquad (26)$$

For a given $A_l(E,x)$ this is a Fredholm integral equation for $B_l(E,x)$. [It is Fredholm because the kernel is a finite function and the domain of integration is finite.] Since we are assuming the two-body on-shell amplitudes τ are known, we need only give A to obtain B. The usual phenomenological guess for f is a constant; one might then think that $A=$constant would be the corresponding correct guess. Although any choice of A will generate a B by (26) that satisfies unitarity, only if the unitarity effect is small (that is, only if the Neumann series generated by iteration of (26) converges rapidly) will the choice $A=$constant satisfy analyticity (approximately) as well. So far we do not have enough experience with solving (26) to know if the Neumann series to (26) always converges, but clearly there are cases when it will not converge rapidly. $A_l(E,x)$ and $(E-\frac{3}{2}x)^{1/2}\times B_l(E,x)$ are essentially the real and imaginary parts (really the dispersive and absorptive parts) of an analytic function in the cut plane with the cut in x going from $\frac{2}{3}E$ to $-\infty$. These two parts are related by a dispersion relation and by the choice of driving terms, that is, by subtraction. A simple choice of these, in general, does not correspond to a simple choice for A and B.

The relation of A, B, and analyticity can be seen in a simple two-body example. Consider a two-body process in which a weak initial channel leads to a strong one.[3] Classic examples are photo-pion production $(\gamma+N\rightarrow\pi+N)$ in particle physics and photodisintegration of the deuteron $(\gamma+d\rightarrow n+p)$ in nuclear physics. Let t_{ab} be the amplitude for this process in an appropriate eigen channel where b is the strong process. Assume further that only the a and b channels are energetically allowed. Unitarity applied to t_{ab} gives schematically (to lowest order in the weak process)

$$\text{Im}\,t_{ab}=(\sigma)^{1/2}\,t_{ab}t^*_{bb}\ , \qquad (27)$$

where $(\sigma)^{1/2}$ is the relative momentum in the strong channel (σ is the corresponding kinetic energy) and comes from the phase space. The elastic t matrix is normalized to

$$t_{bb}=\frac{e^{i\delta}\sin\delta}{(\sigma)^{1/2}}. \tag{28}$$

Equation (27) says that t_{ab} can be written

$$t_{ab}=M_{ab}e^{i\delta}\ , \tag{29}$$

where M_{ab} is real. Any choice of M_{ab} will satisfy the algebraic constraint (27). A common choice is

$$M_{ab}=N_{ab}\sin\delta\ , \tag{30}$$

where N_{ab} is real, and constant or slowly varying. This is often called Watson's theorem, but it clearly is neither a theorem nor generally valid. In fact, (27) implies that t_{ab} has a square-root branch cut in σ starting at $\sigma=0$. One might try to exploit this by writing

$$t_{ab}=A'_{ab}+i(\sigma)^{1/2}B'_{ab}\ , \tag{31}$$

where A' and B' are real, that is they have no unitarity cut. This corresponds to (25). Putting this in (27) with (28) gives

$$t_{ab}=A'_{ab}e^{i\delta}/\cos\delta\,. \tag{32}$$

This form is superior to (30) for weak interactions where it gives $t_{ab}\to A'_{ab}$ rather than $t_{ab}\to 0$, but it clearly is inadequate in other cases, for example in the case of a final state resonance where it gives not the usual Breit-Wigner form, but rather $t_{ab}\to\infty$ on resonance ($\delta=\frac{1}{2}\pi$). Of course A'_{ab} can be chosen in (32) to make (32) exactly correct in any theory, just as $A_l(E,x)$ can be chosen in (26) to make it correct. The point is that those cases where in the limit of weak final state interaction we expect t_{ab} in the two-body case or f in the three-body case to go over to a simple form, for example constant, do not correspond to choosing A in (25) or A'_{ab} in (31) to be correspondingly simple in the presence of strong final state interactions. Rather, they correspond to a simple choice of driving term in a dispersion relation. To study this we turn to implementation of the unitarity constraint including analyticity.

B. With analyticity

In the two-body example we wish to exploit the fact that unitarity does not just give the imaginary part of t_{ab}, but tells us that in fact t_{ab} has a branch cut in σ from 0 to ∞, with discontinuity given by (27). It may have other singularities in σ, but those are not directly required by unitarity. If we assume $t_{ab}(\sigma)$ goes to zero sufficiently rapidly as $\sigma\to\infty$, we can exploit this information to write a dispersion relation (really Cauchy's theory)

$$t_{ab}(\sigma)=A'_{ab}(\sigma)+\frac{1}{\pi}\int_0^\infty\frac{d\sigma'\,\mathrm{Im}t_{ab}(\sigma')}{\sigma'-\sigma}\ , \tag{33}$$

where the driving term A'_{ab} does now represent any simple ideas we may have about the primitive mechanism for the transition that is $t_{ab}\to A'_{ab}$ as the rescattering is turned off. If we substitute (27) and (28) in (33) we get the well-known singular integral equation whose solution has been given by Muskhelishili and Omnes.[6] The usual formulation of the problem, Eq. (33), corresponds to including only the unitarity cut in t_{ab} and neglecting left-hand cuts (except those in A') that, though present in a Schrödinger treatment of the process, are not required by unitarity. It is also easy to see that with A'_{ab} chosen as constant in (33), the real part of the solution for t_{ab} will not be constant if the final state interactions are important, which is why choosing A'_{ab} as constant in (31) and (32) will not work.

These same considerations apply to the three-body case. Equation (26) can be used to implement the unitarity constraint, but the choice of $A_l(E,x)$ is difficult unless the final state interactions are unimportant. To employ simple choices of driving terms, one must include analyticity as we did in (33) for the two-body case. As we have seen, unitarity forces $\langle k|f_l(E)|p\rangle$ to have a cut in p^2 in the interval $-\infty<p^2\le\frac{2}{3}E$. If we assume f has no other cuts, in particular if we assume it has no left-hand cuts in the subenergy σ due to potential contributions [just as we assumed no left-hand cuts in the two-body case to write (33)], and if we assume f goes to zero sufficiently rapidly at large p^2 to drop the contour at infinity, we can use Cauchy's theorem to write the dispersion relation

$$\langle k|f_l(E)|q\rangle=\langle k|R_l(E)|q\rangle-\frac{1}{\pi}\int_{(2/3)E}^{-\infty}\frac{dp^2}{p^2-q^2}\mathrm{Disc}\langle k|f_l(E)|p\rangle\ , \tag{34}$$

where $\langle k|R_l(E)|q\rangle$ is the l-wave projection of the driving term, the part of f that has no unitarity cut. There are two technical impediments to the direct use of (23) in (34). First, as we discussed in Sec. III, while (34) requires the discontinuity for $-\infty<p^2\le\frac{2}{3}E$, (23) is only valid for $0\le p^2\le\frac{2}{3}E$. Hence some analytic continuation is required. A simple way to do that continuation is to substitute (23) into (34) and interchange the order of the p' and p^2 integration. For small p' the argument of the δ function will have its zeros for $0\le p^2\le\frac{2}{3}E$. Having done the p^2 integral for small p', the expression may be continued to all p'. Equation (34)

then becomes

$$\langle k|f_l(E)|q\rangle = \langle k|R_l(E)|q\rangle + \frac{1}{2}\int \frac{p'^2 dp'}{(2\pi)^3}\langle k|f_l(E)|p'\rangle \tau(E-\tfrac{3}{2}p'^2)B_l(q,p')\ , \qquad (35)$$

where

$$B_l(q,p') = \int_{(2/3)E}^{-\infty} \frac{dp^2}{p^2-q^2}\int_{-1}^{1} dz\, P_l(z)\delta(p^2+p'^2+pp'z-\tfrac{1}{2}E)\ . \qquad (36)$$

The argument of the δ function has roots at

$$p_\pm = \tfrac{1}{2}\{-p'z \pm [2E-p'^2(4-z^2)]^{1/2}\} \equiv \tfrac{1}{2}(-p'z\pm\gamma). \qquad (37)$$

It should be recalled that E has a small positive imaginary part, and under these circumstances only the root with the positive square root is on the first sheet of the p^2 plane. Using the standard rules for doing the integral of a δ function, we get for (36)

$$B_l(q,p') = -\int_{-1}^{1} \frac{P_l(z)(-p'z+\gamma)dz}{\gamma[q^2-\frac{1}{4}(-p'z+\gamma)^2]}\ . \qquad (38)$$

For even l, $P_l(z)=P_l(-z)$, and hence for even l we can symmetrize the integrand in (38). A few lines of algebra and the definition of γ from (37) then give

$$B_l(q,p') = \int_{-1}^{1} \frac{P_l(z)dz}{\frac{1}{2}E-q^2-p'^2-qp'z} \qquad (39)$$

for even l. For odd l we came to our second technical impediment. The partial wave projection makes $\langle k|f_l(E)|q\rangle$ an even function of q for even l, but for odd l it is odd in q [recall that at threshold $\langle k|f_l(E)|q\rangle \sim q^l$]. Hence (34) is not a correct form for the analytic structure of the spectator function for odd l, since then $\langle k|f_l(E)|q\rangle$ has an additional kinematic branch cut in q^2 in the interval $0\le q^2<\infty$. This difficulty is circumvented by studying $q\langle k|f_l(E)|q\rangle$ for odd l. The extra factor of q removes the kinematic cut. The amplitude $q\langle k|f_l(E)|q\rangle$ can be treated by Cauchy's theorem and after interchanging the p' and p^2 integrals, using the δ function, and exploiting the fact that $P_l(z)=-P_l(-z)$ for odd l, one gets after some algebra precisely Eq. (35) with B_l, as in (39). Hence the result is true for even and odd l. The partial wave projection may now be undone to give the linear integral equation

$$\langle \vec{k}|f(E)|\vec{q}\rangle = \langle\vec{k}|R(E)|\vec{q}\rangle + \int \frac{d^3p'}{(2\pi)^4}\frac{\langle\vec{k}|f(E)|\vec{p}'\rangle\tau(E-\frac{3}{2}p'^2)}{E-2p'^2-2q^2-2\vec{p}'\cdot\vec{q}}\ . \qquad (40)$$

This is just the usual separable interaction linear integral equation for the spectator function in the special case of unit vertex function. The vertex functions were excluded when the assumption of no left-hand cuts in the subenergy was made to write the dispersion relation of Eq. (34). The denominator of the integrand in (40) is just the well-known particle exchange denominator. Equation (40) is represented graphically in Fig. 2. $\langle\vec{k}|R(E)|\vec{q}\rangle$ is the driving term for the process. It is easy to see that in a simple $2\to3$ (or a $1\to3$) decay process, the usual Born term has the correct features, in particular no unitarity cut, to be $\langle\vec{k}|R(E)|\vec{q}\rangle$ so long as the two-body state in the incident channel is stable. Since no mention has been made so far of Schrödinger dynamics, that state can be a bound state or an "elementary" particle. If $\tau(\epsilon)$ in (40) has a stable two-body state pole, (40) can also be continued in the usual way to represent two-body elastic scattering from that state, as well as the breakup process. If the two-body state in τ comes from an elementary particle, Eq. (40) is a Fredholm equation, but if it is a bound state in a potential it is not, since in that case the appropriate traces of the kernel do not converge at the upper limit.[11] In the usual Schrödinger theory the vertex functions provide that convergence. One might try to correct this deficiency by introducing a subtraction in the dispersion relation (34). A useful point to make that subtraction would be at the pair subenergy corresponding to the two-body bound state. One would then have a Fredholm integral equation for the breakup in which the elastic scattering for the stable two-body bound state appears as part of the inhomogeneous term. This would allow one to examine explicitly the question of what additional information is contained in the breakup process over elastic scattering. Calculations to date with separable potentials for the n-d system

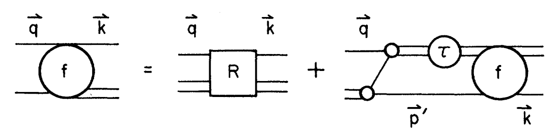

FIG. 2. Diagrammatic representation of the dynamical equation implementing unitarity and analyticity, Eq. (40).

would lead one to believe that the answer is "very little."[7] An alternative would be to use (40) to generate the Neumann series. For reasonable forms of $\langle\vec{k}|R(E)|\vec{q}\rangle$ the integrals in that case all converge, even without vertex functions. One could then regroup that series by the method of Padé approximates. Clearly much work remains to be done to investigate the usefulness of this and related approaches.

It is interesting that implementation of unitarity and analyticity leads directly to a linear scattering integral equation so close in form to the equation generated by the Schrödinger-Lippmann-Schwinger-Faddeev approach. Of course (40) requires only on-shell information, as does any unitarity-analyticity approach, but it does require $\tau(\epsilon)$ for negative ϵ, that is on-shell but for negative energies. This is familiar in the Faddeev approach. Its occurrence here reflects the fact that this "unphysical region" information is required in the three-body problem by the most general principle of quantum mechanics. One might try to improve convergence in (40) and include more physical information by taking $t(\epsilon)$ to have both a right-hand unitarity and a left-hand potential cut, since it was only f, not τ, that we assumed not to have such a cut. But in fact this will spoil the total energy analyticity of (40). It is remarkable that we have come so far as (40) with only subenergy analyticity and unitarity, but now we must invoke total energy analyticity. If we allow $t(\epsilon)$ to have a left-hand cut then because of the way τ appears in (40), the integrals will run over that cut and will introduce a complex component to the integral, even if $E<0$. But clearly for E below the lowest scattering threshold the spectator amplitudes must be purely real and hence $t(\epsilon)$ is not allowed to have a left-hand cut. Hence in terms of this parametrization consistency (i.e. E analyticity) requires that if $\langle k|f_1|q\rangle$ has no left-hand cut, $\tau(\epsilon)$ must not have any either.

Many of these problems can be avoided by using the parametrization for decay of (16b), which involves not $\tau(\epsilon)$ but $D(\epsilon)$. This leads to the discontinuity relation (20b) or, in terms of the simple three-boson case, (21b). We make a partial wave decomposition of (21b) as in (23), assume the cut structure for $\langle k|F_l(E)|q\rangle$ required by unitarity alone, just as in (34) (at least for even l), and arrive at the integral equation (corresponding to 35)

$$\langle k|F_l(E)|q\rangle=\langle k|R_l(E)|q\rangle+\frac{1}{2}\int\frac{p'^2dp'}{(2\pi)^3}\frac{\langle k|F_l(E)|p'\rangle}{D(E-\frac{3}{2}p'^2)}\tilde{B}_l(q,p'),\tag{41}$$

where for even l we can write, corresponding to (36),

$$\tilde{B}_l(q,p')=\frac{1}{2}\int_{-1}^{1}dz\,P_l(z)\int_{(2/3)E}^{-\infty}\frac{dp^2}{p^2-q^2}N(E-\tfrac{3}{2}p^2)[\delta(p^2+p'^2+pp'z-\tfrac{1}{2}E)+\delta(p^2+p'^2-pp'z-\tfrac{1}{2}E)].\tag{42}$$

We can explicitly do the p^2 integral to get

$$\tilde{B}_l(q,p')=-\frac{1}{2}\int_{-1}^{1}dz\,P_l(z)\Big|\frac{1}{\gamma}\left[\frac{-p'z+\gamma}{q^2-\frac{1}{4}(-p'z+\gamma)^2}N(E-\tfrac{3}{8}(-p'z+\gamma)^2)+\frac{p'z+\gamma}{q^2-\frac{1}{4}(p'z+\gamma)^2}N(E-\tfrac{3}{8}(p'z+\gamma)^2)\right].\tag{43}$$

We want to be able to continue (43) to all p' or to study it for $E<0$; in both cases we seemingly encounter trouble because γ has a branch point and hence becomes complex. This is the same difficulty we found before when we used τ without an N. In fact, it is not really there. Because of the symmetrization in z, the quantity in square brackets in (43) is actually only a function of γ^2, hence it may be continued to large p' and/or to negative E without fear of meeting spurious branch points or, what is the same thing, spurious imaginary parts. This cancellation of terms odd in γ is most easily effected when the explicit analytic form of $N(\epsilon)$ is known. It cannot be done if N is only known numerically for real positive ϵ. But if $t(\epsilon)$, the on-shell two-body t matrix, is known for all ϵ, $N(\epsilon)$ can also be determined for negative ϵ. Since $N(\epsilon)$ has only a left-hand cut, we can then write

$$N(\epsilon)=\frac{1}{\pi}\int_{-\infty}^{-\mu^2}\frac{\rho(y)dy}{y-\epsilon},\tag{44}$$

where ρ is the discontinuity of N across the left-hand cut and is known if $\tau(\epsilon)$ is known for all negative E. $-\mu^2$ is the "threshold" of that cut. In terms of (44), (42) becomes

$$\tilde{B}_l(q,p')=\frac{1}{2\pi}\int_{-1}^{1}dz\,P_l(z)\int_{(2/3)E}^{-\infty}\frac{dp^2}{p^2-q^2}\int_{-\infty}^{-\mu^2}\frac{\rho(y)dy}{y+\frac{3}{2}p^2-E}[\delta(p^2+p'^2+pp'z-\tfrac{1}{2}E)+\delta(p^2+p'^2-pp'z-\tfrac{1}{2}E)].\tag{45}$$

The denominators in (45) can be split by partial fractions according to

$$\frac{1}{(p^2-q^2)(y+\frac{3}{2}p^2-E)}=\frac{2}{3}\left(\frac{1}{p^2-q^2}-\frac{1}{p^2-\frac{2}{3}(E-y)}\right)\left(\frac{1}{q^2-\frac{2}{3}(E-y)}\right). \tag{46}$$

Using (46) in (45) with (44) and the techniques used to obtain (39), we get

$$\tilde{B}_l(q,p')=\int_{-1}^{1}dz\,P_l(z)\left[\frac{N(E-\frac{3}{2}q^2)}{\frac{1}{2}E-p'^2-q^2-p'qz}+\frac{1}{\pi}\int_{-\infty}^{-\mu^2}\frac{dy\,\rho(y)}{(E-\frac{3}{2}q^2-y)(\frac{2}{3}y-p'^2-\frac{1}{6}E-p'[\frac{2}{3}(E-y)]^{1/2}z)}\right], \tag{47}$$

which shows explicitly that the continuation to large p' causes no problems. Equation (47) seems to contain new problems. For positive or negative E, q^2 runs from 0 to ∞ and $N(E-\frac{3}{2}q^2)$ in the first term will require N on its cut and can therefore introduce problems. Similarly, the $E-\frac{3}{2}q^2-y$ denominator in the second integral of (47) can vanish for E positive or negative. In fact, these two difficulties exactly cancel. They both arise from the common denominators factor in (46) of $E-\frac{3}{2}q^2-y$, which in turn arises from the partial fractions decomposition. In fact, (46) is not singular when this denominator vanishes, because the terms in the brackets cancel them. Hence there are no singularities or imaginary parts arising in (47) at that point, but rather they cancel. The second denominator in the second integral in (47) cannot vanish if $E>\mu^2$ since $y<-\mu^2$. There is also no singularity associated with the square root in this factor so long as $E>-\mu^2$. For $E<-\mu^2$ these terms will produce singularities and hence complex parts, but these are expected since now we are considering E more negative than the range of the force, and even the on-shell two-body t matrix becomes complex there.

Equation (47) is valid for all l just as (39) is, and may be used in (41). The partial wave projection is not simply "undone" in this case because in one term of (47) we have $p'qz$, but in the other $p'[\frac{2}{3}(E-y)]^{1/2}z$; but one can write these as $\vec{p}'\cdot\vec{q}$ and $\vec{p}'\cdot\hat{q}[\frac{2}{3}(E-y)]^{1/2}$, where $\hat{q}$ is a unit vector, and then one obtains

$$\langle\vec{k}|F(E)|\vec{q}\rangle=\langle\vec{k}|R(E)|\vec{q}\rangle+\int\frac{d^3p'}{(2\pi)^4}\frac{\langle\vec{k}|F(E)|\vec{p}'\rangle}{D(E-\frac{2}{3}p'^2)}\times\left[\frac{N(E-\frac{3}{2}q^2)}{E-2p'^2-2q^2-2\vec{p}'\cdot\vec{q}}-\frac{1}{2\pi}\int_{-\infty}^{-\mu^2}\frac{dy\,\rho(y)}{(E-\frac{3}{2}q^2-y)(\frac{2}{3}y-p'^2-\frac{1}{6}E-\vec{p}'\cdot\hat{q}[\frac{2}{3}(E-y)]^{1/2})}\right]. \tag{48}$$

Equation (48), or the set of (41) with (47), forms the new set of integral equations for the spectator amplitudes. For reasonable choice of N they will be Fredholm, that is, have convergent traces. Just as in the other case their convergence can be increased by making a subtraction, for example to relate the three-body amplitude to the elastic scattering of one from a bound state of two. Equation (48) is not so simple as (40), but it contains more places for input and hopefully does a better job of describing the physics. The true usefulness of either form must clearly await a study of numerical examples.

V. DISCUSSION

We have seen how the usual Watson approximation for the treatment of three-body final states fails when there is significant overlap of the pairwise final state interaction. This failure can be both exhibited and quantified by using only unitarity. We have also shown how the usually neglected constraints of unitarity can be implemented through analyticity to give a set of coupled linear integral equations for the final state amplitudes. The equations are constructed using only on-shell information and the cut structure required by unitarity and hence are an embodiment of the minimal constraints required by the general principles of quantum mechanics. At the same time the equations are very similar in form to the separable interaction equations for the three-body problem. Thus the separable interaction equations, rather than being detailed dynamical equations, are barely more than this minimum. Given the remarkable success of the separable interaction approach to a wide range of three-body situations, it seems that these situations are little more than manifestations of these minimal constraints and hence contain little detailed dynamical insight. This point has been made before,[7] but our work here casts new light on it as well as illuminating the content of the separable interaction approach. From a theoretical point of view, it is also remarkable that implementation of unitarity and analyticity, the two central pillars of S-matrix theory, should lead to a linear scattering integral equation of the Lippmann-Schwinger type. Technically this arises because we study subenergy analyticity and hence obtain linear unitarity relations. If we had considered total energy analyticity, we would have found nonlinear unitarity constraints and all the complexity they imply.

The usefulness of the methods we have presented here remains to be tested. The unitarity constraint can be tested directly on existing parametrizations to check their validity. For example, the recent large study of $\pi+N\rightarrow\pi+\pi+N$ was made neglecting the unitarity constraint.[12] That neglect is probably unjustified and we are presently testing their amplitudes in our unitarity equations.[13] More interesting is the question of using the integral equations to generate a new phenomenology, particularly using the freedom to make subtractions so that breakup and elastic scattering can be united. Here the question of usefulness depends on the success of trial calculations and favorable cases. We are presently embarking on a program of doing such calculations. Hopefully others will as well.

It is clear that though our analysis has been given in the language of the three-body case, the general results, the existence of square-root subenergy singularities due to unitarity, the possibility of writing linear integral equations to implement the unitarity constraint, etc., all also apply to four or more particles. It would be interesting to study overlapping final state interactions in these cases.

APPENDIX: UNITARITY CONSTRAINT FOR IDENTICAL PARTICLES

For a final state involving three identical particles ($\hbar=2m=1$) the decomposition corresponding to (16) may be written

$$\langle\vec{k}\rho|T_{2,3}|\vec{p}_1\alpha_1,\vec{p}_2\alpha_2,\vec{p}_3\alpha_3\rangle=(2\pi)^3\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3)\frac{1}{2}\sum_{\substack{ijk\\lmt}}\frac{\langle\vec{k}\rho|f|\vec{p}_i\alpha_i,ltm\rangle}{|q_{jk}|^l}\tau_{l,t}(2q_{jk}^2)C^t_{\alpha_j\alpha_k}Y_{lm}(\hat{q}_{jk}) \tag{A1a}$$

$$=(2\pi)^3\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3)\frac{1}{2}\sum_{\substack{ijk\\lmt}}\frac{\langle\vec{k}\rho|F|\vec{p}_i\alpha_i,ltm\rangle}{D_{l,t}(2q_{jk}^2)}|q_{jk}|^lC^t_{\alpha_j\alpha_k}Y_{lm}(\hat{q}_{jk}) \tag{A1b}$$

for bosons,

$$\langle\vec{k}\rho|T_{2,3}|\vec{p}_1\alpha_1,\vec{p}_2\alpha_2,\vec{p}_3\alpha_3\rangle=(2\pi)^3\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3)\frac{1}{2}\sum_{\substack{ijk\\lmt}}\epsilon_{ijk}\frac{\langle\vec{k}\rho|f|\vec{p}_i\alpha_i,ltm\rangle}{|q_{jk}|^l}\tau_{l,t}(2q_{jk}^2)C^t_{\alpha_j\alpha_k}Y_{lm}(\hat{q}_{jk}) \tag{A1c}$$

$$=(2\pi)^3\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3)\frac{1}{2}\sum_{\substack{ijk\\lmt}}\epsilon_{ijk}\frac{\langle\vec{k}\rho|F|\vec{p}_i\alpha_i,ltm\rangle}{D_{l,t}(2q_{jk}^2)}|q_{jk}|^lY_{lm}(\hat{q}_{jk})C^t_{\alpha_j\alpha_k} \tag{A1d}$$

for fermions, where it is convenient to take the sums to run over all ijk and the factor of $\frac{1}{2}$ has been introduced to keep the f and F normalized as in (16). With our convention for identical particles, $T_{3,3\,\mathrm{discon}}$ can be written

$$\langle\vec{p}_1\alpha_1,\vec{p}_2\alpha_2,\vec{p}_3\alpha_3|T_{3,3\,\mathrm{discon}}|\vec{p}_1'\alpha_1',\vec{p}_2'\alpha_2',\vec{p}_3'\alpha_3'\rangle$$

$$=\frac{1}{12}\sum_{\substack{ijk\\abc\\ltm}}(2\pi)^3\delta(\vec{p}_a-\vec{p}_c')(2\pi)^3\delta(\vec{p}_b+\vec{p}_c-\vec{p}_j'-\vec{p}_k')\delta_{\alpha_a,\alpha_i}Y^*_{lm}(\hat{q}_{bc})C^t_{\alpha_b\alpha_c}\tau_{l,t}(2q_{jk}^2)Y_{lm}(\hat{q}_{jk})C^t_{\alpha_j\alpha_k} \tag{A2a}$$

for bosons,

$$=\frac{1}{12}\sum_{\substack{ijk\\abc\\ltm}}(2\pi)^3\delta(\vec{p}_a-\vec{p}_i')(2\pi)^3\delta(\vec{p}_b+\vec{p}_c-\vec{p}_j'-\vec{p}_k')\delta_{\alpha_a,\alpha_i'}\epsilon_{ijk}\epsilon_{abc}Y^*_{lm}(\hat{q}_{bc})C^t_{\alpha_b\alpha_c}\tau_{l,t}(2q_{jk}^2)Y_{lm}(\hat{q}_{jk})C^t_{\alpha_j'\alpha_k'} \tag{A2b}$$

for fermions. These have the correct symmetry so long as the odd and even l are correctly associated with the symmetry of the $C^t_{\alpha\alpha'}$. We now can substitute (A1) and (A2) in the unitarity relation (14). As before, the left-hand side can be written $(\mathrm{Disc}f)\tau^* + f\,\mathrm{Im}\tau$ [or $(\mathrm{Disc}F)1/D^* + F\,\mathrm{Im}(1/D)$]. On the right the symmetry can be used to greatly reduce the forest of indices. One set of terms cancel on the left and right by two-body unitarity. (Again, insuring that they do cancel is a good check on any calculation.) The remaining terms are equal by symmetry. Cancelling the τ^* or $1/D^*$ on both sides these give, after some simplification and noting that for identical particles $\vec{q}_{jk} = \frac{1}{2}(\vec{p}_j - \vec{p}_k)$,

$$\mathrm{Disc}\langle \vec{k}\rho | f | \vec{p}\alpha, ltm\rangle = -2\pi q^l \sum_{\substack{\beta\beta' \\ l't'm'}} \int \frac{d^3p'}{(2\pi)^3} \frac{\langle \vec{k}\rho | f | \vec{p}'\beta', l't'm'\rangle}{|\vec{p} + \frac{1}{2}\vec{p}'|^{l'}} \tau_{l',t'}(2(\vec{p} + \tfrac{1}{2}\vec{p}')^2) Y_{l'm'}(\widehat{p + \tfrac{1}{2}p'}) Y^*_{lm}(\widehat{p' + \tfrac{1}{2}p})$$
$$\times C^{t'}_{\alpha\beta} C^{t}_{\beta'\beta} \delta(E - 2p^2 - 2p'^2 - 2\vec{p}\cdot\vec{p}') \qquad \text{(A3a)}$$

or

$$\mathrm{Disc}\langle \vec{k}\rho | F | \vec{p}\alpha, ltm\rangle = -2\pi \frac{N_{l,t}(2q^2)}{q^l} \sum_{\substack{\beta\beta' \\ l't'm'}} \int \frac{d^3p'}{(2\pi)^3} \frac{\langle \vec{k}\rho | F | p'\beta, l't'm'\rangle |\vec{p} + \frac{1}{2}\vec{p}'|^{l}}{D_{l',t'}(2(\vec{p} + \frac{1}{2}\vec{p}')^2)} Y_{l'm'}(\widehat{p + \tfrac{1}{2}p'})$$
$$\times Y^*_{lm}(\widehat{p' + \tfrac{1}{2}p}) C^{t'}_{\alpha\beta} C^{t}_{\beta'\beta} \delta(E - 2p^2 - 2p'^2 - 2\vec{p}\cdot\vec{p}') \qquad \text{(A3b)}$$

for bosons, where

$$E = \tfrac{3}{2}p^2 + 2q^2 . \qquad \text{(A4)}$$

Not surprisingly, (A3) is exactly the same as (20) where the identity of the particles is used. For fermions the same calculation gives precisely the same result as (A3) except that the sign of the right-hand side is reversed. This is the well-known sign difference between boson and fermion for exchange graphs and arises because (A3) is essentially the imaginary or singular part of such a graph.

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