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Simple relation between the asymmetry and longitudinal polarization of the recoil nucleus in muon capture

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An interesting relation $\alpha - 2P_L = 1$ is obtained between the asymmetry parameter α and the longitudinal polarization P_L of the recoil nucleus in the reaction $\mu^- + A(J_i^{\pi_i}) \rightarrow B(J_f^{\pi_f}) + \nu_\mu$, in which the initial nuclear spin is zero. It is shown algebraically that this relation is independent of both the nuclear structure and the values of the coupling constants of the muon capture interaction with or without inclusion of second class current terms.

I. INTRODUCTION

Wolfenstein[1] has considered a specific muon capture reaction $\mu^- + {}^{12}C(0^+) \rightarrow {}^{12}B(1^+) + \nu_\mu$ and he has derived the following expressions[2] for the asymmetry parameter α (obtained with polarized muons) and the longitudinal polarization P_L (obtained with unpolarized muons) of the recoil nucleus[3]:

$$\alpha = -\frac{G_A^2 + 2G_AG_P - G_P^2}{3G_A^2 + G_P^2 - 2G_AG_P}, \tag{1}$$

$$P_L = -\frac{2G_A^2}{3G_A^2 + G_P^2 - 2G_AG_P}. \tag{2}$$

From Eqs. (1) and (2) Palffy[4] and his collaborators[5] have extracted an interesting relation

$$\alpha - 2P_L = 1. \tag{3}$$

It may be pointed out that the expressions for α and P_L obtained by Wolfenstein are for the specific nuclear transition ${}^{12}C(0^+) \rightarrow {}^{12}B(1^+)$ taking into account only the momentum independent terms in the muon capture interaction and considering only "allowed" nuclear transitions. The purpose of this article is to show that the relation (3) is true for the restricted nuclear transition $0^{\pi_i} \rightarrow J_f^{\pi_f}$ with or without parity change ($\pi_i\pi_f = \pm 1$) and is independent of both the nuclear structure and the values of the muon capture coupling constants. It is true even when the second-class current terms are included in the muon capture interaction. The Fujii-Primakoff effective Hamiltonian[6] is used in our treatment and momentum dependent terms of order $1/M$ in the muon capture interaction are also included, M being the nucleon mass. Also the complete set of nuclear tensor operators has been taken into account. However, it is found that for a more general transition $J_i^{\pi_i} \rightarrow J_f^{\pi_f}$ (with $J_i, J_f \neq 0$), the relation (3) is not valid.

II. ASYMMETRY OF THE RECOIL NUCLEUS

The asymmetry parameter α is defined such that the angular distribution $\Lambda(\theta)$ of the recoil nucleus is given by

$$\Lambda(\theta) \propto 1 - \alpha(\vec{P} \cdot \hat{\nu}),$$

where $\vec{P}$ denotes the muon polarization at the instant of capture, $\hat{\nu}$, the direction of neutrino emission which is opposite to that of nuclear recoil, and θ, the angle between $\vec{P}$ and nuclear recoil. This asymmetry with respect to the direction of muon polarization is a consequence of parity violation in weak interactions and has been studied in great detail by earlier workers.[7,8] Using the Fujii-Primakoff effective Hamiltonian,[6] an expression for the asymmetry parameter α can be derived and it is given below:

$$\alpha = \mathfrak{D}/\mathfrak{B}, \tag{4}$$

where

$$\begin{aligned} 4\pi\mathfrak{D} = &-G_A^2 b_1 + (G_P^2 + 2G_A^2 - 2G_AG_P)b_2 \\ &+ 2(G_P - G_A)(g_A/M)b_3 - 2G_A(g_V/M)b_4 \\ &+ G_V^2 b_5 - 2G_V(g_V/M)b_6, \end{aligned} \tag{5}$$

$$\begin{aligned} 4\pi\mathfrak{B} = &G_A^2 b_1 + (G_P^2 - 2G_AG_P)b_2 \\ &+ 2(G_P - G_A)(g_A/M)b_3 + 2G_A(g_V/M)b_4 \\ &+ G_V^2 b_5 - 2G_V(g_V/M)b_6. \end{aligned} \tag{6}$$

The quantities b_1, b_2, b_3, and b_4 involve nuclear matrix elements and are given by Eqs. (30)–(33) of Ref. 9, hereafter referred to as DPS. The quantities b_5 and b_6 are given below[10,11]:

$$b_5 = 16\pi^2[J_f]^2|I(J_f, 0, J_f, 0, J_f)|^2, \tag{7}$$

$$\begin{aligned} b_6 = &16\pi^2[J_f]^2 I(J_f, 0, J_f, 0, J_f) \\ &\times \sum_l (i)^{l+3-J_f} C(J_f 1 l, 00) \\ &\times I(l, 1, J_f, 0, J_f), \end{aligned} \tag{8}$$

where

$$[K]=(2K+1)^{1/2}\ , \tag{9}$$

and the quantity $I(l,q,\lambda,N,J_f)$ is the reduced nuclear matrix element defined by

$$I(l,q,\lambda,N,J_f) = \left\langle J_f \left\| \sum_n j_l(\nu r_n)\varphi(r_n)\{(Y_l(\hat{r}_n)\times\nabla_q)_\lambda\times\sigma_N\}_{J_f} \right\| 0 \right\rangle. \tag{10}$$

The summation is over the nucleon index n. $\varphi(r)$ represents the radial muon wave function in $1s$ orbit.

III. LONGITUDINAL POLARIZATION

It is expected that the recoil nucleus in the capture reaction $\mu^- + A \to B + \nu_\mu$ should possess longitudinal polarization even when the muons are unpolarized. It can be thought of as a result of parity violation in muon capture interaction and the consequent negative Helicity of the neutrino.

If $\vec{J}$ denotes the nuclear spin, the longitudinal polarization P_L of the recoil nucleus resulting from the capture of unpolarized muons is the expectation value[12] of the operator $(-\vec{J}\cdot\hat{\nu})$:

$$P_L = \frac{\mathrm{Tr}(-\vec{J}\cdot\hat{\nu}\rho_f)}{\mathrm{Tr}(\rho_f)}\ , \tag{11}$$

where ρ_f is the density matrix for the final nuclear state resulting from the capture of unpolarized muons. If t is the nuclear transition operator, the density matrix ρ_f is defined such that its element is given by

$$(\rho_f)_{M_f,M_f'} = \sum_{M_i,M_i'} \langle J_f M_f | t | J_i M_i\rangle\ (\rho_i)_{M_i,M_i'} \times \langle J_f M_f' | t | J_i M_i'\rangle^*, \tag{12}$$

where ρ_i is the density matrix describing the initial nucleus. Then the muon capture rate is simply the trace of the density matrix in the spin space of the final nucleus:

$$\Lambda = \mathrm{Tr}\rho_f\ . \tag{13}$$

The density matrix ρ_f completely describes the spin orientation of the final nucleus[9,13] which can be represented conveniently by a set of parameters $\langle T_K^{m_K}\rangle$ defined by

$$\langle T_K^{m_K}\rangle = \frac{\mathrm{Tr}(T_K^{m_K}\rho_f)}{\mathrm{Tr}(\rho_f)}\ , \tag{14}$$

where $T_K^{m_K}$ denotes a spherical tensor operator of rank K in the spin space of the final nucleus and satisfies the normalization condition

$$\mathrm{Tr}(T_K^{\dagger m_K} T_{K'}^{m_{K'}}) = (2J_f+1)\delta_{K,K'}\delta_{m_K,m_{K'}}\ , \tag{15}$$

subject to the restriction $0 \leq K \leq 2J_f$.

Using the known results[14,15]

$$\langle J_f \| T_K \| J_f\rangle = (2K+1)^{1/2}\ , \tag{16}$$

and

$$\langle J_f \| J \| J_f\rangle = \sqrt{\eta}\ , \tag{17}$$

with

$$\eta = J_f(J_f+1)\ , \tag{18}$$

Eq. (11) can be rewritten as

$$P_L = -(\eta/3)^{1/2}\frac{\mathrm{Tr}(\vec{T}_1\cdot\hat{\nu}\rho_f)}{\mathrm{Tr}(\rho_f)}\ . \tag{19}$$

The traces occurring in Eq. (19) can be evaluated following the method outlined in DPS. In the notation of DPS, the terms which contribute to the numerator are

$$-iG_A{}^2\hat{\nu}\cdot(\vec{M}_2\times\vec{M}_2^*),\quad G_A(g_V/M)\{\vec{M}_2\cdot\vec{M}_3^*+\text{c.c.}\},$$
$$(G_P-G_A)G_V\{(\hat{\nu}\cdot\vec{M}_2)M_1^*+\text{c.c.}\}\ ,$$
$$-G_P(g_V/M)\{(\hat{\nu}\cdot\vec{M}_2)(\hat{\nu}\cdot\vec{M}_3^*)+\text{c.c.}\},$$

and

$$G_V(g_A/M)\{M_4M_1^*+\text{c.c.}\}\ ,$$

where c.c. denotes the complex conjugate. M_I ($I=1$ to 4) are the nuclear matrix elements between the initial spin state $|J_iM_i\rangle$ and the final spin state $|J_fM_f\rangle$:

$$M_I = \left\langle J_fM_f \left| \sum_n t_I(n) \right| J_iM_i\right\rangle\ , \tag{20}$$

where the summation is over the nucleon index n. The transition operator for the nucleus is given by[16]

$$t_I = e^{-i\hat{\nu}\cdot\vec{r}}\varphi(r)\mathcal{O}_I\ , \tag{21}$$

with

$$\mathcal{O}_1 = 1,\quad \mathcal{O}_2=\vec{\sigma},\quad \mathcal{O}_3=\vec{p},$$

and

$$\mathcal{O}_4 = \vec{\sigma}\cdot\vec{p}\ . \tag{22}$$

$\vec{\sigma}$ is the Pauli operator for the nucleon, and $\vec{p}(=-i\vec{\nabla})$ is the momentum operator for the proton. $\varphi(r)$ represents the radial muon wave function in $1s$ orbit.

Let us now consider the case when the initial spin is zero. In this case, as shown in the Appendix, the terms involving M_1 and the term $(\hat{\nu}\cdot\vec{M}_2)(\hat{\nu}\cdot\vec{M}_3^*)$ do not contribute to $\mathrm{Tr}(-\vec{J}\cdot\hat{\nu}\rho_f)$. The only two terms that contribute to the longitudinal polarization are $-iG_A{}^2\hat{\nu}\cdot(\vec{M}_2\times\vec{M}_2^*)$ and $G_A(g_V/M)\{\vec{M}_2\cdot\vec{M}_3^*+\text{c.c.}\}$. Following DPS, the traces are evaluated and the results are given below:

$$P_L = -(\mathcal{C}/\mathcal{B}) , \quad (23)$$

where

$$\pi\mathcal{C} = (\eta/3)^{1/2}[G_A{}^2 c_1 + 1.5 G_A(g_V/M)a_4] , \quad (24)$$

with

$$c_1 = 4\pi^2\sqrt{6} \sum_{l,l',L} (i)^{l'-l}(-1)^L[J_f]^3[L][l][l']C(l1L,00)C(Ll'1,00)W(J_f1L1,l1)W(J_f11l',LJ_f) \times I(l,0,l,1,J_f)I(l',0,l',1,J_f) . \quad (25)$$

The quantity a_4 is given by Eq. (28) of DPS and $\mathcal{B}$ by Eq. (6).

IV. RELATION $\alpha - 2P_L = 1$

From Eqs. (4) and (23) we have

$$\alpha - 2P_L = (\mathcal{D} + 2\mathcal{C})/\mathcal{B} . \quad (26)$$

Substituting the expressions for $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, we find that the right hand side of Eq. (26) becomes unity when

$$G_A{}^2[4(\eta/3)^{1/2}c_1 + b_2] + G_A(g_V/M)(3\eta)^{1/2}a_4 = G_A{}^2 b_1 + G_A(g_V/M)b_4 . \quad (27)$$

This relation is independent of the muon capture coupling constants if

$$4(\eta/3)^{1/2}c_1 + b_2 = b_1 , \quad (28)$$

$$(3\eta)^{1/2}a_4 = b_4 . \quad (29)$$

It may be noted that Eq. (28) arises from momentum independent terms and Eq. (29) from momentum dependent terms so that one would arrive at the same result whether the momentum dependent terms are included or not.

Using the table of Racah coefficients[11] one can obtain algebraically the following relation between the Racah coefficients:

$$[\eta(2J_f+1)]^{1/2}W(J_fJ_fJ_f \pm 1J_f;11) = \sqrt{6}\;W(J_fJ_f \pm 111;1J_f) . \quad (30)$$

Using Eq. (30) and the table of C coefficients,[11] we prove algebraically that Eqs. (28) and (29) are true. The correctness of the Eqs. (28)–(30) has also been checked numerically. It may be emphasized that we have neither used a nuclear model nor the numerical values for the coupling constants of the muon capture interaction. From Eqs. (26)–(30), we get the relation (3) and this relation is independent of nuclear models as well as the muon capture coupling constants. It is true even when the second class current terms are included in the muon capture interaction. Since we have used the Fujii-Primakoff effective Hamiltonian which is correct only to terms of order $1/M$ where M is the nucleon mass, the relation (3) is proved in this valid approximation.

V. CONCLUSION

It is really interesting to note that a simple relation exists between the asymmetry of nuclear recoils arising from the capture of *completely polarized* muons (the asymmetry is αP and this becomes α for a completely polarized muon beam) and the longitudinal polarization of the recoil nucleus arising from the capture of *unpolarized* muons. This relation is independent of both the nuclear structure and the values of the coupling constants of the muon capture interaction. It is true even when second-class currents exist. The relation (3) can be verified experimentally by a careful measurement of α and P_L in muon capture by different nuclei for which the initial nuclear spin is zero. Alternatively, this relation is useful in checking the correctness of measurements of α and P_L experimentally. In conclusion, we wish to reiterate that this relation is of greater significance in view of the proposal by Louvain-Saclay collaboration[5,17] to measure simultaneously the asymmetry, average polarization and longitudinal polarization of the recoil nucleus in the reaction $\mu^- + {}^{12}C(0^+) \to {}^{12}B(1^+) + \nu_\mu$. Since their experimental setup does not allow direct measurement of α,[4,18] the only way they can determine α is by the use of relation (3).

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APPENDIX

In this Appendix, it is shown that the terms involving M_1 and the term $(\hat{\nu}\cdot\vec{M}_2)(\hat{\nu}\cdot\vec{M}_3^*)$ do not contribute to $\mathrm{Tr}(-\vec{J}\cdot\hat{\nu}\rho_f)$ when the initial spin J_i is zero.

(a) M_1 terms. Let us take a term containing

M_1 and show that it does not contribute to $\mathrm{Tr}(-\vec{J}\cdot\hat{\nu}\rho_f)$ when $J_i=0$. Expanding the transition operator t_I into tensor operators and using the method outlined in DPS, one can obtain

$$\mathrm{Tr}\{(\vec{J}\cdot\hat{\nu})(\hat{\nu}\cdot\vec{M}_2)M_1^*\}=-4\pi\sqrt{\eta}\,[J_f]^3[J_i]^{-2}\sum_{\lambda,l,l'}(i)^{l'-l}[\lambda]C(\lambda 1 l,00)C(\lambda 1 l',00)W(\lambda J_i 1 J_f;J_f l')$$
$$\times\mathcal{G}(l,0,l,1,\lambda)\mathcal{G}(l',0,l',0,l')\ , \tag{A1}$$

where

$$\mathcal{G}(l,q,\mathfrak{L},N,L)=\left\langle J_f\left\|\sum_n j_l(\nu r_n)\varphi(r_n)\{(Y_l(\hat{r}_n)\times\nabla_q)_{\mathfrak{L}}\times\sigma_N)\}_L\right\|J_i\right\rangle\ . \tag{A2}$$

When the initial spin J_i is zero, $l'=J_f=\lambda$. Hence the parity C coefficient $C(\lambda 1 l',00)$ vanishes and as a consequence the term $(\hat{\nu}\cdot\vec{M}_2)M_1^*$ does not contribute. In a similar manner, one can easily prove that M_1 terms do not contribute to $\mathrm{Tr}(-\vec{J}\cdot\hat{\nu}\rho_f)$ when the initial spin J_i is zero.

(b) The term $(\hat{\nu}\cdot\vec{M}_2)(\hat{\nu}\cdot\vec{M}_3^*)$. It can be proved that

$$\mathrm{Tr}\{(\vec{J}\cdot\hat{\nu})(\hat{\nu}\cdot\vec{M}_2)(\hat{\nu}\cdot\vec{M}_3^*)\}=4\pi\sqrt{\eta}\,[J_f]^3[J_i]^{-2}$$
$$\times\sum_{\lambda,\lambda',l,l'}\{(i)^{l'+1-l}[\lambda]C(\lambda 1 l,00)C(\lambda' 1 l',00)C(\lambda 1\lambda',00)W(\lambda J_i 1 J_f;J_f\lambda')$$
$$\times\mathcal{G}(l,0,l,1,\lambda)\times\mathcal{G}(l',1,\lambda',0,\lambda')\}\ . \tag{A3}$$

When the initial spin J_i is zero, $\lambda=J_f=\lambda'$ and hence the parity C coefficient $C(\lambda 1\lambda',00)$ is identically zero. Therefore, the term $(\hat{\nu}\cdot\vec{M}_2)(\hat{\nu}\cdot\vec{M}_3^*)$ does not contribute to $\mathrm{Tr}(-\vec{J}\cdot\hat{\nu}\rho_f)$ when J_i is zero.

Note added in proof: On seeing this work, Dr. J. Bernabeu (CERN) has shown that the relation $\alpha-2P_L=1$ is a consequence of rotational invariance and the definite helicity of the neutrino. We are grateful to Dr. L. Palffy for sending us a copy of the manuscript of Dr. J. Bernabeu.

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[3]Unless otherwise mentioned, the notations used in Refs. 6 and 9 are generally followed. Also the system of units in which $\hbar=c=$ muon mass $=1$ is followed throughout.

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[10]For angular momentum coefficients and reduced matrix elements, we follow the notation and definition of M. E. Rose, Ref. 11.

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