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Why the Hauser-Feshbach formula works*

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It is shown that flux conservation requires substantial channel-channel correlations of resonance amplitudes. These, together with the effects of level-level correlations and other terms conspire to cancel the large additive corrections to the Hauser-Feshbach formula for the fluctuation cross section. The remaining multiplicative corrections become negligible for nonelastic cross sections when many channels are open. In other cases, approximation formulas provide estimates that are adequate for most purposes. However, the Bohr independence hypothesis is not always satisfied when fewer than about 20 channels are open. The cross section correlation width is shown to differ markedly from the average width. The use of the former for estimating the Hauser-Feshbach denominator is found to be justified. All of these results are verified by means of statistical model calculations of resonance parameters and of cross sections.

I. INTRODUCTION

The well-known Hauser-Feshbach formula for the average compound nucleus cross section is a computationally simple and widely used tool for the analysis of nuclear reactions.[1] This use has been eminently successful and consistent with experimental data, except for some well-known and well understood deviations. Remarkably, however, in the 37 years of its existence, no derivation has been found which can explain the general validity of this formula. In this regard we know little more today than Bethe[2] did in 1937 when he derived the formula under the explicit restrictions that all partial width to level spacing ratios be small, and that there be no correlations between the partial widths of different channels. An expansion in powers of the partial width to spacing ratios reveals many higher order terms that increase rapidly in magnitude as the channel transmission coefficients increase and that seem to defy a simple summation.[3] An approach based upon strict unitarity and analyticity of the S matrix led to a simple Hauser-Feshbach-type expression.[4] However, this expression contains terms that depend on the magnitudes of very complicated correlations among the resonance parameters, and these are extremely difficult to estimate reliably. The simple estimates of these correlation effects that were proposed in Ref. 4 are, as we shall see below, not adequate from the point of view of derivation, they are however in many instances empirically successful in predicting observable deviations from the simple Hauser-Feshbach formula.

A widely stated model justification for the Hauser-Feshbach formula considers that Bohr's compound nucleus principle of independence of formation and decay applies to averages over resonances. This assumption is difficult to justify in the realm of large width to spacing ratios, where the "lifetime" $(\hbar/\Gamma)$ of the "compound nucleus" is short compared to the intrinsic period $(h/\overline{D})$ of its internal state. Such model considerations might suggest an enhancement of the compound elastic cross section by a factor of the order of $\Gamma/\overline{D}$, rather than the familiar correlation enhancement that is expected to be limited to a factor of between 2 and 3.[5] Moreover, the proportionality of average reaction cross sections to the transmission coefficients of the exit channels is suspect in the domain of large transmission coefficients T because of the general breakdown of the linear relationships between T and channel resonance parameters when T becomes large.[6,7]

Finally there is the often expressed speculation that the resonance interference terms cancel out on the average, leaving only the Hauser-Feshbach contribution. However, the resonance interference contributions are not the only significant additions to the Hauser-Feshbach term when transmission coefficients are not small. There are terms which do depend on resonance-resonance correlations of various kinds, terms which depend on channel-channel correlations, and finally there are non-Hauser-Feshbach terms that remain important in the absence of all correlations.[3,4] It is our principal purpose here to show how all these different kinds of terms cancel to leave a remainder that is generally quite close to the Hauser-Feshbach prediction. We present numerical evidence to show under what circumstances this does happen, and to explain why therefore the Hauser-Feshbach formula, together with some simple modifications is so widely successful.

A second purpose is to present evidence to justify the widely employed computational version of the Hauser-Feshbach formula, in which the sum over all transmission coefficients is replaced by 2π times the ratio of the correlation width to the average level spacing.

Both of these explanations will be seen to depend on the demonstrated presence of rather strong channel-channel correlations of the partial widths of overlapping resonances, even in the absence of direct reactions (the latter are here not considered).

II. FLUCTUATION CROSS SECTION

We consider an S matrix of the form[4]

$$S = S^0 + S^p, \tag{1}$$

where S^0 is assumed to be energy independent and the energy dependent part has the form

$$S^p_{cd} = -i \sum_\mu \frac{g_{\mu c} g_{\mu d}}{E - E_\mu + \frac{1}{2} i \Gamma_\mu}. \tag{2}$$

Its energy average is

$$\bar{S}^p = -(\pi/\bar{D})\langle g_\mu \times g_\mu \rangle = \tfrac{1}{2}(\bar{S} - \bar{S}^{*-1}), \tag{3}$$

where $\bar{D}$ is the mean spacing of the E_μ and the last expression follows from the required unitarity of S.[8] It follows then also that

$$S^0 = \tfrac{1}{2}(\bar{S} + \bar{S}^{*-1}). \tag{4}$$

Throughout we use an unlabeled bracket $\langle\ \rangle$ to indicate an average over the index μ, and a labeled bracket $\langle\ \rangle_{\rm av}$ or a bar to indicate an energy average.

The integrated fluctuation (or compound nucleus) cross section is defined in appropriate units as

$$\sigma^{\rm fl}_{cd} = \langle |S_{cd} - \bar{S}_{cd}|^2 \rangle_{\rm av} = \langle |S^p_{cd}|^2\rangle_{\rm av} - |\bar{S}^p_{cd}|^2, \tag{5}$$

and again from unitarity we have

$$\sum_d \sigma^{\rm fl}_{cd} = 1 - \sum_d |\bar{S}_{cd}|^2 \equiv T_c \quad (0 \le T_c \le 1). \tag{6}$$

The aim is now to express $\langle |S^p_{cd}|^2\rangle_{\rm av}$ in terms of the elements of $\bar{S}$ and simple statistical properties of the μ-dependent parameters of Eq. (2) that are believed to be universally valid, to insert this expression into Eq. (5) and to see if the Hauser-Feshbach equation

$$\sigma^{\rm HF}_{cd} = \frac{T_c T_d}{\sum_f T_f}, \tag{7}$$

or some simple modification results.

When Bethe's condition[2] that all $T_c \ll 1$ is violated, and particularly when some T_c are close to unity, then this program faces considerable difficulties. This fact is illustrated by the following circumstances. When $\bar{S}$ is diagonal (no direct reactions), it follows from Eq. (3) that $\langle g_{\mu c} g_{\mu d}\rangle = 0$ for $c \ne d$ and that

$$|\bar{S}^p_{cc}|^2 = (\pi/D)^2 |\langle g_{\mu c}{}^2\rangle|^2 = \frac{1}{4}\frac{T_c^2}{1 - T_c}. \tag{8}$$

This expression becomes very large for T_c close to unity and it exceeds T_c whenever $T_c > 0.8$. It follows that for large values of T_c, the value of $\sigma^{\rm fl}_{cc}$ in Eq. (5) emerges as a small difference between two large terms. Consequently the demands on the accuracy of our evaluation of $\langle |S^p_{cc}|^2\rangle_{\rm av}$ become severe. A similar statement can apply to the evaluation of $\sigma^{\rm fl}_{cd}$ when channels c and d are coupled by a direct reaction so that $\bar{S}_{cd}$ does not vanish.

Proceeding, nevertheless, with the evaluation of Eq. (5) one obtains[4]

$$\sigma^{\rm fl}_{cd} = (2\pi/\bar{D})\langle |g_{\mu c}|^2 |g_{\mu d}|^2/\Gamma_\mu\rangle - M_{cd}, \tag{9}$$

where

$$M_{cd} = 2|\bar{S}^p_{cd}|^2 - \frac{2\pi i}{\bar{D}} \left\langle \sum_{\nu \ne \mu} \frac{g_{\nu c} g_{\nu d} g^*_{\mu c} g^*_{\mu d}}{(E_\mu - E_\nu) + \frac{1}{2} i(\Gamma_\mu + \Gamma_\nu)} \right\rangle. \tag{10}$$

The first term in Eq. (9) looks like the Hauser-Feshbach formula. The second term M_{cd} contains the $|\bar{S}^p_{cd}|^2$ of Eq. (5), plus a second term of equal magnitude, thus exacerbating the difficulty mentioned below Eq. (8). One might hope, of course, that the second term of M, which depends on the magnitude of level-level correlations, would exactly cancel the first term, which does not depend on such correlations. Indeed, the estimate of M given in Ref. 4 leads to such a cancellation in the limit of large $\Gamma/\bar{D}$. However, it has been pointed out by Weidenmüller[9] that such a cancellation leads to a contradiction because the insertion of (9) into Eq. (6) gives

$$T_c = (2\pi/\bar{D})\langle N_\mu |g_{\mu c}|^2\rangle - \sum_d M_{cd}, \tag{11}$$

where

$$N_\mu \equiv \sum_c |g_{\mu c}|^2/\Gamma_\mu \ge 1. \tag{12}$$

Comparison with Eq. (8) leads to the requirement that

$$\sum_d M_{cd} \ge T_c((1 - T_c)^{-1/2} - 1), \tag{13}$$

which becomes large for T_c close to unity. Equation (13) implies an upper bound on the level correlation dependent second term of Eq. (10) that would always be satisfied if the term vanished. Therefore Eq. (13) is insufficient to prove the existence of short range level-level correlations (compare with Ref. 9 and footnote 8.)

III. *M* CANCELLATION

To inquire how the fluctuation cross section can be given by the Hauser-Feshbach formula, despite the discouraging results (8) and (13) we introduce the notation[4]

$$\Theta_{\mu c}=(2\pi/\overline{D})N_\mu\,|g_{\mu c}|^2\,,\qquad \Theta_\mu=\sum_c \Theta_{\mu c}, \tag{14}$$

which yields

$$\langle\Theta_{\mu c}\rangle = T_c+\sum_d M_{cd}\,, \tag{15a}$$

$$\sigma_{cd}^{\rm fl}=\frac{\langle\Theta_{\mu c}\rangle\langle\Theta_{\mu d}\rangle}{\langle\Theta_\mu\rangle}\,G_{cd}C_{cd}-M_{cd}\,, \tag{15b}$$

where

$$G_{cd}\equiv\left\langle\frac{\Theta_{\mu c}\Theta_{\mu d}}{\Theta_\mu}\right\rangle\Big/\frac{\langle\Theta_{\mu c}\Theta_{\mu d}\rangle}{\langle\Theta_\mu\rangle}\,, \tag{16a}$$

$$C_{cd}\equiv\frac{\langle\Theta_{\mu c}\Theta_{\mu d}\rangle}{\langle\Theta_{\mu c}\rangle\langle\Theta_{\mu d}\rangle}=1+2\rho_{cd}(\nu_c\nu_d)^{-1/2}\,, \tag{16b}$$

where the channel correlation coefficient ρ_{cd} is defined as

$$\rho_{cd}=(C_{cd}-1)(C_{cc}-1)^{-1/2}(C_{dd}-1)^{-1/2}\,, \tag{16c}$$

and where it is assumed that the distribution law of the $\Theta_{\mu c}$ is given by a χ^2 distribution with ν_c degrees of freedom (ν_c may have any integer or non-integer positive value.)

We now apply this to the case of n competing statistically equivalent channels. By this we mean that all $T_c=T$, all $\nu_c=\nu$, all $\rho_{cd}=\rho$ for distinct c and d. We then write for all distinct c and d

$$\begin{aligned}&C_{cc}\equiv C=1+2/\nu\,,\\&C_{cd}\equiv D=1+2\rho/\nu\,,\\&M_{cc}\equiv M\,,\\&M_{cd}\equiv P\,,\\&G_{cc}=G_{cd}=\frac{n}{C+(n-1)D}\,.\end{aligned} \tag{17}$$

Upon substitution into Eq. (15b) we obtain then

$$\sigma_{cc}^{\rm fl}=\frac{CT}{C+(n-1)D}-\frac{n-1}{C+(n-1)D}\,(DM-CP)\,, \tag{18a}$$

$$\sigma_{cd}^{\rm fl}=\frac{DT}{C+(n-1)D}+\frac{1}{C+(n-1)D}\,(DM-CP)\,. \tag{18b}$$

This is to be compared with the Hauser-Feshbach prediction

$$\sigma_{cc}^{\rm HF}=\sigma_{cd}^{\rm HF}=\frac{T}{N}\,, \tag{19}$$

and with the width-fluctuation corrected Hauser-Feshbach formula with uncorrelated partial widths

$$\sigma_{cc}^{\rm WF}=\frac{C'T}{C'+n-1}\,,\qquad \sigma_{cd}^{\rm WF}=\frac{T}{C'+n-1}\,. \tag{20}$$

We see immediately that if

$$\frac{P}{M}=\frac{D}{C}\,, \tag{21}$$

then Eq. (18) yields the width fluctuation corrected Hauser-Feshbach formula

$$\sigma^{\rm fl}=\sigma^{\rm WF}\quad\text{with } C'=\frac{C}{D}\,. \tag{22}$$

We shall refer to the condition (21) as *M cancellation,* and we hypothesize that *M* cancellation holds as a consequence of *S*-matrix unitarity and that a corresponding condition holds also for statistically inequivalent channels. Before turning to the justification of this hypothesis we discuss some of its consequences.

In the case of diagonal $\overline{S}$, the most plausible assumptions regarding the statistical properties of the $|g_{\mu c}|^2$ and of the $\Theta_{\mu c}$ would lead one to expect no channel-channel correlations and hence $D=1$ and $P=0$. In consequence of Eq. (13), *M* cancellation could then not occur, and Eq. (18) would predict large deviations from Hauser-Feshbach. In fact, for sufficiently large n, the absence of channel-channel correlations would imply a negative elastic fluctuation cross section for any reasonable value of ν. To avoid such negative cross sections would, for large T and n, require the assumption that $\nu<2(1-T)^{1/2}/n$. Such very small values of ν would imply huge statistical dispersions of the partial widths. This would contradict the existence of stable average cross sections for reasonable and physically relevant averaging intervals.

The effect of direct reactions upon the fluctuation cross section has been generally assumed to arise from the correlations of the $|g_{\mu c}|^2$ between directly coupled channels, which arises from the correlation of the $g_{\mu c}$ as determined by Eq. (3).[10-12] The fact that the $|g_{\mu c}|^2$ are already correlated by the unitarity condition complicates the discussion of the direct effect. This will be treated in a future publication.

The presence of channel-channel correlations and Eq. (22) explains why for large Γ/D the elastic enhancement C' can be limited to a factor of 2,[5] while from the known width distributions[6,7] one expects C to be considerably greater than 2. The reason must be that C/D approaches a value of 2, which, as we shall see, appears to be correct.

IV. NUMERICAL RESULTS

The *M*-cancellation hypothesis was tested numerically by means of a modified version of the

MATDIAG program.[13] In this procedure R-matrix parameters are generated randomly to yield a specified average S matrix.[14] The $\gamma_{\mu c}$ are selected independently to be normally distributed with zero means.[15] The R-matrix level spacings are selected according to the Wigner distribution.[16] The program then carries out the level matrix inversion procedure to produce the corresponding S-matrix pole parameters E_μ, Γ_μ, and $g_{\mu c}$ of Eq. (2) and from them the quantities M, P, C, and D are calculated as given by Eqs. (10), (16), and (17), as well as many other properties of the resonance parameters.

As an example, we performed 5 statistically independent calculations, each with 100 resonances, for the case of 10 statistically equivalent uncoupled channels with transmission coefficients $T=0.75$. We obtained the following results.

$$C=4.64\pm 0.38,$$

$$D=2.23\pm 0.14,$$

from which we find that

$$\nu=0.55\pm 0.06,$$

$$\rho=0.34\pm 0.06,$$

$$C'=2.08\pm 0.22.$$

For comparison, the Porter-Thomas distribution[15] would yield $C=3$ ($\nu=1$) and Satchler's[5] normal distribution would give $C=2$ ($\nu=2$). The $\Theta_{\mu c}$, and also the $|g_{\mu c}|^2$, fluctuate much more severely than either of these predictions. The channel-channel correlation ρ and its effect D are pronounced. What happens is that a "broad" resonance μ tends to have large $|g_{\mu c}|^2$ for all channels, while a "narrow resonance" tends to have small $|g_{\mu c}|^2$. The compound elastic enhancement factor C' is consistent with Satchler's value of 2.

From the same calculations we obtained

$$M=0.64\pm 0.10,$$

$$P=0.32\pm 0.02.$$

These are very large numbers when compared to the Hauser-Feshbach contribution of only 0.075. Our results are consistent with M cancellation because

$$\frac{P}{M}=0.50\pm 0.08,\quad \frac{D}{C}=0.48\pm 0.05.$$

Figure 1 shows a comparison of the P/M and D/C ratios obtained in this and other similar calculations. The quoted uncertainties are calculated from the sample dispersions under the assumptions of uncorrelated normally distributed variables. These results are generally consistent with M cancellation and suggest that the elastic enhancement factor $C'=C/D=M/P$ varies from about 3 for small $\sum T$ to about 2 for large $\sum T$, as expected.[4]

The channel-channel correlation coefficients ρ are plotted in Fig. 2. They are seen to depend primarily on the transmission coefficient T and rise from zero at small values of T to around 0.5 at $T\approx 1$ in an almost linear fashion. These correlations in the $\Theta_{\mu c}$ are primarily due to channel-channel correlations in the $|g_{\mu c}|^2$, rather than due to the level normalizations N_μ.

The substantial magnitudes of P clearly also im-

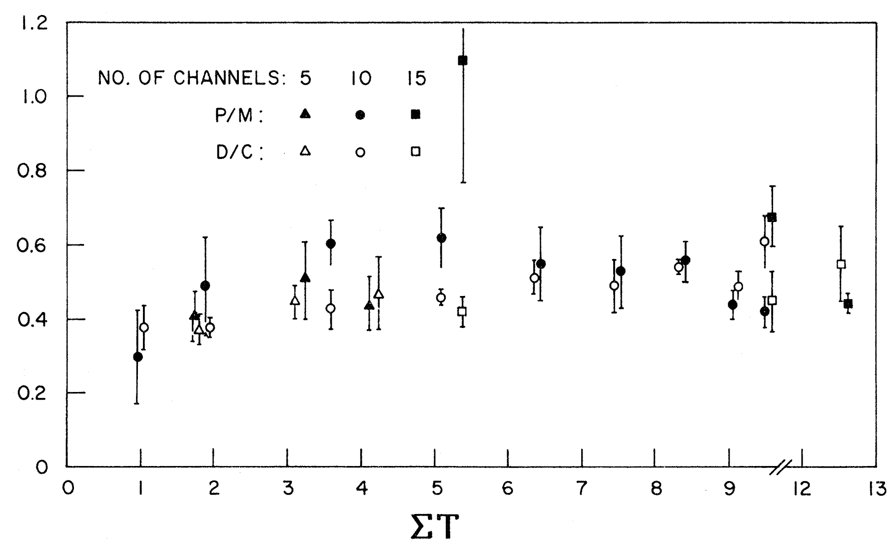

FIG. 1. Comparisons of statistically computed ratios P/M (black points) and D/C (open points) for statistically equivalent channels with transmission coefficient sums ΣT. (Program MATDIAG.)

ply correlations among the $g_{\mu c}$, belonging to different resonance terms μ. However these level-level correlations are very weak and the resulting contributions to P arise from the cumulative effects of the sum over many slightly correlated terms ν in Eq. (10). The modified MATDIAG program calculates correlations of $g_{\mu c}^2$, $|g_{\mu c}|^2$, and $\Theta_{\mu c}$ between neighboring resonance poles. In no case was it possible to establish statistically significant nearest neighbor level-level correlations, in marked contrast to the very pronounced channel-channel correlations.

V. OTHER REPRESENTATIONS OF S

The parametrization of the fluctuations of S that is given in Eqs. (1) and (2) has the virtue that because of its simple analytic form it is relatively easy to average. On the other hand, the statistical properties of the parameters, in particular the correlations which they must satisfy because of the unitarity condition are, as we have seen, very complicated. Nevertheless, we have seen that because of M cancellation the effects of these correlations on average cross sections are quite limited. This suggests that Eqs. (1) and (2) are the wrong parametrization for our purpose and that we should look for another representation, one that is manifestly unitary with no hidden and complicated correlations, but also one that can be averaged. Representation by an R matrix satisfies the first requirement but not the second. At present we are not aware of a representation that satisfies both of these requirements.

One interesting attempt in this direction is the work of Kawai, Kerman, and McVoy.[10] These authors construct a representation of the S matrix that looks just like Eqs. (1) and (2), except that the parameters E_μ, Γ_μ, and $g_{\mu c}$ are all energy dependent. In that case the results of Eqs. (3) and (4) no longer hold and Kawai *et al.* were able to choose the energy dependence so that $S^0 = \bar{S}$ and $\bar{S}^p = 0$; thus overcoming the difficulty associated with our Eq. (8). However, the correct averaging of σ in the Kawai representation presents difficulties that are probably as formidable as those encountered in our Sec. II. All parameters must be evaluated anew at each energy point in the averaging interval. Instead of this, Kawai *et al.* suppress the energy dependences of their parameters within their averaging interval. The resulting approximate S matrix with constant pole parameters is unitary at most at one energy. Its deviation from unitarity elsewhere in the averaging interval is a consequence of the requirement that $S^0 = \bar{S}$ in violation of Eq. (4), and it is not connected with any possible dynamical effects, such as the presence of the thresholds of new reaction channels. Averages based on such an approximation are questionable because the effect on averages of small deviations from unitarity increases rapidly as transmission coefficients approach unity.

In addition Kawai *et al.*[10] adopt statistical as-

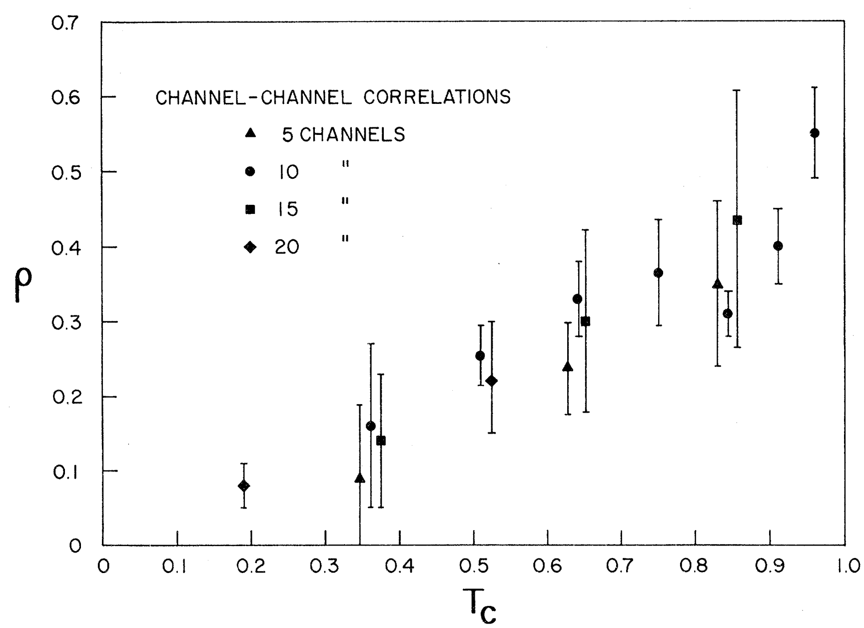

FIG. 2. Statistically computed channel-channel correlation coefficients ρ, as defined in Eq. (16c), for statistically equivalent channels, each having transmission coefficient T_c. (Program MATDIAG.)

sumptions, which are claimed to remove all level-level correlations, and therefore also the second term of M in Eq. (1) is made to vanish. It appears unlikely, however, that these assumptions can also remove the channel-channel correlations discussed in Sec. III above. However no such effect in the first Hauser-Feshbach term of Eq. (9) is included in the Kawai average.

Interestingly enough, Kawai *et al.* finally arrived at a formula which yields almost identical numerical results as the width fluctuation corrected Hauser-Feshbach formula σ^{WF} with all $\nu_c = 2$. They claim validity for this result for large $\Gamma/\bar{D}$, and indeed the formula is very good in that limit for diagonal $\bar{S}$ (see next section), presumably because of compensations among the neglected terms that are akin to M cancellation. However, when applied to compound processes that compete with direct reactions, their formula fails.[12,17]

VI. APPROXIMATION FORMULAS

Even if we assume that M cancellation holds exactly we are still faced with specifying n channel distribution coefficients ν_c and $n(n-1)/2$ channel-channel correlation coefficients ρ_{cd}, in addition to the transmission coefficients T_c in order to calculate fluctuation cross sections. For most practical purposes it is sufficient, however, to take the channel-channel correlation into account by means of effective correlation enhancement factors C_c' in the manner of Eq. (22) and corresponding effective $\nu_c' = 2/(C_c' - 1)$. Dropping the primes, we are then left with the familiar width fluctuation corrected Hauser-Feshbach formula[3,18,19]:

$$\sigma_{cd}^{\mathrm{WF}} = \frac{(1 + 2\delta_{cd}/\nu_d) G_{cd} T_c T_d}{\sum_f T_f}, \tag{23}$$

where G_{cd} can be obtained by numerical evaluation of the integral[20,21]

$$G_{cd} = \int_0^\infty dt \prod_f \left(1 + \frac{2t\nu_f^{-1} T_f}{\sum_g T_g}\right)^{-(\frac{1}{2} + \delta_{fc} + \delta_{fd})}. \tag{24}$$

The integral (24) is easy enough to evaluate numerically by digital computer. However another approximation method is also available.

$$\sigma_{cd}^{\mathrm{app}} = X_c X_d + 2\delta_{cd} X_d^2/\nu_d, \tag{25a}$$

$$T_c = X_c \sum_d X_d + 2X_c^2/\nu_c. \tag{25b}$$

For $\nu_c = 2$ this is identical to the result that Kawai, Kerman and McVoy[10] obtained in the limit of large $\Gamma/\bar{D}$. For arbitrary ν_c it is identical to the formula of Tepel, Hofmann, and Weidenmüller,[22] who used the notation $V_c = X_c \sum_d X_d$. For n statistically equivalent channels Eqs. (25) give identical results as Eq. (23). When different channels have different T_c and ν_c, then Eq. (25b) must first be solved for the X_c which are then substituted into (25a). It is doubtful that this procedure is numerically preferable to evaluating the integral (24). However, Tepel *et al.*[22] have discovered a simple approximate solution of Eq. (25b) which is quite accurate in most instances. According to this

$$X_c \approx \frac{T_c}{1 + (2T_c/\nu_c \sum T)} \left(\sum_d \frac{T_d}{1 + (2/\nu_d)(T_d/\sum T)}\right)^{-1/2}. \tag{25c}$$

In order to test the adequacy of these approximation formulas we have performed a number of statistical model cross section calculations using the computer program STASIG.[23] In this program a statistical R matrix is generated in exactly the same way as in MATDIAG, but instead of performing a level matrix inversion, the channel matrix is inverted to compute the S matrix at regular energy intervals. From this energy dependent cross sections are computed which are then averaged and statistically analyzed in a variety of other ways. Many internal checks are performed including the direct energy averaging of the S matrix for comparison with the input average S matrix.

Effective tests of the approximation formulas required channels with differing transmission coefficients. Accordingly we chose cases in which half the channels (called α channels) had very weak transmission $T_\alpha \approx 0.1$, and half the channels (called β channels) had very strong transmission $T_\beta \approx 0.91$. Calculations were performed with the total number of channels n varying from 4 to 30. The resulting averages of the statistically generated average cross sections are listed in Table I for the elastic fluctuation cross sections in the α and β channels and for the nonelastic cross sections. The probable errors in the last places of these averages are given in parentheses. These statistical average cross sections are compared in Table I with the uncorrected Hauser-Feshbach result (HF) and with the result obtained with the width fluctuation corrected formula (WF) of Eq. (23) and with the approximation (app) of Eq. (25). In the latter two cases the ν_c were chosen so as to agree with the elastic enhancements of the statistically generated average cross sections

$$1 + 2/\nu_\alpha = \bar{\sigma}_\alpha^{\mathrm{el}} / \bar{\sigma}_{\alpha\alpha'}^{\mathrm{ne}},$$

and correspondingly for β.

The predictions of the width fluctuation corrected formula (23) and the approximation (25) are within 3% of one another throughout Table I. We conclude that both formulas may be used interchangeably in estimating compound nucleus cross sections, and

TABLE I. Average elastic and nonelastic fluctuation cross sections obtained by statistical model calculations and from Eqs. (23), (25), and (7) for n competing channels, half of which have $T_c \approx 0.1$ (α channels) and half of which have $T_c \approx 0.91$ (β channels, for $n = 4$ the value is 0.94).

n		$\bar{\sigma}^{el}_{\alpha}$	$\bar{\sigma}^{el}_{\beta}$	$\bar{\sigma}^{ne}_{\alpha\alpha'}$	$\bar{\sigma}^{ne}_{\alpha\beta}$	$\bar{\sigma}^{ne}_{\beta\beta'}$
4	Statistical	0.0186(13)	0.583(11)	0.007 27(12)	0.0380(16)	0.282(6)
	WF, Eq. (23)	0.0157	0.580	0.006 15	0.0400	0.280
	app, Eq. (25)	0.0157	0.571	0.006 15	0.0412	0.276
	HF, Eq. (7)	0.0050	0.425	0.004 98	0.0460	0.425
8	Statistical	0.0081(4)	0.352(5)	0.003 35(6)	0.0206(4)	0.160(6)
	WF	0.0070	0.350	0.002 89	0.0211	0.160
	app	0.0069	0.349	0.002 86	0.0213	0.158
	HF	0.0025	0.206	0.002 46	0.0225	0.206
12	Statistical	0.0045(2)	0.260(5)	0.001 98(7)	0.0141(2)	0.115(3)
	WF	0.0041	0.260	0.001 82	0.0143	0.115
	app	0.0041	0.259	0.001 80	0.0143	0.114
	HF	0.0016	0.138	0.001 60	0.0149	0.138
16	Statistical	0.0031(3)	0.219(6)	0.001 22(4)	0.0106(2)	0.088(2)
	WF	0.0032	0.219	0.001 27	0.0105	0.088
	app	0.0032	0.219	0.001 26	0.0105	0.088
	HF	0.0011	0.104	0.001 14	0.0109	0.104
20	Statistical	0.0024(2)	0.172(6)	0.001 01(3)	0.008 45(15)	0.073(2)
	WF	0.0024	0.172	0.000 99	0.008 47	0.073
	app	0.0023	0.172	0.000 99	0.008 48	0.073
	HF	0.0009	0.083	0.000 91	0.008 68	0.083
30	Statistical	0.0015(2)	0.110(6)	0.000 67(2)	0.005 68(12)	0.050(1)
	WF	0.0015	0.110	0.000 65	0.005 70	0.050
	app	0.0015	0.110	0.000 65	0.005 70	0.050
	HF	0.0006	0.054	0.000 62	0.005 78	0.054

that Eq. (25) with the Tepel approximation (25c) is certainly computationally more convenient. The agreement between these formulas and the statistically generated fluctuation cross sections is generally also very good. Only when few channels are open are there appreciable discrepancies which amount to less than 20% in our four channel case. When many channels are open, the uncorrected Hauser-Feshbach formula (7) also gives a good account of the nonelastic fluctuation cross section.

VII. BOHR HYPOTHESIS AND ELASTIC ENHANCEMENTS

One question of interest is how well the approximation formulas represent cross section ratios. For this purpose we have plotted in Fig. 3 the ratio of average nonelastic fluctuation cross sections

$$R = \bar{\sigma}_{\alpha\alpha'}\,\bar{\sigma}_{\beta\beta'}/(\bar{\sigma}_{\alpha\beta})^2\,,$$

obtained from the data of Table I. The bars indicate averages over all computed cross sections of the same type. Evidently $R \approx 1$ for more than 20

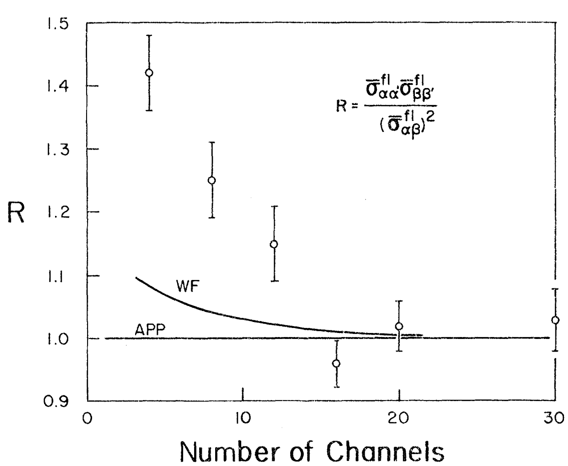

FIG. 3. The ratio R of statistically computed nonelastic fluctuation cross sections for equal numbers of α channels with $T_\alpha \approx 0.0975$ and β channels with $T_\beta \approx 0.91$. ($T_\beta = 0.94$ for the point $N = 4$.) The curves labeled WF and app give the ratios R computed from Eqs. (23) and (25), respectively. (Program STASIG.)

competing open channels, thus satisfying the modified "Bohr assumption" of Tepel *et al.*[22] However, for smaller channel numbers we observe significant deviations from unity and these deviations are substantially larger than those predicted by the width fluctuation correction formula. Thus caution is advisable in the application of the approximation formulas to cross section ratios.

Ratios such as R have been measured experimentally in situations where reactions competing with many channels dominate.[24] When angular momentum and isospin conservation effects are taken into account these experiments are found to be consistent with $R=1$. To observe deviations from $R=1$ would require the comparison of channels with large differences in transmission coefficients and small numbers of competing channels.

The use of either Eqs. (23) or (25) requires a knowledge of the effective degree of freedom index ν_c for each channel. Tepel *et al.*[22] suggested that each ν_c depends only upon the transmission coefficient for that channel in a specific way such that ν_c goes from 1 to 2 as T_c goes from 0 to 1. Our experience with the program STASIG does not support this simple dependence. Figure 4 shows the effective elastic enhancements C and the ν indices obtained from the STASIG results in Table I, as

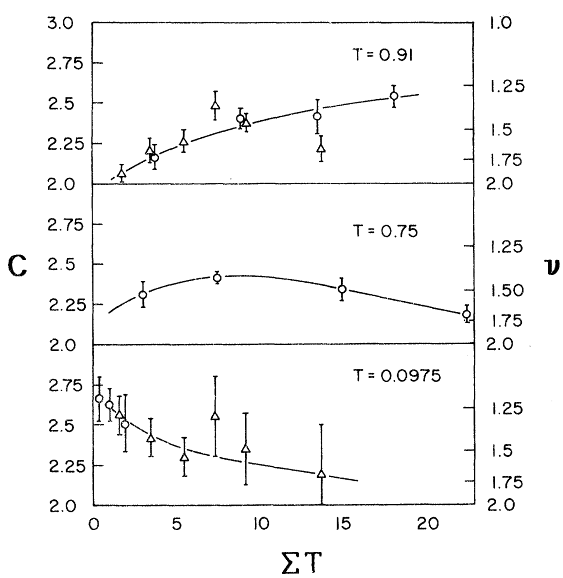

FIG. 4. Effective elastic enhancement factors C and distribution indices ν, for various channel transmission coefficients T and transmission coefficient sums ΣT, as obtained from statistically computed cross sections. The circles refer to calculations in which all channels have the same T, the triangles refer to the calculations described in Table I. The curves are free hand indications of the trends exhibited by the points. (Program STASIG.)

well as from STASIG calculations that were made with statistically equivalent channels for cases with $T_c=0.1$, 0.75, and 0.91 and with up to 30 channels. It is apparent from Fig. 3 that the effective ν_c depends strongly both on T_c and also on the channel sum of transmission coefficients $\sum T$, and possibly upon further details of the distribution of transmission factors.

The results are consistent with the hypothesis that ν_c approaches 2 as $\sum T$ becomes large. However, this approach is evidently considerably slower for large T_c than for small T_c. On the other hand, the effect of moderate variations in ν_c (say between 1.5 and 2.0) upon nonelastic cross sections is not very large when the number of channels is large. Therefore, the assumption that $\nu_c=2$ for all cases with $N>20$ will not introduce serious errors in nonelastic cross sections. The results from Table I are also consistent with the statement that the ν_c for all competing channels are the same when $\sum T$ is greater than 5. In view of the limited accuracies of the approximation formulas (23) and (25), interpolations based on Fig. 3 are probably adequate for most cases, particularly where nonelastic cross sections are concerned.

VIII. WIDTHS AND FLUCTUATIONS

In situations where very many channels are open it is often impractical to estimate the transmission coefficients for all channels, or even to identify all open channels correctly. In such cases it has become customary to replace the sum of transmission coefficients $\sum T$ which occurs in all of the approximation formulas (7), (23) and (25) by[25]

$$\sum_c T_c \cong 2\pi\Gamma^{\text{corr}}/\bar{D}, \tag{26}$$

Here $\bar{D}$ is the mean spacing of resonances as estimated from theoretical level density formulas and Γ^{corr} is the correlation width obtained by fitting the energy autocorrelation function $C(E)$ of the cross section to the formula[26]

$$C(E)=\frac{C(0)}{1+(E/\Gamma^{\text{corr}})^2}. \tag{27}$$

This formula comes from cross section fluctuation theory[26] where it is assumed that in the case of very many open channels all widths become equal to the average width $\bar{\Gamma}$ and it is then shown that $\bar{\Gamma}$ is identified with the Γ^{corr} of Eq. (27).

This procedure is open to question on two counts. Because of the strong channel-channel correlations it is no longer obvious that the widths become equal in the many channel limit. Indeed, numerical evidence that puts this assumption in question has already been presented.[7] But if the widths are not approximately equal the cross section fluctuation theory becomes suspect and with it Eq. (27).

Secondly, even if it turned out that $\overline{\Gamma} = \Gamma^{\rm corr}$, it has already been shown that Eq. (26) would be incorrect and we should instead write[6,7,27]

$$\sum_c \ln\frac{1}{1-T_c} = 2\pi\overline{\Gamma}/\overline{D}. \quad (28)$$

Nevertheless, the use of Eqs. (26) and (27) in conjunction with the Hauser-Feshbach formula (7) has lead to generally consistent results.[25] In order to understand this, we have studied the distribution of widths obtained from MATDIAG calculations and also the cross section auto correlation functions obtained from corresponding STASIG calculations. Figure 5 shows a typical width distribution histogram for the case of 20 open channels with all $T = 0.91$. The distribution of widths is seen to be very broad and very skewed. There are no very narrow widths at all. A peak in the distribution occurs near the minimum width (in this case about $\overline{\Gamma}/3$), and a long tail reaches to values of several times the average width $\overline{\Gamma}$. The average width $\overline{\Gamma}$ agrees well with the prediction of Eq. (28). The correlation width $\Gamma^{\rm corr}$ predicted by Eq. (26) lies at the low width peak of the distribution. Also shown in Fig. 5 on the same scale of widths are the points obtained by averaging all 190 independent auto correlation functions of the statistically computed elastic and nonelastic cross sections. Fitting these points with the formula (27) produces a correlation width $\Gamma^{\rm corr}$ which is in good agreement with the prediction of Eq. (26). The numerical results for $\overline{\Gamma}$, $\Gamma^{\rm corr}$, and Eqs. (26) and (28) are shown in Table II for this case and three others, as well.

TABLE II. Average widths and correlation widths obtained from statistical model calculations with n statistically equivalent channels having transmission coefficients T_c, compared to the predictions of Eqs. (26) and (27).

n	T	nT Eq. (26)	$2\pi\Gamma^{\rm corr}/\overline{D}$ (STASIG)	$-n\ln(1-T)$ Eq. (27)	$2\pi\overline{\Gamma}/\overline{D}$ (MATDIAG)
20	0.75	15.0	14.7	27.7	31.4
30	0.75	22.5	20.7	41.6	43.7
10	0.91	9.1	9.2	24.1	26.1
20	0.91	18.2	17.4	48.2	54.4

The conclusion to be drawn is this. The widths are not narrowly distributed and fluctuation theory is therefore not well founded. Nevertheless, the cross section auto correlation function is well represented by Eq. (27) for small values of the argument. However, $\Gamma^{\rm corr}$ is not related to the average width $\overline{\Gamma}$, but rather to the minimum width. Moreover, this $\Gamma^{\rm corr}$ satisfies Eq. (26), while $\overline{\Gamma}$ satisfies Eq. (28).

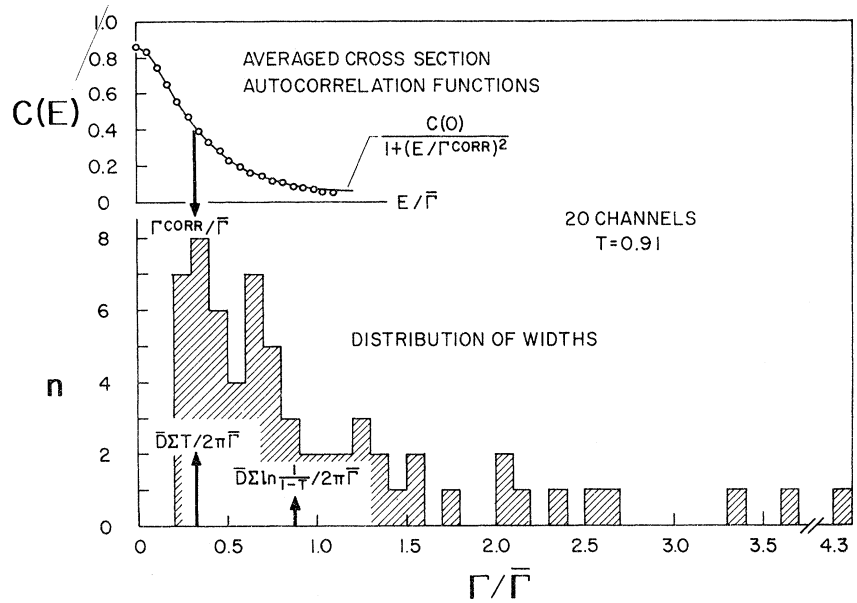

FIG. 5. Histogram of the number n of statistically computed widths falling within each width interval for the case of 20 statistically equivalent channels, each having $T = 0.91$. The points show the average of statistically computed cross section auto correlation functions for the same case. The results are compared with the predictions of Eqs. (26), (27), and (28). (Programs MATDIAG and STASIG.)

IX. CONCLUSIONS

We have shown that flux conservation (S-matrix unitarity) requires channel-channel correlations of the resonance amplitudes of reactions even when no direct reactions are present. By means of statistical model calculations of resonance parameters we have demonstrated these channel-channel correlations, as well as the effects of weaker level-level correlations. We have also shown that the magnitudes of these correlations conspire to cancel the very large non-Hauser-Feshbach terms in the expression for the fluctuation cross section, at least approximately (M cancellation). The remaining multiplicative corrections to the Hauser-Feshbach formula become unimportant for nonelastic cross sections when the number of open channels n is large. For the evaluation of elastic fluctuation cross sections and for nonelastic cross sections when n is small, two approximate formulas are available whose results we have compared with averages of statistically generated cross sections. These approximations can be used to obtain fairly reliable results when used in conjunction with empirically determined channel distribution indices ν_c. Caution in the use of the approximation formulas is indicated, when cross section ratios are to be determined.

The successful use of the cross section correlation width in estimating the denominator of the Hauser-Feshbach formula is explained by the results of statistical model calculations of width distributions and cross section correlation functions. We have shown that for large T the correlation width is close to the minimum width and much less than the average width and that $2\pi\Gamma^{\text{corr}}/\overline{D} \approx \sum T$.

We have raised the question of finding a better representation of the reaction amplitude which might yield all of these results more directly without the cancellation of large contributions to the average cross section and without complicated nonlinear relationships between resonance parameters and transmission coefficients, and we have argued that no such improved representation is as yet available.

The implications of these results for fluctuation cross sections that compete with direct reactions and for the theories of precompound processes and of cross section fluctuations will be discussed in future publications.

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For each S-matrix sample containing between 500 and 2000 poles, STASIG computes cross sections at a large number of energies and then averages them over energy. In contrast the calculations of Ref. 22 obtain cross sections at only one energy for each S-matrix sample containing typically 100 poles, and average them over many such samples. The difference between our conclusions and those of Ref. 22 appear to be due in part to the difference in averaging procedures and in part to the difference in sample sizes.

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