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Relativistic and retardation effects in the Thomas-Reiche-Kuhn sum rule for a bound particle*

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Relativistic and retardation corrections of lowest order are evaluated for the photoabsorption Thomas-Reiche-Kuhn (TRK) sum rule of a particle bound in a potential. It is found that the retardation corrections partially cancel and the remainder is canceled by part of the relativistic corrections. The latter corrections depend on the Lorentz-transformation properties of the potential. The complete integrated photoabsorption cross section including first-order relativistic corrections, is found to be equal to the high-energy limit of the Compton amplitude calculated by Goldberger and Low, a result which is the dispersion-relation form of the TRK sum rule. A heuristic derivation of the latter results is presented.

Following the pioneering work of Kramers[1] and Kronig[2] on the effect of causality on the index of refraction of a medium and the suggestion of Kronig[3] that the effect of causality on the S matrix may be important, Gell-Mann, Goldberger, and Thirring[4] derived the analytic properties of the forward scattering amplitude of photons on any system. They used the strong requirement of microcausality in the form that the Green's function of the physical system must vanish when the arguments have a spacelike separation. Their result, that the forward scattering amplitude is analytic in the upper-half plane of the photon-energy variable ω, allowed them to write a dispersion relation for the amplitude in the usual way.

Because an unsubtracted dispersion relation for the non-spin-flip amplitude (the only one in which we are interested) is inconsistent with the low-energy theorem for Compton scattering and the optical theorem, and because of convergence difficulties, they wrote a once-subtracted dispersion relation:

$$\mathrm{Re} f(\infty) = \mathrm{Re} f(0) + \frac{\omega^2}{2\pi^2} P \int_0^\infty \frac{\sigma_{\rm abs}(\omega')}{\omega'^2 - \omega^2}\, d\omega' \quad . \tag{1}$$

The symbol P denotes the principal value integral and the optical theorem for photoabsorption has been used to eliminate the imaginary part of f. Of particular interest is the $\omega \to \infty$ limit, if the integral in Eq. (1) converges:

$$\mathrm{Re} f(\infty) = \mathrm{Re} f(0) - \frac{1}{2\pi^2} \int_0^\infty \sigma_{\rm abs}(\omega') d\omega' \quad . \tag{2}$$

This sum rule for photoabsorption now depends on the difference of the real part of the scattering amplitude at infinite and zero energies. The latter amplitude is easy to calculate and is known from general principles, while the former is difficult to calculate.[5]

Our purpose is to examine the relationship (2) for the special case of a single particle bound in a potential. The dynamics will be treated semirelativistically, that is, including $(v/c)^2$ corrections to the usual nonrelativistic dynamics. This will allow us to calculate $\mathrm{Re}[f(0)]$ and $\int \sigma_{\rm abs}$, while the remaining quantity $\mathrm{Re}[f(\infty)]$ will be taken from the work of Goldberger and Low.[5] It should be noted that because our treatment is semirelativistic it is not possible to neglect the effect of pair production in the static field of the potential (virtual pair production and annihilation are already included in the $\vec{A}^2/2m$ "seagull" term). For this reason, the scattering amplitude of Eqs. (1) and (2) is really the amplitude for scattering of light by the particle bound in the potential (our "atom") minus the amplitude for light scattering in the static potential field in the absence of the bound particle. This is fully discussed by Erber.[6]

Our model is a Dirac particle of mass m bound in a potential $V_s + \beta V_v$, where V_s is a world-scalar potential and V_v is a vector potential (e.g., a Coulomb potential). We also introduce the vector electromagnetic potential $\vec{A}$.

$$i\frac{\partial \psi}{\partial t} = H\psi = \vec{\alpha} \cdot (\vec{p} - e\vec{A}) + \beta(m + V_s) + V_v \psi \quad . \tag{3}$$

Performing a Foldy-Wouthuysen reduction[7] we find to order $(1/m)^3$ [reckoning a potential as $(1/m)$ for a weakly-bound system] and defining $\vec{\pi} = \vec{p} - e\vec{A}$

$$i\frac{\partial \phi}{\partial t} = H'\phi \ , \tag{4a}$$

$$H' = m + \frac{\vec{\pi}^2}{2m} - \frac{\vec{\pi}^4}{8m^3} + V_s + V_v + \frac{1}{8m^2}\vec{\nabla}^2 V_v - \frac{1}{8m^2}\{\vec{\pi}\,;\{\vec{\pi},\, V_s\}\} + \frac{1}{4m^2}\vec{\sigma} \cdot [\vec{\nabla}(V_v - V_s) \times \vec{\pi} + 2e V_s \vec{\nabla} \times \vec{A}] + H_{\rm mag} + H_{\rm so} \tag{4b}$$

plus additional terms of order $(1/m^3)$ independent of V which vanish for constant $\vec{A}$ and which will not be needed. The quantities H_{mag} and H_{so} are the usual magnetic and electromagnetic spin-orbit interactions.[8] For a spinless particle a similar form results, without the spin-dependent terms and the $\vec{\nabla}^2 V_v$ term.

The integrated photoabsorption cross section can be written in the form

$$S = \frac{1}{2\pi^2\alpha}\int_0^\infty \sigma_{\text{abs}}(\omega)d\omega$$

$$= 2\sum_N \frac{1}{\omega_N}|\langle N|\vec{\epsilon}\cdot\vec{J}(\vec{k}_N)|0\rangle|^2 , \tag{5}$$

where ω_N is the energy of the state $|N\rangle$ relative to the state $|0\rangle$, i.e., the eigenvalues of H with $\vec{A}=0$ (denoted H_0), $\vec{\epsilon}$ is the photon polarization vector, $\vec{k}_N$ is the photon momentum corresponding to an energy ω_N ($\vec{k}_N=\omega_N\hat{k}$), and in terms of the current operator $\vec{J}(\vec{x})$

$$\vec{J}(\vec{k}) = \int d^3x\,\vec{J}(\vec{x})e^{i\vec{k}\cdot\vec{x}} . \tag{6}$$

Because ω_N is a quantity presumably of order $(1/m)$ we can expand the exponential and keep terms of second order in the expansion. In these "retarded" terms we use the nonrelativistic current operator and include the spin (magnetic) part. We obtain

$$\vec{J}(\vec{k}) = \frac{1}{m}\vec{p} + \Delta\vec{J} + \frac{i}{2m}\{\vec{p},\vec{k}\cdot\vec{r}\}$$
$$+\frac{i}{2m}\vec{\sigma}\times\vec{k} - \frac{1}{4m}\{\vec{p},(\vec{k}\cdot\vec{r})^2\} - \frac{1}{2m}(\vec{\sigma}\times\vec{k})\vec{k}\cdot\vec{r} , \tag{7}$$

where the second term $(\Delta\vec{J})$ is the unretarded current from the relativistic corrections of order $(1/m^3)$. The third term is the (retarded) $E2$ and $M1$ contribution, the next term is the magnetic $M1$ term while the last two are retarded $E1$ terms (plus other multipoles which do not contribute). Calculating those parts of Eq. (5) which involve retardation terms in the standard way[9] we find

$$S_{\text{ret}} = -\langle 0|\frac{1}{m^2}(\hat{k}\cdot\vec{r})^2(\vec{\epsilon}\cdot\vec{\nabla})^2 V_0|0\rangle$$
$$+\langle 0|\frac{1}{m^3}(\vec{\epsilon}\cdot\vec{p})^2 + \frac{1}{m^2}(\hat{k}\cdot\vec{r})^2(\vec{\epsilon}\cdot\vec{\nabla})^2 V_0|0\rangle , \tag{8}$$

where $V_0 = V_s + V_v$ and the two terms are the retarded $E1$ and $E2$ contributions [the $M1$ parts vanish as does the spin-orbit contribution mentioned below Eq. (4b)]. The unretarded current is obtained from Eq. (4b) by using $\vec{J}_0 \equiv -\delta H'/\delta\vec{A}|_{\vec{A}=0} = i[H_0,\vec{r}]$; one obtains

$$\vec{J}_0 = \frac{\vec{p}}{m} + \Delta\vec{J} = \frac{\vec{p}}{m} - \frac{\vec{p}^2\vec{p}}{2m^3} - \frac{\{\vec{p},V_s\}}{2m^2} . \tag{9}$$

Using this current operator one easily finds[9] the unretarded contribution to S (denoted S_0)

$$S_0 = \langle 0|[\vec{r}\cdot\vec{\epsilon},[H_0,\vec{r}\cdot\vec{\epsilon}]]|0\rangle$$
$$= \frac{1}{m} - \langle 0|\frac{1}{m^3}(\vec{p}\cdot\vec{\epsilon})^2 + \frac{\vec{p}^2}{2m^3} + \frac{1}{m^2}V_s|0\rangle . \tag{10}$$

In order to check the consistency of our calculation, we also calculate the Compton amplitude at zero photon energy, which is known to vanish for this model[4] because of general principles. This amplitude is the sum of the "seagull" contribution and the part due to virtual excitation of the particle in the potential. By means of a standard calculation the latter contribution to f can be shown to be given by S_0. The seagull piece is determined in the standard way to be

$$f_{\text{SG}} = -\langle 0|\left(\vec{\epsilon}\cdot\frac{\delta}{\delta\vec{A}}\right)^2 H'|_{\vec{A}=0}|0\rangle$$
$$= -\langle 0|\left(\vec{\epsilon}\cdot\frac{\partial}{\partial\vec{p}}\right)^2 H_0|0\rangle$$
$$= -\langle 0|[\vec{r}\cdot\vec{\epsilon},[H_0,\vec{r}\cdot\vec{\epsilon}]]|0\rangle . \tag{11}$$

Thus f_{SG} is the negative of S_0 and their sum, which is the zero energy Compton amplitude, vanishes, as it must.

There is considerable cancellation among the various parts of $S(S_{\text{ret}}$ and $S_0)$. The potential terms cancel in S_{ret}. The remaining term cancels a part of Eq. (10) and the complete result from Eq. (2) is

$$-\frac{f(\infty)}{\alpha} = S \equiv S_{\text{ret}} + S_0 \cong [m + \langle\vec{p}^2/2m\rangle + \langle V_s\rangle]^{-1} . \tag{12}$$

Note that V_v does not contribute to S. The result (12) for $f(\infty)$ was originally derived by Goldberger and Low from first principles [see their Eqs. (2.15) and (2.18) for the case of a spin-0 particle and the analytic results 3.2 and 3.4 for the special case of a spin-$\frac{1}{2}$ particle in a $1/r$ potential, which can be shown to agree with (12)]. In an early paper,[9] the result (12) for the sum rule was worked out for the special case of a vector potential. We see that to the order we have worked it out, Eq. (2) is self-consistent, our sum rule results equalling the $f(\infty)$ calculated by Goldberger and Low.

The relationship (12) is unlike the original assumption of Gell-Mann, Goldberger, and Thirring, which was that S should be just $1/m$. We can rather easily see why this assumption was unjustified. The relativistic increase of mass with velocity seen in Eq. (4b) means that the Compton amplitude should be $1/E \simeq 1/m - p^2/2m^3$ rather than just $1/m$, since the bound particle is moving with a velocity $\approx \vec{p}/m$ when struck by the photon. Another way of stating the result is that the effective mass of a particle with a momentum-dependent Hamiltonian is given by $1/m_{\text{eff}} \simeq 2(\partial/\partial\vec{\pi}^2)H'|_{\vec{A}=0} \simeq 1/m - \vec{p}^2/2m^3 - V_s/m^2$ which gives the kinetic mass increase term and an additional term proportional to the scalar potential. This term, which is also seen in Eq. (11) was originally discussed in a similar context by Lipkin and Tavkhelidze.[10] In Eq. (3) we see directly that $(m+V_s)$ can be regarded as an effective mass. When a reduction is made of Eq. (3) in powers of $(1/m)$, one effectively expands in terms of the quantity $m+V_s$. The contribution of V_s to the magnetic moment interaction is explicit in Eqs. (4b), $(\vec{\sigma}\cdot\vec{B}/2m)(V_s/m)$. For all these equivalent reasons one expects V_s to enter into the Thompson cross section. Indeed, both $\vec{p}^2/2m$ and V_s corrections appear in the relativistic corrections to the nuclear magnetic moment operator[11] and the pion-absorption operator[12] for the reason just mentioned.

In summary, we wish to make four points. (1) The potential parts of the retardation corrections cancel out but the kinetic part of the retardation results remains. (2) The relativistic corrections are probably *at least* as large as the retardation corrections. For a barely-bound particle, interacting with a scalar potential the complete correction cancels, the relativistic correction just cancelling the retardation term. For a particle interacting with a vector potential, the relativistic correction is 2.5 times as large as the retardation correction. (3) The relativistic corrections depend on the dynamical details. (4) Because of the results of Ref. 1 the various pieces of the dispersion relations (2) can be calculated and are consistent with each other.

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