Isospin splitting of the giant resonance. I. Isodoublets

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We study the isospin splitting of the giant resonance of $T=\frac{1}{2}$ nuclei with sum rules. The particular isospin geometry involved in this case permits a number of simplifications in the isospin sum rules and several largely model independent relations are obtained and discussed. It turns out that for several light isodoublets the "symmetry" energy U tends to be smaller that $60(T+1)A^{-1}$ MeV.

[NUCLEAR STRUCTURE $T=\frac{1}{2}$ nuclei; calculated isospin splitting of $E1$ resonance with sum rules, model independent results discussed.]

I. INTRODUCTION

This paper arose out of a study of the isospin dynamics involved in the dipole isovector excitation of nuclei. In this paper we present results which have emerged in the case of $T=\frac{1}{2}$ nuclei, whereas in a subsequent paper we shall analyze the more subtle case of $T>\frac{1}{2}$ nuclei.

Since an isospin structure has been predicted in the giant dipole photoexcitation of nuclei, a massive experimental effort has been made to determine the isospin of photonuclear resonance (for successes and failures see the review[1] of Paul) and a considerable amount of data and corroborating evidence for the existence of the two T components in the giant dipole state is now available. The distribution of dipole strength and the displacement among the two components has been estimated by different methods. The data essentially follow the theoretical prediction; however, both the experimental and the theoretical situation requires more detailed work. In light nuclei a significant amount of dipole strength is concentrated in the T-upper component, thereby favoring the experimental observation of the dipole isospin splitting; however, the dipole state itself is influenced strongly by nuclear structure aspect and does not yet have the simplicity observed in the heavier nuclei (in these last the giant resonance has essentially a Lorentzian shape; however, problems arise in the identification of the T-upper component).

In general it can be said that the systematics appear to demonstrate that isospin splittings exist in the light mass region also, but difficulties arise in the appropriate location of the strength. On the other hand the different[2,3] theoretical predictions which in the medium and heavy region are in good agreement [the so-called 60 MeV law emerges in this region; i.e., the isospin splitting ΔE of the dipole states of a target T is essentially predicted by the formula $\Delta E=60(T+1)A^{-1}$ MeV] are quite different for light nuclei and the prediction from isospin sum rules tends to be smaller than 60 MeV.

The object of this work is to perform a careful and, as far as possible, model independent analysis of this problem both for the experimentalists faced with the interpretation of the data and for those theoreticians undertaking different shell model calculations to test the consistency of their models.

To realize our program we use the isospin sum rules approach[4] of Leonardi and Rosa-Clot. The essence of the method has been illustrated in different contexts but in the present work we will focus on $T=\frac{1}{2}$ nuclei for which a number of simplifications are allowed. Furthermore, some model dependent results of the previous analysis are avoided with the help of new and unexplored isospin sum rules. The case of $T>\frac{1}{2}$ nuclei will be treated separately in a forthcoming paper.

Some important specific results that have emerged are as follows:

(i) An upper limit is found for the isospin splitting of $T=\frac{1}{2}$ nuclei which is largely *model independent;* as a consequence we obtain

(ii) for $A=11$, 13, 15, and 17 the isospin splitting tends to fall to lower values in respect to the 60 MeV law. Available shell model calculations in ^{13}C largely exceed this upper limit. It is argued that the failure of the model to achieve consistent isospin splitting is caused by using wave functions which fail to get a reasonable nuclear matter rms radius.

An essential preliminary to our program is the evaluation of the parameters entering the dipole sum rules and this is carried out in the next three sections (II–IV).

In Secs. V and VI we apply our formalism to several $T=\frac{1}{2}$ nuclei and the numerical results will be discussed and compared with other calculations. In Sec. VII we treat the special case of the $A=3$ system for which further simplifications are allowed due to the spatial symmetry of the wave function. Finally in Sec. VIII we will anticipate some of the reasons which make the isospin splitting of the $T>\frac{1}{2}$ nuclei a more complex and, from several aspects, a rather interesting problem.

II. ISOSPIN SUM RULES

In the past Leonardi and Rosa-Clot[4] have set up a theory to evaluate the parameters entering in the isospin structure of the giant resonance based on isospin sum rules.

We recall here the most significant results. For economy of words we will refer to the q-energy weighted sum rules as the σ_{q-1} sums, these being the symbols we used in the text. For example $\sigma_{-1}(q=0)$ determines the bremsstrahlung-weighted integrated cross section $\sigma_{-1}\equiv\int E^{-1}\sigma(E)dE$, σ_0 determines the integrated absorption cross section $\sigma_0\equiv\int\sigma(E)dE$. Two critical quantities used in the past for characterizing the effect of the dipole operator $D_3^{\pm}=\frac{1}{2}\sum_i \tau_{i3}^{\pm}x_i$ on the ground state of the target $|0\rangle\equiv|TT_3\rangle$ are the quantities σ_{-1} and σ_0 defined by the equations:

$$\frac{137}{4\pi^2}\sigma_{q-1}=\sum_n E_n^q|\langle n|D_3|TT_3\rangle|^2 =\langle TT_3|D_3H^qD_3|TT_3\rangle\,, \tag{1}$$

where $H|n\rangle=E_n|n\rangle$, $E_0=0$, $q=0$, 1, and closure has been used for the last equality. Another interesting quantity (considered by Levinger and Bethe[5] a long time ago) is the σ_1 sum which determines $\int\sigma(E)E\,dE$, which correspond to $q=2$ in (1).

The sums in (1) are unrestricted in the isospin indices so that the isovector operator D admixes states with $T'=T-1$, T, $T+1$. To set up the isospin restricted sum rules $\sigma_{-1}(T')$, $\sigma_0(T')$ one sums over states $|n,\ T',\ T_3'\rangle$ of isospin T' only. An essential step in our program is to analyze the unexplored restricted isospin sum rules $\sigma_1(T')$ too; as we shall see in Sec. IV these will be extremely useful to obtain model independent upper limits on the isospin splitting.

The separation of sum rules for different isospins of the states excited by the dipole operator is very conveniently made by expressing the right-hand side of (1) as a sum of its isoscalar ($\equiv\sigma^s_{q-1}$), isovector ($\equiv\sigma^v_{q-1}$), and isotensor (σ^t_{q-1}) parts.

This has the great merit of facilitating separation of the model dependent parts (those involving two and three body operators) from the model independent (those involving one body operators).

In the special case of $T=\frac{1}{2}$ nuclei the isotensor part is strictly 0 for geometrical reasons so that $\sigma^s_{q-1}=\sigma_{q-1}$. Furthermore the $T-1=T'$ channel does not exist. This greatly simplifies our analysis.

Once the proper isospin geometry is carried out one obtains the following set of independent sum rules:

$$\sigma_{q-1}(T+1)\equiv\int E^{q-1}\sigma(E,\,T+1)dE=\frac{1}{T+1}(\sigma_{q-1}-T\sigma^v_{q-1})\,, \tag{2}$$

$$\sigma_{q-1}(T)\equiv\int E^{q-1}\sigma(E,\,T)dE=\frac{T}{T+1}(\sigma_{q-1}+\sigma^v_{q-1})\,, \tag{3}$$

where

$$\sigma^v_{q-1}=\frac{4\pi^2}{137}\langle 0|(D\otimes H^q\otimes D)_{\text{isovector part}}|0\rangle\ .$$

Furthermore, we define the relative intensity x_q of the $T+1$ fragment of the dipole excitation and the energy splitting ΔE of the $T+1$ and T giant resonances as

$$x_q=\frac{\sigma_{q-1}(T+1)}{\sigma_{q-1}},\qquad \Delta E=\frac{\sigma_0(T+1)}{\sigma_{-1}(T+1)}-\frac{\sigma_0(T)}{\sigma_{-1}(T)}$$

x_q and ΔE are easily determined once σ_{q-1} and σ^v_{q-1} are known; in fact one has (for $T=\frac{1}{2}$ nuclei)

$$x_q=\frac{1}{T+1}-\left(\frac{T}{T+1}\right)\frac{\sigma^v_{q-1}}{\sigma_{q-1}}\ , \tag{4}$$

$$\Delta E=(T+1)\frac{\sigma_0}{\sigma_{-1}}\left(\frac{\sigma^v_{-1}}{\sigma_{-1}}-\frac{\sigma^v_0}{\sigma_0}\right)\left[\left(1+\frac{\sigma^v_{-1}}{\sigma_{-1}}\right)\left(1-T\frac{\sigma^v_{-1}}{\sigma_{-1}}\right)\right]^{-1}; \tag{5}$$

explicitly one has

$$\sigma_{-1}=\frac{4\pi^2}{137}\langle 0|D_3D_3|0\rangle\ , \tag{6}$$

$$\sigma^v_{-1}=\frac{2\pi^2}{137}\,\frac{1}{T_3}\langle 0|[D^-,D^+]|0\rangle\ , \tag{7}$$

$$\sigma_0=\frac{2\pi^2}{137}\langle 0|[D_3,[H,D_3]]|0\rangle\ , \tag{8}$$

$$\sigma^v_0=\frac{\pi^2}{137}\,\frac{1}{T_3}\langle 0|\{[D^-,H],D^+\}-\{[D^+,H],D^-\}|0\rangle\ , \tag{9}$$

$$\sigma_1=\frac{2\pi^2}{137}\langle 0|\{[D_3,H],[H,D_3]\}|0\rangle\ , \tag{10}$$

$$\sigma_1{}^v=\frac{2\pi^2}{137}\,\frac{1}{T_3}\langle 0|[[H,D^+],[H,D^-]]|0\rangle\ . \tag{11}$$

III. EVALUATION OF SUM RULES

In this section we focus our attention on the evaluation of the quantities

$$\sigma^{v}_{q-1}/\sigma_{q-1}\ .$$

For the $q=0$ case this ratio may be considered known from experiments. In fact, using $[\tau^+,\tau^-]=-2\tau_3$,

$$\frac{3}{2}\frac{137}{\pi^2}\sigma^{v}_{-1}\equiv r_v{}^2=\frac{3}{2T_3}\langle TT_3|\sum_i x_i{}^2\tau_{3i}|TT_3\rangle$$

$$=\left(\langle r_p{}^2\rangle+\frac{N\epsilon}{2T}\right), \qquad (12)$$

where $r_v{}^2$ is the isovector rms radius and $\langle r_p{}^2\rangle$ is the rms radius of the protons, N is the number of the neutrons and $\epsilon=\langle r_n{}^2\rangle-\langle r_p{}^2\rangle$. For light nuclei the approximation $\epsilon=0$ is rather justified so that

$$\sigma^{v}_{-1}=\frac{2}{3}\frac{\pi^2}{137}\langle r^2{}_p\rangle\ .$$

σ_{-1} is measured for several nuclei; so that to the extent in which $\langle r_p{}^2\rangle$ and σ_{-1} are known quantities the amount of the upper fragment x_0 proves to be model independent for $T=\frac{1}{2}$ nuclei. For the evaluation of ΔE we need both x_0 and x_1; in the following we will analyze x_1. Working out the algebra of Eqs. (8) and (9) one obtains ($\hbar=c=1$)

$$\sigma_0=\frac{2\pi^2}{137}\left(\frac{NZ}{mA}-\frac{1}{4}\left\langle\sum_{i<j}(x_i-x_j)^2(\tau_{3i}-\tau_{3j})^2\right.\right.$$
$$\left.\left.\times V(r_{ij})(MP^x_{ij}-HP^\tau_{ij})\right\rangle\right)\ , \qquad (13)$$

where we have used

$$H=\sum_i^A\frac{p_i^2}{2m}+\sum_{i<j}V_{ij}\ ,$$

$$V_{ij}=(W+MP^x_{ij}+BP^\sigma_{ij}-HP^\tau_{ij})V(r_{ij})\ .$$

This is the well known T-R-K sum rule.[5] On the other hand we have

$$\frac{137}{2\pi^2}\sigma^v_0=\frac{1}{2m}\left(1-\frac{1}{A}\right)-\frac{1}{4T_3}\left\langle\sum_{i<j}(x_i-x_j)^2(\tau_{3i}+\tau_{3j})V(r_{ij})(MP^x_{ij}-HP^\tau_{ij})-\frac{i}{4mT_3}\left\langle\sum_{k\neq t}p_kx_t(\tau^+_k\tau^-_t-\tau^+_t\tau^-_k)\right\rangle\right.$$
$$\left.+\frac{1}{4T_3}\left\langle\sum_{t\neq i,j}V(r_{ij})(MP^x_{ij}-HP^\tau_{ij})x_t(x_i-x_j)(\tau^+_t\tau^-_i-\tau^-_t\tau^+_i-\tau^+_t\tau^-_j+\tau^-_t\tau^+_j)\right\rangle\ . \qquad (14)$$

These quantities are rather complicated and model dependent and the evaluation of them must be based on models. In the following section we estimate these quantities with some models.

Let us first take a wave function which is a product (i.e., nonantisymmetric) of single particle plane wave functions. This means replacing single particle spatial states by plane waves within a sphere equal to the nuclear radius R.[6] (The short range nature of the two-body force justifies this approximation.) This gives

$$\frac{137}{2\pi^2}\sigma_0=\frac{1}{m}\frac{NZ}{A}-NZ(2M+H)J_E\ , \qquad (15)$$

where

$$J_E=(\tfrac{4}{3}\pi R^3)^{-1}\frac{1}{2}\int(x_1-x_2)^2V(r_{12})C^2(K_Fr_{12})d\vec{r}_{12}\ .$$

K_F is the usual Fermi momentum,

$$C(x)=\frac{3}{x^2}\left(\frac{\sin x}{x}-\cos x\right)$$

and $R=r_0A^{1/3}$.

Similarly one has

$$\frac{137}{2\pi^2}\sigma^v_0=\frac{1}{2m}\frac{A-1}{A}-\frac{A-1}{2}(2M+H)J_E \qquad (16)$$

and this gives

$$\frac{\sigma_0{}^v}{\sigma_0}=\frac{A-1}{2NZ}\ ; \qquad (17)$$

for $T=\frac{1}{2}$ nuclei this reduces to $2/A+1$.[7]

So far we have ignored the effect of the antisymmetrization on the wave function. The quantity σ_0, involving a summation over neutron-proton pairs only, is unaffected by Pauli effects. The quantity σ^v_0 involves sums over n-n and p-p pairs so that it is sensitive to the antisymmetrization effects and a quantitative estimate of these effects is imperative.

Taking into account the antisymmetrization of the wave function and disregarding three-body correlations one has

$$\frac{137}{2\pi^2}\sigma^v_0=\frac{1}{2m}\frac{A-1}{A}-\frac{A-1}{2}(2M+H)$$
$$\times J_E\left[1-\frac{(M+2H)J_D}{(2M+H)J_E}\right], \qquad (18)$$

where

$$J_D = \left(\frac{4\pi R^3}{3}\right)^{-1} \frac{1}{2} \int (x_1 - x_2)^2 V(r_{12}) d\vec{r}_{12}$$

for a Yukawa shape and a Rosenfeld mixture one has that

$$\chi \equiv 1 - \frac{2H+M}{2M+H} \frac{J_D}{J_E}$$

ranges from −2 to −1.5 for a potential of 1.4 and 1.3 fm range, respectively, and $r_0 = 1.2$. Conversely a Hamada-Johnston (H-J) potential gives for this ratio $\chi = 0.86$.[8]

Introducing now the dipole enhancement factor

$$k = -mA(2M+H)J_E$$

so that

$$\sigma_0 = \frac{2\pi^2}{137} \frac{NZ}{mA}(1+k) .$$

We can express the ratio σ_0^v/σ_0 as

$$\frac{\sigma_0^v}{\sigma_0} = \frac{2}{A+1}\left(\frac{1+\chi k}{1+k}\right) . \qquad (19)$$

From Eq. (19), remembering that $k \simeq 0.4/0.8$, one sees that whereas the Hamada-Johnston potential does not sensitively modify the uncorrelated result, a Rosenfeld mixture with Pauli effects gives for the ratio σ_0^v/σ_0 a significant reduction. A further remark is that in the framework of a pure harmonic oscillator model calculation $\sigma_0^v = \hbar\omega\sigma_{-1}^v$.

There is no doubt that these results are rather model dependent. Looking at the different contributions to σ_0^v [Eq. (14)] the potential part of σ_0^v strongly depends on the potential assumed (as mentioned before a smooth dependence is expected from using a Fermi gas ground state, due to the short range of the potential), whereas the *two-body* part of the kinetic contribution which is 0 with a Fermi gas model could depend on the ground state assumed.

It would be tempting to consider the result obtained with a realistic nucleon-nucleon interaction as the more reliable; however, for our purpose (to give as far as possible model independent constraints to the isospin structure of the giant resonance in light nuclei) these results are to be improved.

In the next section we overcome the previous difficulties with the help of new isospin sum rules.

IV. NEW ISOSPIN SUM RULES

We have seen that whereas σ_{-1}^v has a simple interpretation, so that the ratio $\sigma_{-1}^v/\sigma_{-1}$ may be considered known from experiments, x_1 is rather model dependent. In the following we want to circumvent this difficulty.

Keeping in mind that the $T_>$ fragment is qualitatively higher in energy than the $T_<$ and looking at formula (4) one easily realizes that

$$x_0 \lesssim x_1 \lesssim x_2$$

or

$$\frac{\sigma_{-1}^v}{\sigma_{-1}} \gtrsim \frac{\sigma_0^v}{\sigma_0} \gtrsim \frac{\sigma_1^v}{\sigma_1}$$

so that one obtains an upper limit for x_1 if x_2 is known. In the following we study σ_1^v/σ_1, showing that this ratio may be estimated in a model independent way: One obtains the result

$$\frac{\sigma_0^v}{\sigma_0} \gtrsim \frac{\sigma_1^v}{\sigma_1} \geqslant \frac{2}{A} .$$

Working out the algebra of Eqs. (10) and (11) one obtains

$$\frac{137}{4\pi^2}\sigma_1 = \frac{1}{6m}\left[\left\langle \sum_i \frac{p_i^2}{2m}\right\rangle + \left\langle \sum_{ij} \frac{(\vec{p}_i \cdot \vec{p}_j)}{2m}(\tau_{3i}\tau_{3j})\right\rangle\right]$$
$$- \frac{i}{8m}\left\langle \sum_{i<j} (\tau_{3i} - \tau_{3j})^2 (p_{xi} - p_{xj})(x_i - x_j) V(r_{ij})(MP_{ij}^x + HP_{ij}^\sigma P_{ij}^x)\right\rangle$$
$$+ \frac{1}{4}\left\langle \sum_{i<j} (\tau_{3i} - \tau_{3j})^2 (x_i - x_j)^2 V^2(r_{ij})(MP_{ij}^x + HP_{ij}^\sigma P_{ij}^x)^2\right\rangle \qquad (20)$$

and

$$\frac{137}{4\pi^2}\sigma_1^v = \frac{1}{6mT_3}\left\langle \sum_i \frac{\tau_{3i}}{2m} p_i^2\right\rangle - \frac{i}{8mT_3}\left\langle \sum_{i<j} (\tau_{3i} + \tau_{3j})[(p_{xi} - p_{xj})(x_i - x_j) V(r_{ij})](MP_{ij}^x + HP_{ij}^\sigma P_{ij}^x)\right\rangle$$
$$+ \frac{1}{4T_3}\left\langle \sum_{i<j} (\tau_{i3} + \tau_{j3})(x_i - x_j)^2 V^2(r_{ij})(MP_{ij}^x + H_{ij}^\sigma P_{ij}^x)^2\right\rangle . \qquad (21)$$

The first sum rule has been analyzed a long time ago by Bethe, Levinger, and Kent.[5,9] So far the second one has not been explored.

Using once more a Wigner-Seitz procedure for the terms involving the potential we obtain the following results:

$$\frac{137}{4\pi^2}\sigma_1=\frac{A}{6m}T_{av}+\frac{1}{6m}\left\langle\sum_{ij}\frac{\vec{p}_i\cdot\vec{p}_j}{2m}(\tau_{3i}\tau_{3j})\right\rangle -\frac{1}{4m}NZ(2M+H)G_E +\frac{NZ}{2}[2(M^2+H^2)+2MH]L_D\ , \quad (22)$$

$$\frac{137}{4\pi^2}\sigma^v_1=\frac{1}{m}\frac{T_{av}}{3}-\frac{A-1}{8m}(2M+H)G_E\left(1-\frac{2H+M}{2M+H}\frac{G_D}{G_E}\right) +\frac{A-1}{4}[2(M^2+H^2)+2MH]L_D \times\left\{1-\frac{[4MH+(M^2+H^2)]L_E}{[2(M^2+H^2)+2MH]L_D}\right\}, \quad (23)$$

where T_{av} is the mean kinetic energy per nucleon

$$G_D=\frac{6}{4\pi R^3}\int\frac{\partial}{\partial x_{12}}[x_{12}V(r_{12})]d\vec{r}_{12}\ ,$$

$$G_E=(\text{same as above})\int(\text{same})\cdot C^2(k_F r_{12})d\vec{r}_{12}\ ,$$

$$L_D=\frac{1}{4\pi R^3}\int V^2(r_{12})(r_{12})^2 d\vec{r}_{12}\ ,$$

$$L_E=\frac{1}{4\pi R^3}\int V^2(r_{12})C^2(k_F r_{12})(r_{12})^2 d\vec{r}_{12}\ .$$

It is clear that if the effects of the antisymmetrization were not taken into account on σ_1 [this is equivalent to disregarding the correlations $(\vec{p}_i\cdot\vec{p}_j)$ between the momenta] and on σ^v_1 [this is equivalent to assuming the term in the curly brackets in Eq. (23) is equal to 1] the ratio σ^v_1/σ_1 would be simply $2/A$. The effect of Pauli correlations on σ_1 is to *decrease* the uncorrelated value through the contribution $\sum_{i\neq j}(\vec{p}_i\cdot\vec{p}_j)\tau_{3i}\tau_{3j}$. (This contribution is certainly negative, as discussed by Levinger and Kent in Ref. 5). The effect of the antisymmetrization on σ^v_1 is model dependent but easily controlled. In fact G_D is rigorously zero for a wide variety of nuclear potentials. (Yukawa and square well shape included.) Furthermore L_E is obviously less than L_D (for any potential) and both for a Yukawa and square well potential one has $L_D\simeq 2L_E$.

Finally the ratio $[4MH+(M^2+H^2)]/[2(M^2+H^2)+2MH]$ is less than $\frac{1}{2}$ ($=\frac{1}{2}$ for $H=0$) for any reasonable choice of the interaction; in particular for realistic potentials like H-J and effective potentials of Rosenfeld type this ratio is negative and the net effect of the antisymmetrization is a (small) increase of the last term of Eq. (23).

Combining all these results one obtains the rather remarkable fact that

$$\frac{\sigma^v_1}{\sigma_1}\gtrsim\frac{2}{A} \quad (24)$$

and this last inequality is essentially *model independent*.

V. DISCUSSION

Coming back to our formula (5) one finds that for $T=\frac{1}{2}$ nuclei

$$\Delta E\lesssim\frac{3}{2}\frac{\sigma_0}{\sigma_{-1}}\left(\frac{\sigma^v_{-1}}{\sigma_{-1}}-\frac{2}{A}\right)\left[\left(1+\frac{\sigma^v_{-1}}{\sigma_{-1}}\right)\left(1-\frac{1}{2}\frac{\sigma^v_{-1}}{\sigma_{-1}}\right)\right]^{-1}\ . \quad (25)$$

Let us first show that the last factor in this formula is rather constant and ranges from 0.90 to 0.95 for different A and different choice of σ_{-1}.

In fact $\sigma^v_{-1}=2\pi^2/137\langle r^2\rangle/3=0.048\,\frac{3}{5}r_0^2A^{2/3}$ fm^2 and $\sigma_{-1}=\alpha A^{4/3}$ fm^2 with $r_0=1.2$ and $0.03\lesssim\alpha\lesssim0.04$ so that $A^{-2/3}\lesssim\sigma^v_{-1}/\sigma_{-1}\lesssim1.4A^{-2/3}$.

One concludes that for $6\lesssim A\leqslant 40$ the last factor in Eq. (25) varies from 0.90 to 0.95 so that one has

$$\Delta E\lesssim\frac{3}{2}\frac{1}{A}\left[\frac{\sigma_0}{\sigma_{-1}}\left(A\frac{\sigma^v_{-1}}{\sigma_{-1}}-2\right)0.95\right]\ . \quad (26)$$

The factor in parentheses is usually indicated as U in the literature.

This formula clearly shows us how the isospin splitting in $T=\frac{1}{2}$ nuclei depends on the giant resonance position and on $\sigma^v_{-1}/\sigma_{-1}$, so that a consistency condition must be satisfied between x_0 from (4) and ΔE from Eq. (25). As a consequence measurements on x_0 must be consistent with $\Delta E(x_0)$.

Our numerical results are shown in the table for a wide variety of σ_{-1} and σ_0. The radii are taken from experiments when available, in other cases an interpolation formula is used. Some comments on our assumptions are in order here.

Measurements[10] of σ_{-1} and σ_0 for $A\lesssim40$ from 10 MeV to the pion threshold indicate $\sigma_{-1}\approx(0.3-0.35)A^{4/3}$ mb and $\sigma_0\simeq(60NZ/A)\,(1.5\sim2)$ mb MeV. One should keep in mind, however, that the measured photocross section $\sigma(E)$ suffers uncertainty in the low energy region thereby introducing an uncertainty on σ_{-1} (this last quantity strongly depends on the low energy region due to the $1/E$ factor).

From these measurements after the subtraction of the so-called electronic cross section (theoretically computed) one obtains that $\sigma(E)$ tend to be negative below 15 MeV. As a consequence σ_{-1} is clearly underestimated in these cases. (This is certainly the case for ^{9}Be and ^{12}C.) This suggests that σ_{-1} could be larger than $(0.3-0.35)A^{4/3}$ mb for light nuclei.

As far as σ_0 is concerned, the uncertainty in the high energy tail of the electronic cross section is still significant and estimated to be about 10% thereby introducing an even larger uncertainty in the enhancement factor of the classical sum rule.

As a consequence of these remarks, final conclusions can be drawn when experiments are in a shape which allows a more detailed comparison. However, for say $A=11$, 13, and 15 one should conclude that the isospin splitting is definitely smaller than $U=40$ MeV.[11]

As a final remark we must point out that the isospin splitting measurements on the dipole photoexcitation are carried out in an energy region up to 30–35 MeV, whereas our formulas are obtained using total integrated cross sections (σ_0, σ_{-1}) up to π threshold so that our theoretical prediction should be compared with care with a similar formula obtained with *measured* cross sections truncated up to 30–40 MeV. However, in $T=\frac{1}{2}$ nuclei to the extent in which in the high energy tail of the $\sigma(E)$ there is no *dominance* of the $T_<$ (lower) fragment of the dipole (conversely rather general arguments support the contrary), one expects that ΔE measured $\lesssim \Delta E$ as defined from Eq. (26).

VI. COMPARISON WITH MODEL CALCULATIONS

Any detailed model calculation should be consistent with our sum rules. In particular the following check should be satisfied. Once σ_0, σ_{-1}, $\sigma_{-1}(T+1)$, and $\sigma_0(T+1)$ are calculated:
(i) the ratio $[\sigma_{-1}(T+1)]/\sigma_{-1}$ should be consistent with the ratio obtained from (4) using the "model" value of σ_1^v;
(ii) the ratio σ_0^v/σ_0 extracted (a) from Eq. (4) using the calculated ratio $[\sigma_0(T+1)]/\sigma_{-1}$ and (b) from Eq. (5) using the calculated σ_0, σ_{-1}, $\sigma_{-1}(T+1)$, and $\sigma_0(T+1)$ should be the same, and in fair agreement with a direct model calculation through formulas (9) and (14). Furthermore, any "realistic" model should give for σ_{-1}^v the "experimental" root mean square radius and $\sigma_0^v/\sigma_0 \lesssim 2/A$.

The check (ii) involving both the wave functions and the Hamiltonian interaction is obviously a more severe and significant consistency test than test (i).

A detailed "continuum" shell model calculation[12] is available for the isospin structure of the giant resonance of ^{13}C.

The results of these calculations are $\sigma_{-1}=0.4A^{4/3}$ mb,

$$\sigma_0 = 1.35\,\frac{NZ}{A}\,60\ \text{MeV mb},\quad \frac{\sigma_{-1}(T+1)}{\sigma_{-1}}=0.56\ \text{fm},$$

$$\frac{\sigma_0(T+1)}{\sigma_0}=0.65\ \text{fm}\ ,$$

$$\sigma_{-1}^v = 0.048\times 8.41\ \text{fm}^2,\quad \text{and } U=68\ \text{MeV}.$$

As a general remark we observe that this value of U is larger than our upper limits. [Our upper limits are obtained with $\sigma_{-1}=0.32A^{4/3}$ and $\sigma_0=1.7$ (T-R-K). Using σ_{-1} and σ_0 as in Ref. 12 the limits become $U\simeq 9$ MeV, i.e., one order of magnitude smaller than $U \simeq 68$ MeV.]

The first consistency check (i) is quite satisfactory but with the second we are in trouble; in fact σ_0^v/σ_0 from (4) and (5) are 0.049 and 0.065, respectively, with more than 30% of difference. The last step of check (ii) is more difficult to work out. As mentioned in Sec. III, χ in formula (19) is very model dependent thereby introducing a model dependence on ΔE as obtained from Eq. (5). Unfortunately calculations are available only with a Soper mixture. Finally one finds that σ_{-1}^v as calculated from the model is much larger than one would expect from our knowledge of the nuclear matter distribution. This is an important point and looking at the formulas one is led to suspect that in order to get reliable answers on the isospin structure of the giant resonance a microscopic model should properly predict the neutrons and protons rms radii, two aspects intimately connected. One should of course keep in mind that continuum shell model calculations have great merit in other contexts.

Turning to the isospin structure of the giant resonance more promising results are obtained from phenomenological models in which a symmetry energy term is inserted in the problem from the beginning both in the single particle energies of the "active" particles and in the residual interaction.[13]

VII. THREE NUCLEONS

Among the $T=\frac{1}{2}$ nuclei the $A=3$ system is a very special case.[14] For these nuclei the spatial part ψ of the wave function is expected to be mainly totally symmetric and as a consequence the Pauli principle plays no role in our sum rules (6, 9, 10, and 11) so that the only important correlations are those arising from the centre of mass condition. Using the properties of ψ and the relation $(\sum_i \vec{r}_i)^2=0$ one obtains the following simple results:

$$-\left\langle \sum r_i^2 \right\rangle = -A\langle r^2\rangle = \left\langle \sum_{ij} (\vec{r}_i\cdot\vec{r}_j)\right\rangle$$

$$= A(A-1)\langle \vec{r}_1\cdot\vec{r}_2\rangle$$

TABLE I. Values of U (symmetry energy) as calculated from (25) for different choices of α, β, r_{ch}, and A. The parameters α and β are defined from $\sigma_{-1} = \alpha A^{4/3}$ mb, $\sigma_0 = \beta 60(NZ/A)$ MeV mb.

	Symmetry energy U in MeV			
$A(r_{ch}$ fm)	$\alpha=0.32$ $\beta=1.6$	$\alpha=0.32$ $\beta=1.7$	$\alpha=0.35$ $\beta=1.6$	$\alpha=0.35$ $\beta=1.7$
9(2.42)	57	61	42	45
11(2.42)	46	49	34	21
13(2.32)	30	31	21	22
15(2.40)	46	49	35	37
15(2.60)	39	42	30	32
17(2.60)	48	51	37	39
19(2.80)	53	56	40	42
21(2.85)	51	55	39	41
23(2.95)	54	58	42	45
25(2.98)	53	56	41	43
27(2.93)	47	50	36	38
27(3.10)	58	62	45	48
31(3.07)	50	53	37	41
41(3.51)	64	67	50	53

and

$$\left\langle \sum_{ij} (\vec{r}_i \cdot \vec{r}_j)(\tau_{i3}\tau_{j3}) \right\rangle = (4T^2 - A)\langle \vec{r}_1 \cdot \vec{r}_2 \rangle = \frac{4T^2 - A}{A-1}\langle r^2 \rangle ,$$

so that

$$\sigma_{-1} = \frac{\pi^2}{137}\frac{\langle r^2\rangle}{3}\left(\frac{4NZ}{A-1}\right) \tag{27}$$

and

$$\frac{\sigma^v_{-1}}{\sigma_{-1}} = \frac{1}{2}\frac{r^2_v}{\langle r^2\rangle} . \tag{28}$$

A similar procedure may be carried out in computing the potential energy contribution to σ^v_0 and σ_0 and σ^v_1 and σ_1.

Using

$$\left\langle \sum_{i\neq j} (\tau_{i3} - \tau_{j3})^2 \right\rangle = 8NZ ,$$

$$\left\langle \sum_{i\neq j} (\tau_{i3} + \tau_{j3}) \right\rangle = 4T_3(A-1) ,$$

$$\left\langle \sum_{k\neq t} (\tau^+_k \tau^-_t - \tau^+_t \tau^-_k) \right\rangle = 0 ,$$

and

$$\left\langle \sum_{t\neq i,j} (\tau^+_t \tau^-_i - \tau^-_t \tau^+_i - \tau^+_t \tau^-_j + \tau^-_t \tau^+_j) \right\rangle = 0 ,$$

one obtains

$$\sigma_0 = \frac{2\pi^2}{137}\left(\frac{NZ}{mA} - \frac{NZ}{3}(M + \tfrac{1}{2}H)\langle r^2 V_{\rm eff}\rangle\right) , \tag{29}$$

$$\sigma^v_0 = \frac{2\pi^2}{137}\left[\frac{1}{2m}\left(\frac{A-1}{A}\right) - \frac{A-1}{6}(M-H)\langle r^2 V_{\rm eff}\rangle\right] , \tag{30}$$

where

$$V_{\rm eff}(r) = \tfrac{1}{2}(V(r)_{\rm singlet} + V(r)_{\rm triplet}) = V(r_{12}) = V(r_{13}) = V(r_{23}) .$$

Similarly one obtains

$$\sigma_1 = \frac{4\pi^2}{137}\left(\frac{2NZ}{9(A-1)}T_{\rm av} + \frac{NZ}{3}(M^2 + H^2 + MH)\langle r^2 V_{\rm eff}{}^2\rangle\right) , \tag{31}$$

$$\sigma^v_1 = \frac{4\pi^2}{137}\left(\frac{1}{9m}T_{\rm av} + \frac{A-1}{6}(M^2 + H^2 - 2MH)\langle r^2 V^2_{\rm eff}\rangle\right) , \tag{32}$$

so that for $A=3$ one has

$$\frac{\sigma^v_0}{\sigma_0} = \frac{1}{2}\left(1 + \frac{\frac{3}{2}H\langle r^2 V_{\rm eff}\rangle}{(1/m) - (M + \frac{1}{2}H)\langle r^2 V_{\rm eff}\rangle}\right) \tag{33}$$

and

$$\frac{\sigma^v_1}{\sigma_1} = \frac{1}{2}\left(1 - \frac{3MH\langle r^2 V_{\rm eff}{}^2\rangle}{(1/m)(T_{\rm av}/3) + (M^2 + H^2 + MH)\langle r^2 V_{\rm eff}{}^2\rangle}\right). \tag{34}$$

In formulas (12) and (28), r^2_v and $\langle r^2\rangle$ are the mean square isovector radius and radius, respectively, of the nucleus considering point nucleons. Measurements[15] available from electron scattering data include obviously finite size effects: Whereas for heavier nuclei these effects are small and to some extent they can be subtracted out, in the case of $A=3$ system both the *neutron* and the *proton* form factors play an important role in calculating the value of $r_v{}^2$ and $\langle r^2\rangle$ from the charge mean square radius $r_{\rm ch}{}^2$. The charge symmetry and the spatial symmetry of ψ would suggest that for $A=3$ $\langle r_n{}^2\rangle=\langle r_p{}^2\rangle$, i.e., $r_v{}^2=\langle r^2\rangle$. The relation between $r_{\rm ch}^2$ and r_i^2 may be written as

$$r_{\rm ch}{}^2=\left\langle \frac{1}{Z}\sum_i \left[\left(\frac{1+\tau_{3i}}{2}\right)(r_i{}^2+a) +\left(\frac{1-\tau_{3i}}{2}\right)(s)\right]\right\rangle ,$$

where a and s are interpreted as the mean square charge radius of the proton and neutron, respectively, (proportional to the charge form factors at $q^2=0$). Using[15] $a=0.64$ fm^2 and $s=-0.126$ fm^2 one obtains

$$r_p(^3\mathrm{H})\simeq 1.59\pm 0.05 \text{ fm}, \quad (r_{\rm ch}(^3\mathrm{H})=1.70\pm 0.05 \text{ fm}) ,$$

$$r_p(^3\mathrm{He})\simeq 1.70\pm 0.05 \text{ fm}, \quad (r_{\rm ch}(^3\mathrm{He})=1.87\pm 0.05 \text{ fm}) ,$$

so that the effect of the form factors is to make $r_n{}^2(^3\mathrm{H})[=r_p{}^2(^3\mathrm{He})]$ quite close to $r_p{}^2(^3\mathrm{H})$ within the experimental errors and the Coulomb effects. This implies that

$$\tfrac{1}{2}\lesssim \frac{\sigma_{-1}^v}{\sigma_{-1}}\lesssim 0.65 .$$

On the other hand σ_0^v/σ_0 may be estimated from (33) for a wide variety of potentials and it ranges from 0.46 (Yamaguchi nonlocal potential) to 0.59 (local effective potential) so that $U\lesssim 8$ MeV and the isospin splitting in $A=3$ system is less than 4 MeV.

VIII. $T>\frac{1}{2}$ nuclei

In this last section we briefly comment on nuclei with $T>\frac{1}{2}$.

From a theoretical point of view the isospin structure of the giant resonance, when analyzed with the sum rules approach of Leonardi and Rosa-Clot, exhibit quite different characteristics, following the case of $T=\frac{1}{2}$ and $T>\frac{1}{2}$ nuclei, so that we have preferred to treat the two cases separately. In this paper we have been concerned with $T=\frac{1}{2}$ nuclei; in a forthcoming paper we will discuss in detail the $T>\frac{1}{2}$ case. Let us here anticipate some interesting differences.

In the case of $T>\frac{1}{2}$ the term σ_{q-1}^t [the isotensor part of the sum rules (2, 3)] is different from zero, and in our formulas (4) and (5) one must take into account the effect of σ_{q-1}^t. (However, the formulas given in the previous sections for σ_{q-1} and σ_{q-1}^v are generally valid for any T nucleus.)

As a geometrical counterpart of being $\sigma_{q-1}^t=0$ one has that *three* isospin channels $(T+1,\ T,\ T-1)$ are generally excited for $T>\frac{1}{2}$ nuclei, by an *isovector* dipole excitation.

With this in mind we can define the average excitation energies of the various isospin channels $\sigma_0(T')=E_{T'}\sigma_{-1}(T')$ thereby introducing two types of isospin splitting, namely $\Delta E^+=E_{T+1}-E_T \equiv (T+1)\Delta^+$ and $\Delta E^-=E_T-E_{T-1}\equiv T\Delta^-$. The quantities $\Delta^++\Delta^-$ and $\Delta^+-\Delta^-$ have an interesting physical meaning. The first one is related to the so-called *isovector* potential and a microscopic basis for this explanation has been given by several authors. The second one is related to an *isotensor* potential the meaning of which has been discussed in Ref. 16. Till now little attention has been paid to this term (tacitly assuming that $\Delta^+-\Delta^-=0$) but several reasons suggest that this term could be of the same order of magnitude as the isovector.

It turns out[17] that an interesting relation connects $\Delta^+-\Delta^-$, $\Delta^++\Delta^-$, and σ_{-1}^t.

Finally for large excess neutron nuclei ($T\gg\frac{1}{2}$) the approximation $\sigma_{-1}^v\simeq r_p{}^2$ is predictably rough. σ_{-1}^v is related to the difference between the proton and neutron radii and extremely sensitive to them [see Eq. (12)], i.e.,

$$\sigma_{-1}^v=\frac{1}{137}\frac{2\pi^2}{3}\left(\langle r_p{}^2\rangle+\frac{N\epsilon}{2T}\right) .$$

This fact makes reasonable the suggestion that from a knowledge of σ_{-1}^v we can obtain valuable information about the mean square radius of the *neutron* distribution.

All these quantities, namely, $\sigma_{-1}{}^t$, $\Delta^+-\Delta^-/\Delta^++\Delta^-$, and neutron-proton distribution differences (σ_{-1}^v) are deeply interconnected and make the isospin structure of the giant resonance of $T>\frac{1}{2}$ nuclei an extremely interesting topic.

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