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Four-body forces in nuclear matter

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The binding energy contributions to nuclear matter from four-body forces have been estimated. The four nucleon potential is written in coordinate space, and two intermediate nucleons are treated as spectators to give an effective two-body potential. Two-body correlations are included between all four nucleons and, following a coordinate transformation, two of the six dimensions for the spectator nucleons can be integrated analytically. Spin-isospin dependence is extracted and the remaining integrations are done numerically to give a one-pion-exchange-potential–like effective potential similar to the three-body force effective potentials of Loiseau, Nogami, and Ross. Binding energy calculations can then be done in the usual manner. As the number of nucleons increases the effect of exchanges of the spectator nucleons must also be considered. When this is taken into account, the over-all contribution to the energy of nuclear matter is approximately 0.1 MeV attraction.

[NUCLEAR STRUCTURE Many-body forces, calculated binding E of four-body force; includes correlations; effective two-body potentials.]

I. INTRODUCTION

Recent calculations of the binding energy contributions from three-body forces in nuclear matter have shown that the results are of the order of 2 MeV or more[1,2] and are therefore quite large in comparison to other higher order effects involving two-body forces. It therefore becomes important to consider both more complex three-body effects and higher order many-body effects. In this paper we calculate the binding energy contributions from the simplest irreducible four-body force, shown in Fig. 1. There are more complex four-body forces, just as there are more complex three-body forces than the simplest irreducible three-body scattering diagram. The latter diagram, however, gives the dominant contribution to the binding energy from three-body forces, and it is for this reason we have chosen the simplest irreducible four-body diagram for the estimates presented here.

Nucleons 3 and 4 in Fig. 1 are treated as spectator nucleons and are allowed to exchange only among themselves. Our method, which is an extension of the method of Loiseau, Nogami, and Ross[3] (LNR), enables an effective two-body potential to be derived. This has an OPEP-like (one-pion-exchange potential) spin and isospin dependence for nucleons 1 and 2, and has both central and tensor parts. Binding energy calculations can then be done in the usual way.[2,3]

In Sec. II we derive the direct three pion exchange four-body potential, in which the spectator

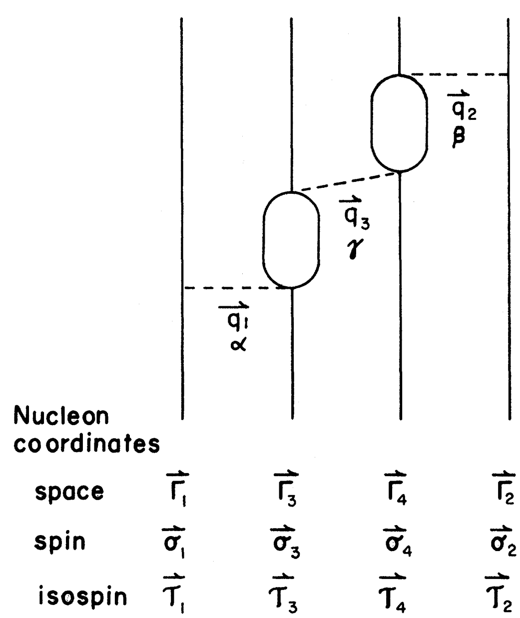

FIG. 1. The three pion exchange four-body force.

nucleons 3 and 4 do not exchange. Section III discusses the contribution to the potential from exchange of nucleons 3 and 4. The total four-body potential is used to derive an effective two-body potential in Sec. IV. Some details of this calculation are relegated to the Appendix. Finally, in Sec. V we give the contribution of the four-body potential to the binding energy in nuclear matter.

II. DIRECT THREE-PION-EXCHANGE FOUR-BODY POTENTIAL

The scattering matrix for the three-pion-exchange four-body potential (3PEP) shown in Fig. 1 can be evaluated following the method of Miyazawa,[4] giving

$$S(3,4)=2\pi i\delta(0)\sum_{\alpha\beta\gamma}\int\int\int d\vec{q}_1 d\vec{q}_2 d\vec{q}_3 \frac{4\pi f^2\tau_{1\alpha}\tau_{2\beta}}{(2\pi)^9\mu^2} e^{i\vec{q}_1\cdot\vec{r}_1-i\vec{q}_2\cdot\vec{r}_2} \frac{\vec{\sigma}_1\cdot\vec{q}_1\vec{\sigma}_2\cdot\vec{q}_2 KK'(q_1^2)KK'(q_2^2)K'(q_3^2)}{(q_1^2+\mu^2)(q_2^2+\mu^2)(q_3^2+\mu^2)}$$

$$\times\langle\beta\vec{q}_2|B_0^{(4)}|\gamma\vec{q}_3\rangle\langle\gamma\vec{q}_3|B_0^{(3)}|\alpha\vec{q}_1\rangle, \tag{2.1}$$

where $f^2=0.08$ is the πN-coupling constant, μ is the pion mass ($0.7\ \text{fm}^{-1}$), and K and K' are the vertex and propagator form factors, as in three-body calculations.[2,3] The reduced pion-nucleon scattering amplitude (i.e., the amplitude with the forward propagating intermediate nucleon removed as discussed for example by McKellar and Rajaraman[5]), shown as a "blob" in Fig. 1, is given by Miyazawa as $S_{\pi N}$, which is related to the more convenient B amplitude of Eq. (2.1) by

$$\langle\alpha\vec{q}_1|S_{\pi N}^{(3)}|\beta\vec{q}_2\rangle\equiv 2\pi i\delta(q_{10}-q_{20})\langle\alpha\vec{q}_1|B_{q_{10}}^{(3)}|\beta\vec{q}_2\rangle. \tag{2.2}$$

For the term where nucleons 3 and 4 do not exchange, so that $\beta=\gamma$, the B matrix elements which appear in (2.1) reduce to[6]

$$\langle\beta\vec{q}_2|B_0^{(4)}|\gamma\vec{q}_3\rangle=\delta_{\beta\gamma}\,2(A+B)K(q_2^2)K(q_3^2)\vec{q}_2\cdot\vec{q}_3 e^{i(\vec{q}_2-\vec{q}_3)\cdot\vec{r}_4} \tag{2.3}$$

and a similar expression for $B_0^{(3)}$. We use this direct term to define a three-pion-exchange four-body potential $W_{3\text{PE}}$, which can now be written as

$$W_{3\text{PE}}=\frac{-4\pi f^2\vec{\tau}_1\cdot\vec{\tau}_2}{(2\pi)^9\mu^2}[2(A+B)]^2\int\int\int d\vec{q}_1 d\vec{q}_2 d\vec{q}_3 e^{i\vec{q}_1\cdot(\vec{r}_1-\vec{r}_3)}e^{i\vec{q}_2\cdot(\vec{r}_4-\vec{r}_2)}e^{i\vec{q}_3\cdot(\vec{r}_3-\vec{r}_4)}$$

$$\times\frac{K^2K'(q_1^2)}{q_1^2+\mu^2}\frac{K^2K'(q_2^2)}{q_2^2+\mu^2}\frac{K^2K'(q_3^2)}{q_3^2+\mu^2}\vec{\sigma}_1\cdot\vec{q}_1\vec{q}_1\cdot\vec{q}_3\vec{q}_3\cdot\vec{q}_2\vec{q}_2\cdot\vec{\sigma}_2\ . \tag{2.4}$$

Using the relative coordinates of Fig. 2 in (2.4) allows us to write

$$\vec{q}_1=-i\mu\vec{\nabla}_U,\quad \vec{q}_2=-i\mu\vec{\nabla}_V,\quad \vec{q}_3=-i\mu\vec{\nabla}_W\ , \tag{2.5}$$

so that the integrals over $\vec{q}_1\vec{q}_2\vec{q}_3$ can be done giving

$$W_{3\text{PE}}(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)=\frac{4\pi f^2\vec{\tau}_1\cdot\vec{\tau}_2}{(2\pi)^9\mu^2}[2(A+B)]^2(2\mu\pi^2)^3\mu^6\mathcal{Y}(\vec{U},\vec{V},\vec{W})\ , \tag{2.6}$$

where

$$\mathcal{Y}(\vec{U},\vec{V},\vec{W})=\tfrac{1}{27}\{\vec{\sigma}_1\cdot\vec{\sigma}_2(\bar{U}\bar{V}\bar{W}-\tilde{U}\bar{V}\bar{W}-\bar{U}\tilde{V}\bar{W}-\bar{U}\bar{V}\tilde{W}+2\tilde{U}\tilde{V}\tilde{W})+S_{12}(\underline{\hat{U}})\tilde{U}(\bar{V}-\tilde{V})(\bar{W}-\tilde{W})+S_{12}(\underline{\hat{V}})\tilde{V}(\bar{W}-\tilde{W})(\bar{U}-\tilde{U})$$

$$+S_{12}(\underline{\hat{W}})\tilde{W}(\bar{U}-\tilde{U})(\bar{V}-\tilde{V})+9\vec{\sigma}_1\cdot\underline{\hat{U}}\vec{\sigma}_2\cdot\underline{\hat{W}}\underline{\hat{U}}\cdot\underline{\hat{W}}\tilde{U}\tilde{W}(\bar{V}-\tilde{V})+9\vec{\sigma}_1\cdot\underline{\hat{W}}\vec{\sigma}_2\cdot\underline{\hat{V}}\underline{\hat{W}}\cdot\underline{\hat{V}}\tilde{W}\tilde{V}(\bar{U}-\tilde{U})$$

$$+9\vec{\sigma}_1\cdot\underline{\hat{U}}\vec{\sigma}_2\cdot\underline{\hat{V}}[\underline{\hat{U}}\cdot\underline{\hat{V}}(\bar{W}-\tilde{W})+3\underline{\hat{U}}\cdot\underline{\hat{W}}\underline{\hat{W}}\cdot\underline{\hat{V}}\tilde{W}]\tilde{U}\tilde{V}\}\frac{e^{-U}}{U}\frac{e^{-V}}{V}\frac{e^{-W}}{W}\ . \tag{2.7}$$

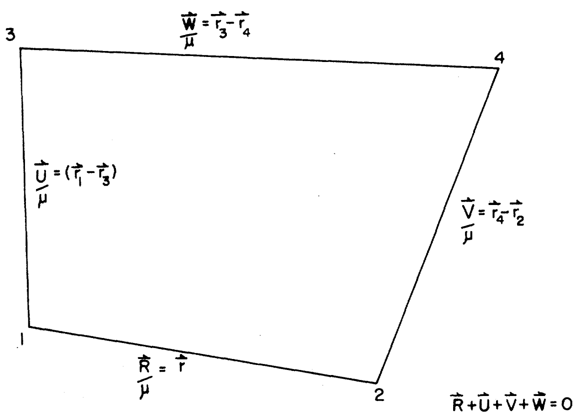

FIG. 2. Relative coordinates for the four nucleons.

$\bar{U}\bar{V}\bar{W}$ and $\tilde{U}\tilde{V}\tilde{W}$ are defined by

$$\bar{U}=1-\zeta\,\frac{\eta^2}{\mu^2}\;e^{-(\eta/\mu-1)U}, \tag{2.8}$$

$$\tilde{U}=\left(1+\frac{3}{U}+\frac{3}{U^2}\right)-\zeta\,\frac{\eta^2}{\mu^2}\left(1+\frac{3\mu}{\eta U}+\frac{3\mu^2}{\eta^2U^2}\right)e^{-(\eta/\mu-1)U}, \tag{2.9}$$

and similar expressions for $\bar{V}\bar{W}$ and $\tilde{V}\tilde{W}$. ζ and η are the parameters of the form factors of Table I.

In Sec. III a spin-isospin average of the scattering matrix calculated from (2.1) with nucleons 3 and 4 exchanged is shown to be about $-\frac{1}{4}$ of the direct term. As this type of exchange, where "spectator nucleons" permute among themselves, has no analog in three-body forces, we will include its effect approximately by introducing a factor $\frac{3}{4}$ into the effective two-body potential discussed in Sec. IV.

III. CONTRIBUTION OF SPECTATOR EXCHANGE TO THE FOUR-BODY POTENTIAL

In this section we compare $W_{3\mathrm{PE}}$ [Eq. (2.4)] derived from Fig. 1 in the direct case, where there is no isospin flip of nucleons 3 and 4, to the potential derived from Fig. 1 with an exchange of nucleons 3 and 4. Referring back to Eq. (2.1), we must now calculate

$$\sum_{\alpha\beta\gamma}\tau_{1\alpha}\tau_{2\beta}\,\langle\beta\vec{q}_2|\,B_0^{(4)}\,|\,\gamma\vec{q}_3\rangle\langle\gamma\vec{q}_3|\,B_0^{(3)}\,|\,\alpha\vec{q}_1\rangle$$

$$=\sum_{\alpha\beta\gamma}MK(q_2{}^2)K^2(q_3{}^2)K(q_1{}^2)$$

$$\times\,e^{i(\vec{q}_2-\vec{q}_3)\cdot\vec{r}_4}\,e^{i(\vec{q}_3-\vec{q}_1)\cdot\vec{r}_3} \tag{3.1}$$

for an exchange of nucleons 3 and 4.

We first extract the spin-isospin dependence of this expression using (2.2). For convenience, we

TABLE I. $NN\pi$ form factors:

$$\frac{K^2(q^2)K'(q^2)}{q^2+\mu^2}=\frac{1}{q^2+\mu^2}+\frac{\zeta}{q^2+\eta^2},$$

where ζ and η are given in the table. Original references may be found in Ref. 3.

Form factor	ζ	$(\eta/\mu)^2$
I	0	...
II	0.72	5.73
III	1	10

write

$$M=\tau_{1\alpha}\tau_{2\beta}M_1M_2\,, \tag{3.2}$$

where

$$M_1=A\tau_{4\beta}\tau_{4\gamma}\vec{\sigma}_4\cdot\vec{q}_3\vec{\sigma}_4\cdot\vec{q}_2+A\tau_{4\gamma}\tau_{4\beta}\vec{\sigma}_4\cdot\vec{q}_2\vec{\sigma}_4\cdot\vec{q}_3$$
$$+(A\leftrightarrow B,\ \beta\leftrightarrow\gamma)\,, \tag{3.3}$$

$$M_2=A\tau_{3\gamma}\tau_{3\alpha}\vec{\sigma}_3\cdot\vec{q}_1\vec{\sigma}_3\cdot\vec{q}_3+A\tau_{3\alpha}\tau_{3\gamma}\vec{\sigma}_3\cdot\vec{q}_3\vec{\sigma}_3\cdot\vec{q}_1$$
$$+(A\leftrightarrow B,\ \alpha\leftrightarrow\gamma)\,. \tag{3.4}$$

In the following section we construct an effective two-body potential for the four-body potential by averaging over the two spectator nucleons. To simplify the relations given in this section we will at this stage compute the spin-isospin average of the exchange term. The resulting four-body potential is still directly comparable with that of Sec. II [Eq. (2.4)], since that is independent of the spin and isospin of the spectator nucleons.

We can simplify M_1:

$$M_1=(A-B)\epsilon_{\beta\gamma\epsilon}\tau_{4\epsilon}\bar{\delta}_{\beta\gamma}2i\sum_{ijk}\epsilon_{ijk}\sigma_{4k}q_{3i}q_{2j}$$
$$+(A+B)\delta_{\beta\gamma}2\vec{q}_3\cdot\vec{q}_2\,, \tag{3.5}$$

where

$$\bar{\delta}_{\beta\gamma}=1-\delta_{\beta\gamma} \tag{3.6}$$

and ϵ_{ijk} is the usual permutation symbol. A similar expression holds for M_2.

We can expand the product M_1M_2 using (3.5), calculate isospin exchange averages, and refactorize the result, giving

$$\langle M\rangle_{\substack{\text{iso}\\\text{ave}}\ \substack{\text{exch}\\3,4}}=\tau_{1\alpha}\tau_{2\beta}[P_1P_2\tfrac{1}{2}\delta_{\epsilon\phi}+Q_1Q_2\tfrac{1}{2}]\,, \tag{3.7}$$

where

$$P_1=(A-B)\epsilon_{\beta\gamma\epsilon}\bar{\delta}_{\beta\gamma}2i\sum_{ijk}\epsilon_{ijk}\sigma_{4k}q_{3i}q_{2j}\,,$$

$$P_2=(A-B)\epsilon_{\gamma\alpha\phi}\bar{\delta}_{\gamma\alpha}2i\sum_{i'j'k'}\epsilon_{i'j'k'}\sigma_{3k'}q_{1i'}q_{3j'}\,,$$

$$Q_1=(A+B)\delta_{\beta\gamma}2\vec{q}_3\cdot\vec{q}_2\,, \tag{3.8}$$

$$Q_2=(A+B)\delta_{\gamma\alpha}2\vec{q}_3\cdot\vec{q}_1\,.$$

We must now calculate the spin exchange average of (3.9). We find

$$\langle P_1P_2\rangle_{\substack{\text{spin}\\ \text{ave}}\ \substack{\text{exch}\\ 3,4}} = -4(A-B)^2\epsilon_{\beta\gamma\epsilon}\epsilon_{\gamma\alpha\phi}\bar{\delta}_{\beta\gamma}\bar{\delta}_{\gamma\alpha}\left(\tfrac{1}{2}\vec{q}_3\cdot\vec{q}_1\vec{q}_2\cdot\vec{q}_3-\tfrac{1}{2}q_3^2\vec{q}_2\cdot\vec{q}_1\right). \tag{3.9}$$

Q_1Q_2 is spin independent, so its spin exchange average simply gives a factor $\frac{1}{2}$. Collecting these results together and summing over α, β, and γ then gives

$$\sum_{\alpha\beta\gamma}\langle M\rangle_{\substack{\text{spin}\\ \text{exch}}\ \substack{\text{isospin}\\ \text{ave }3,4}} = \vec{\tau}_1\cdot\vec{\tau}_2[(A+B)^2\vec{q}_1\cdot\vec{q}_3\vec{q}_3\cdot\vec{q}_2-4(A-B)^2(\vec{q}_1\cdot\vec{q}_3\vec{q}_3\cdot\vec{q}_2-q_3^2\vec{q}_1\cdot\vec{q}_2)]. \tag{3.10}$$

This expression should be compared with the direct average of M which appears in (2.4):

$$\sum_{\alpha\beta\gamma}\langle M\rangle_{\substack{\text{spin}\\ \text{direct}}\ \substack{\text{isospin}\\ \text{ave }3,4}} = \vec{\tau}_1\cdot\vec{\tau}_2 4(A+B)^2\vec{q}_1\cdot\vec{q}_3\vec{q}_3\cdot\vec{q}_2. \tag{3.11}$$

The second term in our exchange result (3.10) is proportional to $4(A-B)^2$. From Miyazawa we have

$$\frac{4(A-B)^2}{(A+B)^2}=\frac{1}{4}. \tag{3.12}$$

In addition, the angle integrations of $\vec{q}_3$ in the coefficient will produce some cancellation. For example, if there is no other dependence on $\vec{q}_3$ the term in parentheses integrates over $\hat{q}_3$ to

$$-\tfrac{2}{3}4\pi q_3^2\vec{q}_1\cdot\vec{q}_2. \tag{3.13}$$

The second term in Eq. (3.10) is therefore at least a factor of 4 smaller than the first term. The first term is exactly one quarter of the direct result (3.11). Allowing for a relative factor (−1) in the exchange matrix element gives the factor $\frac{3}{4}$ in Eq. (4.1) for the effective two-body potential.

Exchange effects such as this have no analog in the three-body force case. They become more important for N-body forces as N increases, as we show elsewhere.[7]

IV. EFFECTIVE TWO-BODY POTENTIAL

If we restrict our attention to scattering where the intermediate nucleons (3 and 4 in Fig. 1) are not involved in any other processes, and can therefore be thought of as "spectators," the four-body potential (2.6) reduces to an effective two-body potential, given by

$$V_{3\text{PE}}(r_{12})=\frac{3}{4}\int\int W_{3\text{PE}}(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)\rho(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)d\vec{r}_3d\vec{r}_4, \tag{4.1}$$

where ρ is the density distribution of the four nucleons as determined by two-body correlations, and the factor $\frac{3}{4}$ discussed in Sec. III is due to permutations of the spectator nucleons.

This is an extension of the effective potential method used for three-body calculations[1-3] and is an essential step in obtaining a tractable calculation for the dominant second order tensor terms in the binding energy. As in the three-body case,[2] we have trivial equality of the first order terms $\langle W_{3\text{PE}}\rangle=\langle V_{3\text{PE}}\rangle$, while the validity of the approximation of replacing $W_{3\text{PE}}$ by $V_{3\text{PE}}$ in second order has yet to be tested in detail.

To proceed with the calculation of (4.1), we initially allow any correlation function ρ which is a function of the six scalar internucleon distances only. We now transform to the dimensionless coordinate system $UVWXY\phi$ shown in Fig. 3:

$$\int\int_{\vec{r}_3\vec{r}_4\text{ over all space}} d\vec{r}_3d\vec{r}_4=\mu^{-6}\int_{\tilde{D}_1+\tilde{D}_2} J\,dU\,dV\,dW\,dX\,dY\,d\phi, \tag{4.2}$$

where J is the Jacobian for this transformation, and $\tilde{D}_1$ and $\tilde{D}_2$ are disjoint identical domains corresponding to nucleon 4 respectively above and below the plane of nucleons 1, 2, and 3 in Fig. 3. The Jacobian J is given by

$$J=\frac{1}{6}\frac{UVWXY}{R}\frac{1}{\Omega}, \tag{4.3}$$

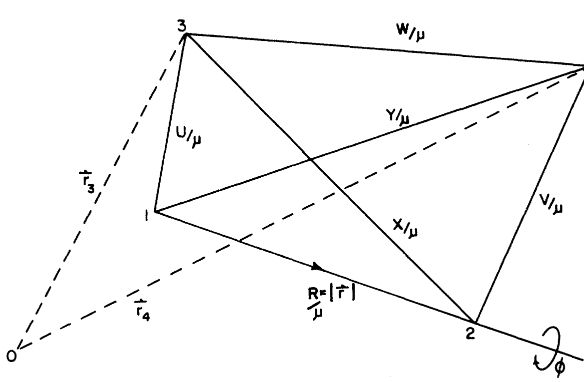

FIG. 3. Coordinates for the integrals in calculating the effective potential.

where $\mu^{-3}\Omega$ is the volume of the tetrahedron, i.e.,

$$\Omega = \tfrac{1}{12}\{R^2W^2(V^2+X^2+U^2+Y^2-R^2-W^2)+X^2Y^2(U^2+V^2+R^2+W^2-X^2-Y^2)+U^2V^2(R^2+X^2+W^2+Y^2-U^2-V^2)$$
$$-X^2W^2V^2-R^2V^2Y^2-U^2X^2R^2-U^2W^2Y^2\}^{1/2}\,. \qquad (4.4)$$

As the only ϕ dependence of the integrand is in $W_{3\rm PE}$ and therefore in $\mathcal{Y}(\vec{U},\vec{V},\vec{W})$ of Eq. (2.7), we now direct our attention to this expression for the first integration. All the ϕ dependence is in the tensor terms $(S_{12}(\hat{\underline{U}})$, etc.) and the terms such as $\vec{\sigma}_1\cdot\hat{\underline{U}}\vec{\sigma}_2\cdot\hat{\underline{V}}$. The terms like $S_{12}(\hat{\underline{U}})$ integrate simply to give

$$\int S_{12}(\hat{\underline{U}})d\phi = 2\pi S_{12}(\hat{\underline{r}})P_2(\hat{\underline{U}}\cdot\hat{\underline{r}})\,. \qquad (4.5)$$

We use spherical tensor recoupling relations[8] to write

$$\vec{\sigma}_1\cdot\hat{\underline{U}}\vec{\sigma}_2\cdot\hat{\underline{V}} = \tfrac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2\hat{\underline{U}}\cdot\hat{\underline{V}}+\tfrac{1}{2}(\vec{\sigma}_1\times\vec{\sigma}_2)\cdot(\hat{\underline{U}}\times\hat{\underline{V}})$$
$$+\underline{T}_2(\vec{\sigma}_1\vec{\sigma}_2)\cdot\underline{T}_2(\hat{\underline{U}}\hat{\underline{V}})\,. \qquad (4.6)$$

The first and third terms of (4.6) integrate to give

$$\int \vec{\sigma}_1\cdot\hat{\underline{U}}\vec{\sigma}_2\cdot\hat{\underline{V}}\,d\phi = \tfrac{2}{3}\pi[\vec{\sigma}_1\cdot\vec{\sigma}_2\hat{\underline{U}}\cdot\hat{\underline{V}}+$$
$$+S_{12}(\hat{\underline{r}})(\tfrac{3}{2}\hat{\underline{U}}\cdot\hat{\underline{r}}\hat{\underline{V}}\cdot\hat{\underline{r}}-\tfrac{1}{2}\hat{\underline{U}}\cdot\hat{\underline{V}})]\,. \qquad (4.7)$$

The second term is nonzero, as $\hat{\underline{U}}$ and $\hat{\underline{V}}$ are not in general coplanar; however, the contribution from this term is proportional to $\hat{\underline{U}}\times\hat{\underline{V}}\cdot\hat{\underline{r}}$ and is therefore of opposite sign in the two domains $\tilde{D}_1$ and $\tilde{D}_2$, so that it vanishes when these are added.

In all other cases, the contributions in $\tilde{D}_1$ and $\tilde{D}_2$ are equal and simply add to give a factor of 2, so that the integration with respect to ϕ gives

$$\int_{\tilde{D}_1+\tilde{D}_2}\mathcal{Y}(\vec{U}\vec{V}\vec{W})d\phi = 4\pi[F_c\vec{\sigma}_1\cdot\vec{\sigma}_2+F_tS_{12}(\hat{\underline{r}})]\,, \qquad (4.8)$$

where F_c and F_t are functions of $UVWXY$ and r, given by

$$F_c = \tfrac{1}{27}\{\overline{U}\overline{V}\overline{W}-\tilde{U}\overline{V}\overline{W}-\overline{U}\tilde{V}\overline{W}-\overline{U}\overline{V}\tilde{W}+2\tilde{U}\tilde{V}\tilde{W}+3\alpha^2(\overline{U}-\tilde{U})\tilde{V}\tilde{W}+3\beta^2\tilde{U}(\overline{V}-\tilde{V})\tilde{W}+3\gamma^2\tilde{U}\tilde{V}(\overline{W}-\tilde{W})+9\alpha\beta\gamma\tilde{U}\tilde{V}\tilde{W}\}$$
$$\times\frac{e^{-U}}{U}\frac{e^{-V}}{V}\frac{e^{-W}}{W} \qquad (4.9)$$

and

$$F_t = \tfrac{1}{27}\{\tfrac{1}{2}(3\delta^2-1)\tilde{U}(\overline{V}-\tilde{V})(\overline{W}-\tilde{W})+\tfrac{1}{2}(3\epsilon^2-1)(\overline{U}-\tilde{U})\tilde{V}(\overline{W}-\tilde{W})+\tfrac{1}{2}(3\kappa^2-1)(\overline{V}-\tilde{V})(\overline{V}-\tilde{V})\tilde{W}$$
$$+\tfrac{1}{2}(3\epsilon\kappa-\alpha)\alpha(\overline{U}-\tilde{U})\tilde{V}\tilde{W}+\tfrac{1}{2}(3\delta\kappa-\beta)\beta\tilde{U}(\overline{V}-\tilde{V})\tilde{W}+\tfrac{1}{2}(3\delta\epsilon-\gamma)[\gamma(\overline{W}-\tilde{W})+3\beta\alpha\tilde{W}]\tilde{U}\tilde{V}\}\frac{e^{-U}}{U}\frac{e^{-V}}{V}\frac{e^{-W}}{W}\,, \qquad (4.10)$$

where

$$\alpha = \frac{X^2-V^2-W^2}{2VW}\,,\qquad \delta = \frac{X^2-U^2-R^2}{2RU}\,,$$
$$\beta = \frac{Y^2-U^2-W^2}{2UW}\,,\qquad \epsilon = \frac{Y^2-R^2-V^2}{2RV}\,, \qquad (4.11)$$
$$\gamma = \frac{R^2+W^2-X^2-Y^2}{2UV}\,,\qquad \kappa = \frac{U^2+V^2-X^2-Y^2}{2RW}\,,$$

and $\overline{U}\overline{V}\overline{W}$ and $\tilde{U}\tilde{V}\tilde{W}$ are defined by (2.8) and (2.9).

The expression (4.1) now reduces to

$$V_{3\rm PE}(r) = \vec{\tau}_1\cdot\vec{\tau}_2[\vec{\sigma}_1\cdot\vec{\sigma}_2V_c^{3\rm PE}(r)+S_{12}(\hat{\underline{r}})V_t^{3\rm PE}(r)]\,, \qquad (4.12)$$

where

$$V_{c,t}^{3\rm PE}(r) = \frac{3}{4}f^2\mu\frac{[2\bar{\rho}(A+B)]^2}{4\pi}\mathcal{G}_{c,t}$$
$$=0.333\mathcal{G}_{c,t}\ \mathrm{MeV} \qquad (4.13)$$

and

$$\mathcal{G}_{c,t} = \int_0^\infty dU\int_0^\infty dV\int_{W_L}^{W_H}dW\int_{X_L}^{X_H}dX\int_{Y_L}^{Y_H}dY\,J\Theta F_{c,t}\,, \qquad (4.14)$$

where $\bar{\rho}$ is the average density of nuclear matter, and

$$\Theta = \frac{\rho(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)}{\bar{\rho}^2}\,. \qquad (4.15)$$

We now specialize to the cutoff correlation, when

$$\Theta = \theta(R-\mu c)\theta(U-\mu c)\theta(V-\mu c)\theta(W-\mu c)$$
$$\times\theta(X-\mu c)\theta(Y-\mu c)\,, \qquad (4.16)$$

where c is the cutoff distance (i.e., any two nucleons are separated by at least c).

The limits of integration are given by

$$\begin{aligned} W_L &= \max\{0, U-V-R, V-R-U, R-U-V\}, \\ W_H &= U+V+R, \\ X_L &= \max\{|U-R|, |V-W|\}, \\ X_H &= \min\{U+R, V+W\}, \\ Y_L &= (M-P)^{1/2}, \\ Y_H &= (M+P)^{1/2}, \end{aligned} \qquad (4.17)$$

with

$$\begin{aligned} M &= \tfrac{1}{2}(X^2+R^2+U^2+V^2+W^2) + \frac{(U^2-R^2)(V^2-W^2)}{2X^2}, \\ P &= \frac{1}{2X^2}(\Delta\{XRU\}\Delta\{XVW\})^{1/2}. \end{aligned} \qquad (4.18)$$

The notation $\Delta\{abc\}$ is defined as

$$\begin{aligned} \Delta\{abc\} &= 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \\ &= 16 \text{ (area of triangle ``}abc\text{'')}^2. \end{aligned} \qquad (4.19)$$

Inspecting (4.4), we see that the expression in curly parentheses is a quadratic in Y^2. We can rewrite J as

$$J = \frac{UVW}{R} \frac{2Y}{[(Y_H{}^2 - Y^2)(Y^2 - Y_L{}^2)]^{1/2}}, \qquad (4.20)$$

and noting that Y dependence in F_c and F_t is also

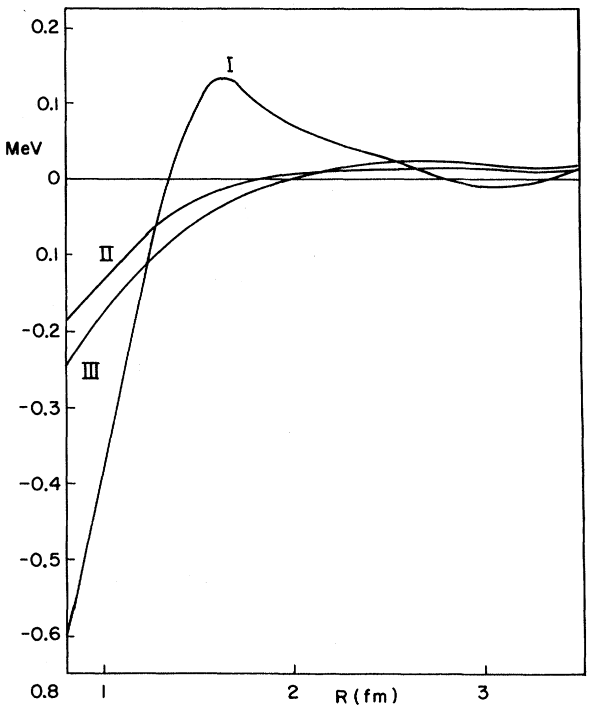

FIG. 5. 3PEP central potential, for cutoff at 0.8 fm, form factors I, II, and III.

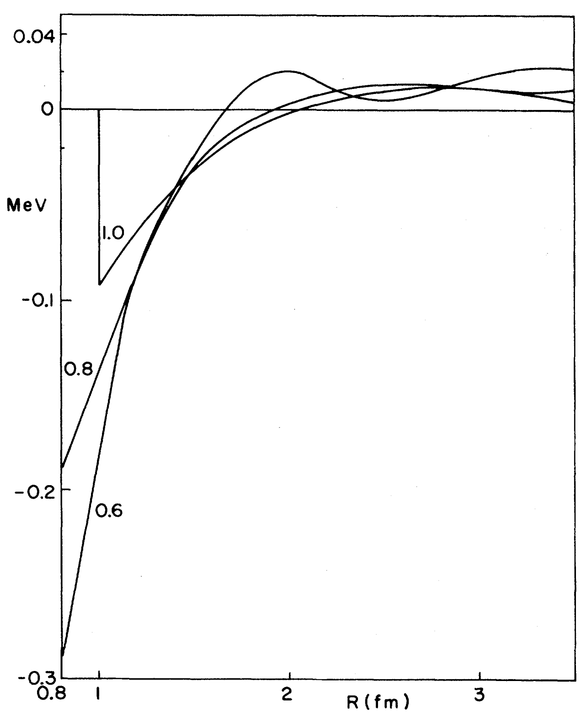

FIG. 4. 3PEP central potential, for form factor II with cutoff at 0.6, 0.8, and 1.0 fm.

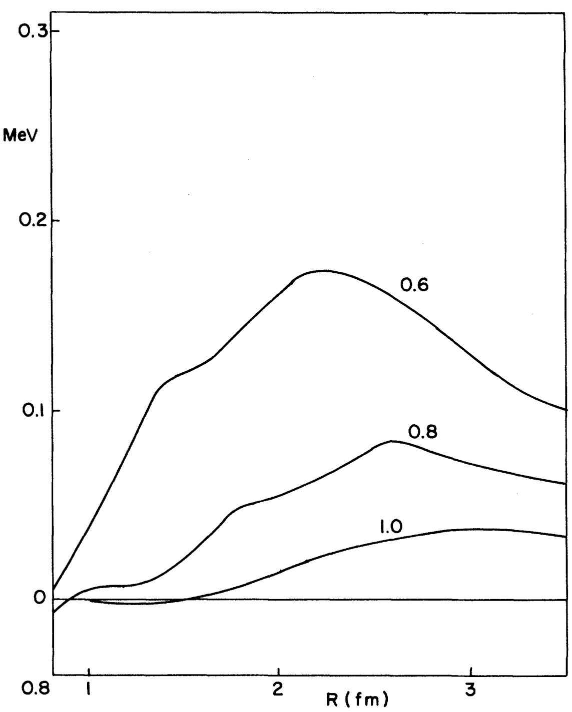

FIG. 6. 3PEP tensor potential, for form factor II, with cutoffs at 0.6, 0.8, and 1.0 fm.

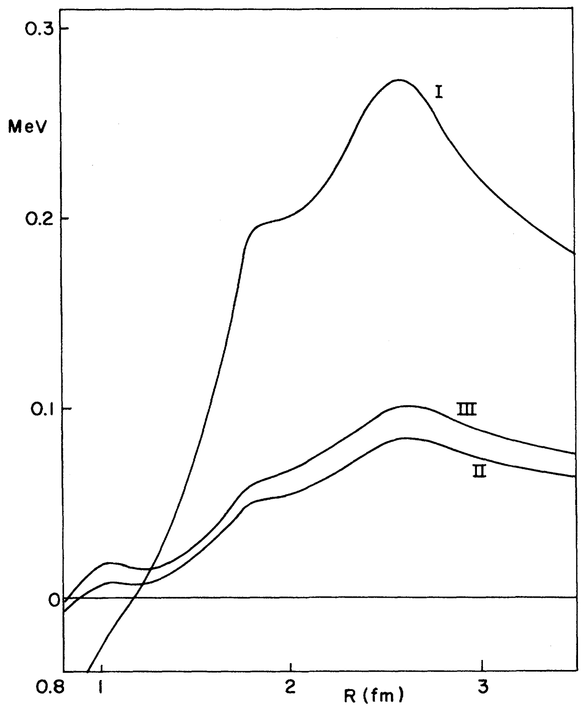

FIG. 7. 3PEP tensor potential, for cutoff at 0.8 fm, form factors I, II, and III.

quadratic in Y^2, the integration over Y can be done analytically. The algebraic expressions resulting are quite lengthy and are not included here.[9]

There are four more integrations, over U, V, W, and X, needed to compute $\mathcal{G}_c$ and $\mathcal{G}_t$ and hence the effective potential. These cannot be done analytically, but can be simplified because the final integrand is symmetric under interchange of U and V. We defer the proof of this symmetry to the Appendix. The resulting effective potentials are shown in Figs. 4 to 7 for three different form factors and various values of the internucleon cutoff distance c. Predictably, the four-body potentials depend strongly on the cutoff distance used in the correlation function $\rho(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)$ and only mildly on the choice of form factors, provided a realistic form factor (II or III) is used. Following the comparison of cutoff correlations to smooth Reid-hard-core correlations in Blatt and McKellar,[2] we take the potentials with a cutoff at 0.8 fm and form factors II or III as being most realistic.

We should emphasize the scale. At typical nucleon-nucleon separations the effective potential is of the order of tenths of an MeV.

V. BINDING ENERGY CONTRIBUTION OF 3PEP

Having calculated an effective two-body potential, we can now compute its binding energy contributions to nuclear matter in first and second order. Using the notation of LNR,[3] the contributions are

$$\Delta E^{(1)} = \langle V^{3\mathrm{PE}}\rangle / N\,, \tag{5.1}$$

$$\begin{aligned}\Delta E^{(2)} &= \left[2\left\langle V^{\mathrm{OPEP}}\frac{Q}{e}V^{3\mathrm{PE}}\right\rangle + \left\langle V^{3\mathrm{PE}}\frac{Q}{e}V^{3\mathrm{PE}}\right\rangle + 2\left\langle V^{2\mathrm{PE}}\frac{Q}{e}V^{3\mathrm{PE}}\right\rangle\right]\Big/ N \\ &= \Delta E_1^{(2)} + \Delta E_2^{(2)} + \Delta E_3^{(2)}\,.\end{aligned} \tag{5.2}$$

N is the nucleon number, Q is the Pauli operator, e is an energy denominator, and

$$V^{\mathrm{OPEP}}(r) = \frac{f^2\vec{\tau}_1\cdot\vec{\tau}_2}{2\pi^2\mu^2}\int d\vec{q}\,\frac{\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q}}{q^2+\mu^2}K^2(q^2)K'(q^2)e^{i\vec{q}\cdot\vec{r}}\,. \tag{5.3}$$

The terms $\Delta E_2^{(2)}$ and $\Delta E_3^{(2)}$ are negligible compared to the dominant terms $\Delta E^{(1)}$ and $\Delta E_1^{(2)}$, which are themselves small compared to three-body results.

Table II gives the results for each of the form factors of Table I. In the first column we show the results of a calculation with $c = 0.8$ fm and with no form factors used in OPEP (5.3). This column should be compared with the corresponding three-

TABLE II. Binding energy contributions of the four-body force to nuclear matter (MeV).

		No form in OPEP	Form in OPEP		
		$c = 0.8$ fm	$c = 0.6$ fm	$c = 0.8$ fm	$c = 1.0$ fm
Form factor	$\Delta E^{(1)}$	0.1	0.2	0.1	0.1
I	$\Delta E_1^{(2)}$	−1.0	−4.8	−1.0	−0.2
(No form)	Total	−0.9	−4.6	−0.9	−0.1
	$\Delta E^{(1)}$	0.16	0.26	0.16	0.09
II	$\Delta E_1^{(2)}$	−0.28	−0.68	−0.22	−0.06
	Total	−0.12	−0.42	−0.06	0.03
	$\Delta E^{(1)}$	0.25	0.45	0.25	0.13
III	$\Delta E_1^{(2)}$	−0.35	−0.85	−0.31	−0.09
	Total	−0.10	−0.40	−0.06	0.04

body results of Table III, which were calculated in Ref. 2 and are about 7% of those results.

The remaining three columns of Table III contain the results for cutoffs $c=0.6$, 0.8, and 1.0 fm when form factors are included in (5.3). The size of the binding energy contributions decreases as the cutoff distance c increases, as might be expected because of the short range nature of the four-body force. Form factor I is unrealistic, as explained in LNR. With a cutoff at 0.8 fm, the results for both form factors II and III are 0.06 MeV attraction. These numbers should be compared with the corresponding three-body binding energies of 1.3 and 1.6 MeV attraction for form factors II and III, respectively. This represents a convergence ratio of approximately 0.05 over-all.

In three-body calculations,[2] the results with internucleon cutoffs at 0.8 fm and OPEP used in second order perturbation were very close to the results obtained using the Reid-hard-core potential, both to derive internucleon correlations and as the potential in the second order terms. We therefore believe that the present results with the cutoff at 0.8 fm should give a realistic estimate of the effects of the four-body force in nuclear matter, if a hard-core two-body force is used to derive the two-body correlations.

VI. CONCLUSION

Our results indicate that including forces which involve more and more nucleons gives rapidly decreasing contributions to the energy of nuclear matter. This is as one would expect, since the probability of finding four nucleons within a pion Compton wavelength of each other is roughly $\mu^{-3}\bar{\rho} \approx \frac{1}{2}$ as much as the probability of finding three nucleons that close together, ignoring correlations.

The fact that we find a four-body force contribution to the energy of order 0.1 MeV indicates that efforts to refine three-body force calculations beyond this level of accuracy are unreasonable unless four-body force contributions, and possibly more complex three-body force contributions, are included.

ACKNOWLEDGMENTS

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TABLE III. Binding energy contributions of the three-body force to nuclear matter (MeV). In this table $\Delta E^{(1)} = \langle V^{2PE}\rangle /N$; $\Delta E_1^{(2)} = 2\langle V^{OPEP}(Q/e)V^{2PE}\rangle /N$; $\Delta E_2^{(2)} = \langle V^{2PE}(Q/e)V^{2PE}\rangle /N$. These results are taken from Refs. 1 and 2.

		LNR (no form in OPEP) $c=d=0.8$ fm	LNR (form in OPEP) $c=d=0.8$ fm
Form factor	$\Delta E^{(1)}$	1.3	1.3
I	$\Delta E_1^{(2)}$	−6.0	−6.0
(No form)	$\Delta E_2^{(2)}$	−0.5	−0.5
	Total	−5.2	−5.2
	$\Delta E^{(1)}$	1.0	1.0
II	$\Delta E_1^{(2)}$	−2.5	−2.2
	$\Delta E_2^{(2)}$	−0.1	−0.1
	Total	−1.6	−1.3
	$\Delta E^{(1)}$	1.3	1.3
III	$\Delta E_1^{(2)}$	−3.0	−2.7
	$\Delta E_2^{(2)}$	−0.2	−0.2
	Total	−1.9	−1.6

APPENDIX

Writing (4.14) as

$$\mathcal{G}_{c,t} = \theta(R-\mu c)\iint_{\mu c}^{\infty} dU\,dV\,f_{c,t}(U,V) \tag{A1}$$

defines the function $f_{c,t}(U,V)$, which is now shown to be symmetric under an interchange of U and V. This roughly halves the computing time for $\mathcal{G}_c$ and $\mathcal{G}_t$.

Using equations (4.14) and (4.16) we can rewrite $f_{c,t}$ as

$$f_{c,t}(U,V) = \int_{W_L}^{W_H} \theta(W-\mu c)k(U,V,W)dW\,, \tag{A2}$$

where

$$k(U,V,W) = \iint_{\substack{X_L \le X \le X_H \\ Y_L \le Y \le Y_H}} dX\,dY\,j_{UVW}(X,Y) \tag{A3}$$

and

$$j_{UVW}(X,Y) = F_{c,t}J\theta(X-\mu c)\theta(Y-\mu c)\,. \tag{A4}$$

Equation (4.17) shows that W_L and W_H are unaffected by an interchange of U and V. Hence (A2) gives

$$f_{c,t}(V,U) = \int_{W_L}^{W_H} \theta(W-\mu c)k(V,U,W)dW\,. \tag{A5}$$

To evaluate $k(V,U,W)$ we must examine the effect of $U \leftrightarrow V$ on the domain limits X_L, X_H, Y_L, Y_H, regarded as functions of U, V, and W. An inspection of Fig. 3 shows that the simultaneous interchange of labels $U \leftrightarrow V$ and $X \leftrightarrow Y$ leaves a tetrahedron with the same labels RUX, RVY, XVW, and YUW bounding the four triangular faces. The effect of an interchange between U and V on equation (A3) is therefore

$$k(V,U,W) = \iint_{\substack{Y_L \le X \le Y_H \\ X_L \le Y \le X_H}} dX\,dY\, j_{VUW}(X,Y). \qquad \text{(A6)}$$

From Eqs. (4.3), (4.4), (4.9), (4.10), and (4.11), we see that the right-hand side of (A4) is unaffected by the simultaneous interchange $U \leftrightarrow V$, $X \leftrightarrow Y$, causing $\alpha \leftrightarrow \beta$, $\delta \leftrightarrow \epsilon$, $\overline{U} \leftrightarrow \overline{V}$, and $\tilde{U} \leftrightarrow \tilde{V}$ while γ, κ, $\overline{W}$, and $\tilde{W}$ remain unchanged. Hence

$$j_{VUW}(Y,X) = j_{UVW}(X,Y). \qquad \text{(A7)}$$

Changing the notation X and Y for the dummy integration variables to Y and X and using (A7) in (A6) then gives the right-hand side of (A3), so that

$$k(V,U,W) = k(U,V,W). \qquad \text{(A8)}$$

Substituting (A8) into (A5) completes the proof.

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