Threshold pion production in pion-nucleus collisions: A simple estimate

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The Goldhaber-Teller model generalized to spin-isospin vibrations is used to provide a simple estimate for the total cross section for threshold pion production in pion-nucleus collisions ($\pi^- A_Z \rightarrow \pi^- \pi^+ A_{Z-1}$) in the case of nuclei with $N = Z$. Cross sections are calculated using the threshold approximant to the production amplitude for single nucleons consisting of pion pole plus contact terms alternately derived from the phenomenological Lagrangian theory and the current-commutator theory of multiple-pion production. The threshold approximant in the latter theory fits the experimental pion-production data on protons poorly and, in the case of ^{4}He, the lightest target nucleus we consider, that theory predicts cross sections about a factor of 2 smaller than those similarly calculated with phenomenological Lagrangian input. On the other hand, for ^{12}C and ^{16}O, the predictions of the alternative threshold theories are in essential agreement. In the case of the current-commutator theory, the Goldhaber-Teller predictions for ^{4}He and ^{16}O are consistent with those of the particle-hole model obtained earlier by Eisenberg after quadrupling his calculated values to compensate for an omitted factor of 2 in his production amplitude. While the cross sections for the (π, 2π) reaction in nuclei are still expected to be quite small, the prospect for their accessibility seems reasonably improved.

NUCLEAR REACTIONS ^{4}He(π^-, $\pi^-\pi^+$), ^{12}C(π^-, $\pi^-\pi^+$), ^{16}O(π^-, $\pi^-\pi^+$); threshold pion production cross section calculated with a spin-isospin-generalized Goldhaber-Teller model with phenomenological Lagrangian and current-commutator inputs and compared to particle-hole model predictions.

Threshold pion production in pion-nucleus collisions was studied some time ago by Eisenberg[1] as a possibly promising mechanism for the preferential excitation of the $J^\pi = 0^-$ spin-isospin oscillations in closed-shell nuclei. The numerical applications there[1] of the theory (to ^{4}He and ^{16}O), for which a local approximation[1] to the current-commutator amplitude for soft-pion production (by soft pions)[1,2] and the particle-hole formalism[3] of closed-shell nuclei are input, indicated that the corresponding *partial* excitation cross sections (as well as those for the giant resonance 1^- and 2^-, $T = 1$ states) are expected to be quite small [ranging from the order of, at most, tenths of a microbarn in the case of the heavier target nucleus (^{16}O) to a few microbarns in the case of the lighter nucleus (^{4}He) for incident pion kinetic energies in the neighborhood of threshold (170 MeV $< T_\pi^{\text{lab}} \lesssim 250$ MeV)].[4] On the other hand, with the operation of the meson factories at LAMPF, in Canada, and in Zürich, such experiments become practicable, so that it may be more appropriate at this preexperimental stage to give estimates for the (π, 2π) *total* cross section for collective excitation in nuclei which are based on a more tractable and less detailed collective model. The Goldhaber-Teller (G-T) model[5] generalized to collective spin-isospin vibrations[6] affords just such a simple collective approach[7] with the additional advantage of allowing one to focus attention on the patent differences in the threshold approximants to the alternative soft-pion formulations[1,8] and their associated predictive consequences for threshold production in *nuclei*.

Thus we note that in the phenomenological Lagrangian approach to single-pion production, in which the soft-pion amplitude is straightforwardly calculated from the effective Lagrangian,[8-10]

$$\mathcal{L} = \sum_{j=1}^{3} \mathcal{L}_{\pi N}^{(j)} + \mathcal{L}_{\pi\pi}, \tag{1}$$

$$\mathcal{L}_{\pi N}^{(1)} = (g/2M)\,\bar{\psi}\gamma_\mu\gamma_5\vec{\tau}\,\psi\cdot\partial^\mu\vec{\phi}, \tag{2a}$$

$$\mathcal{L}_{\pi N}^{(2)} = -(g/2M)^2(g_V/g_A)^2\,\bar{\psi}\gamma_\mu\vec{\tau}\,\psi\cdot\vec{\phi}\times\partial^\mu\vec{\phi}, \tag{2b}$$

$$\mathcal{L}_{\pi N}^{(3)} = -(g/2M)^3(g_V/g_A)^2\,\bar{\psi}\gamma_\mu\gamma_5\vec{\tau}\,\psi\cdot\partial^\mu\vec{\phi}\,\phi^2, \tag{2c}$$

$$\mathcal{L}_{\pi\pi} = (g/2M)^2(g_V/g_A)^2 \times[-\phi^2(\partial^\mu\phi)^2 + \tfrac{1}{2}(1-\tfrac{1}{2}\xi)m_\pi^{\,2}(\phi^2)^2], \tag{2d}$$

the threshold approximant (which we indicate by ~) for the experimentally favored value[11] $\xi = 0$ composed of *just pion-pole plus contact terms* (Figs. 1 and 2)[12]

$$T^{\text{Phenom. Lagrangian}}_{N(p)\to N'(p')+\pi(q_1,\alpha)+\pi(q_2,\beta)+\pi(q_3,\gamma)} \sim -\, i(2\pi)^4\delta(Q-p+p')(1/2f_\pi{}^3)(g_A/g_V)\bar{u}(p')$$
$$\times\Big\{\tfrac{1}{2}\tau_\alpha\gamma_5\Big[\not{q}_1-\frac{4M}{Q^2-m_\pi{}^2}(Q\cdot q_1-q_2\cdot q_3-m_\pi{}^2)\Big]\delta_{\gamma\beta}$$
$$+\tfrac{1}{2}\tau_\beta\gamma_5\Big[\not{q}_2-\frac{4M}{Q^2-m_\pi{}^2}(Q\cdot q_2-q_3\cdot q_1-m_\pi{}^2)\Big]\delta_{\gamma\alpha}$$
$$+\tfrac{1}{2}\tau_\gamma\gamma_5\Big[\not{q}_3-\frac{4M}{Q^2-m_\pi{}^2}(Q\cdot q_3-q_1\cdot q_2-m_\pi{}^2)\Big]\delta_{\alpha\beta}\Big\}u(p)\,, \tag{3}$$

specialized in the case[13] of $\pi^-p\to\pi^-\pi^+n$ to

$$T^{\text{Phenom. Lagrangian}}_{\pi^-p\to\pi^-\pi^+n} \sim -\, i(2\pi)^4\delta(Q-p+p')(1/\sqrt{2}f_\pi)^3(g_A/g_V)$$
$$\times\bar{u}(p')\gamma_5\Big\{(\not{Q}-\not{q}_2)-\frac{4M}{Q^2-m_\pi{}^2}[(Q-q_2)^2-2m_\pi{}^2]\Big\}u(p)\,, \tag{4}$$

comprises the *major share of the threshold amplitude,* i.e., one has at threshold

$$\frac{a^{-+n}_{\text{approx}}}{a(-+n)}=\left(\frac{1.2}{1.36}\right)=0.88\,, \tag{5}$$

where[8] $\sigma^{\pi^-p\to\pi^-\pi^+n}_{\text{Threshold}}=|a(-+n)|^2k^2\times$(phase space). On the other hand, in the current-commutator theory,[2] the analogous $\xi=0$ threshold approximant (in this case the local equal-time-commutator terms)

$$T^{\text{Curr. commutators}}_{N(p)\to N'(p')+\pi(q_1,\alpha)+\pi(q_2,\beta)+\pi(q_3,\gamma)} \sim -\, i(2\pi)^4\delta(Q-p+p')(1/\sqrt{2}f_\pi)^3\sqrt{2}\,[\delta_{\beta\gamma}(Q+q_1)^\mu\langle n(p')|A^\alpha_\mu|p(p)\rangle$$
$$+\delta_{\gamma\alpha}(Q+q_2)^\mu\langle n(p')|A^\beta_\mu|p(p)\rangle+\delta_{\alpha\beta}(Q+q_3)^\mu\langle n(p')|A^\gamma_\mu|p(p)\rangle]\,, \tag{6}$$

with

$$\langle n(p')|A^\alpha_\mu|p(p)\rangle=(g_A/g_V)\bar{u}(p')\tfrac{1}{2}\tau^\alpha\gamma_5\left(\gamma_\mu-\frac{2MQ_\mu}{Q^2-m_\pi{}^2}\right)u(p) \tag{7}$$

specialized to $\pi^-p\to\pi^-\pi^+n$,

$$T^{\text{Curr. commutators}}_{\pi^-p\to\pi^-\pi^+n} \sim -\, i(2\pi)^4\delta(Q-p+p')(1/\sqrt{2}f_\pi)^3(g_A/g_V)\bar{u}(p')\gamma_5\Big[(3\not{Q}-\not{q}_2)-\frac{2M}{Q^2-m_\pi{}^2}Q\cdot(3Q-q_2)\Big]u(p)\,, \tag{8}$$

accounts for a *much smaller fraction of the threshold amplitude.* Indeed, one finds for the ratio of threshold cross sections for the two "pole + contact term" approximants [Eqs. (4) and (8)],

$$\frac{\sigma^{\text{Curr. commutators}}_{\text{Threshold}}(\pi^-p\to\pi^-\pi^+n)}{\sigma^{\text{Phenom. Lagrangian}}_{\text{Threshold}}(\pi^-p\to\pi^-\pi^+n)}=\tfrac{1}{2}\,,$$

so that the good agreement with threshold data reported by Chang[2] must depend as much on the inclusion in this approach[2] of the "recoil corrections" provided by typically omitted baryon-pole terms.[14]

Note that it is the threshold approximant of the current-commutator theory[2] which Eisenberg[1] uses in his estimate of threshold pion production in nuclei.[1] *Unfortunately, his expression (11)* [cf. our nonrelativistic form Eq. (10) below] *is too small by a factor of 2.*[15] *Consequently, all the production cross section estimates there*[1] *must be multiplied by a factor of 4.* This factor of 4 has been supplied in our plots (Figs. 4 and 5) of summed partial production cross sections derived from the relevant tables of Ref. 1.

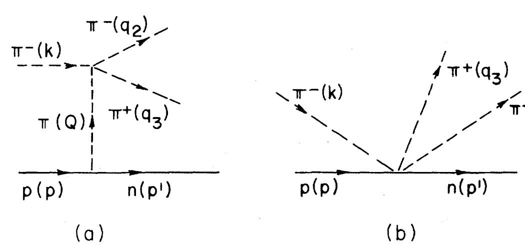

FIG. 1. Important diagrams in the reaction $\pi^-p\to\pi^-\pi^+n$ at threshold in the phenomenological Lagrangian theory. The contribution from the double equal-time commutators (the threshold approximation of Ref. 1) of the current-commutator theory (Ref. 2) is likewise expressible as a sum of (a) pion-pole and (b) contact graphs.

To apply the alternative threshold approximants Eqs. (4) and (8) to nuclei, one next makes the reduction to the nonrelativistic forms [neglecting corrections $O(1/M)$]

$$T^{\text{Phenom. Lagrangian}}_{\pi^- p \to \pi^-\pi^+ n} \sim -i(2\pi)^4\delta(Q-p+p')\frac{(g_A/g_V)}{(\sqrt{2}f_\pi)^3}\zeta^{(\sigma')\dagger}\left\{(\vec{Q}-\vec{q}_2)\cdot\vec{\sigma}-\frac{2\vec{Q}\cdot\vec{\sigma}}{Q^2-m_\pi^2}[(Q-q_2)^2-2m_\pi^2]\right\}\zeta^{(\sigma)}, \tag{9}$$

$$T^{\text{Curr. commutators}}_{\pi^- p \to \pi^-\pi^+ n} \sim -i(2\pi)^4\delta(Q-p+p')\frac{(g_A/g_V)}{(\sqrt{2}f_\pi)^3}\zeta^{(\sigma')\dagger}\left[(3\vec{Q}-\vec{q}_2)\cdot\vec{\sigma}-\frac{\vec{Q}\cdot\vec{\sigma}}{Q^2-m_\pi^2}(3Q^2-q_2\cdot Q)\right]\zeta^{(\sigma)}. \tag{10}$$

One might be led by the close analogy here to the one-nucleon input for the calculation of threshold pion electroproduction in nuclei[16] to include the additional contribution associated with the N^{*+} tail[16,17] [the pertinent graph is displayed in Fig. 3][18]:

$$T^{N^*\text{ tail}}_{\pi^- p \to \pi^-\pi^+ n} \sim -i(2\pi)^4\delta(Q-p+p')\frac{1}{m_\pi-\omega_R+\frac{1}{2}i\Gamma}\frac{(1/\sqrt{3})^3C_3(0)g_A^*}{m_\pi f_\pi^3}\zeta^{(\sigma')\dagger}$$
$$\times[\vec{k}\cdot\vec{\sigma}(\vec{k}\cdot\vec{q}_3)+\tfrac{1}{3}\vec{k}\cdot\vec{\sigma}(\vec{q}_2\cdot\vec{q}_3)+\tfrac{2}{3}\vec{q}_2\cdot\vec{\sigma}(\vec{k}\cdot\vec{q}_3)-\tfrac{1}{3}\vec{q}_3\cdot\vec{\sigma}(\vec{k}^2+\vec{k}\cdot\vec{q}_2)-\tfrac{1}{3}i\vec{q}_2\cdot(\vec{q}_3\times\vec{k})]\zeta^{(\sigma)}. \tag{11}$$

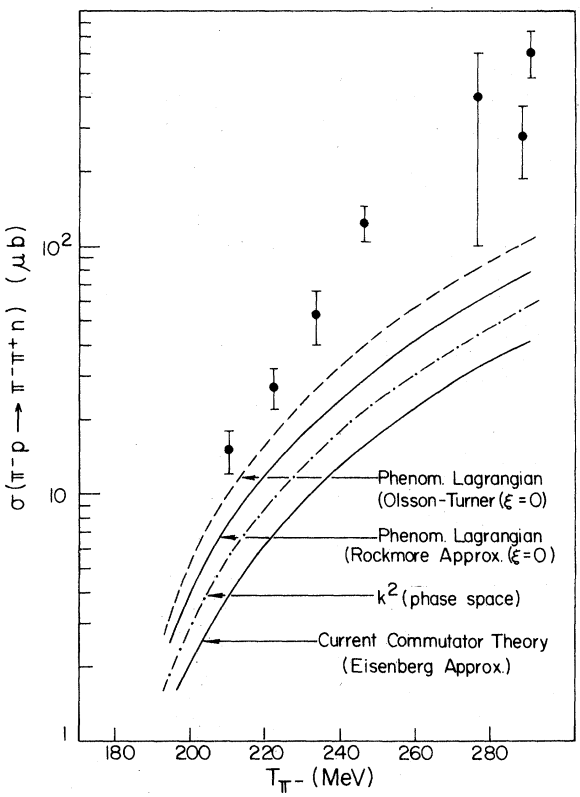

FIG. 2. Various theoretical threshold predictions for the reaction $\pi^- p \to \pi^-\pi^+ n$ versus T_{π^-}, incident pion laboratory kinetic energy, compared with experimental data points (cf. Ref. 8). The solid curves were calculated from the threshold approximation consisting of the sum of the two graphs of Fig. 1. The dashed curve shows the improved agreement between theory and experiment which results when one includes the contributions from *all* graphs in the phenomenological Lagrangian theory and is obtained (Ref. 8) by multiplying the quantity k^2(phase space) (the dashed-dotted reference curve in the figure) by $|a(-+n)|^2=1.85$ (for $\xi=0$).

This contribution has been evaluated in the center-of-momentum frame of the outgoing π^+ and recoiling neutron ($\vec{q}_3+\vec{p}'=0$) in the adiabatic limit of its nonrelativistic reduction. However, since this contribution produces an effect on the production cross section from a single nucleon of less than 10% in the threshold region (cf. Table I), we neglect it in the application to nuclei.

Using the alternative nonrelativistic single-nucleon amplitudes [Eqs. (9) and (10)], we next evaluate the pion production cross section in nuclei leading to states of collective excitation in the generalized[6,19] Goldhaber-Teller model.[5] In this simple model, the cross section for the reaction

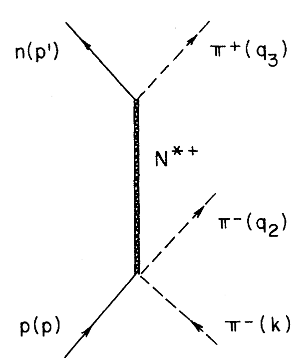

FIG. 3. Contribution to the process $\pi^- p \to \pi^-\pi^+ n$ in which the π^+ is produced via an intermediate N^{*+}. The threshold amplitude consisting of the sum of the diagrams of Figs. 1 and 2 is the exact analog here of the threshold amplitude for electroproduction used by Čzyz and Walecka (Ref. 17).

$\pi^- A_Z \to \pi^-\pi^+ A^*_{Z-1}$ is given by[20]

$$d\sigma = \frac{(2\pi)^4}{v_r} \sum_{M'} \int \delta((\vec{k} - \vec{q}_1 - \vec{q}_2) - \vec{Q})\, \delta((\omega - \omega_1 - \omega_2) - \delta)\, |\mathcal{H}|^2 \frac{d\vec{q}_1 d\vec{q}_2 d\vec{Q}}{(2\pi)^9 8\omega_1\omega_2\omega}, \tag{12}$$

where

$$\mathcal{H} = \vec{\mathfrak{a}} \cdot \vec{\mathfrak{M}} \tag{13}$$

with

$$\vec{\mathfrak{M}} = \left(\Phi^\dagger_{Z-1} \sum_{i=1}^{A} e^{i\vec{Q}\cdot\vec{r}_i} \vec{\sigma}^{(i)} \underline{\tau}^{(i)}_- \Phi_Z\right), \tag{14}$$

and

$$\vec{\mathfrak{a}}^{\text{Phenom. Lagrangian}} \cong \frac{(g_A/g_V)}{(\sqrt{2}\, f_\pi)^3}\left[1 - \frac{2(Q - q_2)^2 - 4m_\pi^2}{Q^2 - m_\pi^2}\right]\vec{Q}, \tag{15a}$$

$$\vec{\mathfrak{a}}^{\text{Curr. commutators}} \cong \frac{(g_A/g_V)}{(\sqrt{2}\, f_\pi)^3}\left(3 - \frac{3Q^2 - q_2 \cdot Q}{Q^2 - m_\pi^2}\right)\vec{Q}. \tag{15b}$$

Because of the increased sensitivity of our results to the shape of the charge distribution we take for the ground-state charge form factor

$$F(\vec{Q}) = \int d^3r\, e^{i\vec{Q}\cdot\vec{r}} \rho_0(\vec{r}), \tag{16}$$

which enters into the matrix elements for the $0^+ \to 1^-(M')$ transition

$$\langle 1^- M'm' | \mathcal{H} | 0^+ \rangle = i\sqrt{2}\, \mathfrak{a}_i\, \delta^{(s)}_{im'} |\vec{Q}|\, F(\vec{Q}) \times \left(\frac{2\pi}{3AM\delta}\right)^{1/2} Y_1^{M'}(\hat{Q})^*, \tag{17}$$

the harmonic oscillator result[21]

$$F(Q) = Z\left(1 - \frac{(Z-2)}{6Z} a^2 Q^2\right) e^{-(1/4)a^2Q^2} \tag{18}$$

with length parameters[22]

$$a(^4\text{He}) = 1.31 \text{ fm}, \quad a(^{12}\text{C}) = 1.64 \text{ fm}, \quad a(^{16}\text{O}) = 1.79 \text{ fm}, \tag{19}$$

rather than the simpler "dipole" fit of Ref. 6. Straightforward manipulation[6,19] yields an expression for the production cross section

$$d\sigma = \frac{(g_A/g_V)^2}{(\sqrt{2}\, f_\pi)^6} \frac{1}{MA\delta} \frac{1}{(2\pi)^5 8k} \times \int \frac{d\vec{q}_1 d\vec{q}_2 d\vec{Q}}{\omega_1\omega_2} Q^4 |F(Q)|^2 |a(- +n; Q)|^2 \times \delta((\vec{k} - \vec{q}_1 - \vec{q}_2) - \vec{Q})\, \delta((\omega - \omega_1 - \omega_2) - \delta), \tag{20}$$

which requires only a *single* numerical integration

TABLE I. $\dfrac{\Delta\sigma_{N^{*+}\text{tail}}(\pi^- p \to \pi^-\pi^+ n)}{\sigma^{\text{Phenom. Lagrangian}}_{\text{Threshold Approx.}}(\pi^- p \to \pi^-\pi^+ n)} \equiv R.$

T_{π^-} (MeV)	R
200	0.063
220	0.072
260	0.092

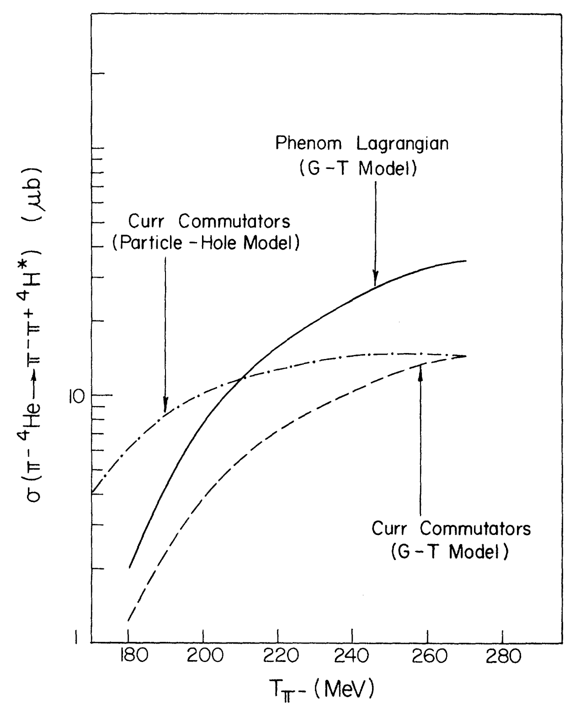

FIG. 4. Theoretical threshold $^4\text{He}(\pi^-, \pi^-\pi^+)^4\text{H}^*$ cross sections in the particle-hole and generalized Goldhaber-Teller models as a function of T_{π^-}. The dashed curves are calculated using Eisenberg's approximation to the current-commutator theory (Ref. 2). In the case of the particle-hole model prediction (Ref. 1), a sum over the cross sections for excitation of the giant resonance 0^-, 1^-, and 2^-, $T=1$ states, an additional factor of 4 omitted in that work has been supplied. Note the sizable difference between the predictions of the two threshold production theories in the generalized Goldhaber-Teller model.

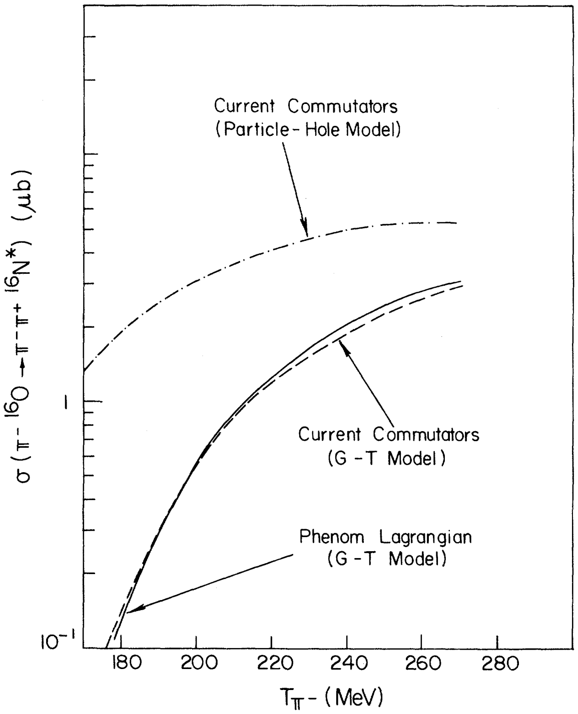

FIG. 5. Theoretical threshold $^{16}O(\pi^-, \pi^-\pi^+)^{16}N^*$ cross sections in the particle-hole and generalized Goldhaber-Teller models. In the generalized Goldhaber-Teller model, the predictions of the two pion-production theories do not appear to differ significantly in this case.

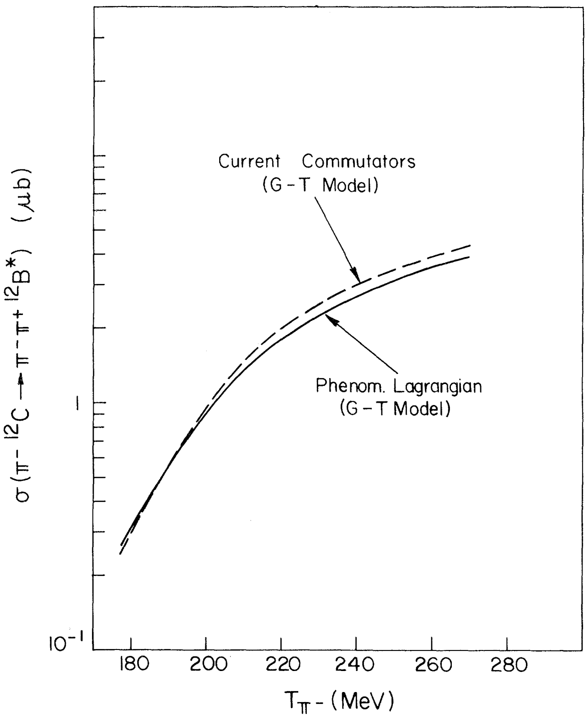

FIG. 6. Theoretical predictions for threshold $^{12}C(\pi^-, \pi^-\pi^+)^{12}B^*$ cross sections in the generalized Goldhaber-Teller model.

once the squared amplitudes $|a(-+n;Q)|^2$,

$$|a^{\text{Phenom. Lagrangian}}(-+n;Q)|^2 = \left(1 - \frac{4\omega_2\delta}{\vec{Q}^2 + m_\pi^2 - \delta^2}\right)^2, \tag{21a}$$

$$|a^{\text{Curr. commutators}}(-+n;Q)|^2 = \left(\frac{3m_\pi^2 + \delta\omega_2}{\vec{Q}^2 + m_\pi^2 - \delta^2}\right)^2, \tag{21b}$$

have been fitted with functions of the form $f(Q) = A_1 e^{-a_1 Q^2} + A_2 Q^2 e^{-a_2 Q^2}$. The predictions of the generalized Goldhaber-Teller model for the three target nuclei with $N=Z$, ^{4}He, ^{16}O, and ^{12}C, are displayed in Figs. 4, 5, and 6 and are consistent[23] within a factor of 2 to 3 in the neighborhood of threshold with the predictions of the particle-hole model used by Eisenberg[1] after the necessary "renormalization" of the numbers calculated there.[1] (Moreover, since it has been remarked that particle-hole model calculations using harmonic oscillator wave functions frequently overestimate cross sections by a factor of 2, the predictions of the two models may not actually be so far apart.[23,24] For ^{4}He, the lightest target nucleus considered, one finds the interesting result that the predicted cross sections for the two alternative soft-pion theories differ by a factor of about 2. We conclude that while the cross sections for the $(\pi, 2\pi)$ reaction in nuclei are still expected to be quite small, the prospect for their accessibility seems reasonably improved.

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[2] L.-N. Chang, Phys. Rev. 162, 1497 (1967).

[3] V. Gillet and N. Vinh Mau, Nucl. Phys. 54, 321 (1964).

[4] Furthermore, initial- and final-state pion interactions are not taken into account in the calculation of Ref. 1, so that cross sections for the $(\pi, 2\pi)$ process may be reduced by as much as a factor of 8 from the soft-pion limit predictions. In this connection see H. Überall, B. A. Lamers, C. W. Lucas, and A. Nagl, Phys. Lett. 44B, 324 (1973).

[5] M. Goldhaber and E. Teller, Phys. Rev. 74, 1046 (1948).

[6] H. Überall, Phys. Rev. 137, B502 (1965).

[7] Note that this nuclear model has previously served with qualitative success in a like capacity in the calculation of the total cross section for neutrino absorption in ^{12}C leading to states of collective vibration (Ref. 6). [The relevant comparison of that calculation with total cross sections obtained in the one-particle–one-hole Brown theory by summing over nine $T=1$ levels has been given by F. J. Kelly and H. Überall, Phys. Rev. 158, 987 (1967).]

[8] M. G. Olsson and L. Turner, Phys. Rev. Lett. 20, 1127 (1968); Phys. Rev. 181, 2141 (1969).

[9] S. Weinberg, Phys. Rev. Lett. 18, 188 (1967); J. Schwinger, Phys. Lett. 24B, 473 (1967); R. Dashen and M. Weinstein, Phys. Rev. 183, 1261 (1969).

[10] ξ is the symmetry-breaking parameter introduced by Olsson and Turner (Ref. 8) to measure the deviation from the σ model in the soft-pion calculation of the mass-shell π-π amplitude. The same parameter ξ must necessarily enter into the threshold amplitude for $\pi N \rightarrow \pi\pi N$ through the peripheral (π-π scattering) graph [Fig. 1(a)].

[11] It is concluded in Ref. 8 that available data for the reaction $\pi^- p \rightarrow \pi^- \pi^+ n$ near threshold tends to support the value $\xi = 0$ [Weinberg's Lagrangian (Ref. 9)]. It should be pointed out that while the expressions in Olsson and Turner (the first of Refs. 8) are correct, *the plot of the threshold cross section prediction from all graphs*, including nucleon pole terms (their Fig. 2), *is in error;* the correct plot for $\xi = 0$ is given in our Fig. 2. On the other hand, their plot of Q^2(phase space) [k^2(phase space) in our notation] versus T_π (Fig. 1 of the second of Refs. 8) *is correct.* This bothersome discrepancy was also noted by E. Lomon in Measurement of the Cross Section for $\pi^- p \rightarrow \pi^- \pi^+ n$ with a Magnetic Spectrometer, edited by G. A. Rebka, Jr., *et al.*, LAMPF Research Proposal, August 31, 1971 (unpublished). J. S. Kovacs and W. F. Long, Phys. Rev. D 1, 1333 (1970), keep *all* graphs involving pions and nucleons [while validly neglecting those which involve the formation of a (3, 3) isobar] and, making none of the typical threshold approximations of Ref. 8, conclude that the pion-production total cross section data below $T_\pi \simeq 330$ MeV is best fitted by a chiral model in which the symmetry-breaking term transforms as a rank-two chiral tensor. (This chiral model corresponds to the choice $\xi = -2$ for which the scattering length $a_2 = 0$.) However, *as one approaches threshold* one finds the predictions of the rank-two chiral tensor model and the Weinberg model ($\xi = 0$) converging *equally* to the experimental data; the one from above, the other from below.

[12] The convenient temporary characterization of the production reaction as $N(p) \rightarrow N'(p') + \pi(q_1, \alpha) + \pi(q_2, \beta) + \pi(q_3, \gamma)$ results in the simplifying Bose symmetrization of expression (3); we usually write $k = -q_1$. (See Figs. 1 and 3.) We take $f_\pi = [(g/M)(g_v/g_A)]^{-1} \simeq 82$ MeV. (Cf. the discussion in the second of Refs. 8.)

[13] The projection operators $\mathcal{P}_{j;J}$ ($j = \frac{1}{2}, \frac{3}{2}$; $J = 0, 1, 2$) relevant to this specialization, which satisfy the relations

$$\delta_{\beta\alpha}\tau_\gamma = \mathcal{P}_{1/2,0} - \sqrt{2}\mathcal{P}_{1/2,1} + \tfrac{1}{2}\sqrt{2}\mathcal{P}_{3/2,1} + \tfrac{1}{2}\sqrt{10}\mathcal{P}_{3/2,2},$$

$$\delta_{\gamma\alpha}\tau_\beta = \mathcal{P}_{1/2,0} + \sqrt{2}\mathcal{P}_{1/2,1} - \tfrac{1}{2}\sqrt{2}\mathcal{P}_{3/2,1} + \tfrac{1}{2}\sqrt{10}\mathcal{P}_{3/2,2},$$

$$\delta_{\beta\gamma}\tau_\alpha = 3\mathcal{P}_{1/2,0},$$

are to be found in Ref. 2.

[14] Recall that in contrast to the phenomenological Lagrangian theory it is noted in Ref. 2 that the π-π interaction in this application of the current-commutator theory turns out to be small, i.e., the residue of the pion pole is *appreciably reduced* from the contribution of the corresponding on-shell amplitude.

[15] With some effort it can be further verified that Eisenberg's resultant expression (14) for the production cross section in the case of a pure particle-hole configuration is similarly too small by a factor of 4.

[16] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. 111, 329 (1958).

[17] W. Čzyz and J. D. Walecka, Nucl. Phys. 51, 312 (1964). It is amusing that, following Čzyz and Walecka, one finds in the Fermi gas model an $\eta^{5/2}$ dependence for the S-wave production cross section $d^2\sigma/d\omega_3\, d\Omega_3$ near threshold $[\eta = (\omega - \omega_3 - m_\pi)/m_\pi]$ for $\eta < k(2p_F - k)/(2M^* m_\pi)$, where $p_F \simeq \frac{1}{4}M$, $M^* = \frac{3}{5}M$.

[18] This contribution emerges from the N^*-pole analysis of the amplitude (Ref. 2),

$$\frac{-i(\frac{2}{3})}{(\sqrt{2}f_\pi)^3}(k+q_2)_\mu q_{3\nu} \int d^4x\, d^4z\, e^{-i(k-q_2)\cdot x + i q_3 \cdot z} \times \langle n(p') | T(V_3^\mu(x), A^\nu_{(1-i2)}(z)) | p(p) \rangle .$$

As in Ref. 2, we take $C_3(0) = 0.345$, $g_A^* \simeq 1.44$. A. J. Dufner and Y. S. Tsai, Phys. Rev. 168, 1801 (1968), have obtained the parameter $C_3(0)$ from Chew-Low static theory. They find

$$C_3{}^2(0) = \frac{(M + m_\pi)m_\pi{}^2}{M} \frac{8}{3} \left(\frac{\mu_p - \mu_n}{4M}\right)^2,$$

where M is the proton mass, with $C_3(0) = 2.2(m_\pi/M)$. The resonance parameters $\omega_R = 2.2\, m_\pi$ and $\frac{1}{2}\Gamma = 0.8\, m_\pi$ are taken from Ref. 17.

[19] For further discussion and applications of this model, see H. Überall, Phys. Rev. 139, B1239 (1965); Nuovo Cimento 41B, 25 (1966); Springer Tracts in Modern Physics 49, 1 (1969); F. J. Kelly, L. J. McDonald, and H. Überall, Nucl. Phys. A139, 329 (1969).

[20] The "average" choice $\delta_J \cong \delta$ with $\delta = 22.5$ MeV (δ_J = excitation energy of A^*_{z-1} measured from the A_z ground state) enables the simplifying use of the uncoupled expressions of Ref. 6.

[21] L. R. B. Elton, *Nuclear Sizes* (Oxford U. P., London, 1961).

[22] Our values for $a(^4He)$ and $a(^{12}C)$ drive from Ref. 21; that of $a(^{16}O)$ from R. W. Sharp and L. Zamick, Nucl. Phys. A208, 130 (1973).

[23] F. J. Kelly and H. Überall, Phys. Rev. 158, 987 (1967). Cf. Fig. 4 of this reference in particular.

[24] H. Überall, B. A. Lamers, J. B. Langworthy, and F. J. Kelly, Phys. Rev. C 6, 1911 (1972).