# Covariant pion-nucleus optical potential*

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We present a detailed evaluation of the leading term (triangle diagram) of the pion-nucleus optical potential. The development uses a covariant multiple scattering theory introduced previously. It is shown how recent phenomenological studies of the pion-nucleon (off-shell) scattering amplitudes may be incorporated into this theory. Further, various problems associated with the transformation of the pion-nucleon scattering amplitudes from the center of mass of the $\pi$-$N$ system to the center of mass of the $\pi$-*nucleus* system are resolved in this approach.

## I. INTRODUCTION

Recently, there has been a great deal of interest in the calculation of the pion-nucleus interaction. A large number of the published works in this area rely on the nonrelativistic multiple scattering theory of Watson.[1] Recently, in a series of papers we have presented a new covariant multiple scattering theory which we hope to apply to the problem of pion-nucleus interactions.[2-4] We feel that our method has a number of important advantages over the more conventional approaches. For example, the use of relativistic kinematics is integral to our scheme. Also, there is no need to assume a potential interaction between the pion and nucleon as is the case in the nonrelativistic theories; in principle, we can also incorporate production and absorption amplitudes in our analysis. Binding effects, off-shell effects, and the dynamics of the target particles are all treated properly.

In this work we concentrate on the analysis of the leading contribution to the pion-nucleus optical potential—the single-scattering term, or "triangle diagram." In Sec. II we review our covariant scattering theory and in Sec. III we present the analytical expressions for the triangle diagram. Our results involve the covariant (off-shell) pion-nucleon scattering amplitude, and in Sec. IV we show how recent phenomenological studies of $\pi$-$N$ amplitudes may be incorporated into our analysis. The last step requires that an interpretation of the phenomenological amplitudes be made such that they have significance in the covariant theory. In Sec. V and VI we discuss the transformation of the scattering amplitudes from the $\pi$-$N$ center-of-mass frame to the $\pi$-nucleus center-of-mass frame. This problem has been discussed by several authors recently and represents a source of some confusion. We believe our treatment is unambiguous and serves to resolve the problems which arise in other noncovariant calculations. Finally, Sec. VII contains some of our conclusions and some suggestions for future work.

## II. REVIEW OF COVARIANT SCATTERING THEORY

A relativistic dynamical calculation of projectile-nucleus scattering can be achieved with the aid of a two-particle covariant equation, as we have discussed previously.[2-4] In this approach, each nuclear bound state is treated as an elementary particle, that is, each of these states is described by a field and has its own propagator. The invariant amplitude $M$ for the scattering of a pion (particle 1) by a nucleus (particle 2) then satisfies the integral equation

$$M = K + KGM \ , \tag{2.1}$$

where $K$ is an irreducible kernel and $G = G_1 G_2$ is the propagator which propagates both particles. Equation (2.1) is equivalent to the set of equations

$$M = U + UgM \ , \tag{2.2}$$

$$U = K + K(G - g)U \ , \tag{2.3}$$

where $g$ is a new Green's function. A first order approximation ($U = K$) yields an *approximate* equation for $M$.

In our previous work[2,3] we describe several possible choices for $g$. Each of these reduces Eq. (2.1) to a three-dimensional covariant integral equation. Owing to the absence of a strong interaction theory for the fundamental pion-nucleon interaction, calculation of the pion-nucleus scattering necessarily contains some phenomenological aspects. In particular, the pion-nucleon scattering amplitudes and production (or absorption) amplitudes appearing in an analysis of the diagrams for pion-nucleus scattering cannot be calculated from current theories. Instead, we must rely on the information provided by phenomenological analyses of these amplitudes.[5] Since up to this date *only* $\pi$-$N$ scattering processes have received wide attention, it is sensible to use a covariant scheme which requires the off-shell pion-nucleon *scattering* amplitudes as the sole theoretical input in calculations.

Since the nuclei for which we would construct optical potentials are, in general, much heavier than the pion, it is reasonable to limit our discussion to the case in which the heavy nucleus is always on its mass shell. We therefore use the following Green's function (for integral spin targets):

$$g_6(k\,|\,W)=\left(\frac{1}{2\omega_{\vec{k}}2E_{A,\vec{k}}}\right)\delta(k^0-\Delta_6^{\pi,A}(\vec{k}))\times\frac{1}{2W-(\omega_{\vec{k}}+E_{A,\vec{k}})+i\eta}\ , \tag{2.4}$$

where

$$\Delta_6^{\pi,A}(\vec{k})=W+\tfrac{1}{2}L-E_{A,\vec{k}} \tag{2.5}$$

and

$$L=(M_A{}^2-M_\pi{}^2)/2W\ . \tag{2.6}$$

In Eq. (2.4), $2W$ is the center-of-mass energy of the pion-nucleus system, $M_A$ is the mass of the target and $\omega_{\vec{k}}$ and $E_{A,\vec{k}}$ are the relativistic pion and target energies, respectively. (Note that $g_6$ represents the first part of the Green's function $g_5$ defined in Ref. 2, modified for the case of a spin-zero target.) We remark that $\Delta_6^{\pi,A}(\vec{k}_0)=0$ if $|\vec{k}_0|$ corresponds to the value appropriate to the asymptotic condition such that $2W=\omega_{\vec{k}_0}+E_{A,\vec{k}_0}$.

Using reduction techniques we have discussed previously, we see that Eq. (2.1) takes the form:

$$\langle\vec{p}'\,|\,\overline{M}_6(W)\,|\,\vec{p}\rangle=\langle\vec{p}'\,|\,\overline{U}_6(W)\,|\,\vec{p}\rangle+\int d\vec{k}\langle\vec{p}'\,|\overline{U}_6(W)\,|\,\vec{k}\rangle\frac{R_6(\vec{k})}{2W-(\omega_{\vec{k}}+E_{A,\vec{k}})+i\eta}\langle\vec{k}\,|\,\overline{M}_6(W)\,|\,\vec{p}\rangle\ . \tag{2.7}$$

Here,

$$\langle\vec{p}'\,|\,\overline{M}_6(W)\,|\,\vec{p}\rangle\equiv\langle\vec{p}',\,p'^0=\Delta_6^{\pi,A}(\vec{p}')\,|\,M(W)\,|\,\vec{p},\,p^0=\Delta_6^{\pi,A}(\vec{p})\rangle\ , \tag{2.8}$$

$$\langle\vec{p}'\,|\,\overline{U}_6(W)\,|\,\vec{p}\rangle\equiv\langle\vec{p}',\,p'^0=\Delta_6^{\pi,A}(\vec{p}')\,|\,U(W)\,|\,\vec{p},\,p^0=\Delta_6^{\pi,A}(\vec{p})\rangle\ , \tag{2.9}$$

and

$$R_6(\vec{k})=1/(2\omega_{\vec{k}}2E_{A,\vec{k}})\ . \tag{2.10}$$

[Equation (2.10) is appropriate for a spin-zero target.]

Finally, using the notation of Appendix A, we define a $T$ matrix and a $V$ matrix through

$${}_{\rm NR}\langle\vec{p}'\,|\,T_6(W)\,|\,\vec{p}\rangle_{\rm NR}=R_6{}^{1/2}(\vec{p}')\langle\vec{p}'\,|\,\overline{M}_6(W)\,|\,\vec{p}\rangle R_6{}^{1/2}(\vec{p})\ , \tag{2.11}$$

and

$${}_{\rm NR}\langle\vec{p}'\,|\,V_6(W)\,|\,\vec{p}\rangle_{\rm NR}=R_6{}^{1/2}(\vec{p}')\langle\vec{p}'\,|\,\overline{U}_6(W)\,|\,\vec{p}\rangle R_6{}^{1/2}(\vec{p})\ , \tag{2.12}$$

$$\simeq R_6{}^{1/2}(\vec{p}')\langle\vec{p}'\,|\,\overline{K}_6(W)\,|\,\vec{p}\rangle R_6{}^{1/2}(\vec{p})\ . \tag{2.13}$$

We find that the $T$ matrix satisfies the equation

$${}_{\rm NR}\langle\vec{p}'\,|\,T_6(W)\,|\,\vec{p}\rangle_{\rm NR}={}_{\rm NR}\langle\vec{p}'\,|\,V_6(W)\,|\,\vec{p}\rangle_{\rm NR}+\int d\vec{k}\,\frac{{}_{\rm NR}\langle\vec{p}'\,|\,V_6(W)\,|\,\vec{k}\rangle_{\rm NR}}{2W-(\omega_{\vec{k}}+E_{A,\vec{k}})+i\eta}\times{}_{\rm NR}\langle\vec{k}\,|\,T_6(W)\,|\,\vec{p}\rangle_{\rm NR}\ . \tag{2.14}$$

This last integral equation is of the Lippmann-Schwinger form and is a covariant dynamical equation. We recall that the indices of the $T$, $V$, $M$, $U$, and $K$ matrices in Eqs (2.5) to (2.14) refer explicitly to the particular reduction scheme we have considered, that is, the one based on $g_6$ of Eq. (2.4).

In the next section we consider the construction of the optical potential ${}_{\rm NR}\langle\vec{p}'\,|\,V_6(W)\,|\,\vec{p}\rangle_{\rm NR}$ in the approximation indicated in Eq. (2.13).

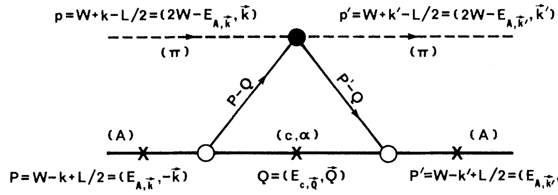


FIG. 1. The single-scattering diagram. The dashed, the light, and the heavy lines represent, respectively, the pion, the nucleon, and the various nuclei. The $\pi$-$N$ off-shell scattering amplitude is denoted by the filled circle, while the nuclear vertex interactions are represented by the open circles. The $\alpha$ denotes the unoccupied orbit in $C$. The cross denotes a particle which has been placed on its mass shell.

## III. REDUCTION OF THE TRIANGLE DIAGRAM

The most important contribution to the pion-nucleus optical potential comes from the single-scattering term (or "triangle diagram")—see Fig. 1. Using our diagram rules, summarized in Appendix A, and underlining the momenta of those particles that have been placed on their mass shell, we have

$$(2\pi)^3\langle p',\underline{P}'|\overline{K}_6(W)|p,\underline{P}\rangle \equiv \langle \vec{k}'|\overline{K}_6(W)|\vec{k}\rangle$$
$$=\sum_{\alpha,C}\int d^4Q\left(\frac{i}{2\pi}\right)[(2\pi)^3\langle \underline{P}'|\Gamma|Q,P'-Q\rangle]G_N(P'-Q)G_{C,\alpha}(Q)$$
$$\times[(2\pi)^3\langle p',P'-Q|M_{\pi N}(\sqrt{s})|p,P-Q\rangle]G_N(P-Q)[(2\pi)^3\langle Q,P-Q|\Gamma|\underline{P}\rangle] \qquad (3.1)$$

with $s=(p'+P'-Q)^2=(p+P-Q)^2$.

In Eq. (3.1) we use the notation $\langle p',P'-Q|M_{\pi N}(\sqrt{s})|p,P-Q\rangle$, in keeping with the notation of Fig. 1; however, *the momentum-conserving δ functions have been removed* and this expression represents the invariant matrix in the *relative* momenta. That is $[(2\pi)^3\langle p',P'-Q|M_{\pi N}(\sqrt{s})|p,P-Q\rangle] \equiv\langle k_c'|M_{\pi N}(\sqrt{s})|k_c\rangle$, where the $k_c$ and $k_c'$ are the relative four-momenta in the center of mass of the $\pi$-$N$ system (see Fig. 2). Similar remarks apply to the vertex functions $\Gamma$ and to the quantity $\langle p'\underline{P}'|\overline{K}_6(W)|p,\underline{P}\rangle$. Again $[(2\pi)^3\langle p',\underline{P}'|\overline{K}_6(W)|p,\underline{P}\rangle] \equiv\langle\vec{k}'|\overline{K}_6(W)|\vec{k}\rangle$, where $\vec{k}$ and $\vec{k}'$ are the relative momenta in the pion-*nucleus* center-of-mass system (see Fig. 2). The operators which have not had the δ function extracted are distinguished by a caret, i.e., $\hat{S}$, $\hat{M}$, etc.—see Eqs. (A2) and (A9) of Appendix A. The propagators $G$ in Eq. (3.1) are related to the usual Feynman propagators $S_F$ by the relation $S_F=iG/(2\pi)^4$. Consequently, in Eq. (3.1),

$$G_N(P-Q)=[\gamma\cdot(P-Q)-M_N+i\epsilon]^{-1}\ . \qquad (3.2)$$

Also, $G_{C,\alpha}(Q)$ is the corresponding Feynman propagator for the nucleus $C$, multiplied by $-i(2\pi)^4$. The label $\alpha$ denotes the particular state of the nucleus $C$.

Equation (3.1) involves a four-dimensional integration. Since the nucleus $C$ is much more massive than either the nucleon or the pion, it is reasonable to neglect its off-mass-shell aspect (see Appendix B for justification) and to write the corresponding propagator as

$$G_{C,\alpha}(Q)\simeq -2\pi i[N_{C,\alpha}(\vec{Q})]^{-1}\delta(Q^0-E_{C,\alpha,\vec{Q}})\Lambda^{(+)}_{C,\alpha}(\vec{Q}) \qquad (3.3)$$

with

$$\Lambda^{(+)}_{C,\alpha}(\vec{Q})=\sum_r \chi^{\alpha,r}(\vec{Q})\overline{\chi}^{\alpha,r}(\vec{Q}) \qquad (3.4)$$

representing the projection operator for positive energy spinors. Here the summation runs over the spin projections and $N_{C,\alpha}(\vec{Q})=E_{C,\alpha}(\vec{Q})/M_{C,\alpha}$, the nucleus $C$ being necessarily of half-integer spin when the nucleus $A$ is of spin zero. With this on-shell approximation for the spectator nucleus, Eq. (3.1) becomes

$$(2\pi)^3\langle p'\underline{P}'|\overline{K}_6(W)|p,\underline{P}\rangle=\sum_{C,\alpha,r}\int d^3Q[(2\pi)^3\langle\underline{P}'|\Gamma|\underline{Q},P'-Q\rangle]\chi^{\alpha,r}(\vec{Q})\left(\frac{M_{C,\alpha}}{E_{C,\alpha,\vec{Q}}}\right)^{1/2}G_N(P'-Q)$$
$$\times[(2\pi)^3\langle p',P'-Q|M_{\pi N}(\sqrt{s})|p,P-Q\rangle]G_N(P-Q)\overline{\chi}^{\alpha,r}(\vec{Q})\left(\frac{M_{C,\alpha}}{E_{C,\alpha,\vec{Q}}}\right)^{1/2}$$
$$\times[(2\pi)^3\langle\underline{Q},P-Q|\Gamma|\underline{P}\rangle]\,. \qquad (3.5)$$

This equation is still covariant; however, it contains only a three-dimensional integration, the value of $Q^0$ being determined as $Q^0=(\vec{Q}^2+M_{C,\alpha}{}^2)^{1/2}$. Equation (3.5) may be simplified by introducing wave functions defined as follows:

$$\Psi^{\alpha,r}(\underline{Q},P-Q|\underline{P})=G_N(P-Q)\overline{\chi}^{\alpha,r}(\vec{Q})[(2\pi)^3\langle\underline{Q},P-Q|\Gamma|\underline{P}\rangle]$$
$$=(2\pi)^{3/2}\langle\vec{Q},\alpha,r|\psi(0)|\vec{P}\rangle \qquad (3.6)$$

and

$$\bar{\Psi}^{\alpha,r}(\underline{Q}, P'-Q|\underline{P}') = [(2\pi)^3 \langle \underline{P}'|\Gamma|\underline{Q}, P'-Q\rangle]\chi^{\alpha,r}(\vec{Q})G_N(P'-Q)$$
$$= (2\pi)^{3/2}\langle \vec{P}'|\bar{\psi}(0)|\vec{Q}, \alpha, r\rangle \ . \tag{3.7}$$

To obtain some guidance as to the kinematic structure of the matrix elements of the interpolating fields $\psi(0)$ and $\bar{\psi}(0)$, it is useful to expand these matrix elements in a special set of spinor functions to be defined below. We write (suppressing the spin indices in the state vectors)

$$\langle \vec{Q}|\psi(0)|\vec{P}\rangle = \langle \vec{Q}|\psi^{(+)}(0)|\vec{P}\rangle + \langle \vec{Q}|\psi^{(-)}(0)|\vec{P}\rangle$$
$$= \int \frac{d\vec{k}}{(2\pi)^{3/2}}\left(\frac{M_N^*}{|k^0|}\right)^{1/2} \sum_s \left[\langle \vec{Q}|\hat{B}_s(k)|\vec{P}\rangle u^s(k) + \langle \vec{Q}|\hat{D}^\dagger_{-s}(-k)|\vec{P}\rangle v^{-s}(-k)\right] , \tag{3.8}$$

where $k$ is a four-vector, $k \equiv P - Q$. We also write

$$\langle \vec{Q}|\hat{B}_s(k)|\vec{P}\rangle = \delta(\vec{k} - \vec{P} + \vec{Q})\langle \vec{Q}|B_s(P-Q)|\vec{P}\rangle ,$$
$$\langle \vec{Q}|\hat{D}^\dagger_{-s}(-k)|\vec{P}\rangle = \delta(\vec{k} - \vec{P} + \vec{Q})\langle \vec{Q}|D^\dagger_{-s}(-P+Q)|\vec{P}\rangle \ ,$$

so that

$$\langle \vec{Q}|\psi(0)|\vec{P}\rangle = \frac{1}{(2\pi)^{3/2}}\left(\frac{M_N^*}{|P^0 - Q^0|}\right)^{1/2} \sum_s \left[\langle \vec{Q}|B_s(P-Q)|\vec{P}\rangle u^s(P-Q)\right.$$
$$\left. + \langle \vec{Q}|D^\dagger_{-s}(-P+Q)|\vec{P}\rangle v^{-s}(-P+Q)\right] \ . \tag{3.9a}$$

Also, we have

$$\langle \vec{P}'|\bar{\psi}(0)|\vec{Q}\rangle = \langle \vec{P}'|\bar{\psi}^{(+)}(0)|\vec{Q}\rangle + \langle \vec{P}'|\bar{\psi}^{(-)}(0)|\vec{Q}\rangle$$
$$= \frac{1}{(2\pi)^{3/2}}\left(\frac{M_N^*}{|P^0 - Q^0|}\right)^{1/2} \sum_s \left[\langle \vec{P}'|B^\dagger_s(P'-Q)|\vec{Q}\rangle \bar{u}^s(P'-Q)\right.$$
$$\left. + \langle \vec{P}'|D_{-s}(-P'+Q)|\vec{Q}\rangle \bar{v}^{-s}(-P'+Q)\right] \ . \tag{3.9b}$$

In Eqs. (3.8) and (3.9), $M_N^*$ is a parameter which we can term the "off-shell mass." The value for this parameter is given by $(M_N^*)^2 = (P^0 - Q^0)^2 - (\vec{P} - \vec{Q})^2$. The expansion above is only considered meaningful if matrix elements of the field are taken between physical states $\langle \vec{Q}|$ and $|\vec{P}\rangle$ such that $(P-Q)^2 > 0$. This insures that $M_N^*$ is real. Indeed, in the situation considered here, this condition will hold for the relevant matrix elements. Further, the spinors $u^s(k)$ and $v^s(k)$ are to be obtained from the conventional "on-shell" spinors by the following prescription. The "on-shell" spinor $u^s(\vec{k})$ may be considered as a function of $\vec{k}$, $E_{N,\vec{k}}$, and $M_N$, where $M_N$ is the nucleon mass and $E_{N,\vec{k}} = (\vec{k}^2 + M_N^2)^{1/2}$. To obtain the "off-shell" spinor $u^s(k)$, replace $\vec{k}$, $E_{N,\vec{k}}$, and $M_N$ by $\vec{k}$, $|k^0|$, and $M_N^*$. Similarly, we define $v^s(-k)$ by replacing, in $v^s(\vec{k})$, $\vec{k}$, $E_{N,\vec{k}}$, and $M_N$ by $-\vec{k}$, $|k^0|$, and $M_N^*$. These off-shell spinors have the same completeness and orthogonality relations as the convention-

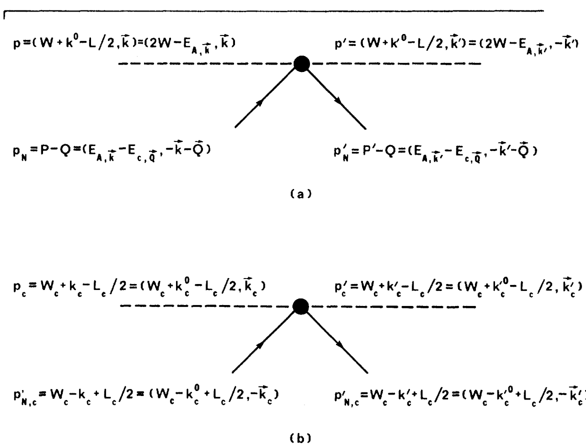


FIG. 2. Kinematics of the pion-nucleon amplitude. (a) The pion-nucleon scattering amplitude in the pion-*nucleus* center-of-mass frame — see Fig. 1. (b) The pion-nucleon scattering amplitude in the pion-*nucleon* center-of-mass frame. Note that $L_c = (M_N^2 - M_\pi^2)/2W_c$.

al spinors, which we denote as $u^s(\vec{k})$ and $v^s(\vec{k})$.

In writing Eqs. (3.8) and (3.9) we are using the fact that the off-shell spinors defined here are a complete set for the expansion of a four-component object such as that appearing at the left-hand side of these equations. Under a Lorentz transformation such as

$$L(\vec{P})|\vec{0}\rangle = |\vec{P}\rangle \qquad (3.10)$$

the field of Eq. (3.8) changes according to

$$L^{-1}(\vec{P})\psi_\mu(0)L(\vec{P}) = S_{\mu\nu}(\vec{P})\psi_\nu(0)\ . \qquad (3.11)$$

Consequently the wave function defined by Eq. (3.6) may be written in terms of the relative momentum in the rest frame of the initial target nucleus (see Fig. 3)

$$\begin{aligned}\Psi_\mu^{\alpha,r}(\underline{Q}, P-Q|\underline{P}) &= \Psi_\mu^{\alpha,r}(\vec{Q}_R)\\ &= S_{\mu\nu}(\vec{P})\langle -\vec{Q}_R, \alpha, r|\psi_\nu(0)|\vec{0}\rangle(2\pi)^{3/2}\\ &= S_{\mu\nu}(\vec{P})[G_N(Q_N)]_{\nu\rho}\bar{\chi}_\lambda^{\alpha,r}(-\vec{Q}_R)\\ &\quad\times[(2\pi)^3\langle \underline{P_R - Q_N}, Q_N|\Gamma|\underline{P_R}\rangle]_{\lambda\rho}\,,\end{aligned} \qquad (3.12)$$

or using Eq. (3.8) and restoring the spin indices

$$\Psi_\mu^{\alpha,r}(\vec{Q}_R) = \int d^3k\left(\frac{M_N^*}{|k^0|}\right)^{1/2} S_{\mu\nu}(\vec{P})\sum_s[\langle -\vec{Q}_R, \alpha, r|B_s(k)|\vec{0}\rangle u_\nu^s(k) + \langle -\vec{Q}_R, \alpha, r|D_{-s}^\dagger(-k)|\vec{0}\rangle v_\nu^{-s}(-k)]\ . \qquad (3.13)$$

Now we have chosen the four-vector $k$ such that its zeroth component is in correspondence with the four-momentum of the off-shell nucleon in Fig. 3(b). Therefore we have $k\equiv Q_N = (M_A - E_{C\vec{Q}_R}, \vec{Q}_R)$. This specification then determines the appropriate value of the mass parameter through the relation $M_N^* = [(k^0)^2 - (\vec{k})^2]^{1/2}$. We can note that the binding energy of the nucleon in the nucleus $A$ makes $M_N^*$ different from $M_N$. (Note further that the nuclei $C$ and $A$ are on their mass shells.)

If the shell model is used for specifying the nuclear structure, we can replace the set of quantum numbers $(\alpha, r)$ by the set $(nIM)$, where $n$ specifies the specific shell model configuration of the spectator nucleus $C$.

It follows that [see Fig. 3(b)]

$$\begin{aligned}\langle -\vec{Q}_R, nIM|\hat{B}_s(k)|\vec{0}\rangle &= \delta^3(\vec{k}-\vec{Q}_R)\sum_{M_L}(-1)^{L-M_L}\left(\frac{2I+1}{2L+1}\right)C_{M\ s\ M_L}^{I\ 1/2\ L}\\ &\quad\times\left(\frac{|k^0|}{M_N^*}\right)^{1/2}\psi_{n(I1/2)LM_L}^{(+)}(\vec{k})\ ,\end{aligned} \qquad (3.14)$$

with

$$\psi_{n(I1/2)LM_L}^{(+)}(\vec{k}) = \left(\frac{M_N^*}{|k^0|}\right)^{1/2} N_C^{1/2}(\vec{k})N_A^{1/2}(\vec{0})\phi_{n(I1/2)LM_L}^{(+)}(\vec{k})\ , \qquad (3.15)$$

$$\psi_{n(I1/2)LM_L}^{(+)}(\vec{k}) = \psi_{n(I1/2)L}^{(+)}(|\vec{k}|)Y_{LM_L}(\hat{k})\ , \qquad (3.16)$$

and

$$\phi_{n(I1/2)LM_L}^{(+)}(\vec{k}) = \phi_{n(I1/2)L}^{(+)}(|\vec{k}|)Y_{LM}(\hat{k})\ . \qquad (3.17)$$

We note that the value of $L$ is determined by parity, and also $M_L = M+s$.

Similarly, we have

$$\begin{aligned}&\langle -\vec{Q}_R, nIM|\hat{D}_{-s}^\dagger(-k)|\vec{0}\rangle\\ &\quad=\delta^{(3)}(\vec{k}-\vec{Q}_R)\sum_{M_{L'}}(-1)^{L'-M_{L'}}\left(\frac{2I+1}{2L+1}\right)^{1/2}(-1)^{1/2+s}C_{M\ s\ M_{L'}}^{I\ 1/2\ L'}\left(\frac{|k^0|}{M_N^*}\right)^{1/2}\psi_{n(I1/2)L'M_{L'}}^{(-)}(\vec{k})\ ,\end{aligned} \qquad (3.18)$$

with $M_{L'} = M+s$.

By virtue of Eqs. (3.14) and (3.18), the nuclear wave function of Eq. (3.13) becomes

$$\begin{aligned}&\Psi_\mu^{n(I1/2)Ms}(\vec{Q}_R)\\ &\quad=\sum_{M_L}(-1)^{L-M_L}\left(\frac{2I+1}{2L+1}\right)^{1/2}C_{M\ s\ M_L}^{I\ 1/2\ L}\psi_{n(I1/2)LM_L}^{(+)}(\vec{Q}_R)S_{\mu\nu}(\vec{P})u_\nu^s(Q_N)\\ &\qquad+(-1)^{1/2+s}\sum_{M_{L'}}(-1)^{L'-M_{L'}}\left(\frac{2I+1}{2L+1}\right)^{1/2}C_{M\ s\ M_{L'}}^{I\ 1/2\ L'}\psi_{n(I1/2)L'M_{L'}}^{(-)}(\vec{Q}_R)S_{\mu\nu}(\vec{P})v_\nu^{-s}(-Q_N)\ .\end{aligned} \qquad (3.19)$$

Since no information is available concerning the negative frequency nuclear wave function except for the deuteron, we shall retain only the first term of Eq. (3.19). Similar calculations allow us to write Eq. (3.7) as follows:

$$\Psi_{\mu'}^{n(I\,1/2)M_{s'}^*}(\vec{Q}_R)=\sum_{M_L}(-1)^{L-M_L}\left(\frac{2I+1}{2L+1}\right)^{1/2}C_{M\ s'\ M_L}^{I\ 1/2\ L}\,\psi_{n(I\,1/2)LM_L}^{(+)*}(\vec{Q}_R')\bar{u}_{\nu'}^{s'}(Q_N')S_{\nu'\mu'}{}^{-1}(\vec{P}')$$
$$+\sum_{M_L}(-1)^{1/2+s'+L-M_L}\left(\frac{2I+1}{2L+1}\right)^{1/2}C_{M\ s'\ M_L}^{I\ 1/2\ L}\,\psi_{n(I\,1/2)LM_L}^{(-)*}(-\vec{Q}_R')\bar{v}_{\nu'}^{-s'}(-Q_N')S_{\nu'\mu'}{}^{-1}(\vec{P}')\ . \tag{3.20}$$

Again, only the first term will be retained in the present analysis.

Using these covariant reductions, Eq. (3.5) becomes

$$[(2\pi)^3\langle p',\,\underline{P}'\,|\,\overline{K}_6(W)\,|\,p,\,\underline{P}\rangle]=\langle\vec{k}'\,|\,\overline{K}_6(W)\,|\,\vec{k}\rangle$$
$$=\sum_{nIL}\int d\vec{Q}\sum_{s''s'}\bar{u}^{s''}(P'-Q)[(2\pi)^3\langle p',\,P'-Q\,|\,M_{\pi N}(\sqrt{s}\,)\,|\,p,\,P-Q\rangle]u^{s'}(P-Q)$$
$$\times\left(\frac{M_{nI}}{E_{nI,\vec{Q}}}\right)^{1/2}\rho^{(+)n(I\,1/2)L}_{s',s''}(\vec{Q}_R,\vec{Q}_R')\left(\frac{M_{nI}}{E_{nI,\vec{Q}}}\right)^{1/2}\ . \tag{3.21}$$

In Eq. (3.21) the invariant density is defined by

$$\rho_{s',s''}^{(+)n(I\,1/2)L}(\vec{Q}_R,\vec{Q}_R')=\sum_{M,M_L,M_L'}(-1)^{M_L+M_L'}\left(\frac{2I+1}{2L+1}\right)C_{M\ s'\ M_L'}^{I\ 1/2\ L}\,C_{M\ s''\ M_L}^{I\ 1/2\ L}\,\psi_{n(I\,1/2)LM_L'}^{(+)}(\vec{Q}_R)\psi_{n(I\,1/2)LM_L}^{(+)*}(\vec{Q}_R') \tag{3.22}$$

and is related to the conventional (nonrelativistic) density $\tilde{\rho}$ by the general relation

$$\rho_{s's''}^{n(I\,1/2)L}(\vec{Q}_R,\vec{Q}_R')=N_{nI}^{1/2}(\vec{Q}_R)N_N^{*-1/2}(\vec{Q}_R)N_A{}^{1/2}(\vec{0})$$
$$\times\tilde{\rho}_{s',s''}^{n(I\,1/2)L}(\vec{Q}_R,\vec{Q}_R')N_{nI}^{1/2}(\vec{Q}_R')N_N^{*-1/2}(\vec{Q}_R')N_A{}^{1/2}(\vec{0})\ . \tag{3.23}$$

In the present case $N_{nI}(\vec{Q}_R)=(E_{nI,\vec{Q}_R}/M_{nI})$; $N_N^*(\vec{Q}_R)=(Q_N^0/M_N^*)$ and $N_A(\vec{0})=2M_A$. Equation (3.23) is the direct consequence of the relation between the invariantly normalized states and nonrelativistically normalized states.

From Eqs. (3.21) and (2.13), we obtain the first-order optical potential

$${}_{\rm NR}\langle\vec{k}'\,|\,V(W)\,|\,\vec{k}\rangle_{\rm NR}=(2E_{A,\vec{k}'})^{-1/2}(2\omega_{\vec{k}'})^{-1/2}$$
$$\times\langle\vec{k}'\,|\,\overline{K}_6(W)\,|\,\vec{k}\rangle(2E_{A,\vec{k}})^{-1/2}(2\omega_{\vec{k}})^{-1/2}\ . \tag{3.24}$$

If we recall our notational conventions, we have (see Fig. 2)

$$\bar{u}^{s''}(P'-Q)[(2\pi)^3\langle p',P'-Q\,|\,M_{\pi N}(\sqrt{s}\,)\,|\,p,P-Q\rangle]u^{s'}(P-Q)$$
$$=\bar{u}^{s''}(p_{N,c}')\langle\vec{k}_c',\,k_c'^0\,|\,M_{\pi N}(\sqrt{s}\,)\,|\,\vec{k}_c,\,k_c^0\rangle u^{s'}(p_{N,c})\ , \tag{3.25}$$

where we have made the dependence on $k_c'^0$ and $k_c^0$ explicit. In Sec. IV we will relate the $M$ matrix in Eq. (3.21) to the $T$ matrices written in the space of two-dimensional spinors.

Let us now consider a standard approximation in which the nucleon is placed on its mass shell. This is accomplished by placing $k_c^0=\Delta_6^{\pi N}(\vec{k}_c)$ and $k_c'^0=\Delta_6^{\pi N}(\vec{k}_c')$—see Sec. IV. In this case we can introduce the $T$ matrix in our formalism through the use of the equation

$$\bar{u}^{s''}(-\vec{k}_c')\langle\vec{k}_c',\,k_c'^0=\Delta_6^{\pi N}(\vec{k}_c)\,|\,M_{\pi N}(\sqrt{s}\,)\,|\,\vec{k}_c,\,k_c^0=\Delta_6^{\pi N}(\vec{k}_c)\rangle u^s(-\vec{k}_c)$$
$$=(2\omega_{\vec{k}_c'})^{1/2}(E_{N,\vec{k}_c'}/M_N)^{1/2}\bar{u}^{s''}(-\vec{k}_c')_{\rm NR}\langle\vec{k}_c'\,|\,T_{\pi N}(\sqrt{s}\,)\,|\,\vec{k}_c\rangle_{\rm NR}\,u^{s'}(-\vec{k}_c)(2\omega_{\vec{k}_c})^{1/2}(E_{N,\vec{k}_c}/M_N)^{1/2}\ . \tag{3.26}$$

Then

$${}_{\rm NR}\langle\vec{k}'\,|\,V(W)\,|\,\vec{k}\rangle_{\rm NR}\simeq\sum_{nIL}\int d\vec{Q}\sum_{s''s'}\bar{u}^{s''}(-\vec{k}_c')_{\rm NR}\langle\vec{k}_c'\,|\,T_{\pi N}(\sqrt{s}\,)\,|\,\vec{k}_c\rangle_{\rm NR}\,u^{s'}(-\vec{k}_c)$$
$$\times\left(\frac{\omega_{\vec{k}_c'}\omega_{\vec{k}_c}E_{N,\vec{k}_c'}E_{N,\vec{k}_c}}{\omega_{\vec{k}'}\omega_{\vec{k}}E_{N,\vec{P}'-\vec{Q}}E_{N,\vec{P}-\vec{Q}}}\right)^{1/2}\tilde{\rho}_{s's''}^{n(I\,1/2)L}(\vec{Q}_R,\vec{Q}_R')\left(\frac{M_A{}^2E_{nI,\vec{Q}_R'}E_{nI,\vec{Q}_R}E_{N,\vec{P}'-\vec{Q}}E_{N,\vec{P}-\vec{Q}}}{E_{A,\vec{k}'}E_{A,\vec{k}}E_{nI,\vec{Q}}^2\,E_{N,\vec{Q}_R}E_{N,\vec{Q}_R'}}\right)^{1/2} \tag{3.27}$$

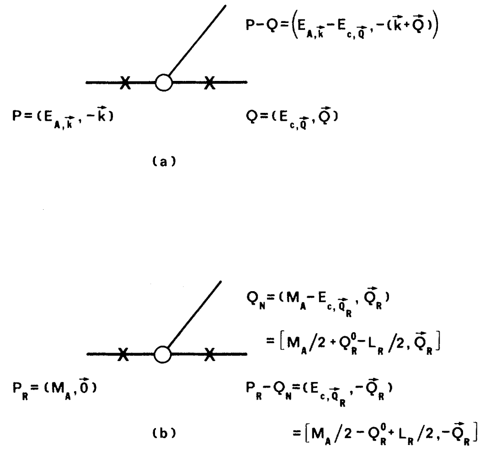


FIG. 3. Kinematic variables for the vertex functions in (a) the $\pi$-nucleus center-of-mass frame and (b) the rest frame of the target nucleus. Note $L_R=(M_A{}^2-M_N{}^2)/M_A$.

where $\vec{k}_c'$ and $\vec{k}_c$ are, respectively, the final and initial pion momenta in the c.m. frame of the pion-nucleon system.

Our result differs substantially from the conventional one based on Watson theory in the following aspects:

(1) The optical potential ${}_{\rm NR}\langle\vec{k}'|V(W)|\vec{k}\rangle_{\rm NR}$ in Eq. (3.24) can be used in a *covariant* integral equation.

(2) The total energy of the $\pi$-$N$ system, $\sqrt{s}$, can be calculated by four-momentum conservation and depends therefore on the binding of the nucleon. Consequently, our $\pi$-$N$ scattering amplitude contains dynamical off-shell effects.

(3) Relativistic kinematics are fully integrated into our method of calculation. Note that the kinematic factor at the right of the density matrix in Eq. (3.27) is absent in the conventional treatments where the nuclei are treated nonrelativistically.

The *detailed* dynamical effects implied in our approach will be discussed in the later sections. However, it is worth noting the following point at this stage.

(4) In Eqs. (3.26) and (3.27) we have considered the approximation in which the nucleon is placed on its mass shell. This leads to Eq. (3.27), which may readily be compared to the standard result of nonrelativistic multiple scattering theory. The more general result is contained in Eq. (3.21), where the nucleon is off its mass shell. This kinematic freedom will be important in carrying out transformations from the $\pi$-nucleus center-of-mass frome to the $\pi$-$N$ center-of-mass frame. This feature will be discussed in some detail in the following section.

## IV. USE OF PHENOMENOLOGICAL *T* MATRICES

In this section we discuss the invariant scattering amplitude and indicate how phenomenological information may be used in its construction. We may denote the invariant off-shell $\pi$-$N$ amplitude in Eq. (3.21) by

$$\mathfrak{F}_{fi}^{s''s'}=\bar{u}^{s''}(P'-Q)[(2\pi)^3\langle p', P'-Q|M_{\pi N}(\sqrt{s})\times|p, P-Q\rangle]u^{s'}(P-Q)\,. \qquad (4.1)$$

We may also define the amplitude $F_{fi}^{s''s'}$ such that

$$\mathfrak{F}_{fi}^{s''s'}=\left(\frac{\sqrt{s}}{2\pi^2[M_N^*M_N'^*]^{1/2}}\right)\times F_{fi}^{s''s'}(p', P'-Q; p, P-Q|\sqrt{s})\,, \qquad (4.2)$$

and in the case in which all particles are on their mass shells we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{fi}^{s''s'}=|F_{fi}^{s''s'}|^2\,. \qquad (4.3)$$

In turn, $F_{fi}$ may be written

$$F_{fi}^{s''s'}(p', P'-Q; p, P-Q|\sqrt{s})=-\frac{(M_N^*M_N'^*)^{1/2}}{4\pi\sqrt{s}}\bar{u}^{s''}(P'-Q)\{[A^{(+)}+\tfrac{1}{2}B^{(+)}\gamma\cdot(p'+p)]+(\vec{I}\cdot\vec{\tau})_{fi}[A^{(-)}+\tfrac{1}{2}B^{(-)}\gamma\cdot(p'+p)]\}u^{s'}(P-Q)\,, \qquad (4.4)$$

with

$$A^{(+)}=\tfrac{1}{3}[2A_{T=3/2}(s,t,p'^2,p^2,(P-Q)^2,(P'-Q)^2)+A_{T=1/2}(s,t,p'^2,p^2,(P-Q)^2,(P'-Q)^2)]\,,$$

$$A^{(-)}=\tfrac{1}{3}[A_{T=3/2}(s,t,p'^2,p^2,(P-Q)^2,(P'-Q)^2)-A_{T=1/2}(s,t,p'^2,p^2,(P-Q)^2,(P'-Q)^2)]\,, \qquad (4.5)$$

$$s=(p+P-Q)^2=(p'+P'-Q)^2\,, \qquad (4.6)$$

and

$$t=(p-p')^2\,. \qquad (4.7)$$

There are similar definitions for $B^{(+)}$ and $B^{(-)}$.

The indices $f$ and $i$ refer now to the isospin states of the pion-nucleon system. Equation (4.4) is a *generalization of the on-shell invariant* $\pi$-$N$ *amplitude*.[6,7]

Furthermore, in the c.m. frame of the $\pi$-$N$ system we have

$$\mathfrak{F}_{fi}^{s''s'}=\left(\frac{-\sqrt{s}}{2\pi^2[M_N^* M_N'^*]^{1/2}}\right)\chi_{s''}^{\dagger}\{[F_1^{(+)}+F_2^{(+)}(\vec{\sigma}\cdot\hat{k}_c')(\vec{\sigma}\cdot\hat{k}_c)]\delta_{fi}+(\vec{\mathrm{I}}\cdot\vec{\tau})_{fi}[F_1^{(-)}+F_2^{(-)}(\vec{\sigma}\cdot\hat{k}_c')(\vec{\sigma}\cdot\hat{k}_c)]\}\chi_{s'}\ , \tag{4.8}$$

where the quantities $\chi_{s''}^{\dagger}$ and $\chi_{s'}$ are now the nonrelativistic two-component spinors. The $\hat{k}_c'$ and $\hat{k}_c$ are the unit vectors of the final and initial pion momenta. We have also defined[7]

$$F_1^{\pm}=[(|p_{N,c}'^0|+M_N'^*)(|p_{N,c}^0|+M_N^*)]^{1/2}\times[A^{(\pm)}+[\sqrt{s}-\tfrac{1}{2}(M_N^*+M_N'^*)]B^{(\pm)}]/8\pi\sqrt{s}\ , \tag{4.9}$$

$$F_2^{(\pm)}=[(|p_{N,c}'^0|-M_N'^*)(|p_{N,c}^0|-M_N^*)]^{1/2}\times[-A^{(\pm)}+[\sqrt{s}+\tfrac{1}{2}(M_N^*+M_N'^*)]B^{(\pm)}]/8\pi\sqrt{s}\ . \tag{4.10}$$

Using Eq. (A13), we have also in the on-shell limit

$$F_{fi}^{s's}=\frac{2}{M_N}(\omega_{\vec{k}_c'}\omega_{\vec{k}_c}E_{N,\vec{k}_c'}E_{N,\vec{k}_c})^{1/2}T_{fi}^{s''s'} \tag{4.11}$$

the amplitude $T_{fi}^{s''s'}$ is related to the differential cross section by [see Eqs. (A13) and (A14)]

$$\left(\frac{d\sigma}{d\Omega}\right)_{fi}=(2\pi)^4\left(\frac{E_{N,\vec{k}_c}\omega_{\vec{k}_c}}{\sqrt{s}}\right)^2|T_{fi}^{s''s'}|^2\ . \tag{4.12}$$

In terms of the phenomenological partial wave phase shifts, the on-shell $T$ matrix is

$$T_{fi}^{s''s'}=\frac{-\sqrt{s}}{(2\pi)^2E_{N,\vec{k}_c}\omega_{\vec{k}_c}|\vec{k}_c'|}\times\left(s'',f\,\Big|\sum_{m=0}^{3}f_m(s,t)O_m\Big|\,s',i\right)\ , \tag{4.13}$$

with

$$\begin{aligned}
f_0&=\sum_l\{\tfrac{2}{3}[la_{3,2l-1}^l+(l+1)a_{3,2l+1}^l]+\tfrac{1}{3}[la_{1,2l-1}^l+(l+1)a_{1,2l+1}^l]\}P_l(\hat{k}_c'\cdot\hat{k}_c)\ ,\\
f_1&=\sum_l\{\tfrac{2}{3}(a_{3,2l-1}^l-a_{3,2l+1}^l)+\tfrac{1}{3}(a_{1,2l-1}^l-a_{1,2l+1}^l)\}\frac{d}{d(\hat{k}_c'\cdot\hat{k}_c)}P_l(\hat{k}_c'\cdot\hat{k}_c)\ ,\\
f_2&=\sum_l\{\tfrac{1}{3}[la_{3,2l-1}^l+(l+1)a_{3,2l+1}^l]-\tfrac{1}{3}[la_{i,2l-1}^l+(l+1)a_{1,2l+1}^l]\}P_l(\hat{k}_c'\cdot\hat{k}_c)\ ,\\
f_3&=\sum_l[\tfrac{1}{3}(a_{3,2l-1}^l-a_{3,2l+1}^l)-\tfrac{1}{3}(a_{1,2l-1}^l-a_{1,2l+1}^l)]\frac{d}{d(\hat{k}_c'\cdot\hat{k}_c)}P_l(\hat{k}_c'\cdot\hat{k}_c)\ ,
\end{aligned} \tag{4.14}$$

$$O_0=1,\quad O_1=-i\vec{\sigma}\cdot(\hat{k}_c\times\hat{k}_c'),\quad O_2=\vec{\mathrm{I}}\cdot\vec{\tau},\quad O_3=O_1O_2\ , \tag{4.15}$$

and

$$a_{2T,2j}^l(\sqrt{s})=\frac{1}{2i}(e^{2i\delta_{2T,2j}^l(\sqrt{s})}-1)\ . \tag{4.16}$$

The off-shell generalization of Eq. (4.13) consistent with Eq. (4.11) consists in replacing the quantities $(E_{N,\vec{k}_c}\omega_{\vec{k}_c})$, $f_m(s,t)$ and $a_{2T,2j}^l(\sqrt{s})$ by $(E_{N,\vec{k}_c'}E_{N,\vec{k}_c}\omega_{\vec{k}_c'}\omega_{\vec{k}_c})^{1/2}$, $f_m(s,t,p_c{}^2,p_c'{}^2,p_{N,c}{}^2,p_{N,c}'{}^2)$ $=f_m(s,\vec{k}_c'\cdot\vec{k}_c,|\vec{k}_c'|^2,|\vec{k}_c|^2,k_c'^0,k_c^0)$, and $a_{2T,2j}^l(s,|\vec{k}_c'|^2,|\vec{k}_c|^2,k_c'^0,k_c^0)$.

Combining these results we may write

$$F_{fi}^{s''s'}=\left(s'',f\,\Big|\sum_{m=0}^{3}f_m(s,t,\vec{k}_c'^2,\vec{k}_c^2,k_c'^0,k_c^0)\times O_m\Big|\,s',i\right)/(|\vec{k}_c'||\vec{k}_c|)^{1/2}\ , \tag{4.17}$$

where the $\vec{k}_c'^2$ and $\vec{k}_c^2$ can be expressed by the invariant relations[4].

$$\vec{k}_c{}^2=\frac{1}{s}\{[p\cdot(P-Q)]^2-p^2(P-Q)^2\}\ , \tag{4.18}$$

$$\vec{k}_c'^2=\frac{1}{s}\{[p'\cdot(P'-Q)]^2-p'^2(P'-Q)^2\}\ , \tag{4.19}$$

$$k_c^0-\tfrac{1}{2}L_c=\frac{1}{2\sqrt{s}}(p-P+Q)\cdot(p+P-Q)=\frac{p^2-(P-Q)^2}{2\sqrt{s}}\ , \tag{4.20}$$

$$k_c'^0-\tfrac{1}{2}L_c=\frac{1}{2\sqrt{s}}(p'-P'+Q)\cdot(p'+P'-Q)=\frac{p'^2-(P'-Q)^2}{2\sqrt{s}}\ . \tag{4.21}$$

These relations may be verified by evaluating the right-hand sides in the $\pi$-$N$ center-of-mass frame using Fig. 2(b).

As we have seen, the development of our covariant scattering theory requires the knowledge of the pion-nucleon scattering amplitude for particles off their mass shells. On the other hand, there have been a series of phenomenological studies of the pion-nucleon scattering amplitude[5] using a *three-dimensional* wave equation (with suppressed spin indices) of the form:

$$_{\rm NR}\langle\vec{k}|\,T(W)\,|\vec{k}'\rangle_{\rm NR} = {}_{\rm NR}\langle\vec{k}|\,V\,|\vec{k}'\rangle_{\rm NR} + \int \frac{_{\rm NR}\langle\vec{k}|\,V\,|\vec{k}''\rangle_{\rm NR}\,d\vec{k}''\,{}_{\rm NR}\langle\vec{k}''|\,T(W)\,|\vec{k}'\rangle_{\rm NR}}{(2W)-(\omega_{\vec{k}''}+E_{N,\vec{k}''})+i\epsilon} \;. \tag{4.22}$$

(In some of these works the role of inelastic channels in the pion-nucleon system has been studied, also in the context of three-dimensional wave equations with relativistic kinematics.) Among the results of these studies has been the construction of separable potentials which yield $T$ matrices which reproduce the on-shell pion-nucleon amplitudes.

In the separable potential approach to the $\pi$-$N$ scattering, the off-shell partial wave amplitude is assumed to take the form[5]

$$(|\vec{k}_c'||\vec{k}_c|)^{-1/2}a^l_{2T,2j}(|\vec{k}_c'|,|\vec{k}_c|;\sqrt{s}\,)$$
$$= -\frac{(\omega_{\vec{k}_c}\omega_{\vec{k}_c'}E_{N,\vec{k}_c}E_{N,\vec{k}_c'})^{1/2}}{(2\pi)\sqrt{s}}$$
$$\times[\,\lambda^l_{2T,2j}v^l_{2T,2j}(|\vec{k}_c'|)v^l_{2T,2j}(|\vec{k}_c|)/D^l_{2T,2j}(\sqrt{s}\,)]\;, \tag{4.23}$$

$$D^l_{2T,2j}(\sqrt{s}\,) = 1-\lambda^l_{2T,2j}\int\frac{d^3p}{(2\pi)^3}\,\frac{[v^l_{2T,2j}(p)]^2}{\sqrt{s}\,-E_{N,\vec{p}}-\omega_{\vec{p}}+i\epsilon}\;, \tag{4.24}$$

where $\lambda$ is the coupling constant and the $v^l$ are the form factors. In order to make use of these phenomenological amplitudes of Eq. (4.23) we consider an expansion of the function $a^l_{2T,2j}(s,|\vec{k}_c'|^2,|\vec{k}_c|^2,k_c'^0,k_c^0)$ about the values $k_c^0=\Delta_6^{\pi N}(\vec{k}_c)$ and $k_c'^0=\Delta_6^{\pi N}(\vec{k}_c')$, where [recalling Eq. (2.5)]

$$\Delta_6^{\pi N}(\vec{k}) = \tfrac{1}{2}\sqrt{s}\,+\tfrac{1}{2}L-E_{N,\vec{k}}\;. \tag{4.25}$$

In Eq. (4.25) the value of $L$ is that appropriate to the $\pi$-$N$ system, and $E_{N,\vec{k}}=(\vec{k}^2+M_N{}^2)^{1/2}$.

We write

$$a^l_{2T,2j}(s,\,|\vec{k}_c'|^2,\,|\vec{k}_c|^2,\,k_c'^0,\,k_c^0)$$
$$= a^l_{2T,2j}(s,\,|\vec{k}_c'|^2,\,|\vec{k}_c|^2,\;k_c'^0=\Delta_6^{\pi N}(\vec{k}_c'),\;\;k_c^0=\Delta_6^{\pi N}(\vec{k}_c))$$
$$+[k_c'^0-\Delta_6^{\pi N}(\vec{k}_c')]G^l_{2T,2j}(s,\,|\vec{k}_c'|^2,\,|\vec{k}_c|^2)$$
$$+[k_c^0-\Delta_6^{\pi N}(\vec{k}_c)]H^l_{2T,2j}(s,\,|\vec{k}_c'|^2,\,|\vec{k}_c|^2)$$
$$+\cdots\;. \tag{4.26}$$

We now desire to introduce the phenomenlogical amplitudes *via* the identification of the first term on the right of Eq. (4.26) with the quantity given in Eq. (4.23):

$$(|\vec{k}_c||\vec{k}_c'|)^{-1/2}a^l_{2T,2j}(s,|\vec{k}_c'|^2,|\vec{k}_c|^2,\;k_c'^0=\Delta_6^{\pi N}(\vec{k}_c'),\;\;k_c^0=\Delta_6^{\pi N}(\vec{k}_c))$$
$$= -\frac{(\omega_{\vec{k}_c}\omega_{\vec{k}_c'}E_{N,\vec{k}_c}E_{N,\vec{k}_c'})^{1/2}}{(2\pi)\sqrt{s}}\;[\,\lambda^l_{2T,2j}v^l_{2T,2j}(|\vec{k}_c'|)v^l_{2T,2j}(|\vec{k}_c|)/D^l_{2T,2j}(\sqrt{s}\,)]\;. \tag{4.27}$$

The justification of this identification requires two steps. First, we must relate phenomenological amplitudes to the covariant theory by considering some reduction of the four-dimensional to the *three*-dimensional formulation. Second, we must argue that the remaining terms in Eq. (4.26) involving derivatives with respect to $k_c^0$ and $k_c'^0$ are small.

With respect to the first point, we note that we may consider that the solution of Eq. (4.22) be related to the invariant $\pi$-$N$ amplitude by a similar reduction scheme to that used previously in which the heavy particle is put on its mass shell. If we make that particular interpretation of the nonrelativistic amplitude, we may make the identification

$$_{\rm NR}\langle\vec{k}|\,T(W)\,|\vec{k}'\rangle_{\rm NR} = R_6{}^{1/2}(\vec{k})\langle\vec{k},\,k^0=\Delta_6^{\pi N}(\vec{k})|\,M(W)$$
$$\times|\vec{k}',\,k'^0=\Delta_6^{\pi N}(\vec{k}')\rangle R_6{}^{1/2}(\vec{k}')\;, \tag{4.28}$$

where $\Delta_6^{\pi N}(\vec{k})$ has the value given in Eq. (4.25), and $R_6{}^{1/2}(\vec{k})=(M_N/2\omega_{\vec{k}}E_{N,\vec{k}})^{1/2}$. Also, if we make *this* particular interpretation of the phenomenological amplitudes, then the identification made in Eq. (4.27) is correct.

Now it is also clear that Eqs. (4.26) and (4.27) will be useful if all but the first term of Eq. (4.26) are small. While we are not able to calculate the derivatives $G^l_{2T,2j}$ and $H^l_{2T,2j}$ etc., we are able to give a simple expression for the factors $[k^0 - \Delta_6^{\pi N}(\vec{k})]$ and $[k'^0 - \Delta_6^{\pi N}(\vec{k}')]$. In Fig. 2 we indicate the off-shell pion-nucleon amplitude in (a) the pion-*nucleus* center-of-mass system and in (b) the pion-*nucleon* center-of-mass system. In terms of the four-vectors appearing in this figure we have (see Appendix B)

$$k_c^0 = W_c - \tfrac{1}{2}L_c - E_{N,\vec{k}}\left(1 + \frac{p_N{}^2 - M_N{}^2}{\vec{k}_c{}^2 + M_N{}^2}\right)^{1/2}$$
$$\simeq \Delta_6^{\pi N}(\vec{k}_c) - \frac{(M_N^{*2} - M_N{}^2)}{2E_{N,\vec{k}_c}}. \qquad (4.29)$$

Similar expressions are obtained for $k_c'^0$. Now, as remarked previously, for the nucleons in the triangle diagram we have $\eta^2 \equiv |M_N{}^2 - M_N^{*2}|/M_N{}^2 \ll 1$. The corrections to Eq. (4.8) are of the order $\eta(M_N G^l_{2T,2j})$ and $\eta(M_N H^l_{2T,2j})$ and thus we may expect that our approximation is satisfactory if $M_N G^l_{2T,2j} \lesssim 1$, etc.

Having all four particles off their mass shell, the off-shell pion-nucleon amplitude of Eq. (4.26) depends on five independent Lorentz invariants. However, the off-shell amplitude of Eq. (4.23), evaluated in the region corresponding to having both initial and final *nucleons* on the mass shell, depends only on three independent Lorentz invariants. In this sense, Eq. (4.23) is less general than Eq. (4.26). At this point, we emphasize that the use of Eq. (4.23) is only motivated by the observation that all phenomenological analyses of off-shell $\pi$-$N$ scattering amplitude published in the literature have been made on the assumption that the $\pi$-$N$ amplitude depends on four independent variables. (There are three independent variables after the angle projection is made.) Since it is clear that in a covariant theory the relevant off-shell pion-nucleon amplitude does depend on independent variables, future analyses which encompass this aspect and test the validity of the approximation in Eq. (4.23) would be of value.

With these considerations out of the way we may continue our analysis, noting that Eqs. (4.8) and (4.17) imply

$$F_1^{(+)} = [f_0 - (\hat{k}_c' \cdot \hat{k}_c)f_1]/(|\vec{k}_c'||\vec{k}_c|)^{1/2}\,; \quad F_2^{(+)} = f_1/(|\vec{k}_c'||\vec{k}_c|)^{1/2}\,,$$

and (4.30)

$$F_1^{(-)} = [f_2 - (\hat{k}_c' \cdot \hat{k}_c)f_3]/(|\vec{k}_c'||\vec{k}_c|)^{1/2}\,; \quad F_2^{(-)} = f_3/(|\vec{k}_c'||\vec{k}_c|)^{1/2}\,.$$

These relations were obtained previously.[7] They allow the evaluation of the invariant amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ from the knowledge of the phenomenological off-shell amplitudes, Eq. (4.27), using the inverse relations to Eqs. (4.9) and (4.10), viz:

$$A^{(\pm)} = 4\pi\left\{\frac{\sqrt{s} + \frac{1}{2}(M_N^* + M_N'^*)}{[(|p_{N,c}'^0| + M_N'^*)(|p_{N,c}^0| + M_N^*)]^{1/2}}F_1^{(\pm)} - \frac{\sqrt{s} - \frac{1}{2}(M_N^* + M_N'^*)}{[(|p_{N,c}'^0| - M_N'^*)(|p_{N,c}^0| - M_N^*)]^{1/2}}F_2^{(\pm)}\right\}, \qquad (4.31)$$

$$B^{(\pm)} = 4\pi\{[(|p_{N,c}'^0| + M_N'^*)(|p_{N,c}^0| + M_N^*)]^{-1/2}F_1^{(\pm)} + [(|p_{N,c}'^0| - M_N'^*)(|p_{N,c}^0| - M_N^*)]^{-1/2}F_2^{(\pm)}\}.$$

## V. TRANSFORMATION FROM THE $\pi$-$N$ CENTER-OF-MASS FRAME TO $\pi$-NUCLEUS CENTER-OF-MASS FRAME

While the phenomenological (either on-shell or off-shell) $\pi$-$N$ partial wave amplitudes are parametrized and computed in the c.m. frame of the $\pi$-$N$ system, the $\pi$-$N$ amplitude needed in constructing the $\pi$-nucleus optical potential is in the c.m. frame of the latter system. Consequently, one has to know the manner in which the $\pi$-$N$ amplitude is to be transformed. A large number of studies have been made in this connection,[8] and in each of these works a different approximation is implied. A common feature of these studies is that, although the off-shell character of the $\pi$-$N$ amplitude was recognized, one still implicitly put every internal particle on the mass shell when performing angle transformations. This treatment of the off-shell effects, having its root in nonrelativistic scattering theory, is obviously different from ours. One consequence of this on-shell approximation is that the usual Lorentz transformation, appropriate to on-mass-shell particles, cannot be consistently applied for *all* angles when relating the $\pi$-nucleus and the $\pi$-$N$ center-of-mass systems.

In our approach, we allow the intermediate particles to go off their mass shells so that four-momentum is conserved at each interaction element or vertex. The transformation rules are therefore obtained by simply comparing the expression of the *invariant* off-shell amplitude [Eq. (4.4)] in the $\pi$-nucleus c.m. frame with that in the $\pi$-nucleon c.m. frame. Obviously, the rules obtained in this way are not only unambiguous, but are also valid for all angles in both frames. The detailed formulas will be found in Sec. VI. It should be stressed that these improved kinematical features are unaffected by using Eq. (4.27) in place of Eq. (4.26).

We emphasize that we need not and shall not use the "fixed-scatterer approximation" (FSA). However, in view of the fact that the FSA is used in other works where a factorized form, $t(\vec{q})\rho(\vec{q})$, is obtained for the first-order $\pi$-nucleus optical potential, we believe it is worth studying the FSA in a covariant version and comparing our transformation rules with those rules derived by others. We recall that in the FSA the target nucleon is considered at rest (in the laboratory) before the scattering event. According to this model, in the c.m. frame of $\pi$-nucleus system the momenta of the initial and final nucleons to be used in the evaluation of the $\pi$-$N$ amplitude should be taken as $\vec{p}_N = -\vec{k}/N$ and $\vec{p}'_N = -\vec{k}' + (N-1)\vec{k}/N$. Here $\vec{k}$ and $\vec{k}'$ are the initial and final pion momenta (see Fig. 1) and $N$ represents the number of nucleons in the nucleus. We obtain

$$(\mathfrak{F}_{fi}^{s''s'})_{\pi-A}^{\text{FSA}} = [a_{fi} + b_{fi}(\vec{k}\cdot\vec{k}')]\delta_{s''s'} + i(b_{fi}+c_{fi})\langle s''|\vec{\sigma}\cdot(\vec{k}'\times\vec{k})|s'\rangle\,, \tag{5.1}$$

with

$$a_{fi} \equiv -\frac{1}{16\pi^3}[M_N^* M_N'^*(p_N^0 + M_N^*)(p_N'^0 + M_N'^*)]^{-1/2}$$
$$\times\{(p_N^0 + M_N^*)[(p_N'^0 + M_N'^*)A + \tfrac{1}{2}(\vec{k}'^2 - \vec{k}^2)B]_{fi}$$
$$+ (\vec{k}^2/N)[A + \tfrac{1}{2}(p_N^0 + p_N'^0 + M_N^* + M_N'^* - p^0 - p'^0)B]_{fi} + (\vec{k}^2/N^2)[-A + \tfrac{1}{2}(p^0 + p'^0)B]_{fi}\}\,, \tag{5.2}$$

$$b_{fi} \equiv -\frac{1}{16\pi^3}[M_N^* M_N'^*(p_N^0 + M_N^*)(p_N'^0 + M_N'^*)]^{-1/2}(1/N)[-A + \tfrac{1}{2}(p_N^0 + p_N'^0 + M_N^* + M_N'^* + p^0 + p'^0)B]_{fi}\,, \tag{5.3}$$

and

$$c_{fi} \equiv -\frac{1}{16\pi^3}[M_N^* M_N'^*(p_N^0 + M_N^*)(p_N'^0 + M_N'^*)]^{-1/2}(1 - 1/N)(p_N^0 + M_N^*)B_{fi}\,. \tag{5.4}$$

In Eqs. (5.2) to (5.4), we have used the notation

$$A_{fi} \equiv A^{(+)} + (\vec{I}\cdot\vec{\tau})_{fi}A^{(-)}\,, \tag{5.5}$$

$$B_{fi} \equiv B^{(+)} + (\vec{I}\cdot\vec{\tau})_{fi}B^{(-)}\,. \tag{5.6}$$

In what follows all isospin matrix elements will be defined according to Eqs. (5.5) and (5.6)

{Note that Eqs. (5.1)–(5.4) represent a special case of Eq. (6.1) with $\vec{Q} = -[(N-1)/N]\vec{k}$.}

In the c.m. frame of the pion-*nucleon* system, the amplitude $\mathfrak{F}_{fi}^{s''s'}$ is given by Eq. (4.8), which we rewrite as follows:

$$(\mathfrak{F}_{fi}^{s''s'})_{\pi N}^{\text{FSA}} = [a_{fi}^* + b_{fi}^*(\hat{k}_c'\cdot\hat{k}_c)]\delta_{s''s'} + i(b^*)_{fi}\langle s''|\vec{\sigma}\cdot(\hat{k}_c'\times\hat{k}_c)|s'\rangle\,, \tag{5.7}$$

with

$$\hat{k}_c = \vec{k}_c/|\vec{k}_c|\,,\quad \hat{k}_c' = \vec{k}_c'/|\vec{k}_c'|\,, \tag{5.8}$$

$$a_{fi}^* \equiv -\{\sqrt{s}/(2\pi^2[M_N^* M_N'^*]^{1/2})\}(F_1)_{fi}$$
$$\equiv -\{\sqrt{s}/(2\pi^2[M_N^* M_N'^*]^{1/2})\}\times\{f_0 - (\hat{k}_c'\cdot\hat{k}_c)f_1 + (\vec{I}\cdot\vec{\tau})_{fi}[f_2 - (\hat{k}_c'\cdot\hat{k}_c)]\}\,, \tag{5.9}$$

and

$$b_{fi}^* \equiv -\{\sqrt{s}/(2\pi^2[M_N^* M_N'^*]^{1/2})\}(F_2)_{fi}$$
$$= -\{\sqrt{s}/(2\pi^2[M_N^* M_N'^*]^{1/2})\}[f_1 + (\vec{I}\cdot\vec{\tau})_{fi}f_3]\,. \tag{5.10}$$

Owing to the basic assumption of the model, the invariant $M_N^*$ in the above equations is assigned the value corresponding to a target nucleon of zero momentum in the laboratory frame. We have, therefore, $M_N^* = M_A - M_C$, which differs from the free nucleon mass by the binding energy of the target nucleon. Note, however, that in our approach $M_N'^* \neq M_N^*$. The value of the $M_N'^*$ is completely determined by the requirement of four-momentum conservation in the (off-shell) $\pi$-$N$ scattering process.

Equations (5.2)–(5.4), (5.8), and (5.9) relate the coefficients $a$, $b$, and $c$ to the coefficients $a^*$ and $b^*$ via the invariant amplitudes $A$ and $B$. Combining all these relations and defining

$$e_0 = [4s(p_N'^0 + M_N'^*)(p_N^0 + M_N^*)]^{1/2}\,, \tag{5.11}$$

$$d^\pm = [(|p_{N,c}'^0| \pm M_N'^*)(|p_{N,c}^0| \pm M_N^*)]^{1/2}\,, \tag{5.12}$$

$$e_1^\pm = (p_N^0 + M_N^*)/2d^\pm\,, \tag{5.13}$$

$$e_2^\pm = (p_N^0 + p_N'^0)/2d^\pm\,, \tag{5.14}$$

$$e_3^\pm = (p^0 + p'^0)/2d^\pm\,, \tag{5.15}$$

$$e_4^\pm = (M_N^* + M_N'^*)/2d^\pm\,, \tag{5.16}$$

and

$$e_5^\pm = \sqrt{s}/d^\pm\,, \tag{5.17}$$

we obtain the following transformation rules for the coefficients:

$$a_{fi}=e_0\{2\sqrt{s}(p_N'^0+M_N'^*)[e_1^+a_{fi}^*-e_1^-b_{fi}^*]+[(M_N^*+M_N'^*)(p_N'^0+M_N'^*)+\vec{k}'^2-\vec{k}^2][e_1^+a_{fi}^*+e_1^-b_{fi}^*]\}$$
$$+(\vec{k}^2/N)\{[e_2^++e_4^+-(1-1/N)(e_3^+-e_4^+-2e_5^+)]a_{fi}^*+[e_2^-+e_4^--(1-1/N)(e_3^--e_4^-+2e_5^-)]b_{fi}^*\}, \quad (5.18)$$

$$b_{fi}=e_0[(e_2^++e_3^+-2e_5^+)a_{fi}^*+(e_2^-+e_3^-+2e_5^-)b_{fi}^*], \quad (5.19)$$

and

$$c_{fi}=e_0(1-1/N)[e_1^+a_{fi}^*+e_1^-b_{fi}^*]. \quad (5.20)$$

By virtue of Eqs. (5.9) and (5.10), $a_{fi}^*$ and $b_{fi}^*$ are functionals of $f_i$ $(i=0,1,2,3)$ and $(\hat{k}_c'\cdot\hat{k}_c)$.

If only $s$ and $p$ wave scattering is considered in the c.m. frame of the pion-nucleon system, then $f_0$ and $f_2$ are of the form $(\alpha)+(\beta)(\hat{k}_c'\cdot\hat{k}_c)$, while there is no $(\hat{k}_c'\cdot\hat{k}_c)$ dependence in $f_1$ and $f_3$. It is now worth remarking that in our covariant approach all the quantities in Eqs. (5.18) to (5.20) can be expressed *exclusively* in terms of variables defined in the c.m. frame of the pion-nucleus system. In fact, if we define $r(x^2;y^2,z^2)=[x^2-(y+z)^2]^{1/2}[x^2-(y-z)^2]^{1/2}$, one has

$$p_{N,c}'^0=(s+p_N'^2-p'^2)/2\sqrt{s}, \quad (5.21)$$

$$p_{N,c}^0=(s+p_N^2-p^2)/2\sqrt{s}, \quad (5.22)$$

$$(\hat{k}_c'\cdot\hat{k}_c)=\frac{s^2-s(p^2+p'^2+p_N^2+p_N'^2-2t)+(p^2-p_N^2)(p'^2-p_N'^2)}{r(s;p^2,p_N^2)r(s;p'^2,p_N'^2)}, \quad (5.23)$$

where the invariants $s$ and $t$ are defined in Eqs. (4.6) and (4.7). The $p^2$ and $p'^2$ are the squares of initial and final pion four-momenta and they may be different from $M_\pi^2$ for off-shell pions. Similar remarks apply to the nucleons.

In the literature, the invariant amplitude, Eq. (4.4), was not used. Consequently, the transformation rules Eqs. (5.18) to (5.20) were disregarded in those treatments. Furthermore, only Eq. (5.23), *supplemented by the approximation of putting the nucleons and pions on-mass-shell,* was used in transforming $(\hat{k}_c'\cdot\hat{k}_c)$. We emphasize that the invalidity of using the "on-shell" Lorentz transformation scheme for $(\hat{k}_c'\cdot\hat{k}_c)$ may also be considered to be a direct consequence of the invariance of the square of the four-momentum transfer of the pion $t$ when the pion is scattered by a bound nucleon. By comparing the expressions for $t$ in both frames it may be seen that in the case where all four particles are placed on their mass shells, the angle transformation implied by this procedure can only be made meaningful for small angles. For this reason, the results obtained in that procedure are questionable at large angles and not valid at all for *very* large scattering angles, as we have noted at the beginning of this section. This incompatibility is even more serious when we ask how the spin-flip operators $\vec{\sigma}\cdot(\hat{k}_c'\times\hat{k}_c)$ and $\vec{\sigma}\cdot(\hat{k}'\times\hat{k})$ should be transformed. As a matter of fact, these terms were systematically disregarded in the literature. In our procedure, there is no ambiguity in the transformation to be applied for these terms.

A consequence of using the aforementioned inconsistent transformation procedures is that there are then many ways in making approximations for the transformation of angles. These may be thought of as giving different "off-shell extensions." Obviously, these off-shell extensions have nothing to do with dynamics; they are the sole by-product of not using an invariant amplitude. In a future publication we will present some detailed numerical estimates of the errors introduced in using various noncovariant schemes for performing amplitude transformations from the pion-nucleon c.m. frame to the pion-nucleus c.m. frame.

## VI. NEW DYNAMICAL FEATURES

We no longer make use of the fixed-scatterer approximation and now write the invariant pion-nucleon (off-shell) scattering amplitude in the c.m. frame of the pion-nucleus system:

$$\mathfrak{F}_{fi}^{s''s'}=\bar{u}^{s''}(P'-Q)[(2\pi)^3\langle p',P'-Q|M_{\pi N}(\sqrt{s})|p,P-Q\rangle]u^{s'}(P-Q)$$
$$=-\frac{1}{16\pi^3[M_N^*M_N'^*]^{1/2}}\Big(\{\mathfrak{F}_1+\mathfrak{F}_2[(\vec{k}'\cdot\vec{k})+(\vec{k}+\vec{k}')\cdot\vec{Q}]+\mathfrak{F}_3(\vec{k}\cdot\vec{k})+\mathfrak{F}_4(\vec{k}'\cdot\vec{k}')+\mathfrak{F}_5(\vec{Q}\cdot\vec{Q})\}_{fi}\delta_{s''s'}$$
$$+i\langle s''|\vec{\sigma}\cdot[\mathfrak{F}_2(\vec{k}'\times\vec{k})+\mathfrak{F}_5(\vec{k}'-\vec{k})\times\vec{Q}+\mathfrak{F}_6(\vec{k}'+\vec{k})\times\vec{Q}]_{fi}|s'\rangle\Big), \quad (6.1)$$

with $p_N=(P-Q)$, $p_N'=(P'-Q)$,

$$(\mathfrak{F}_1)_{fi}=[(p_N'^0+M_N'^*)(p_N^0+M_N^*)]^{1/2}[A+\tfrac{1}{2}(p^0+p'^0)B]_{fi}, \tag{6.2}$$

$$(\mathfrak{F}_2)_{fi}=\frac{[-A+\frac{1}{2}(p^0+p'^0+p_N^0+p_N'^0+M_N^*+M_N'^*)B]_{fi}}{[(p_N'^0+M_N'^*)(p_N^0+M_N^*)]^{1/2}}, \tag{6.3}$$

$$(\mathfrak{F}_3)_{fi}=\tfrac{1}{2}[(p_N'^0+M_N'^*)/(p_N^0+M_N^*)]^{1/2}B_{fi}, \tag{6.4}$$

$$(\mathfrak{F}_4)_{fi}=\tfrac{1}{2}[(p_N^0+M_N^*)/(p_N'^0+M_N'^*)]^{1/2}B_{fi}, \tag{6.5}$$

$$(\mathfrak{F}_5)_{fi}=\frac{[-A+\frac{1}{2}(p^0+p'^0)B]_{fi}}{[(p_N'^0+M_N'^*)(p_N^0+M_N^*)]^{1/2}}, \tag{6.6}$$

$$(\mathfrak{F}_6)_{fi}=\frac{\frac{1}{2}(p_N'^0+M_N'^*-p_N^0-M_N^*)B_{fi}}{[(p_N'^0+M_N'^*)(p_N^0+M_N^*)]^{1/2}}, \tag{6.7}$$

where $p^0=2W-E_{A,\vec{k}}$ and $p'^0=2W-E_{A,\vec{k}'}$. The matrix elements $A_{fi}$ and $B_{fi}$ are defined by Eq. (5.4).

Using only the large components of the complete wave functions, we have for the single nucleon density

$$\bar{\rho}_{s''s'}^{(+)n(I\,1/2)L}(\vec{Q}_R,\vec{Q}_R')=\sum_{MM_L'M_L''}(-1)^{M_L'+M_L''}\left(\frac{2I+1}{2L+1}\right)C_{M\,s''\,M_L''}^{I\,1/2\,L}Y_{L\,M_L''}^*(\hat{Q}_R')Y_{L\,M_L'}(\hat{Q}_R)\times C_{M\,s'\,M_L'}^{I\,1/2\,L}R_{n(I\,1/2)L}(|\vec{Q}_R'|)R_{n(I\,1/2)L}(|\vec{Q}_R|). \tag{6.8}$$

After introducing Eqs. (6.1) and (3.23) into Eq. (3.21), taking into account Eqs. (3.24) and (6.8), we obtain the final expression for the first-order relativistic $\pi$-nucleus optical potential:

$$\begin{aligned}{}_{\rm NR}\langle\vec{k}'|V(W)|\vec{k}\rangle_{\rm NR}=&\sum_{nIMM_L'M_L''}\sum_{s''s'}\int d\vec{Q}(-1)^{M_L'+M_L''}\left(\frac{2I+1}{2L+1}\right)C_{M\,s''\,M_L''}^{I\,1/2\,L}C_{M\,s'\,M_L'}^{I\,1/2\,L}(-16\pi^3)^{-1}\\
&\times\Big(\{\mathfrak{F}_1+[\vec{k}'\cdot\vec{k}+(\vec{k}+\vec{k}')\cdot\vec{Q}]\mathfrak{F}_2+\mathfrak{F}_3\vec{k}\cdot\vec{k}+\mathfrak{F}_4\vec{k}'\cdot\vec{k}'+\mathfrak{F}_5\vec{Q}\cdot\vec{Q}\}_{fi}\delta_{s''s'}\\
&\quad+i\langle s''|\vec{\sigma}\cdot[\mathfrak{F}_2(\vec{k}'\times\vec{k})+\mathfrak{F}_5(\vec{k}'-\vec{k})\times\vec{Q}+\mathfrak{F}_6(\vec{k}'+\vec{k})\times\vec{Q}]_{fi}|s'\rangle\Big)\\
&\times Y_{L\,M_L''}^*(\hat{Q}_R')Y_{L\,M_L'}(\hat{Q}_R)R_{n(I\,1/2)L}(|\vec{Q}_R'|)R_{n(I\,1/2)L}(|\vec{Q}_R|)\\
&\times\left(\frac{M_A{}^2E_{nI,\vec{Q}_R}E_{nI,\vec{Q}_R'}}{2\omega_{\vec{k}'}2\omega_{\vec{k}}E_{A,\vec{k}'}E_{A,\vec{k}}E_{nI,\vec{Q}}{}^2|Q_N^0||Q_N'^0|}\right)^{1/2}.\end{aligned} \tag{6.9}$$

We note that the presence of $\vec{Q}$ dependence in the pion-nucleon scattering amplitude and the $(nI)$ dependence of the $s$ value of this amplitude, Eq. (4.6), both prevent the factorization of Eq. (6.9). The first dependence will also be obtained in a conventional nonrelativistic scattering theory if the FSA is not used. However, the modification of the $s$ value through binding effects is usually not considered. Indeed, in the *conventional* nonrelativistic scattering theories, the $s$ value for the $\pi$-$N$ system is ambiguous. If we neglect the $(nI)$ dependence of the $\pi$-$N$ amplitude we can carry out the summation on the quantum number $I$ in the Clebsch-Gordan coefficients of Eq. (6.9) and obtain for closed-shell nuclei the result

$$\sum_{IM}C_{M\,s''\,M_L''}^{I\,1/2\,L}C_{M\,s'\,M_L'}^{I\,1/2\,L}=\delta_{M_L',M_L''}\delta_{s''s'}\left(\frac{2L+1}{2I+1}\right),$$

which makes the spin-flip contribution zero for closed-shell nuclear targets. Consequently, one new dynamical feature of the covariant method is that, even for a spin-zero target nucleus, there is a nonzero contribution of the spin-flip amplitudes to the first-order $\pi$-nucleus optical potential. We note further that for the off-shell situation, $|\vec{k}'|\neq|\vec{k}|$ and $p_N'^0\neq p_N^0$. The spin-flip terms can therefore contribute to forward ($\vec{k}'/\!/\vec{k}$) *off-shell* scattering. Only in *on-shell* forward scattering is this contribution identically zero, as may be seen from Eqs. (6.7) and (6.9).

The result for the covariant pion-nucleus optical potential given in Eq. (6.9) is significantly different from the result given in Eq. (3.27), particularly with respect to the improved treatment of the off-shell kinematics as stressed in Sec. V. Clearly, it is important to investigate the numerical consequences of our approach. The full numerical evaluation of Eq. (6.9), incorporating our model for *covariant off-shell dynamics,* will be reported elsewhere.

## VII. CONCLUSION

We have presented a covariant theory of the pion-nucleus optical potential and in this work we have analyzed the most important contribution to this

potential. We emphasize that a series of problems which exist when one uses nonrelativistic multiple scattering theory are resolved in this approach:
(i). There is no need to assume a potential interaction between the pion and the nucleon as in the nonrelativistic theories.
(ii). The off-shell features of the interactions are uniquely specified in the evaluation of the Feynman diagrams.
(iii). The covariance of the theory makes the transformation properties of the amplitudes unambiguous.
(iv). Production and annihilation amplitudes may be included. A paper is in preparation which presents a crossing-symmetric theory of pion-nucleus scattering. In this work production and annihilation appear when one introduces the requirement of crossing symmetry. The analysis results in a nonlinear equation which is a generalization of the Low equation to the scattering from complex systems.
(v). Phenomenological $\pi$-$N$ scattering amplitudes may be used and the various approximations made may be improved as more information becomes available for the *covariant* off-shell amplitudes.

In future work we will compute the cross sections for elastic scattering for pions from various nuclei. In addition, we also plan to investigate various corrections to the approximations considered here. In particular, the modification of the $\pi$-$N$ amplitudes in the presence of the spectator nucleus should be studied.

## APPENDIX A: NORMALIZATION OF STATES AND DIAGRAM RULES FOR NUCLEAR REACTIONS

In this section we describe the diagram rules appropriate to the study of the scattering of a relativistic projectile by a nucleus. We also discuss the connection between the invariant scattering amplitude used in a covariant scattering theory and the noninvariant scattering amplitude appearing in the nonrelativistic analysis. Throughout our discussion we use the system of units with $\hbar = c = 1$, and define the scalar product of two four-momenta as $p_1 \cdot p_2 = p_1^0 p_2^0 - \vec{p}_1 \cdot \vec{p}_2$. For definiteness, we consider the elastic scattering of two particles. Clearly, our analysis can be extended to include inelastic scattering as well as nuclear reactions having more than two particles in the final state.

We use the following Lorentz-invariant normalization for the one-particle states:

$$\langle \vec{p}', \alpha' | \vec{p}, \alpha \rangle = N(\vec{p}) \delta^{(3)}(\vec{p}' - \vec{p}) \delta_{\alpha', \alpha}, \tag{A1}$$

where $\alpha$ and $\alpha'$ represent isospin and (or) spin labels as well as other necessary discrete quantum numbers such as the internal quantum numbers of a nuclear state. The normalization factor $N(\vec{p})$ is chosen to be $(E_{\vec{p}}/m)$ for fermions and $(2E_{\vec{p}})$ for bosons, with $E_{\vec{p}} = (\vec{p}^2 + m^2)^{1/2}$.

We choose to parametrize the $S$ matrix by

$$\hat{S} = 1 - 2\pi i \hat{M}, \tag{A2}$$

$$\langle \vec{p}_1' \vec{p}_2' | \hat{S} | \vec{p}_1 \vec{p}_2 \rangle = N_1(\vec{p}_1) N_2(\vec{p}_2) \delta^{(3)}(\vec{p}_1' - \vec{p}_1) \delta^{(3)}(\vec{p}_2' - \vec{p}_2) - 2\pi i \delta^{(4)}(p_1' + p_2' - p_1 - p_2) \langle \vec{p}' | M(W) | \vec{p} \rangle, \tag{A3}$$

and then obtain the following expression for the differential cross section:

$$(d\sigma)_{fi} = \frac{(2\pi)^4}{v_R} \delta^{(4)}(p_1' + p_2' - p_1 - p_2) d\vec{p}_1' d\vec{p}_2' \frac{|\langle \vec{p}' \alpha_1' \alpha_2' | M(W) | \vec{p} \alpha_1 \alpha_2 \rangle|^2}{N_1(\vec{p}_1') N_2(\vec{p}_2') N_1(\vec{p}_1) N_2(\vec{p}_2)}. \tag{A4}$$

In particular, in the center-of-mass frame, Eq. (A4) can be written as $[\vec{p}_1' = -\vec{p}_2' = \vec{p}';\ \vec{p}_1 = -\vec{p}_2 = \vec{p}]$

$$\left(\frac{d\sigma}{d\Omega}\right)_{fi} = \frac{(2\pi)^4}{v_R} \int \delta(E_{\vec{p}_1'} + E_{\vec{p}_2'} - E_{\vec{p}_1} - E_{\vec{p}_2}) p'^2 dp' \frac{|\langle \vec{p}' \alpha_1' \alpha_2' | M(W) | \vec{p} \alpha_1 \alpha_2 \rangle|^2}{N_1(\vec{p}_1') N_2(\vec{p}_2') N_1(\vec{p}_1) N_2(\vec{p}_2)}, \tag{A5}$$

or

$$\left(\frac{d\sigma}{d\Omega}\right)_{fi} = (2\pi)^4 \mu(\vec{p}) \mu(\vec{p}') \frac{|\langle \vec{p}' \alpha_1' \alpha_2' | M(W) | \vec{p} \alpha_1 \alpha_2 \rangle|^2}{N_1(\vec{p}_1') N_2(\vec{p}_2') N_1(\vec{p}_1) N_2(\vec{p}_2)}, \tag{A6}$$

where $\mu(\vec{p}') \equiv E_{\vec{p}_1'} E_{\vec{p}_2'} / (E_{\vec{p}_1'} + E_{\vec{p}_2'})$. In Eq. (A5), $v_R$ is the relativistic relative velocity of the incoming particles and is equal to $[(p_1 \cdot p_2)^2 - p_1^2 p_2^2]^{1/2} / (E_{\vec{p}_1} E_{\vec{p}_2})$.

In obtaining Eq. (A5), we have exploited the fact that the products $d\vec{p}_1' d\vec{p}_2' / [N_1(\vec{p}_1') N_2(\vec{p}_2')]$ and $v_R N_1(\vec{p}_1) N_2(\vec{p}_2)$ are Lorentz invariants, while in obtaining Eq. (A6) we have used the relation

$$\delta(E_{\vec{p}_1'} + E_{\vec{p}_2'} - E_{\vec{p}_1} - E_{\vec{p}_2}) = \frac{E_{\vec{p}_1'} E_{\vec{p}_2'}}{|\vec{p}'| (E_{\vec{p}_1'} + E_{\vec{p}_2'})} \delta(|\vec{p}'| - |\vec{p}|). \tag{A7}$$

For the *nonrelativistic* analysis, we normalize the one-particle states so that

$$_{\rm NR}\langle \vec{p}', \alpha' | \vec{p}, \alpha \rangle_{\rm NR} = \delta^{(3)}(\vec{p}' - \vec{p}) \delta_{\alpha', \alpha}, \tag{A8}$$

and parametrize the $S$ matrix as follows:

$$\hat{S} = 1 - 2\pi i \hat{\tau}, \tag{A9}$$

or

$$_{NR}\langle \vec{p}_1'\vec{p}_2' | \hat{S} | \vec{p}_1\vec{p}_2 \rangle_{NR} = \delta^{(3)}(\vec{p}_1' - \vec{p}_1)\delta^{(3)}(\vec{p}_2' - \vec{p}_2) - 2\pi i \delta^{(4)}(p_1' + p_2' - p_1 - p_2) \times \langle \vec{p}' | \tau(W) | \vec{p} \rangle . \qquad (A10)$$

This parametrization differs from the one usually used for potential scattering by the presence of an additional factor $\delta^{(3)}(\vec{p}_1' + \vec{p}_2' - \vec{p}_1 - \vec{p}_2)$ in front of the $\tau$ matrix. This $\delta$ function reflects the conservation of the total three-momentum, a conservation law which often is not made explicit in potential scattering theory. Although Eq. (A9) has the same structure as that of Eq. (A2), the matrix element of $\tau$ represents a noninvariant amplitude as a result of the normalization given in Eq. (A8). The differential cross section is

$$(d\sigma)_{fi} = \frac{(2\pi)^4}{v_R} \delta^{(4)}(p_1' + p_2' - p_1 - p_2) d\vec{p}_1' d\vec{p}_2' \times | _{NR}\langle \vec{p}' \alpha_1' \alpha_2' | \tau(W) | \vec{p} \alpha_1 \alpha_2 \rangle_{NR} |^2 . \qquad (A11)$$

In the c.m. frame, Eq. (A11) becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{fi} = \frac{(2\pi)^4}{v_{NR}} \int \delta\left(\frac{\vec{p}'^2}{2\mu} - \frac{\vec{p}^2}{2\mu}\right) p'^2 dp' \times | _{NR}\langle \vec{p}' \alpha_1' \alpha_2' | \tau(W) | \vec{p} \alpha_1 \alpha_2 \rangle_{NR} |^2 , \qquad (A12)$$

or

$$\left(\frac{d\sigma}{d\Omega}\right)_{fi} = (2\pi)^4 \mu^2 | _{NR}\langle \vec{p}' \alpha_1' \alpha_2' | \tau(W) | \vec{p} \alpha_1 \alpha_2 \rangle_{NR} |^2_{|\vec{p}'| = |\vec{p}|} . \qquad (A13)$$

In Eq. (A13) $\mu$ is the reduced mass and $v_{NR} = |\vec{p}|/\mu$.

By comparing the expressions for the cross section we can connect the noninvariant amplitude given by $\tau$ with the Lorentz-invariant amplitude given by $M$. We note the relation

$$_{NR}\langle \vec{p}' \alpha_1' \alpha_2' | \tau(W) | \vec{p} \alpha_1 \alpha_2 \rangle_{NR} = \left(\frac{\mu(\vec{p}')}{\mu}\right)^{1/2} {}_{NR}\langle \vec{p}' \alpha_1' \alpha_2' | T(W) | \vec{p} \alpha_1 \alpha_2 \rangle_{NR} \left(\frac{\mu(\vec{p})}{\mu}\right)^{1/2} , \qquad (A14)$$

where $T(W)$ is the operator we have used in previous work.[2,3] We also recall that we had derived the relation

$$_{NR}\langle \vec{p}' \alpha_1' \alpha_2' | T(W) | \vec{p} \alpha_1 \alpha_2 \rangle_{NR} = R^{1/2}(\vec{p}') \langle \vec{p}' \alpha_1' \alpha_2' | \overline{M}(W) | \vec{p} \alpha_1 \alpha_2 \rangle R^{1/2}(\vec{p}) , \qquad (A15)$$

with $R^{1/2}(\vec{p}) = [N_1(\vec{p}) N_2(\vec{p})]^{-1/2}$, and $2W$ equal to the total energy of the system. For example, in pion-nucleon scattering $N_\pi(\vec{p}) = 2\omega_{\vec{p}}$ and $N_N(\vec{p}) = (E_{N,\vec{p}}/M_N)$; it follows that $R_{\pi N}^{1/2}(\vec{p}) = (M_N / 2\omega_{\vec{p}} E_{N,\vec{p}})^{1/2}$. As we observed previously, Eq. (A15) also provides a basis for relating the off-shell nonrelativistic and Lorentz-invariant amplitudes.

Assuming that each nuclear state can be described by a separate field and therefore has its own propagator, we can summarize the calculation of the $M$ matrix by the following diagram rules:

(1). for each external line of four-momentum $p$ (discrete quantum numbers $\alpha$) entering the diagram, write $|\vec{p}\rangle \chi^{(\alpha)}(\vec{p})$;
(2). for each external line of four-momentum $p'$ (discrete quantum numbers $\alpha'$) leaving the diagram, write $\overline{\chi}^{(\alpha')}(\vec{p}')\langle \vec{p}'|$;
(3). for each internal line, write $(2\pi)^3 |k\rangle S(k) \langle k|$, with $S(k)$ representing the appropriate Feynman propagator for a boson or a fermion;
(4). include a factor $(2\pi)^4 \delta^{(4)}(\sum_i p_i' - \sum_j p_j)$ for each interaction element corresponding to energy and momentum conservation for all the lines joining at the interaction element;
(5). write a factor $(-i)^n$ corresponding to the $n$th order term in the perturbation expansion, with $n$ equal to the number of diagram interaction elements or vertices;
(6). include a factor $(-1)$ for each closed loop of the same fermion;
(7). include a factor $(i/2\pi)$ due to the use of the parametrization in Eq. (A2);
(8). integrate over the four-momenta of all the internal lines;
(9). disregard the remaining $\delta^{(4)}$ function representing the total energy and momentum conservation.

In the above rules $\overline{\chi}^{(\alpha)}(\vec{p})$ represents an appropriate Lorentz-invariant spinor conjugate to $\chi^{(\alpha)}(\vec{p})$. For example, in the spin-$\frac{1}{2}$ case, $\overline{\chi} = \chi^{\dagger}\gamma_0$. We recall that the one-particle states are normalized according to Eq. (A1).

To obtain the $T$ matrix defined in Eq. (A15), we need only replace the rules (1) and (2) by:

(1′). for each external line of four-momentum $p$ (discrete quantum number $\alpha$) entering the diagram, write $|\vec{p}\rangle \chi^{(\alpha)}(\vec{p}) N^{-1/2}(p) = |\vec{p}\rangle_{NR} \chi^{(\alpha)}(\vec{p})$; and
(2′). for each external line of four-momentum $p'$ (discrete quantum number $\alpha'$) leaving the diagram, write $N^{-1/2}(\vec{p}')\overline{\chi}^{(\alpha')}(\vec{p}')\langle \vec{p}'| = \overline{\chi}^{(\alpha)}(\vec{p}')_{NR}\langle \vec{p}'|$.

## APPENDIX B: ESTIMATES OF THE OFF-MASS-SHELL CHARACTER OF THE TARGET CONSTITUENTS

In obtaining Eq. (4.8), we have used the conditions

$$p_N^{\,2} < M_N^{\,2} \qquad (B1)$$

and

$$|(p_N^2 - M_N^2)/(\vec{k}_c^2 + M_N^2)| \ll 1\,, \qquad \text{(B2)}$$

where $\vec{k}_c^2$ is the square of the relative momentum in the c.m. frame of the $\pi$-$N$ system. Now we proceed to prove these two assertions by referring to Fig. 2. Being an invariant, the quantity (see Fig. 3)

$$p_N^2 \equiv (P - Q)^2 = Q_N^2 \qquad \text{(B3)}$$

has the same value in any frame. It follows from Eq. (B3) that

$$p_N^2 = M_A^2 + M_C^2 - 2M_A E_{C,\vec{Q}_R} \simeq (M_A - M_C)^2 - (M_A/M_C)\vec{Q}_R^2\,, \qquad \text{(B4)}$$

where we have used the approximation $E_{C,\vec{Q}_R} \simeq M_C + \vec{Q}_R^2/2M_C$.

Defining the separation energy (or the binding energy of the nucleon) by $|\Delta| = M_N + M_C - M_A > 0$, we obtain from Eq. (B14)

$$p_N^2 - M_N^2 \simeq -[2M_N|\Delta| - |\Delta|^2 + (M_A/M_C)\vec{Q}_R^2] < 0\,, \qquad \text{(B5)}$$

which proves Eq. (B1). To prove our second assertion, Eq. (B2), we note that

$$|(p_N^2 - M_N^2)/(\vec{k}_c^2 + M_N^2)| < |p_N^2 - M_N^2|/M_N^2 \equiv \eta\,. \qquad \text{(B6)}$$

Using Eq. (B5), we have

$$\eta \simeq \frac{2|\Delta|}{M_N} - \left(\frac{|\Delta|}{M_N}\right)^2 + \frac{M_A}{M_C M_N^2}\vec{Q}_R^2\,. \qquad \text{(B7)}$$

In a finite nucleus $|\vec{Q}_R|_{\max} \simeq 250$ MeV/$c$ and $|\Delta|_{\max} \simeq 50$ MeV. Consequently, for heavy nuclei with a large number of nucleons, $\eta_{\max} < 0.15$. For the case of deuteron, $\eta < 0.2$.

On the other hand, if we put the nucleon on its mass shell so as to estimate the maximum off-mass-shell character of the nucleus, $C$, of $(A-1)$ nucleons, then we would have

$$p_N = (E_{N,\vec{Q}_R}, \vec{Q}_R) \qquad \text{(B8)}$$

in the *rest* frame of the nucleus $A$. Using $Q^2 = (P_R - p_N)^2$ we obtain

$$Q^2 = M_A^2 + M_N^2 - 2M_A E_{N,\vec{Q}_R} \simeq (M_A - M_N)^2 - (M_A/M_N)\vec{Q}_R^2\,. \qquad \text{(B9)}$$

Following the same procedure leading Eq. (B4) to Eq. (B7), we see that the off-mass-shell parameter for the nucleus $C$ is

$$\eta'_C \equiv \frac{Q^2 - M_C^2}{M_C^2} \simeq \frac{|\Delta|^2}{M_C^2} - \frac{2|\Delta|}{M_C} - \frac{M_A}{M_C^2 M_N}\vec{Q}_R^2\,. \qquad \text{(B10)}$$

It follows for heavy nuclei ($A \gg 1$) that $\eta'_C \simeq \eta/A$.

## APPENDIX C: NORMALIZATION OF THE SINGLE-PARTICLE WAVE FUNCTIONS

The relativistic single-particle wave function consists of two parts. The first part can be called the "positive-energy" component; it is defined in Eq. (3.14). The second part which is the "negative-energy" component is defined in Eq. (3.18). These two components define the nuclear wave function $\Psi_\mu^{n(I\,1/2)Ms}$ of Eq. (3.19). As a result of using the invariant normalization, all these wave functions have covariant transformation properties.

By calculating the matrix elements of the number operator we can exhibit the appropriate normalization of the functions introduced in Sec. III. We choose to study

$$\langle\vec{P}'|\hat{N}_{0p}|\vec{P}\rangle \equiv \int d\vec{x}\,\langle\vec{P}'|\bar{\psi}(x)\gamma_0\psi(x)|\vec{P}\rangle = AN_A(\vec{P})\delta^{(3)}(\vec{P}' - \vec{P})\,, \qquad \text{(C1)}$$

where $A$ is the number of the target nucleons. By the translational invariance of the nucleon fields we obtain from Eq. (C1)

$$\langle\vec{P}'|\bar{\psi}(0)\gamma_0\psi(0)|\vec{P}\rangle = AN_A(\vec{P})(2\pi)^{-3}\,. \qquad \text{(C2)}$$

For the case with both initial and final target nuclei at rest, we have

$$\langle\vec{0}|\bar{\psi}(0)\gamma_0\psi(0)|\vec{0}\rangle = A(2M_A)(2\pi)^{-3}\,. \qquad \text{(C3)}$$

After the insertion of a complete set of the $(A-1)$-particle states of the nucleus $C$ specified by the quantum numbers $(nIM)$ and $-\vec{Q}_R$, Eq. (C3) reads

$$A = (2M_A)^{-1}(2\pi)^3 \sum_{nIM\mu\nu} \int \frac{d\vec{Q}_R}{N_{nI}(\vec{Q}_R)} \langle\vec{0}|\bar{\psi}_\mu(0)|-\vec{Q}_R, nIM\rangle(\gamma_0)_{\mu\nu}\langle-\vec{Q}_R, nIM|\psi_\nu(0)|\vec{0}\rangle\,. \qquad \text{(C4)}$$

Using, respectively, Eq. (3.19) and its conjugation for the second and the first matrix elements in Eq. (C4), and performing explicitly the summation over spin indices $\mu$ and $\nu$, we obtain

$$A = (2M_A)^{-1} \sum_{nIMLM_L} \int \frac{d\vec{Q}_R}{N_{nI}(\vec{Q}_R)} \left(\frac{E_{N,\vec{Q}_R}}{M_N}\right)\left(\frac{2I+1}{2L+1}\right)\left[|\psi^{(+)}_{n(I\,1/2)LM_L}(\vec{Q}_R)|^2 + |\psi^{(-)}_{n(I\,1/2)LM_L}(\vec{Q}_R)|^2\right] \qquad \text{(C5)}$$

or, after simplifying the $N$ factors,

$$\sum_{nILM_L} \int d\vec{Q}_R[|\phi^{(+)}_{n(I\ 1/2)LM_L}(\vec{Q}_R)|^2 + |\phi^{(-)}_{n(I\ 1/2)LM_L}(\vec{Q}_R)|^2]\left(\frac{2I+1}{2L+1}\right) = A\,. \tag{C6}$$

In obtaining the last equality, we have used Eq. (3.15). In most cases, the component $\phi^{(-)}$ is very small. Recalling that $L$ is determined by parity considerations, we have

$$\sum_{nIM_L} \int d\vec{Q}_R\left(\frac{2I+1}{2L+1}\right)|\phi^{(+)}_{n(I\ 1/2)LM_L}(\vec{Q}_R)|^2 = \sum_{nI}(2I+1)\int_0^\infty Q_R{}^2 dQ_R\,|\phi^{(+)}_{n(I\ 1/2)L}(|\vec{Q}_R|)|^2 = A\,. \tag{C7}$$

If we use the shell model to describe the target, we may identify the intrinsic states $|nIM\rangle$ with *single-hole* states. Introducing a more conventional notation in that case we write $I\to j$, $M\to m_j$, and $L\to l$, and $\phi^{(+)}_{n(I\ 1/2)L}(\vec{Q}_R)\to\phi^{(+)}_{n(l\ 1/2)j}(\vec{Q}_R)$, so that Eq. (C7) becomes

$$\int_0^\infty Q_R{}^2 dQ_R\,|\phi^{(+)}_{n(l\ 1/2)j}(|\vec{Q}_R|)|^2 = 1\,. \tag{C8}$$

In Eq. (C8) the subscript $n$ includes the isospin quantum number as well as specification of the number of radial nodes. From Eq. (C8) we see that the $\phi^{(+)}_{n(l\ 1/2)j}(\vec{Q}_R)$ have the standard normalization of radial shell-model wave functions.

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