Second Born approximation for magnetic multipole electroexcitation*

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The problem of the Coulomb corrections to inelastic electron scattering cross sections is treated in the distorted wave Born approximation (DWBA) using electron wave functions evaluated in the second Born approximation. The formalism is developed for magnetic multipole transitions in a manner analogous to the method developed by Cutler for Coulomb transitions, in which the nuclear and electron physics are represented by separate factors in the integrand of a double integral expression for the cross section. Since the nuclear factor turns out to be simply the inelastic form factor in the first Born approximation, the usual problem of having to specify the transition current density is avoided. The Coulomb correction is considered in detail for $M1$ transitions, and numerical comparison with "exact" DWBA results is presented.

[NUCLEAR REACTIONS Coulomb corrections to $M\lambda$ inelastic electron scattering. Calculated in DWBA using 2nd BA wave functions.]

I. INTRODUCTION

Inelastic electron scattering is a well established method for obtaining spectroscopic data on nuclear states, as well as providing information on the distribution of the various transition currents and charge densities in the nucleus. The gradual improvement of experimental and theoretical techniques in recent years has resulted in demands for data of increasing accuracy. To the experimentalist, this has meant a more diligent understanding and careful application of the many corrections he must usually apply in the process of reducing his data to a form presentable for theoretical interpretation. For example, consideration must be given to the distortion effects of the electron wave functions by the Coulomb field of the nucleus, since the Coulomb distortion can result in cross sections which deviate appreciably from those calculated using the first Born approximation, where the electron wave functions are treated as Dirac plane waves.

The "exact" calculation of the Coulomb distortion effects in inelastic scattering requires a numerical solution of the Dirac equation and such a program, based on a distorted wave Born approximation (DWBA), was initiated by Griffy *et al.*,[1] first for nuclear Coulomb excitation, and later with the inclusion of transverse interactions.[2] The first phase-shift analysis to incorporate magnetization terms in the excitation process was apparently developed by Drechsel,[3] and more recently Chertok, Johnson, and Sakar[4,5] have evaluated the Coulomb corrections for $M1$ and $M2$ excitations. To date, a variety of distorted partial-wave computer codes are available (e.g., GBROW, DUELS, HEINEL) for computing the electroexcitation cross sections.

In all these codes, the interaction is mediated by a single virtual photon, while the electron wave functions are solutions of the Dirac equation in the static Coulomb field of the nucleus. One analysis procedure often employed is to describe the transition density in the DWBA code by some useful model (for example the liquid-drop model of Tassie[6]), calculate the Coulomb correction factors

$$f_c = \frac{(d\sigma/d\Omega)_{\text{exact}}}{(d\sigma/d\Omega)_{\text{1st Born}}}, \tag{1}$$

and reduce the physical cross section to a Born approximation cross section, then reevaluate the transition density parameters and recalculate f_c. After iterating in this manner, one obtains the Tassie parameters and a set of f_c factors which can be used to present the data as Born approximation data. This is convenient for comparison with theoretical cross sections since the latter are usually calculated in the first Born approximation.

The above procedure is not model independent in the sense that the transition charge, current, and magnetization densities must be specified. Drechsel[3] has investigated the dependence of the Coulomb correction factors on the distribution of the transition magnetization and current densities for $M1$ electroexcitation, and finds that those densities which give the same Born approximation cross sections give the same correction factors, for a given ground state charge distribution.

An alternate procedure, also based on DWBA,

has been formulated by Cutler.[7] The interaction matrix element between nuclear states $J_i M_i$ and $J_f M_f$, and electron states Ψ_i and Ψ_f, can be written[8]

$$H_{fi} = \langle \Psi_f J_f M_f | \int d^3r d^3r' [\hat{\rho}(\vec{r}) G \hat{\rho}(\vec{r}') - \hat{\vec{J}}(\vec{r}) \cdot G \hat{\vec{j}}(\vec{r}')] | J_i M_i \Psi_i \rangle \tag{2}$$

in which $\hat{\rho}(\vec{r})$, $\hat{\vec{J}}(\vec{r})$, $\hat{\rho}(\vec{r}')$, and $\hat{\vec{j}}(\vec{r}')$ are, respectively, the nuclear and electron charge and current operators, and G is the free space Green's function for the nuclear Coulomb field:

$$G = \frac{e^{i\omega R}}{R} , \tag{3}$$

where $\vec{R} = \vec{r} - \vec{r}'$ and ω corresponds to the energy exchange. Rather than expanding G in a multipole series as is done in the usual partial-wave analysis, Cutler in effect introduces the Fourier transform of the Green's function

$$G = \frac{1}{2\pi^2} \int \frac{e^{i\vec{q}\cdot\vec{R}}}{q^2 - \omega^2} d^3q \tag{4}$$

into Eq. (2), which permits a complete separation of the coordinates of the electron and nucleus. After some manipulation, the matrix element Eq. (2) can then be expressed as a q integral over the product of the usual Born approximation inelastic form factor and a term which contains the electron physics. Although the electron matrix element could in principle be evaluated exactly by solving the Dirac equation, the formalism readily lends itself to various approximating procedures. For example, Cutler calculated the electron wave functions analytically in the second Born approximation, including the effects of the nuclear charge distribution. The resulting electroexcitation cross sections were thus consistently formulated through terms of order $(\alpha Z)^3$ and can be expressed as

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{2nd Born}} = f_C \left(\frac{d\sigma}{d\Omega}\right)_{\text{1st Born}} , \tag{5}$$

where $f_C = 1 + \alpha Z \beta_\lambda$ and β_λ is independent of Z.

Cutler calculated the correction factors β_λ for longitudinal electric transitions, specifically $C0$, $C1$, and $C2$ by using specific analytic forms for the elastic and inelastic form factors.

One attractive feature of this approach is that no models are required for the ground state charge distribution or the transition densities. An iterative procedure can be used in which the experimental form factors (elastic and inelastic) are inserted as a first estimate to the Born form factors in the expression for β_λ. Further, the β_λ equations are simple enough that they may be computed on the small digital computers available in most laboratories. A disadvantage of the approximation described by Eq. (5) is that it is not valid near the diffraction minima; one must consider terms of order $(\alpha Z)^4$ and higher in the cross section in order to fill in the Born zeros. The reliability of the approximation of course decreases with increasing Z nuclei. Nevertheless, it could be useful, for example, for correcting the low-momentum transfer data used for extracting radiative widths of transitions to the ground state.

In the present work, we have derived the second Born expressions for f_C for magnetic transitions, with particular emphasis on $M1$. The procedure is similar to Cutler's, except here the excitation mechanism is completely transverse. The main results we obtain are a relation for $\beta_{M\lambda}$, expressed as a two dimensional integral, and an approximate version of β_{M1} which involves a one dimensional integration. Numerical evaluations of the Coulomb correction factors are compared with DWBA phase-shift results.

II. SEPARATION OF THE ELECTRON AND NUCLEAR PHYSICS

We wish to develop the DWBA matrix element Eq. (2) into a form in which the nuclear dynamics are expressed by multipole transition matrix elements. In fact, as we shall see, all the nuclear physics in H_{fi} can be represented by a term which turns out to be the usual electroexcitation form factor in the first Born approximation.

We begin by dropping the charge density terms in H_{fi}, since only transverse excitations are being considered. The electron current operator is given by

$$\hat{\vec{j}}(\vec{r}') = \vec{\alpha} \hat{j}(\vec{r}') , \tag{6}$$

where

$$\hat{j}(\vec{r}') = e\delta(\vec{r}' - \vec{r}_e) \tag{7}$$

and $\vec{\alpha}$ is the usual Dirac matrix. Introducing Eq. (6) and the Fourier representation of the Green's function, Eq. (4), into Eq. (2), one obtains

$$H_{fi} = -\frac{1}{2\pi^2} \left\langle \Psi_f J_f M_f \left| \int \frac{d^3q}{q^2 - \omega^2} \left[\int d^3r\, e^{i\vec{q}\cdot\vec{r}} \vec{\alpha} \cdot \hat{\vec{J}}(\vec{r}) \right] \times \left[\int d^3r' e^{-i\vec{q}\cdot\vec{r}'} \hat{j}(\vec{r}') \right] \right| \Psi_i J_i M_i \right\rangle . \tag{8}$$

The nuclear current operator $\hat{\vec{J}}(\vec{r})$ can be decomposed into components transverse and parallel to $\vec{q}$, and by application of the continuity equation, $\hat{\vec{J}}_{\parallel}(\vec{r})$ can be absorbed into the longitudinal interaction. For the transverse interactions, therefore, we have

$$e^{i\vec{q}\cdot\vec{r}}\vec{\alpha}\cdot\hat{\vec{J}}(\vec{r}) = e^{i\vec{q}\cdot\vec{r}}\vec{\alpha}_t\cdot\hat{\vec{J}}_t(\vec{r}) \ . \tag{9}$$

Following Willey,[9] we perform a multipole decomposition

$$\vec{\alpha}_t e^{i\vec{q}\cdot\vec{r}} = -4\pi\sum_{\lambda\mu} i^{\lambda}(-1)^{\mu}\left\{\frac{i}{q^2}\,[\vec{\alpha}\cdot\vec{q}\times\vec{Y}^{\mu}_{\lambda\lambda_1}(\vec{q})]\;[\vec{\nabla}\times j_{\lambda}(qr)\vec{Y}^{-\mu}_{\lambda\lambda_1}(\vec{r})] + [\vec{\alpha}\cdot\vec{Y}^{\mu}_{\lambda\lambda_1}(\vec{q})][j_{\lambda}(qr)\vec{Y}^{-\mu}_{\lambda\lambda_1}(\vec{r})]\right\} \tag{10}$$

and introduce the nuclear electric and magnetic multipole operators[10]

$$\hat{T}^{E}_{\lambda\mu}(q) = \frac{1}{q}\int d^3r\,\vec{\nabla}\times[j_{\lambda}(qr)\vec{Y}^{\mu}_{\lambda\lambda_1}(\vec{r})]\cdot\hat{\vec{J}}(\vec{r}) \ ,$$

$$\hat{T}^{M}_{\lambda\mu}(q) = \int d^3r\, j_{\lambda}(qr)\vec{Y}^{\mu}_{\lambda\lambda_1}(\vec{r})\cdot\hat{\vec{J}}(\vec{r}) \ , \tag{11}$$

where

$$\vec{Y}^{\mu}_{\lambda\lambda_1}(\vec{r}) = \sum_{mm'}(\lambda m 1 m'|\lambda\mu)Y_{\lambda m}(\vec{r})\vec{e}_{m'} \tag{12}$$

are the vector spherical harmonics. For a transition of multipolarity λ, Eqs. (9)–(11) give

$$\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\vec{\alpha}\cdot\hat{\vec{J}}(\vec{r}) = -4\pi\sum_{\mu} i^{\lambda}(-1)^{\mu}\left\{\frac{1}{q^2}[\vec{\alpha}\cdot\vec{q}\times\vec{Y}^{\mu}_{\lambda\lambda_1}(\vec{q})]\,\hat{T}^{E}_{\lambda-\mu}(q) + [\vec{\alpha}\cdot\vec{Y}^{\mu}_{\lambda\lambda_1}(\vec{q})]\,\hat{T}^{M}_{\lambda-\mu}(q)\right\} . \tag{13}$$

We are not concerned with $E\lambda$ transitions and so drop the electric multipole contribution from Eq. (13). On combining Eqs. (7), (8), and (13), one gets

$$H_{fi} = \frac{2e}{\pi}i^{\lambda}\sum_{\mu}(-1)^{\mu}\int\frac{d^3q}{q^2-\omega^2}[\vec{Y}^{\mu}_{\lambda\lambda_1}(\vec{q})\cdot\vec{X}_{m'm}(\vec{q})]\times\langle J_f M_f|\,\hat{T}^{M}_{\lambda-\mu}(q)|J_i M_i\rangle \ , \tag{14}$$

where

$$\vec{X}_{m'm}(\vec{q}) = \int d^3r_e\,\Psi_f^{\dagger}\vec{\alpha}e^{-i\vec{q}\cdot\vec{r}_e}\Psi_i \tag{15}$$

and m', m refer to the final and initial spin directions of the electron. All the electron physics is now isolated in the $\vec{X}_{m'm}$ term, while the nuclear physics is contained in the Born approximation multipole matrix element.

If we use Dirac plane waves for the electron wave functions Ψ_i and Ψ_f, Eq. (15) would yield a δ function connecting the initial and final electron momenta with $\vec{q}$. The $\vec{q}$ integration in Eq. (14) can then be accomplished with the δ function, and we retrieve the usual first Born approximation for H_{fi}.

The nuclear matrix element can be reduced by the Wigner-Eckart theorem:

$$\langle J_f M_f|\,\hat{T}^{M}_{\lambda-\mu}(q)|J_i M_i\rangle = (-1)^{J_f-M_f}\begin{pmatrix} J_f & \lambda & J_i \\ -M_f & -\mu & M_i\end{pmatrix}\times\langle J_f||\,\hat{T}^{M}_{\lambda}(q)||J_i\rangle \ . \tag{16}$$

At this point it is convenient to introduce the inelastic $M\lambda$ form factor $F_{\lambda}(q)$. We define $F_{\lambda}(q)$ such that the inelastic cross section in the first Born approximation is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mu\lambda} = \sigma_M\left(\frac{|q_{\mu}{}^2|}{2q^2}+\tan^2\tfrac{1}{2}\theta\right)|F_{\lambda}(q)|^2 \ , \tag{17}$$

where

$$\sigma_M = \left(\frac{Z\alpha}{2E}\right)^2\frac{\cos^2(\frac{1}{2}\theta)}{\sin^4(\frac{1}{2}\theta)}\frac{1}{\eta} \ ,$$

$$\eta = 1+\frac{2E}{M}\sin^2(\tfrac{1}{2}\theta) \ ,$$

and $q_{\mu}{}^2$ is the square of the four-momentum transfer, θ is the scattering angle, E is the incident energy, M is the nuclear mass, and α is the fine structure constant. Our units are such that $\hbar = c = 1$ and $\alpha = e^2$. The relation between the reduced matrix element and the form factor is, therefore,

$$e^2|F_{\lambda}(q)|^2 = \frac{4\pi}{Z^2}\,\frac{|\langle J_f||\,\hat{T}^{M}_{\lambda}(q)||J_i\rangle|^2}{2J_i+1} \ . \tag{18}$$

Since we are dealing with discrete transitions, $F_{\lambda}(q)$ may be taken to be real.[10]

The cross section requires $|H_{fi}|^2$ averaged over incident electron and nuclear spin directions, and summed over final spin directions. With the definitions given by Eqs. (16) and (18) substituted into Eq. (14), and application of the usual closure relation for 3-j symbols, we find the nuclear spin

average

$$\overline{\sum_{M_i M_f}} |H_{fi}|^2 = \frac{(Ze^2)^2}{\pi^3(2\lambda+1)} \int \frac{d^3q}{q^2-\omega^2} \int \frac{d^3q'}{q'^2-\omega^2} F_\lambda(q') F_\lambda(q) \sum_\mu [\vec{Y}^\mu_{\lambda\lambda 1}(\hat{q}') \cdot \vec{X}_{m'm}(\hat{q}')] * [\vec{Y}^\mu_{\lambda\lambda 1}(\hat{q}) \cdot \vec{X}_{m'm}(\hat{q})] \ . \quad (19)$$

This is as far as we can go without specifying the electron wave functions. In the following sections, the exact wave functions will be replaced by wave functions evaluated in the second Born approximation.

III. ELECTRON WAVE FUNCTIONS IN THE SECOND BORN APPROXIMATION

The derivation given here of the second Born wave functions is similar to Cutler's treatment[7] (see also Ref. 8) and so will be brief. For a more complete discussion, we refer the reader to the preceding references.

The Dirac equation for an electron moving in the external electromagnetic field $A^\mu(r)$ is[11]

$$(\hat{p} - m)\Psi(r) = e\not{A}(r)\Psi(r) \ , \quad (20)$$

where

$$\hat{p} = i\left(\gamma^0 \frac{\partial}{\partial t} + \vec{\gamma} \cdot \vec{\nabla}\right) ,$$

the Dirac matrices are $\gamma^\mu = (\beta, \beta\vec{\alpha})$, and the Feynman slash notation is used:

$$\not{A}(r) = \gamma^\mu A_\mu(r) = \gamma^0 A^0(r) - \vec{\gamma} \cdot \vec{A}(r) \ .$$

The Dirac equation may be written in integral form

$$\Psi(r) = \psi(r) + \int d^4r' S_F(r-r') \not{A}(r') \Psi(r') \ , \quad (21)$$

where the free-particle Feynman propagator is[11]

$$S_F(r-r') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(r-r')}}{k_\mu k^\mu - m^2 + i\epsilon} (\not{k} + m) \quad (22)$$

and the limit $\epsilon \to +0$ is understood. The function $\psi(r)$ is a free wave solution to the Dirac equation with four-momentum $p^\mu = (E, \vec{p})$. The initial electron state $\Psi_i(r)$ contains outgoing spherical waves at infinity while the final state $\Psi_f(r)$ contains incoming waves. For the latter state the propagator is given by Eq. (22) but with ϵ replaced by $-\epsilon$.

We will assume in the following that the field $A^\mu(r)$ of the nucleus does not change appreciably during the excitation, and that the nuclear recoil is negligible. Furthermore, the distortion of the electron wave due to interaction with the vector field $\vec{A}(r)$ will be ignored. With these approximations Eqs. (21) and (22) may be combined to give, for the initial state

$$\Psi_i(\vec{r}) = \psi_i(\vec{r}) + (E_i - i\vec{\alpha} \cdot \vec{\nabla} + \beta m) \times \int d^3r' \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{p^2 - k^2 + i\epsilon} e\phi(\vec{r}') \Psi_i(\vec{r}') \ , \quad (23)$$

where $\phi(\vec{r}')$ is the static Coulomb potential of the nucleus. This potential may be expressed in terms of the (spherical) nuclear charge density $\rho(r)$ by

$$e\phi(\vec{r}') = -Z\alpha \int d^3r'' \frac{\rho(r'')}{|\vec{r}' - \vec{r}''|} e^{-\lambda|\vec{r}'-\vec{r}''|} \quad (24)$$

in which a convergence factor has been introduced to suppress the singularity arising from the infinite range of the Coulomb interaction. At the end of the calculation we take the limit $\lambda \to 0$. The potential can be put in a more convenient form by employing the definition of the elastic scattering form factor $F(q)$ in the first Born approximation. We have

$$F(q) = \int d^3r'' \rho(r'') e^{i\vec{q}\cdot\vec{r}''} \quad (25)$$

or, if we invert the Fourier transforms

$$\rho(r'') = \frac{1}{(2\pi)^3} \int d^3q \, F(q) e^{-i\vec{q}\cdot\vec{r}''} \ . \quad (26)$$

Inserting Eq. (26) into Eq. (24), and carrying out the $\vec{r}''$ integration, we obtain

$$e\phi(\vec{r}') = -4\pi\alpha Z \int \frac{d^3q}{(2\pi)^3} F(q) \frac{e^{-i\vec{q}\cdot\vec{r}'}}{q^2 + \lambda^2} \ . \quad (27)$$

In the preceding expressions, the charge density is normalized to unity:

$$\int d^3r \, \rho(r) = 1 \ .$$

One may question the usefulness of expressing the potential in terms of $F(q)$, since Eq. (27) requires knowledge of the form factor at all values of the momentum transfer, whereas experimental elastic scattering data in most cases do not extend beyond $q = 3$ or 4 fm^{-1}. Fortunately however the final expressions for $\beta_{M\lambda}$ are not very sensitive to the behavior of $F(q)$ for q values much greater than the momentum transfer at which $\beta_{M\lambda}$ is being evaluated.

The substitution of Eq. (27) into Eq. (23) gives an integral equation for the exact wave function. The second Born approximation for $\Psi_i(\vec{r})$ is obtained by replacing $\Psi_i(\vec{r}')$ in the integral by $\psi_i(\vec{r}')$, the plane wave solution of the free-particle Dirac equation. When this is done, the integral over $\vec{r}'$ gives a δ function $\delta(\vec{p}_i - \vec{k} - \vec{q})$, so the integral over $\vec{k}$ becomes trivial and we get

$$\Psi_i(\vec{r}) = e^{i\vec{p}_i\cdot\vec{r}}\left[1+\frac{\alpha Z}{2\pi^2}\int d^3k \frac{(2E_i+\vec{\alpha}\cdot\vec{k})e^{i\vec{k}\cdot\vec{r}}F(k)}{(k^2+\lambda^2)(k^2+2\vec{k}\cdot\vec{p}_i - i\epsilon)}\right]U(\vec{p}_i)\ , \tag{28}$$

where $U(\vec{p})$ is the free Dirac spinor. In obtaining the above, we have replaced the integration variable $\vec{q}$ by $-\vec{k}$ and used the fact that the elastic form factor $F(q)$ is an even function of q. The corresponding approximation for the final state electron wave function is obtained by replacing the initial energy and momentum E_i, $\vec{p}_i$ by the final state quantities, and $\epsilon \to -\epsilon$. The limit $\epsilon \to 0$, $\lambda \to 0$ is understood.

IV. GENERAL EXPRESSION FOR $\beta_{M\lambda}$

It remains now to use the approximate wave functions to calculate the electron function $\vec{X}_{m'm}(\vec{q})$, perform the electron spin average on $|H_{fi}|^2$, and sum over μ. From Eqs. (28) and (15) we have

$$\vec{X}_{m'm}(\vec{q}) = (2\pi)^3\delta(\vec{Q})U(\vec{p}_f)^\dagger\vec{\alpha}U(\vec{p}_i) + 4\pi\alpha Z\frac{F(Q)}{Q^2+\lambda^2}U(\vec{p}_f)^\dagger\left(\vec{\alpha}\,\frac{\vec{\alpha}\cdot\vec{Q}+2E_i}{Q^2+2\vec{Q}\cdot\vec{p}_i - i\epsilon} - \frac{\vec{\alpha}\cdot\vec{Q}-2E_f}{Q^2-2\vec{Q}\cdot\vec{p}_f - i\epsilon}\,\vec{\alpha}\right)U(\vec{p}_i)\ , \tag{29}$$

where $\vec{Q} = \vec{q} - \vec{\Delta}$ and $\vec{\Delta} = \vec{p}_i - \vec{p}_f$ is the three-momentum transfer. Terms of order $(\alpha Z)^2$ have been dropped since the Coulomb correction factor f_C is being calculated to order αZ. On combining Eqs. (29) and (19), one obtains for the integrand a term proportional to $\delta(\vec{Q})\delta(\vec{Q}')$, a cross term of order αZ containing two terms proportional to $\delta(\vec{Q})$ and $\delta(\vec{Q}')$, respectively, and a term proportional to $(\alpha Z)^2$ which will be dropped. The two terms of order αZ can be combined into a single term by interchanging $\vec{q} \rightleftarrows \vec{q}'$ in one term and using the fact that the inelastic form factor $F_\lambda(q)$ is real. Part of the integration over $\vec{q}$ and $\vec{q}'$ is accomplished using the δ functions, and the average over initial and sum over final electron spin projections are done using standard trace techniques.[11] After all this the result may be written

$$\overline{\sum_{\substack{M_i M_f\\ mm'}}} |H_{fi}|^2 = H_1 + \alpha Z H_2\ , \tag{30}$$

where

$$H_1 = (\alpha Z)^2\frac{(4\pi)^3}{2\lambda+1}\frac{F_\lambda^2(\Delta)}{2m^2(\Delta^2-\omega^2)^2}$$
$$\times\sum_\mu \{[\vec{Y}^*(\vec{\Delta})\cdot\vec{p}_f][\vec{Y}(\vec{\Delta})\cdot\vec{p}_i] + [\vec{Y}^*(\vec{\Delta})\cdot\vec{p}_i][\vec{Y}(\vec{\Delta})\cdot\vec{p}_f] + [\vec{Y}^*(\vec{\Delta})\cdot\vec{Y}(\vec{\Delta})](E_iE_f - \vec{p}_i\cdot\vec{p}_f)\}\ , \tag{31}$$

$$H_2 = (\alpha Z)^2\frac{32\pi}{2\lambda+1}\frac{F_\lambda(\Delta)}{m^2(\Delta^2-\omega^2)}\int\frac{d^3q}{q^2-\omega^2}\frac{F_\lambda(q)F(Q)}{Q^2+\lambda^2}$$
$$\times\mathrm{Re}\sum_\mu\left[2\left(\frac{E_i}{b_i}+\frac{E_f}{b_f}\right)\{[\vec{Y}^*(\vec{\Delta})\cdot\vec{Y}(\vec{q})](E_iE_f - \vec{p}_i\cdot\vec{p}_f) + [\vec{Y}^*(\vec{\Delta})\cdot\vec{p}_f][\vec{Y}(\vec{q})\cdot\vec{p}_i] + [\vec{Y}^*(\vec{\Delta})\cdot\vec{p}_i][\vec{Y}(\vec{q})\cdot\vec{p}_f]\}\right.$$
$$+\left(\frac{1}{b_i}+\frac{1}{b_f}\right)\{[\vec{Y}^*(\vec{\Delta})\cdot\vec{Y}(\vec{q})](E_f\vec{p}_i - E_i\vec{p}_f)\cdot\vec{Q} + [\vec{Y}^*(\vec{\Delta})\cdot\vec{Q}](E_i\vec{p}_f - E_f\vec{p}_i)\cdot\vec{Y}(\vec{q})\}$$
$$\left.+\left(\frac{1}{b_i}-\frac{1}{b_f}\right)\{[\vec{Y}(\vec{q})\cdot\vec{Q}](E_f\vec{p}_i + E_i\vec{p}_f)\cdot\vec{Y}^*(\vec{\Delta})\}\right]\ . \tag{32}$$

We have introduced a short notation for the vector spherical harmonics, $\vec{Y}(\vec{q}) = \vec{Y}^\mu_{\lambda\lambda 1}(\vec{q})$, and defined

$$b_i = Q^2 + 2\vec{Q}\cdot\vec{p}_i - i\epsilon\ , \qquad b_f = Q^2 - 2\vec{Q}\cdot\vec{p}_f - i\epsilon\ .$$

Terms in the numerators of Eqs. (31) and (32) proportional to the electron rest mass m have been neglected.

The above expressions are valid for processes

where nuclear recoil is ignored but where the electron loses an energy ω due to the nuclear excitation. Drechsel[3] and Chertok[5] have investigated the effect on f_C of finite energy loss and do not find a strong dependence on ω except for large scattering angles and near the diffraction minimum of the inelastic cross section. Therefore, we will set $\omega = 0$, that is, $E_i = E_f = E$.

If the Ω_q integration of the terms in Eq. (32) containing b_i is done in a coordinate system where the z axis is along $\vec{\Delta}$ and the x axis is along $\vec{p}_i + \vec{p}_f$, while the corresponding integration of the b_f terms is done in a coordinate system where the x axis is along $-\vec{p}_i - \vec{p}_f$,[7] then one sees that H_2 must be invariant under the substitution $\vec{p}_i \to -\vec{p}_f$, $\vec{p}_f \to -\vec{p}_i$. Under this transformation $b_i \longleftrightarrow b_f$, and H_2 simplifies to

$$H_2 = (\alpha Z)^2 \frac{64\pi}{2\lambda+1} \frac{E F_\lambda(\Delta)}{m^2\Delta^2} \int \frac{d^3q}{q^2} \frac{F_\lambda(q) F(Q)}{Q^2+\lambda^2} \frac{1}{q^2 + 2\vec{q}\cdot\vec{p}_f - i\epsilon}$$
$$\times \mathrm{Re} \sum_\mu \{[\vec{Y}^*(\vec{\Delta})\cdot\vec{Y}(\vec{q})]\ (\vec{\Delta}\cdot\vec{q}) + [\vec{Y}^*(\vec{\Delta})\cdot(\vec{p}_i+\vec{p}_f)][\vec{Y}(\vec{q})\cdot(\vec{p}_i+\vec{p}_f)]$$
$$+ [\vec{Y}^*(\vec{\Delta})\cdot\vec{Q}][\vec{Y}(\vec{q})\cdot\vec{Q}] - [\vec{Y}^*(\vec{\Delta})\cdot(\vec{p}_i+\vec{p}_f)][\vec{Y}(\vec{q})\cdot\vec{\Delta}]\} \ . \tag{33}$$

The sum over μ in Eqs. (31) and (33) can be done by using the following identities, evaluated in the coordinate system where the z axis is along $\vec{\Delta}$ and the x axis is parallel to $\vec{p}_i + \vec{p}_f$.

$$\sum_\mu \vec{Y}^*(\vec{\Delta})\cdot\vec{Y}(\vec{q}) = \frac{2\lambda+1}{4\pi} P_\lambda(x) , \tag{34}$$

$$\sum_\mu [\vec{Y}^*(\vec{\Delta})\cdot(\vec{p}_i+\vec{p}_f)][\vec{Y}(\vec{q})\cdot(\vec{p}_i+\vec{p}_f)]$$
$$= \frac{2\lambda+1}{8\pi}(\vec{p}_i+\vec{p}_f)^2 \left[P_\lambda(x) + \frac{P_\lambda^2(x)}{\lambda(\lambda+1)}\cos(2\phi_q)\right] ,$$

$$\sum_\mu [\vec{Y}^*(\vec{\Delta})\cdot\vec{Q}][\vec{Y}(\vec{q})\cdot\vec{Q}]$$
$$= \left(\frac{2\lambda+1}{4\pi}\right) \frac{q\Delta}{\lambda(\lambda+1)} p_\lambda^1(x)[1-x^2]^{1/2} ,$$

$$\sum_\mu [\vec{Y}^*(\vec{\Delta})\cdot(\vec{p}_i+\vec{p}_f)][\vec{Y}(\vec{q})\cdot\vec{\Delta}]$$
$$= -\left(\frac{2\lambda+1}{4\pi}\right) \frac{\Delta|\vec{p}_i+\vec{p}_f|}{\lambda(\lambda+1)} p_\lambda^1(x)\cos\phi_q ,$$

where $x = \hat{q}\cdot\hat{z}$ and $P_\lambda^m(x)$ are the associated Legendre functions.

It is convenient at this point to rewrite some of the vector quantities in terms of unitless vectors. Let

$$\vec{\Delta} = E\vec{\delta} \quad \text{so} \quad \delta = 2\sin(\tfrac{1}{2}\theta) ,$$
$$\vec{q} = E\vec{u} , \tag{35}$$
$$\vec{Q} = \vec{q} - \vec{\Delta} = E(\vec{u} - \vec{\delta}) ,$$

where θ is the electron scattering angle in the laboratory. Combining Eqs. (31), (33), and (34), together with the above definitions, and using

$$\beta_{M\lambda} = H_2/H_1$$

we obtain

$$\beta_{M\lambda} = \frac{4}{\pi^2}\left(\frac{\delta^2}{4+\delta^2}\right) \frac{1}{F_\lambda(\Delta)} \int du\, d\Omega_u \frac{F_\lambda(Eu) F(E|\vec{u}-\vec{\delta}|)}{[(\vec{u}-\vec{\delta})^2+\lambda^2](u^2 + 2\vec{u}\cdot\hat{p}_f - i\epsilon)}$$
$$\times \Big\{ P_\lambda(x)(2\delta u x + 4 - \delta^2) + \frac{2\delta}{\lambda(\lambda+1)} P_\lambda^1(x)[u(1-x^2)^{1/2} + (4-\delta^2)^{1/2}\cos\phi_u]$$
$$+ \frac{(4-\delta^2)}{\lambda(\lambda+1)} P_\lambda^2(x)\cos(2\phi_u) \Big\} . \tag{36}$$

Since $|\vec{u} - \vec{\delta}|$ is independent of ϕ_u for our choice of coordinate axes, we may proceed to integrate Eq. (36) over ϕ_u.

The azimuthal integrals are of the form

$$I_n = \int_0^{2\pi} \frac{\cos(n\phi)d\phi}{u^2 + 2\vec{u}\cdot\hat{p}_f - i\epsilon}$$

which may be expressed as

$$I_n = \int_0^{2\pi} \frac{\cos(n\phi)d\phi}{a + b\cos\phi} , \tag{37}$$

where $n = 0$, 1, or 2, and

$$a = u^2 - u\delta x - i\epsilon ,$$
$$b = u[(4-\delta^2)(1-x^2)]^{1/2} . \tag{38}$$

The integral Eq. (37) can be done by contour integration around the unit circle centered on the origin of the complex plane. With $Z=e^{i\phi}$ we have

$$I_n=\frac{-i}{b}\oint\frac{(Z^n+Z^{-n})dZ}{Z^2+2(a/b)Z+1}\,. \tag{39}$$

The integrand has poles at the origin (if $n>0$) at $Z=Z_1$ and Z_2 where

$$Z_1=-\left(\frac{a}{b}\right)+\left[\left(\frac{a}{b}\right)^2-1\right]^{1/2},$$

$$Z_2=-\left(\frac{a}{b}\right)-\left[\left(\frac{a}{b}\right)^2-1\right]^{1/2}.$$

Note that since $Z_1Z_2=1$, one pole will lie outside the contour unless both Z_1 and Z_2 are on the unit circle. Applying the method of residues one obtains

$$I_0=\frac{2\pi\eta}{(a^2-b^2)^{1/2}}\,,$$

$$I_1=\frac{2\pi}{b}\left[1-\frac{a\eta}{(a^2-b^2)^{1/2}}\right], \tag{40}$$

$$I_2=\frac{2\pi}{b^2}\left[\frac{(2a^2-b^2)\eta}{(a^2-b^2)^{1/2}}-2a\right],$$

where

$$\eta=+1 \quad \text{if} \quad |Z_1|<1\,, \qquad \eta=-1 \quad \text{if} \quad |Z_2|<1\,,$$

$$\eta=0 \quad \text{if} \quad |Z_1|=|Z_2|=1\,.$$

If we now let $\epsilon\to 0$, we find

$$\eta=+1 \quad \text{if} \quad a>0 \quad \text{and} \quad a^2>b^2\,,$$

$$\eta=-1 \quad \text{if} \quad a<0 \quad \text{and} \quad a^2>b^2\,, \tag{41}$$

$$\eta=0 \quad \text{if} \quad b^2>a^2\,.$$

Introducing the azimuthal integrals into Eq. (36), one gets

$$\beta_{M\lambda}=\frac{8}{\pi}\left(\frac{\delta^2}{4+\delta^2}\right)\frac{1}{F_\lambda(\Delta)}\int_0^\infty du\int_{-1}^1 dx\,\frac{F_\lambda(Eu)F[E(u^2+\delta^2-2u\delta x)^{1/2}]}{u^2+\delta^2-2u\delta x+\lambda^2}$$

$$\times\Bigg\{P_\lambda(x)(2\delta ux+4-\delta^2)\frac{\eta}{(a^2-b^2)^{1/2}}+\frac{2\delta P_\lambda^1(x)}{\lambda(\lambda+1)}$$

$$\times\left[\frac{u(1-x^2)^{1/2}\eta}{(a^2-b^2)^{1/2}}+\frac{(4-\delta^2)^{1/2}}{b}\left(1-\frac{a\eta}{(a^2-b^2)^{1/2}}\right)\right]$$

$$+\frac{(4-\delta^2)P_\lambda^2(x)}{b^2\lambda(\lambda+1)}\left[\frac{(2a^2-b^2)\eta}{(a^2-b^2)^{1/2}}-2a\right]\Bigg\}\,, \tag{42}$$

where a and b are defined by Eq. (38) and we have set $\epsilon=0$. Letting $\epsilon\to 0$ at this point introduces an apparent singularity at $u=0$ of order $1/u$ in the integrand, but this is cancelled by the u^λ dependence of $F_\lambda(Eu)$ as $u\to 0$. Furthermore, it can be shown that the integrals across the singularity in $(a^2-b^2)^{-1/2}$, which appears as $a\to b$, are in fact finite.

Equation (42) gives the Coulomb correction to magnetic λ-pole transitions in the second Born approximation, in the limit $m/E\sim 0$ and neglecting any energy loss by the electron. By introducing a suitable approximation, it is possible to perform the x integration and this is illustrated below for magnetic dipole transitions.

V. COULOMB CORRECTION FACTOR FOR MAGNETIC DIPOLE TRANSITIONS

For magnetic dipole transitions we have, in the $\epsilon\to 0$ limit,

$$\beta_{M1}=\frac{8\delta^2}{\pi(4+\delta^2)F_1(\Delta)}$$

$$\times\int_0^\infty du\int_{-1}^1 dx\,\frac{F_1(Eu)F[E(u^2+\delta^2-2u\delta x)^{1/2}]}{u^2+\delta^2-2u\delta x+\lambda^2}$$

$$\times\left[\frac{\eta}{(a^2-b^2)^{1/2}}(\delta ux^2+4x)+\frac{\delta}{u}\right]. \tag{43}$$

In order to do the x integral it is necessary to give an analytic form for the elastic form factor, or replace it by some approximation which specifies the dependence on x. Cutler[7] introduced what he referred to as a "radius approximation"

$$F(q)\approx 1-\tfrac{1}{6}q^2\langle R^2\rangle \tag{44}$$

which he concluded would be appropriate for $\Delta R<2$, where the mean square radius $\langle R^2\rangle=\frac{3}{5}R^2$. The

approximation we adopt is different and perhaps not quite as severe. Since the integrand of Eq. (43) has a maximum value at $x=+1$, we expand the form factor about this point and to first order in x get

$$F(E|\vec{u}-\vec{\delta}|)\approx F_+(u)-xF_-(u),$$

where

$$F_+(u)=\tfrac{1}{2}\{F[E(u+\delta)]+F[E(u-\delta)]\},$$
$$F_-(u)=\tfrac{1}{2}[F[E(u+\delta)]-F[E(u-\delta)]\}. \quad (45)$$

This expression has the advantage that it gives the correct form factor at $x=\pm 1$, does not contain the low-momentum transfer restriction implicit in Eq. (44), and is valid to the extent that $F(q)_{x=1}-F(q)_{x=-1}$ is linear in q^2.

When the above expansion of the elastic form factor is inserted into Eq. (43), the x integration becomes straightforward. We remark only that $a^2-b^2>0$ for all $|x|\leq 1$ if $u>2$, while for $u<2$ we integrate from $x=-1\rightarrow x_1$ and $x_2\rightarrow +1$ where $(a^2-b^2)_{x_i}=0$, and note that

$$\eta=+1 \quad \text{if} \quad u/\delta>1,$$
$$\eta=+1 \quad \text{if} \quad u/\delta<1 \quad \text{and} \quad x<u/\delta,$$
$$\eta=-1 \quad \text{if} \quad u/\delta<1 \quad \text{and} \quad x>u/\delta.$$

Setting $\lambda=0$ one finally obtains

$$\beta_{M1}=\frac{8\delta^2}{\pi(4+\delta^2)F_1(\Delta)}\int_0^\infty du\,F_1(Eu)\Bigg\{\left[F_+(u)-F_-(u)\frac{u^2+\delta^2}{2u\delta}\right]\times\left[\frac{\delta(8+\delta^2+u^2)}{4u^2(u^2-\delta^2)}\ln\left|\frac{u+2}{u-2}\right|+\frac{\delta}{4u}-\frac{\delta}{16}\ln\left|\frac{u+2}{u-2}\right|+\frac{1}{u^2}\ln\left|\frac{u+\delta}{u-\delta}\right|\right]+\frac{F_-(u)}{32}\left[4-3\delta^2+\frac{1}{u}\left(\tfrac{3}{4}\delta^2u^2-u^2-\delta^2+12\right)\ln\left|\frac{u+2}{u-2}\right|\right]\Bigg\}, \quad (46)$$

where $F_\pm(u)$ and δ are defined by Eqs. (45) and (35), and Δ is the three momentum transfer.

The double integral expression for β_{M1} given by Eq. (43), and the approximate single integral relation Eq. (46) represent the main results of the present work. The Coulomb correction factor f_C is related to β_{M1} by Eq. (5).

VI. COMPARISON WITH DWBA CALCULATIONS

Although the approximation for β_{M1} given by Eq. (46) is not a complicated function, some care must be exercised when one evaluates it by numerical integration techniques since singularities exist in the integrand at $u=2$ and $u=\delta$. The integrals over these singularities are finite when they are defined as principal values, that is when one integrates to within ϵ of the singularities, where $\epsilon\rightarrow 0$.

In order to see how Eq. (46) compares with the DWBA analysis of $M1$ electroexcitation cross sections, we have computed f_C using elastic and inelastic form factors based on the charge and current distributions employed by Drechsel[3] and Chertok, Johnson, and Sarkar[4,5] in their partial-wave computations. These calculations use a Fermi distribution to describe the ground state charge density, that is

$$\rho(r)=\frac{\rho_0}{1+e^{(r-c)/z}}, \quad (47)$$

where

$$z=t/4\ln 3$$

and t is the skin thickness. A useful analytic expression for the elastic form factor for the Fermi charge distribution has been derived by Verdier,[12] which to a good approximation is

$$F(q)=\frac{3}{x^3}\left(\frac{y^2\sin x\cosh y}{\sinh^2 y}-\frac{xy\cos x}{\sinh y}\right)\left[1+\left(\frac{\pi}{s}\right)^2\right]^{-1}, \quad (48)$$

where $x=qc$, $y=\pi qz$, and $s=c/z$. This is sufficient for our purposes, for example for $c=3.547$ fm and $t=2.4$ fm we find agreement within $\pm 0.2\%$ with the numerically integrated elastic form factor out to the third diffraction maximum ($q\sim 2.5$ fm^{-1}), except very close to the diffraction minima where the comparison is less favorable.

Two models will be considered for the $M1$ form factor. One model used by Drechsel locates the nuclear $M1$ transition current at a radius R_A and has the form

$$\hat{\vec{J}}(\vec{r})\propto\delta(R_A-r)\vec{Y}^{\mu}_{111}(\hat{r}) \quad (49)$$

so

$$F_{M1}(q)\propto j_1(qR_A). \quad (49')$$

The other model, used by Chertok, Johnson, and Sarkar, is based on the incompressible liquid-drop model. The static charge density is given

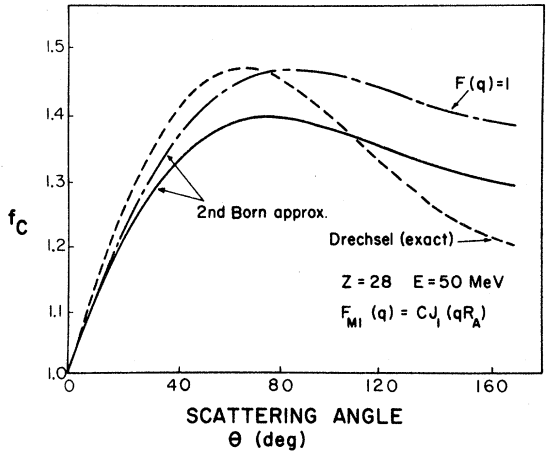

FIG. 1. The Coulomb correction factor $f_c = (d\sigma/d\Omega)/(d\sigma/d\Omega_{BA})$ for an $M1$ transition evaluated in the second Born approximation (solid curve) and in DWBA (dashed curve), for $Z = 28$, $E = 50$ MeV. The effect of replacing the ground state Fermi charge distribution by a point charge, in the second Born calculation, is shown by the curve $F(q) = 1$. Energy loss by the scattered electron has not been included in these calculations.

by Eq. (47), and the corresponding $M1$ transition current operator is

$$\hat{\vec{J}}(\vec{r}) = \frac{d\rho(r)}{dr}\vec{Y}^{\mu}_{111}(\vec{r}), \tag{50}$$

which describes a convection current distributed near the nuclear surface. The form factor which follows from Eq. (50) is

$$F_{M1}(q) \propto qF(q), \tag{50'}$$

where $F(q)$ is the elastic form factor, in this case Eq. (48).

In Fig. 1 we compare f_C, based on Eq. (46), with the DWBA calculations of Drechsel for $Z = 28$ and $E = 50$ MeV. The $M1$ form factor is given by Eq. (49') with $R_A = 4$ fm, and the parameters of the ground state charge density are $c = 4$ fm, $t = 2$ fm. This figure covers momentum transfers $q = 0$–0.5 fm^{-1}, and extends roughly up to the peak of first maxima of $F_{M1}(q)$. The greatest relative discrepancy between the two calculations occurs at the largest scattering angles, where it is almost 7%. It appears that the second Born approximation overestimates β_{M1} at these angles. The influence of the elastic form factor on f_C can be seen by setting $F(q) = 1$ [i.e., $F_+(q) = 1$, $F_-(q) = 0$] in Eq. (46), corresponding to the replacement of the extended nucleus by a point charge. As Fig. 1 shows, the comparison at the forward angles improves slightly, but for large θ the difference approaches 15%.

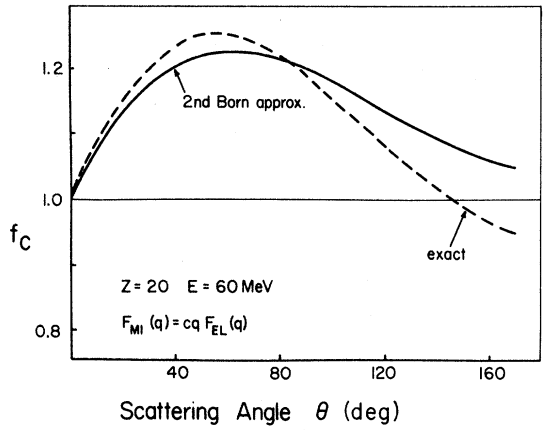

FIG. 2. The Coulomb correction factor f_c for an $M1$ transition evaluated in the second Born approximation (solid curve) and in DWBA (dashed curve), for $Z = 20$, $E = 60$ MeV. The transition is described by a liquid-drop (Tassie) model with a Fermi charge distribution for the ground state. Energy loss by the scattered electron has not been included in these calculations.

In Figs. 2 and 3 we compare f_C with the DWBA results evaluated by the program DUELS for $Z = 20$ at 60 and 140 MeV. The $M1$ form factor is given by Eq. (50') with $c = 3.547$ fm, $t = 2.4$ fm, and the same parameters are used in the elastic form factor. We see from Fig. 2 that again the second Born approximation overestimates f_C at the largest scattering angles, while underestimating it at the forward angles, for momentum transfers well below the diffraction minimum. Figure 3 shows f_C in the region of the $M1$ form factor minimum.

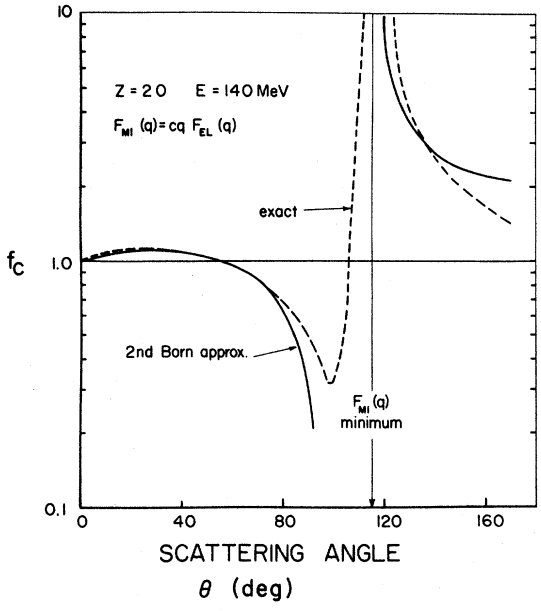

FIG. 3. Same as Fig. 2, except $E = 140$ MeV.

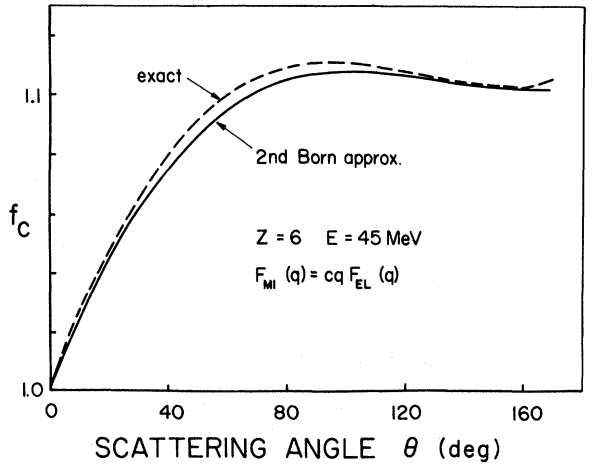

FIG. 4. Same as Fig. 2, except $Z=6$, $E=45$ MeV.

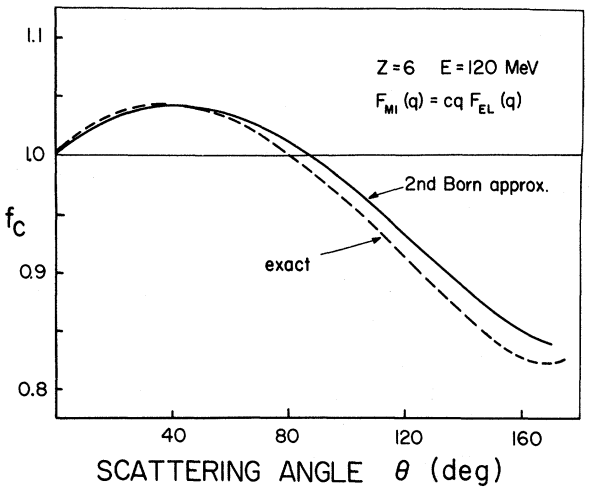

FIG. 5. Same as Fig. 2, except $Z=6$, $E=120$ MeV.

Both the DWBA and the second Born results give $f_C \to +\infty$ as one approaches the minimum from the large angle side, while from the small angle side f_C(second Born) $\to -\infty$. It is in the angular region just below the minimum that the second Born description becomes completely invalid.

Figures 4 and 5 show the correction factors for $Z=6$, $E=45$ and 120 MeV, again using the liquid-drop model but with $c=2.384$ fm and $t=2.4$ fm. The second Born approximation is seen to be in better agreement with the DWBA calculation than in the preceding examples, which is expected since the reliability of the approximation should improve with decreasing Z. Note that even for $Z=6$, the Coulomb correction for $M1$ transitions can be 10% or more.

VII. CONCLUSION

We have derived an expression for the Coulomb distortion factor f_C for magnetic multipole transitions through order αZ using electron wave functions evaluated in the second Born approximation in which energy loss was neglected. The general result is a double integral over the elastic and inelastic form factors computed in the first Born approximation, where the electron wave functions are treated as plane waves. No specific models are necessary in the formalism, and in practice the measured form factors can be used as an initial approximation for the first Born form factors.

The correction factor has been considered in detail for magnetic dipole transitions. An approximate expansion for the elastic form factor was introduced which reduced the double integral in f_C to a single integral. The resulting expression may be numerically evaluated by standard procedures if care is exercised near the singularities of the integrand. Comparison was made of the present approximation with the exact DWBA results for a few nuclei. It was shown that where f_C deviated appreciably from unity, most of the deviation is given by the second Born approximation. For most of the examples considered, the approximation overestimated f_C for far backward angle scattering, the error being the order of 3% to 11%. When one approaches the diffraction minimum of the inelastic cross section, the formalism breaks down, which is not unexpected since the Born minima are not filled in by the second Born approximation. On the other hand, for the lightest nucleus ($Z=6$) and lowest energy (45 MeV), the approximate and exact calculations agreed to within 0.5% at all angles. Therefore, the result of this paper could be most useful for correcting data on low Z nuclei in the momentum transfer range below the diffraction minima, in those situations where DWBA calculations may not be justified or necessary.

The nuclear physics in f_C have been isolated into terms which are simply the transition form factors evaluated in the first Born approximation. This separation of the electron and nuclear physics is not unique to the second Born approximation, and in fact is present in the general DWBA formalism as expressed by Eqs. (19) and (15). A partial wave expansion of the electron wave functions could, in principle, be used to evaluate the electron matrix elements $\vec{X}_{m'm}(\vec{q})$, leading to a DWBA description which does not require specific models for the transition current densities (or charge densities, for longitudinal transitions). To the best of our knowledge, this has not been utilized in any of the existing DWBA computer codes.

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