Core polarization in inelastic scattering and effective charges*

V. R. Brown*
Lawrence Livermore Laboratory, Livermore, California 94550

V. A. Madsen*†‡
Oregon State University, Corvallis, Oregon 97331

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A perturbation theory treatment is made of core-polarization effects in electromagnetic and inelastic scattering transitions due to high-lying collective excitations. Formulas are presented which make the connection between core-polarization parameters in inelastic scattering and effective charges from electromagnetic transitions. These relationships show that there is a natural disparity in neutron and proton polarizations, which arises from the departure (due to the neutron excess) of these high-lying collective excitations from pure isoscalar or isovector character. The resulting isospin-polarization matrix is calculated both from a schematic model and by making a connection with Bohr-Mottelson parametrization of isoscalar and isovector effective charges. The effects of spreading of the isoscalar giant resonance are taken into account in an approximate way which results in a formulation with one parameter free to be determined from empirical electromagnetic effective charges. Numerical results which show the effects of core polarization in electromagnetic transitions, (α,α'), (p,p'), and (n,n') are presented for ^{118}Sn and ^{207}Pb.

[NUCLEAR REACTIONS Inelastic scattering of α, p, n on ^{207}Pb, ^{118}Sn, core polarization effect calculated from electromagnetic effective charges as examples of method developed.]

I. INTRODUCTION

Isoscalar and isovector effective charges are convenient parameters for including core-polarization effects in electromagnetic transitions in nuclei. They have also been applied[1] directly to the isovector and isoscalar amplitudes in inelastic scattering processes. In this paper we examine the connection between core-polarization effects in electromagnetic transitions and in inelastic scattering with special attention paid to the effects of the neutron excess.

In Sec. II, perturbation theory is used to account for the effect of core polarization in a transition calculated with shell-model states. If the giant resonance states are assumed to be pure isoscalar and isovector excitations, it is shown that the resulting effective-charge parameters give equal enhancement for neutron and proton valence transitions in spite of the neutron excess. Lack of purity of these excitations results in a core polarization which is conveniently expressed in terms of an effective-charge matrix $\underline{\epsilon}$. In Sec. III it is shown that a simple generalization of Brown's[2] schematic model gives rise naturally to impure core excitations for nuclei with a neutron excess. In Sec. IV the parameters of the model are evaluated, connection is made with Bohr and Mottelson's[3] electromagnetic effective charges, a reformulation is made to take into account deficiencies in the model by allowing one free parameter, and numerical examples are presented.

II. DERIVATION OF CORE-POLARIZATION PARAMETERS

In this section, perturbation theory is used to include the effects of core excitations on transition rates. The Hilbert space of the nucleus is separated into a model space (the shell model) and the core-excitation space. The model space includes all nucleons but not particle-hole excitations; the core space as well as the model space is assumed already to include the effects of the nuclear two-body interaction, so that giant-resonance particle-hole excitations, which should be nearly independent of shell-model details, are already present in the unperturbed Hamiltonian. Perturbation theory is used to connect the shell-model space with the high-lying core excitations. There is no need at this point to assume that there are only a single isovector and isoscalar giant resonance; the strength of these levels may be spread over many states t. The initial wave function is written using perturbation theory as a shell-model part

plus an excited-core part,

$$\psi_i = \psi_i^{(0)} + \sum_t \frac{\langle f|\mathcal{A}_t V|i\rangle}{E_i - E_f - E_t} \mathcal{A}_t^\dagger \psi_f^{(0)} + \cdots , \quad (1)$$

where other shell-model states than $\psi_f^{(0)}$ are included in perturbation terms not explicitly shown because they do not contribute in first order. In the TDA approximation[4]

$$\mathcal{A}_t^\dagger = \sum_{mi} C_{mi}^n [a_{m1/2}^\dagger b_{i-1/2}^\dagger]_\lambda + \sum_{mi} C_{mi}^p [a_{m-1/2}^\dagger b_{i1/2}^\dagger]_\lambda \quad (2)$$

of Eq. (1) is an operator which creates a collective neutron and proton particle (m)-hole (i) core excitation of angular momentum λ on a model state.

The amplitude for a multipole transition operator is represented by the three diagrams of Fig. 1, parts (b) and (c) representing the core-polarization effects. If the particle-hole components in Eq. (2) are high in energy compared to the final model state, the reduced amplitude can be written approximately as

$$\langle \psi_f \| \mathcal{O}_\lambda \| \psi_i \rangle = \langle \psi_f^{(0)} \| \mathcal{O}_\lambda \| \psi_i^{(0)} \rangle - 2 \sum_t \frac{\langle 0|\mathcal{O}_\lambda \mathcal{A}_t^\dagger|0\rangle}{E_t} \langle \psi_f^{(0)} \| \mathcal{A}_t V \| \psi_i^{(0)} \rangle ; \quad (3)$$

the (b) and (c) terms of Fig. 1 giving nearly equal contributions. In Eq. (3) $|0\rangle$ represents the particle-hole vacuum. The collective particle-hole excitations are gross features of nuclei and are not expected to be affected much by the valence wave functions; we have therefore assumed that their matrix elements are equal in initial and final shell-model states $|\psi_i^{(0)}\rangle$ and $|\psi_f^{(0)}\rangle$. Further, these have each been set to the particle-hole vacuum matrix element of Eq. (3). Although it is not essential to the argument, the rest of the derivation proceeds most easily with the use of a separable two-body interaction, which we take

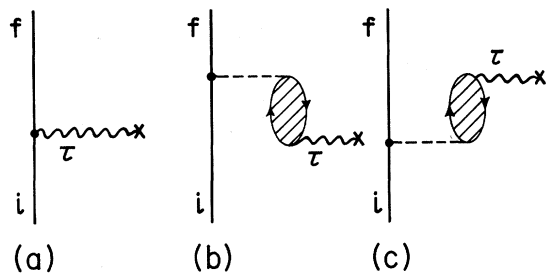

FIG. 1. Diagrams representing core-polarization effects. (a) is a pure shell-model transition due to some external field; (b) and (c) are the core-polarization contributions to the shell-model transition.

TABLE I.

External field	a_0	a_1
Electromagnetic	$\frac{1}{2}$	$-\frac{1}{2}$
(α, α')	V_0	0
(p,p')	V_0	$-V_1$
(n,n')	V_0	V_1

of the form

$$V(1,2) = \sum_\lambda Q_\lambda(1) \cdot Q_\lambda(2)[V_0 + V_1 \vec{\tau}(1) \cdot \vec{\tau}(2)] . \quad (4)$$

The multipole operator of Eq. (3) with explicit isospin dependence is given by

$$\mathcal{O}_{\lambda\mu} = Q_{\lambda\mu}(a_0 + a_1 \tau_z) . \quad (5)$$

Table I gives a_0 and a_1 for various transitions of interest. In inelastic scattering the projectile is thought of as producing an external field on the nucleus analogous to the electromagnetic field. For such cases Eq. (5) represents the nuclear part of a multipole expansion of the two-body interaction. The separable potential results in j-independent effective charges, which make the electromagnetic and inelastic scattering polarization proportional. This correspondence between inelastic scattering and electromagnetic transitions has been tested by Bernstein[5] for the (α, α') and by Schmittroth[6] for nucleon scattering. The nature of the distorted wave functions for the projectile emphasizes the nuclear surface as does r^2 in Q_2 for quadrupole transitions; the proportionality of electromagnetic and inelastic single-particle transitions is probably close enough that for collective states the use of approximations such as Eq. (5) is meaningful, although not necessarily very accurate. In principle particle-exchange effects should be included in the parameters, but, for spatially even nuclear interactions, exchange produces no alteration of Table I. If we were to assume that the important core excitations induced by the interaction with the shell-model space are pure isoscalar and isovector resonances of multipolarity λ, we would get for Eq. (3)

$$\langle \psi_f \| Q_\lambda(a_0 + a_1 \tau_z) \| \psi_i \rangle \approx \sum_\tau a_\tau \langle \psi_f^{(0)} \| Q_\lambda \mathcal{T}_\tau \| \psi_i^{(0)} \rangle \times \left[1 - \frac{2V_\tau}{E_\tau} |\langle 0|\mathcal{A}_\tau Q_\lambda \mathcal{T}_\tau|0\rangle|^2\right] ; \quad (6)$$

the quantity in the bracket is e_τ the effective charge for isospin transfer τ with $\mathcal{T}_0 = 1$ and $\mathcal{T}_1 = \tau_z$.

The difficulty with Eq. (6) is that the effective

charge is independent of whether the nucleons in the model space are neutrons or protons. For the case of α scattering, a pure isoscalar external field, the amplitude enhancement factor due to core polarization is e_0 whether the valence particle in the model space is a neutron or a proton. However, for a nucleus with a large neutron excess there should be a larger enhancement for valence protons than neutrons, because the neutron-proton force is stronger than neutron-neutron or proton-proton forces and because there are more neutrons than protons to polarize. Thus, Eq. (6) is inadequate to describe the effects of core polarization on transition rates.

To display the inadequacy of our assumptions, let us consider the properties of the collective particle-hole excitations Eq. (2). In nuclei with a large neutron excess the neutron and proton particle-hole terms have few if any common particle-hole indices mi. Thus on a microscopic scale the isospin purity of the particle-hole excitations breaks down completely. However, as pointed out by Bohr and Mottelson,[3] on a macroscopic scale the isospin of the giant resonance states is still expected to be fairly pure. The charge-independent interaction [Eq. (4) in our case], when diagonalized among particle-hole states, will give normal modes that are nearly purely isoscalar, in which neutrons and protons vibrate in phase, and isovector, in which they vibrate out of phase. Such an effect is seen in the calculations of Veje[7] on 3^- states, and further, it can be demonstrated explicitly in terms of the schematic model generalized slightly to account for the neutron excess (see Sec. III and Appendix A).

Since isospin breaks down for these excitations on a microscopic scale, the isospin character of the excitations is not conveniently discussed in terms of the wave functions but rather in terms of transition amplitudes. Accordingly we define the multipole matrix element

$$S_\tau^t = \langle t | Q_\lambda \mathcal{T}_\tau | 0 \rangle = \langle 0 | \mathcal{Q}_t Q_\lambda \mathcal{T}_\tau | 0 \rangle \ , \tag{7}$$

where $|t\rangle$ is the collective excitation $\mathcal{Q}_t^\dagger|0\rangle$. If the excitation is pure in isospin, say $t=0$ or $t=1$ for isoscalar and isovector states, respectively, then $S_\tau^t = \delta_{t\tau} S_\tau^\tau$, but generally, as shown in Sec. III and Appendix A, S_τ^t will be nonzero for $t \neq \tau$. In terms of these matrix elements S_τ^t the reduced transition matrix element generalized from Eq. (6) is

$$\langle \psi_f \| Q_\lambda (a_0 + a_1 \tau_z) \| \psi_i \rangle = \sum_\tau a_\tau \left\{ \langle \psi_f^{(0)} \| Q_\lambda \mathcal{T}_\tau \| \psi_i^{(0)} \rangle - 2 \sum_{\tau'} \langle \psi_f^{(0)} \| Q_\lambda \mathcal{T}_{\tau'} \| \psi_i^{(0)} \rangle V_{\tau',\tau\tau'} \right\} , \tag{8}$$

where

$$\sigma_{\tau'\tau} = \sum_t \frac{S_{\tau'}^t S_\tau^t}{E_t} \ . \tag{9}$$

If the excitations t are pure in isospin, we get

$$\sigma_{\tau'\tau} = \frac{|S_\tau^\tau|^2}{E_\tau} \delta_{\tau'\tau} \ , \tag{10}$$

and we have Eq. (6), but if they are not pure we have a different effect. As we see from Eq. (8) an external field transferring τ to the nucleus can transfer $\tau' \neq \tau$ to the model state due to the impurity in the collective excitation.

The form Eq. (9) can be written very conveniently in terms of an effective-charge matrix

$$\underline{\epsilon} = \underline{1} - 2\underline{V}\,\underline{\sigma} \equiv \underline{1} + \underline{\epsilon}^{\rm pol} \ , \tag{11}$$

where $\underline{\sigma}$ is defined by its elements in Eq. (9), $\underline{1}$ is a unit matrix, and $\underline{V}$ is the diagonal matrix

$$\underline{V} = \begin{pmatrix} V_0 & 0 \\ 0 & V_1 \end{pmatrix} . \tag{12}$$

It is also convenient to define a column matrix of the strength parameters from Table I

$$\underline{a} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \tag{13}$$

and an effective-strength column matrix

$$\underline{a}^{\rm eff} = \underline{\epsilon}\,\underline{a} \ . \tag{14}$$

TABLE II. Effective strength parameters calculated from Eq. (15) and Table I including the effects of core polarization in a nucleus with a neutron excess. The parameters $\epsilon_{\tau'\tau}$ are generalizations of the isoscalar ($\tau=0$) and isovector ($\tau=1$) effective charges; τ_z' gives ±1 operating on neutron and proton shell-model states, respectively. In the absence of any polarization effects $\epsilon_{\tau'\tau} = \delta_{\tau'\tau}$.

Transition	Effective strength parameter $a_0^{\rm eff}$	$a_1^{\rm eff}$
Electromagnetic	$\frac{1}{2}(\epsilon_{00} - \epsilon_{01}) \equiv \frac{e_0}{2}$	$-\frac{1}{2}(\epsilon_{11} - \epsilon_{10}) \equiv -\frac{1}{2}e_1$
(α,α')	$\epsilon_{00}V_0$	$\epsilon_{10}V_0$
(n,n')	$\epsilon_{00}V_0 + \epsilon_{01}V_1$	$\epsilon_{10}V_0 + \epsilon_{11}V_1$
(p,p')	$\epsilon_{00}V_0 - \epsilon_{01}V_1$	$\epsilon_{10}V_0 - \epsilon_{11}V_1$

TABLE III. Alternate representation of polarization parameters.

Type of core polarization	Shell-model target nucleon	Polarization parameter
Isoscalar	n	$\delta^{0n}=\epsilon_{00}+\epsilon_{10}$
Isoscalar	p	$\delta^{0p}=\epsilon_{00}-\epsilon_{10}$
n	n	$1+\delta^{nn}=\frac{1}{2}(\epsilon_{00}+\epsilon_{10}+\epsilon_{11}+\epsilon_{01})$
p	n	$\delta^{pn}=\frac{1}{2}(\epsilon_{00}+\epsilon_{10}-\epsilon_{11}-\epsilon_{01})$
n	p	$\delta^{np}=\frac{1}{2}(\epsilon_{00}-\epsilon_{10}-\epsilon_{11}+\epsilon_{01})$
p	p	$1+\delta^{pp}=\frac{1}{2}(\epsilon_{00}-\epsilon_{10}+\epsilon_{11}-\epsilon_{01})$

In terms of $\underline{a}^{\rm eff}$, Eq. (8) can be written

$$\langle \psi_f \| Q_\lambda(a_0+a_1\tau_z)\|\psi_i\rangle = \langle \psi_f^{(0)} \| Q_\lambda(a_0^{\rm eff}+a_1^{\rm eff}\tau_z')\|\psi_i^{(0)}\rangle$$

$$= \langle \psi_f^{(0)} \| Q_\lambda[(\epsilon_{00}a_0+\epsilon_{01}a_1)+(\epsilon_{10}a_0+\epsilon_{11}a_1)\tau_z']\|\psi_i^{(0)}\rangle\,, \tag{15}$$

where in the $\psi_f^{(0)}$, $\psi_i^{(0)}$ matrix elements τ_z' operates only on valence or shell-model wave functions. The matrix $\underline{\epsilon}$ is a generalization of isoscalar and isovector effective charges to nuclei with a neutron excess. In the absence of the neutron excess, $\underline{\epsilon}$ is diagonal.

The expressions for the effective strength parameters appearing in Eq. (15), listed in Table II, illustrate an interesting feature of the polarization parameters in the presence of a neutron excess. In Eq. (15) it is the second index τ on $\epsilon_{\tau'\tau}$ which determines whether a particular parameter contributes to an isoscalar or an isovector transition of the nucleus, since a_0 and a_1 are the strength parameters connected with the external field of isospin τ. On the other hand, it is τ' which determines which isospin is transferred to the shell-model space since it refers to shell-model transitions. For example in α scattering, which is purely isoscalar, Table II contains a term which transfers isospin 1 to the model space. Similarly, we see that in electromagnetic transitions the isoscalar effective charge e_0 contains a term ϵ_{01} which is truly isovector for the nucleus as a whole, and vice versa for e_1. This behavior has come about because of the departure from purity of the collective particle-hole states.

An alternate representation of the polarization parameters is given[8,9] in Table III. The parameters δ^{pp} and δ^{pn} are the core-proton polarization parameters for shell-model protons and neutrons, respectively (they are equal to the proton and neutron electromagnetic polarization charges), δ^{nn} and δ^{np} are the corresponding parameters for core-neutron polarization, and δ^{0n} and δ^{0p} are isoscalar polarization parameters. In Table IV the effective strengths $a^{\rm eff}$ are given in terms of these parameters δ.

In order to see what qualitative effects are expected from Eq. (15) we consider the signs of V_τ and $\sigma_{\tau'\tau}$. The nuclear central-exchange force has $V_0<0$, $V_1>0$. From Eq. (9) it is clear that the parameter $\sigma_{\tau'\tau}$ is positive for $\tau'=\tau$, but $\sigma_{01}=\sigma_{10}$ is also expected to be positive. To see this we

TABLE IV. Effective strengths.

Type of transition	Target nucleon	Effective strength parameter $a_0^{\rm eff}+a_1^{\rm eff}\tau_z'$
(α,α')	n	$V_0\delta^{0n}$
	p	$V_0\delta^{0p}$
(p,p')	n	$V_{pn}(1+\delta^{nn})+V_{pp}\delta^{pn}$
	p	$V_{pp}(1+\delta^{pp})+V_{pn}\delta^{np}$
(n,n')	n	$V_{nn}(1+\delta^{nn})+V_{np}\delta^{pn}$
	p	$V_{np}(1+\delta^{pp})+V_{nn}\delta^{np}$
Electromagnetic	n	δ^{pn}
	p	$1+\delta^{pp}$

can calculate from Eqs. (2) and (7) that

$$S_\tau^t = \sum_{mi} C_{mi}^{n\,*} D_{mi}^n + (-1)^\tau \sum_{mi} C_{mi}^{p\,*} D_{mi}^p \ , \qquad (16)$$

where $D_{mi} = \langle mi|Q_\lambda|0\rangle$ is the amplitude of a particle-hole pair mi. D_{mi} and C_{mi} will tend to have a simple phase relation; in fact in the schematic model

$$C_{mi}^n = c_n^t D_{mi}^n \ ,$$
$$C_{mi}^p = c_p^t D_{mi}^n \ . \qquad (17)$$

For the isoscalar mode c_n^0 and c_p^0 are both positive. Since, because of the neutron excess, there are more neutron than proton terms, S_1^0 as well as S_0^0 will be positive. For the isovector mode $c_n^1 > 0$, $c_p^1 < 0$; clearly S_1^1 will be positive but again because there are more neutron than proton terms S_0^1 will also be positive. The schematic model (see Sec. III) results in $c_n^1 < c_p^1$ to compensate for the neutron excess and make the $t=1$ state nearly purely $\tau=1$, but the compensation falls short leaving the neutron term of Eq. (16) larger. Thus S_τ^t are all positive and so, therefore, are all $\sigma_{\tau'\tau}$. Thus we have from Eq. (11)

$$\epsilon_{00} > 1, \quad \epsilon_{11} < 1, \quad \epsilon_{10} < 0, \quad \epsilon_{01} > 0 \ . \qquad (18)$$

The first two of these inequalities are well known features of the effective charges, the isoscalar results in enhancement and the isovector, retardation. The other two give rise to additional differences between a shell-model neutron and proton.

The results of Table II and the inequalities (18) resolve the problem with α scattering mentioned above. The polarization parameter for target protons, $\epsilon_{00} - \epsilon_{10}$, is greater than that for neutrons, $\epsilon_{00} + \epsilon_{10}$. A similar effect occurs with the inelastic scattering parameters of Tables III and IV, $\delta^{np} = \delta^{pn} - \epsilon_{10} + \epsilon_{01} > \delta^{pn}$ and $\delta^{nn} = \delta^{pp} + \epsilon_{10} + \epsilon_{01} > \delta^{pp}$ [the latter relationship follows from Eq. (11) and the fact that $-V_0 > V_1$]. In each case the polarization parameter for core neutrons is greater than that for protons, due to the neutron excess. This feature in inelastic scattering has been anticipated in Ref. 9.

Bohr and Mottelson[3] have suggested that the form factors for the interaction of a model-space nucleon with the isoscalar and isovector modes may be rather different since the former mode is expected to be a surface vibration and the latter, a volume vibration. Use of the separable interaction forces the form factors to be equal. To compensate for this deficiency an extra parameter γ^t is now introduced into the matrix element of the two-body operator. This parameter removes the restriction that the coupling to the isovector and isoscalar states be proportional to the separable two-body force strength parameters V_1 and V_0. The only change in our formulation to this point is that the polarization parameters $\sigma_{\tau'\tau}$ now contain γ^t;

$$\sigma_{\tau'\tau} = \sum_t \frac{\gamma^t S_\tau^t S_{\tau'}^t}{E_t} \ . \qquad (19)$$

We will see in Sec. III how we can get information about γ^t from the energy shifts of the isoscalar and isovector modes.

III. SCHEMATIC MODEL

The isoscalar and isovector giant resonances are core excitations carrying much of the particle-hole strength. In order to get some information about expected systematics of the polarization parameters $\epsilon_{\tau'\tau}$, we present in this section results of Brown's schematic model of giant resonances[2] generalized slightly to include both V_0 and V_1 interactions, Eq. (4). Details of the calculation are in Appendix A.

The external-field operator has been given in Eq. (5). We write its particle-hole matrix element as

$$\langle mi|Q_{\lambda\mu}(a_0 + a_1\tau_z)|0\rangle = (a_0 \pm a_1) D_{mi} \begin{cases} \text{neutrons} \\ \text{protons} \end{cases} \ . \qquad (20)$$

The schematic model assumption is generalized for the isospin dependent two-body interaction to

$$\langle mi|V|nj\rangle = D_{mi} D_{nj} \times \begin{cases} \alpha & mi, nj \text{ like nucleons} \\ \beta & mi, nj \text{ unlike nucleons} \end{cases} , \qquad (21)$$

where

$$\alpha = V_0 + V_1 = V_{nn} = V_{pp} < 0 \ ,$$
$$\beta = V_0 - V_1 = V_{np} = V_{pn} < 0 \ . \qquad (22)$$

In the degenerate particle-hole energy limit the diagonalization of the Hamiltonian with the interaction Eq. (22) in the particle-hole states gives two states instead of one which split off, the rest remaining degenerate. Of the two states which split off, the energy shifts are given by[10]

$$\Delta E = \frac{y_n + y_p}{2} \pm \tfrac{1}{2}[(y_n - y_p)^2\alpha^2 + 4\beta^2 y_n y_p]^{1/2}$$
$$\approx \frac{y_n + y_p}{2}[\pm|\beta| + \alpha] = (y_n + y_p) \begin{cases} V_1 \\ V_0 \end{cases} , \qquad (23)$$

where

$$y_n = \sum_{mi} |D_{mi}^n|^2 \ ,$$
$$y_p = \sum_{mi} |D_{mi}^p|^2 \ . \qquad (24)$$

Since $|\beta| > |\alpha|$, the energies of one of the states in increased by the interaction and the other is lowered. The upper and lower states are tentatively taken as isovector and isoscalar giant resonances and labeled $t=1$ and $t=0$, respectively.

As in Sec. II a correction can now be made for the use of separable potentials which, along with the dropping of exchange terms, was used to obtain the schematic-model interaction matrix elements Eq. (21). It is assumed that the collective eigenstates are not strongly affected by the use of the separable interaction. The eigenfunctions give for quadrupole states approximately

$$\Delta E_t = \langle t|H|t\rangle - 2\hbar\omega = \langle t|V|t\rangle \ , \tag{25}$$

since in the degenerate model the collective eigenstates of schematic Hamiltonian are also eigenstates of the one-body part of the Hamiltonian with eigenvalue $2\hbar\omega$. The variational principle assures us that the energy shift Eq. (25) is in error only to second order in the error in the eigenstate made by using the separable interaction. If the separable interaction for V is used in Eq. (25), one again gets the result Eq. (23). The finite-range interaction, as in Sec. II, introduces the extra factor γ^t. Thus Eq. (23) becomes

$$\Delta E_t = \gamma^t V_t (y_n + y_p) \ . \tag{26}$$

Of primary interest are the transition amplitudes Eq. (7). In the schematic model, Eq. (16) gives (see Appendix A) in terms of $y_\pm \equiv y_n \pm y_p$

$$\begin{aligned}
S_0^0 &= c_n^0\left[y_+ + \frac{y_-}{2}\left(1-\frac{\alpha}{\beta}\right)\right] \approx \sqrt{y_+} \ , \\
S_1^0 &= c_n^0 \frac{y_-}{2}\left(1+\frac{\alpha}{\beta}\right) \approx \tfrac{1}{2}\sqrt{y_+}\,\frac{y_-}{y_+}\left(1+\frac{\alpha}{\beta}\right) , \\
S_0^1 &= c_n^1 \frac{y_-}{2}\left(1-\frac{\alpha}{\beta}\right) \approx \tfrac{1}{2}\sqrt{y_+}\,\frac{y_-}{y_+}\left(1-\frac{\alpha}{\beta}\right) , \\
S_1^1 &= c_n^1\left[y_+ + \frac{y_-}{2}\left(1+\frac{\alpha}{\beta}\right)\right] \approx \sqrt{y_+} \ ,
\end{aligned} \tag{27}$$

where

$$c_n^t = (y^+)^{-1/2}\left[1 - \frac{y_-}{2y_+} + (-1)^t \frac{\alpha}{2\beta}\frac{y_-}{y_+}\right] \tag{28}$$

is a neutron amplitude parameter defined in Eqs. (17) and (A13). We see from Eq. (27) that the lower state $t=0$ is nearly purely isoscalar, and the upper state is nearly purely isovector; however, each state is slightly impure, the correction being proportional to $y_n - y_p$, which would vanish in the limit $N=Z$. Since S_0^t and S_1^t both have the same sign, the polarization parameters $\sigma_{\tau',\tau}$ Eqs. (9) and (19) are all positive in agreement with the argument given in Sec. II.

It is worthwhile noting that the cross section for excitation of t by $\tau \neq t$ is not very large. For example, isoscalar excitation of the $t=1$ state compared to the $t=0$ is $|S_0^1/S_0^0|^2$, which is of order $|(N-Z)/A|^2$, whereas terms which are being retained[10] in the calculation of polarization are of order $(N-Z)/A$.

IV. EVALUATION OF POLARIZATION PARAMETERS

A. Calculation from the schematic model

The energies of the collective states we take as $60/A^{1/3}$ and $120/A^{1/3}$ for isoscalar[11] and isovector,[12] respectively. Using the standard value of $41/A^{1/3}$ for the shell-model spacing gives us values for the energy shifts

$$\Delta E_0 = -22A^{-1/3} = \gamma^0 V_0 y_+ \ , \tag{29a}$$

$$\Delta E_1 = 38A^{-1/3} = \gamma^1 V_1 y_+ \ . \tag{29b}$$

Using Eqs. (29a) and (29b) and the sum rule Eq. (B9) we have

$$\begin{aligned}
\gamma^t V_t &= \begin{Bmatrix} -22 \\ 38 \end{Bmatrix} \times [\,0.137A^{5/3}(Z^{2/3}+N^{2/3})\,]^{-1} \\
&\approx \begin{cases} 128/A^{7/3}, & t=0 \ , \\ 221/A^{7/3}, & t=1 \ . \end{cases}
\end{aligned} \tag{30}$$

The latter result uses $\langle r^2\rangle \approx \frac{3}{5}(1.2A^{1/3})^2$ as in a uniform matter distribution. The polarization parameters from Eqs. (11), (27), and (28) are now evaluated as

$$\epsilon_{\tau\tau}^{\rm pol} = -2V_\tau \sigma_{\tau\tau} \approx -2V_\tau \gamma^\tau (E_\tau)^{-1}(S_\tau^\tau)^2 \ , \tag{31a}$$

$$\begin{aligned}
\epsilon_{10} &= -2V_1[\,\gamma^0 S_0^0 S_1^0/E_0 + \gamma^1 S_0^1 S_1^1/E_1\,] \\
&\approx V_1 S_1^0 \epsilon_0^{\rm pol}/(V_0 S_0^0) + S_0^1 \epsilon_1^{\rm pol}/S_1^1 \\
&= \frac{V_1}{V_0 - V_1}\frac{y_-}{y_+}[\,\epsilon_{00} - \epsilon_{11}\,] \ ,
\end{aligned} \tag{31b}$$

$$\epsilon_{01} = \frac{V_0}{V_1}\epsilon_{10} = \frac{V_0}{V_0 - V_1}\frac{y_-}{y_+}[\,\epsilon_{00} - \epsilon_{11}\,] \ . \tag{31c}$$

$\epsilon_{\tau\tau}^{\rm pol}$ can be evaluated from Eq. (31a) using Eqs. (29) and (27)

$$\begin{aligned}
\epsilon_{\tau\tau}^{\rm pol} &= -2\frac{(\Delta E_\tau)}{y_+}\,\frac{1}{E_\tau}\,y_+ \\
&= -2\frac{(E_\tau - 2\hbar\omega)}{E_\tau} \ .
\end{aligned} \tag{32}$$

Since E_τ and $\hbar\omega$ both are proportional to $A^{-1/3}$, the diagonal elements $\epsilon_{\tau\tau}$ are simply constants in this no-parameter model.

From Eq. (B9) we can write

$$y_-/y_+ \approx \tfrac{2}{3}(N-Z)/A \ . \tag{33}$$

Equations (31b) and (31c) then give

$$\epsilon_{10} = \frac{2}{3}\frac{V_1}{V_0 - V_1}\frac{N-Z}{A}(\epsilon_{00} - \epsilon_{11}) , \tag{34}$$

$$\epsilon_{01} = \frac{2}{3}\frac{V_0}{V_0 - V_1}\frac{N-Z}{A}(\epsilon_{00} - \epsilon_{11}) . \tag{35}$$

The off-diagonal elements depend on the neutron excess as they must, since it is the neutron excess which produces the lack of isospin purity of the giant-resonance states.

B. Correspondence with Bohr-Mottelson's polarization charges

On the basis of a collective model treatment Bohr and Mottelson have calculated[3] electromagnetic effective charges including the neutron-excess effect we have discussed above. By making appropriate identifications we can parametrize our polarization parameters in terms of theirs, which are given by

$$e^{\rm pol} = \frac{Z}{A}\chi_0(1 + \beta\tau_z') - \tfrac{1}{2}\chi_1(\tau_z' - \xi) , \tag{36}$$

where

$$\xi = (N-Z)/A , \tag{37a}$$

$$\beta = \hat{V}_1\xi/4\hat{V}_0 = V_1\xi/V_0 . \tag{37b}$$

In Eq. (37b) $\hat{V}_0$ and $\hat{V}_1$ are the isoscalar and the Lane-isovector optical-potential strengths. The first term in Eq. (36) is the polarization term due to the isoscalar giant resonance, and the second is due to the isovector giant resonance. As discussed in Sec. III each of these excitations is impure and contributes a minor term (the β and ξ terms) opposite in isospin to its major term.

To compare Eq. (36) with our result Eq. (14) we rewrite the effective strength parameter from the latter result:

$$a^{\rm eff} = [(1 - 2V_0\sigma_{00})a_0 - (2V_0\sigma_{01})a_1] + [(1 - 2V_1\sigma_{11})a_1 - (2V_1\sigma_{10})a_0]\tau_z' . \tag{38}$$

The polarization part is therefore

$$a^{\rm pol} = -2\{[\sigma_{00}a_0 + \sigma_{01}a_1]V_0 + [\sigma_{11}a_1 + \sigma_{10}a_0]V_1\tau_z'\} . \tag{39}$$

Putting in the definition Eq. (9) of $\sigma_{\tau'\tau}$ we rewrite Eq. (39) as

$$a^{\rm pol} = -2\sum_t (a_0 S_0^t + a_1 S_1^t)\frac{1}{E_t} \times (S_0^t V_0 + S_1^t V_1\tau_z') . \tag{40}$$

In this form the first factor is the transition amplitude to the giant state t and the second factor is the polarization amplitude to the state t. Equation (40) still applies to any external field; to compare with Bohr and Mottelson's result we use from Table I electromagnetic strength parameters $a_0 = \frac{1}{2}$, $a_1 = -\frac{1}{2}$ and define

$$S_{\rm em}^t = \tfrac{1}{2}(S_0^t - S_1^t) . \tag{41}$$

Thus (40) can be written

$$a^{\rm pol} \to e^{\rm pol} = -2\sum_t S_{\rm em}^t \frac{1}{E_t}(S_0^t V_0 + S_1^t V_1\tau_z') . \tag{42}$$

Comparing with Eq. (36) we obtain the identification

$$S_{\rm em}^0 S_0^0/E_0 = -\frac{1}{2V_0}\frac{Z}{A}\chi_0 , \tag{43a}$$

$$S_{\rm em}^0 S_1^0/E_0 = -\frac{1}{2V_1}\frac{Z}{A}\chi_0\beta , \tag{43b}$$

$$S_{\rm em}^1 S_1^1/E_1 = \frac{1}{2V_1}\tfrac{1}{2}\chi_1 , \tag{43c}$$

$$S_{\rm em}^1 S_0^1/E_1 = -\frac{1}{2V_0}\tfrac{1}{2}\chi_1\xi . \tag{43d}$$

From Eqs. (43) it follows that

$$S_1^0/S_0^0 = (V_0/V_1)\beta , \tag{44a}$$

$$S_0^1/S_1^1 = -(V_1/V_0)\xi , \tag{44b}$$

and, therefore, that

$$S_{\rm em}^0 \equiv \tfrac{1}{2}(S_0^0 - S_1^0) = \tfrac{1}{2}S_0^0\left(1 - \frac{V_0}{V_1}\beta\right) , \tag{45a}$$

$$S_{\rm em}^1 \equiv \tfrac{1}{2}(S_0^1 - S_1^1) = -\tfrac{1}{2}S_1^1\left(1 + \frac{V_1}{V_0}\xi\right) . \tag{45b}$$

From Eqs. (10, (43), and (45) it is easy to obtain the result

$$\sigma_{00} = -\left(1 - \frac{V_0}{V_1}\beta\right)^{-1}\left(\frac{1}{V_0}\frac{Z}{A}\chi_0\right) - \frac{V_1}{V_0}\xi\left(1 + \frac{V_1}{V_0}\xi\right)^{-1}\left(\frac{\chi_1\xi}{2V_0}\right) \approx -(1-\xi)^{-1}\frac{1}{V_0}\frac{Z}{A}\chi_0 , \tag{46a}$$

$$\sigma_{11} = -\frac{V_0}{V_1}\beta\left(1 - \frac{V_0}{V_1}\beta\right)^{-1}\left(\frac{1}{V_1}\frac{Z}{A}\chi_0\beta\right) - \left(1 + \frac{V_1}{V_0}\xi\right)^{-1}\left(\frac{\chi_1}{2V_1}\right) \approx -\left(1 + \frac{V_1}{V_0}\xi\right)^{-1}(\chi_1/2V_1) , \tag{46b}$$

$$\sigma_{10}=\sigma_{01}=-\left(1-\frac{V_0}{V_1}\beta\right)^{-1}\left(\frac{1}{V_1}\frac{Z}{A}\chi_0\beta\right)$$
$$+\left(1+\frac{V_1}{V_0}\xi\right)^{-1}\left(\frac{\chi_1\xi}{2V_0}\right)$$
$$\approx-(Z\chi_0/A-\chi_1/2)(\xi/V_0)$$
$$\approx\xi\left(\sigma_{00}-\frac{V_1}{V_0}\sigma_{11}\right). \tag{46c}$$

From Eqs. (11) and Eq. (46) it follows that

$$\epsilon_{00}^{\rm pol}=2(Z/A)\chi_0(1-\xi)^{-1}=\chi_0\,, \tag{47a}$$
$$\epsilon_{11}^{\rm pol}=\chi_1\left(1+\frac{V_1}{V_0}\xi\right)^{-1}, \tag{47b}$$
$$\epsilon_{10}=\frac{N-Z}{A}\frac{V_1}{V_0}(\epsilon_{00}-\epsilon_{11})\,, \tag{47c}$$
$$\epsilon_{01}=\frac{N-Z}{A}(\epsilon_{00}-\epsilon_{11})\,. \tag{47d}$$

These results are very similar to Eqs. (32), (34), and (35). $\epsilon_{00}^{\rm pol}$ is a constant, but $\epsilon_{11}^{\rm pol}$ varies slowly with neutron excess. The off-diagonal elements are somewhat larger than calculated from the schematic model.

C. One-parameter formulation

Having made the connection with Bohr-Mottelson's electromagnetic polarization charges we now have, in principle, in Table II and Eqs. (14) and (47) a no-parameter formulation of polarization for inelastic scattering. However, in actual fact the giant resonances are fractionated, and, because of the energy denominators in the expressions for effective charges, any part of the strength split off to lower energies will have an increased effect on the polarizations. This spreading will be more serious for the isoscalar than isovector excitation because its center lies at a much lower energy. As a result, isoscalar polarizations are commonly much greater than given by the Bohr-Mottelson effective charges. We therefore want to account for the spreading effect for the isoscalar state by leaving one free parameter.

In the perturbation theory result, presented in Sec. II, the four polarization parameters $\epsilon_{\tau'\tau}$ can be represented in terms of the three parameters $\sigma_{\tau'\tau}$. Since σ_{11} contains only a small effect from the lower giant state, we take it to be given correctly by Eq. (32). The other parameters σ_{01} and σ_{00} will be strongly affected by the fractionation, so we must include its effect. From Eq. (19)

$$\sigma_{01}=\sum_t\frac{\gamma^tS_1^tS_0^t}{E_t}$$
$$=\sum_{\text{isoscalar }t}\frac{\gamma^tS_1^tS_0^t}{E_t}+\sum_{\text{isovector }t}\frac{\gamma^tS_1^tS_0^t}{E_t}$$
$$\approx\left\langle\frac{S_1^0}{S_0^0}\right\rangle\sum_{\text{isoscalar }t}\frac{\gamma^t(S_0^t)^2}{E_t}+\left\langle\frac{S_0^1}{S_1^1}\right\rangle\sum_{\text{isovector }t}\frac{\gamma^t(S_1^t)^2}{E_t}$$
$$\approx\left\langle\frac{S_1^0}{S_0^0}\right\rangle\sigma_{00}+\left\langle\frac{S_0^1}{S_1^1}\right\rangle\sigma_{11}\,, \tag{48}$$

where $\langle S_1^0/S_0^0\rangle$ and $\langle S_0^1/S_1^1\rangle$ are average values of these isoscalar- and isovector-state ratios. If these are taken from the schematic model, Eqs. (31b) and (31c) result. As before $\epsilon_{11}^{\rm pol}$ is taken from Eq. (32), but now ϵ_{00} is left as a free parameter to be determined empirically. If the Bohr-Mottelson formulation is used Eqs. (47b)–(47d) are obtained, ϵ_{00} again being left as a free parameter.

The determination of the free parameter ϵ_{00} in both the schematic and the Bohr-Mottelson treatments can be made by using empirical electromagnetic effective charges. From Table II neutron and proton effective charges are given in terms of $\underline{\epsilon}$ by

$$\left\{\begin{matrix}e_n\\e_p\end{matrix}\right\}=\tfrac{1}{2}(\epsilon_{00}-\epsilon_{01})\mp\tfrac{1}{2}(\epsilon_{11}-\epsilon_{10})\,. \tag{49}$$

Using Eq. (31) of the schematic model and Eq. (49) depending on whether the valence particle is a neutron or a proton, we obtain

$$\epsilon_{00}=[2e_n+\epsilon_{11}(1-y_-/y_+)](1-y_-/y_+)^{-1} \tag{50a}$$

and

$$\epsilon_{00}=\{2e_p-\epsilon_{11}[1+(y_-/y_+)(V_{pp}/V_{np})]\}\left(1-\frac{y_-}{y_+}\frac{V_{pp}}{V_{np}}\right)^{-1}. \tag{50b}$$

Again Eq. (33) can be used for y_-/y_+ and Eq. (32) gives ϵ_{11}. Correspondingly in the Bohr-Mottelson parametrization, using Eqs. (47c) and (47d), we obtain

$$\epsilon_{00}=[2e_n+(2Z/A+V_1\xi/V_0)\epsilon_{11}](2Z/A+V_1\xi/V_0)^{-1} \tag{51a}$$

and

$$\epsilon_{00}=[2e_p-(2N/A+V_1\xi/V_0)\epsilon_{11}](2Z/A-V_1\xi/V_0)^{-1}. \tag{51b}$$

D. Numerical examples

In this section numerical results are presented for the $(p_{3/2}\nu)^{-1}\to(p_{1/2}\nu)^{-1}$ transition in ^{207}Pb and

TABLE V. Effective polarization charges calculated for the $(p_{3/2}\nu)^{-1} \rightarrow (p_{1/2}\nu)^{-1}$ transition in ^{207}Pb and for the excitation of the first 2^+ state in ^{118}Sn. A comparison is made among the no-parameter schematic model (NPSM), the one-parameter schematic model (OPSM), the no-parameter Bohr-Mottelson model (NPBM), and the one-parameter Bohr-Mottelson model (OPBM). Empirical electromagnetic effective charges of $e_n = 0.85$ (Ref. 14) in ^{207}Pb and $e_n = 0.745$ (see text and Ref. 15) in ^{118}Sn are used in the one-parameter models.

Nucleus	Case	ϵ_{00}	ϵ_{01}	ϵ_{11}	ϵ_{10}	e_0	e_1
^{207}Pb $e_n = 0.85$ [a]	NPSM	1.74	0.128	0.366	−0.064	1.61	0.429
	OPSM	2.34	0.184	0.366	−0.092	2.16	0.458
	NPBM	2.02	0.361	0.232	−0.235	1.66	0.467
	OPBM	2.82	0.537	0.232	−0.349	2.28	0.581
^{118}Sn $e_n = 0.745$ [b]	NPSM	1.74	0.093	0.366	−0.047	1.64	0.413
	OPSM	2.02	0.113	0.366	−0.057	1.91	0.423
	NPBM	2.01	0.261	0.274	−0.170	1.75	0.444
	OPBM	2.27	0.304	0.274	−0.197	1.96	0.472

[a] See Ref. 14.
[b] See text and Ref. 15.

for the $0^+ \rightarrow 2^+$ first excited state transition in ^{118}Sn. For the no-parameter schematic model (NPSM) Eq. (30) is used to obtain $V_\tau \gamma^\tau$, the giant-quadrupole isoscalar[11] and isovector resonance energies are taken as $60/A^{1/3}$ and $120/A^{1/3}$, $S_\tau{}^t$ are taken from Eq. (27), and y_n and y_p used in Eq. (30) are taken from the sum-rule results of Eq. (B9). The Kallio-Koltveit equivalent one-Fermi-Yukawa effective-interaction[13] ratio $V_1/V_0 = -0.5$ is used. The $\epsilon_{\tau'\tau}$ are all calculated from Eqs. (31).

In the one-parameter schematic model (OPSM) the empirical effective charges are taken as $e_n = 0.85$[14] for ^{207}Pb and $e_n = 0.745$ for ^{118}Sn. The latter number is calculated by comparison of the cloud-nucleon results of Yoshida[15] with the empirical[16] $B(E2)$. For ^{207}Pb the $(p_{3/2}\nu)^{-1} \rightarrow (p_{1/2}\nu)^{-1}$ amplitude is expected to be an adequate valence wave function and the random-phase-approximation calculations for ^{118}Sn are expected to account adequately for the 51–82 neutron-shell valence wave functions. The parameters ϵ_{11}, ϵ_{10}, and ϵ_{01} are obtained from Eqs. (31) as in the no-parameter calculation, but ϵ_{00} is now obtained from Eq. (50a) using the empirical value of e_n.

In the no-parameter Bohr-Mottelson (NPBM) model we use Eqs. (11) and (46). The parameters of the model used in Ref. 3 are $\chi_0 = 1.0$, $\chi_1 = -0.64$, and $\hat{V}_1/\hat{V}_0 = -2.6$ (corresponding to a ratio $V_1/V_0 = -0.65$ for the two-body interaction). In the one-parameter formulation (OPBM), ϵ_{11} is still calculated from Eqs. (47), but ϵ_{00} is taken from Eq. (51a) in terms of the empirical neutron effective charge. ϵ_{01} and ϵ_{10} are then determined from ϵ_{00} and ϵ_{11} using Eqs. (47c) and (47d).

The results of these sample calculations for polarization parameters $\epsilon_{\tau'\tau}$ are given in Table V. The ϵ_{00}, ϵ_{01}, and ϵ_{10} elements are substantially different in the no-parameter and one-parameter models, showing the inadequacy of the two isolated giant resonances to account for polarization. The results of the schematic model are rather similar to the Bohr-Mottelson model for the diagonal elements of $\underline{\epsilon}$, but the off-diagonal elements are considerably larger in the latter. In spite of this dif-

TABLE VI. The effective polarization parameters of Table III calculated for the $(p_{3/2}\nu)^{-1} \rightarrow (p_{1/2}\nu)^{-1}$ transition in ^{207}Pb and the excitation of the first 2^+ state in ^{118}Sn. A comparison is made between the one-parameter schematic model (OPSM) and the one-parameter Bohr-Mottelson model (OPBM). Empirical electromagnetic charges of $e_n = 0.85$ (Ref. 14) in ^{207}Pb and $e_n = 0.745$ (see text and Ref. 15) in ^{118}Sn are used.

Nucleus	Case	$1+\delta^{nn}$	$1+\delta^{pp}$	δ^{np}	δ^{pn}	δ^{0n}	δ^{0p}
^{207}Pb $e_n = 0.85$ [a]	OPSM	1.40	1.31	1.12	0.849	2.25	2.43
	OPBM	1.62	1.43	1.74	0.851	2.47	3.17
^{118}Sn $e_n = 0.745$ [b]	OPSM	1.22	1.17	0.912	0.742	1.96	2.08
	OPBM	1.33	1.22	1.25	0.748	2.07	2.47

[a] See Ref. 14.
[b] See text and Ref. 15.

ference the electromagnetic effective charges e_0 and e_1 are very nearly the same in both models. This does not mean that the predictions of the models will be the same for other kinds of transitions, since different linear combinations of the $\epsilon_{\tau'\tau}$ are involved.

Table VI gives the polarization parameters of Tables III and IV for the same two nuclei. The effects of the neutron excess are evident from comparisons of δ^{nn} with δ^{pp}, δ^{np} with δ^{pn}, and δ^{0n} with δ^{0p}. Each of these pairs would be equal in the absence of a neutron excess.

It is interesting to compare the results of Table II or Table IV and Table VI for strength parameters with the procedure of Ref. 1, multiplying V_0 by e_0 and V_1 by e_1 to account for the collective enhancement. In the latter procedure (n,n') from a neutron vibration (proton closed shell) or (p,p') from a proton vibration involving the same core have the *same* strength parameter. On the other hand, the strength parameter for (n,n') and (p,p') for neutron and proton nuclei, respectively, with a ^{208}Pb core have strength parameters which differ substantially. The values $V_{np}=3V_{pp}=3V_{nn}$ and use of Table VI and Table IV give $V_{pp}(1+\delta^{pp})+V_{pn}\delta^{np}$ $=1.31V_{pp}+3V_{pp}+3V_{pp}(1.12)=4.67V_{pp}$ as the strength parameter for (p,p') on a proton target and $1.40V_{nn}+3(0.849)V_{nn}=3.95V_{pp}$ as the strength parameter for (n,n') on a neutron target. This difference is a result of the greater polarization of neutrons than of protons in a neutron excess nucleus. The difference is even greater when the Bohr-Mottelson parameters are used. The strength for (p,p') on a proton nucleus is $1.43V_{pp}+3V_{pp}(1.74)$ $=6.65V_{pp}$ and the strength for (n,n') on a neutron nucleus is $1.62V_{nn}+3(0.851)V_{nn}=4.19V_{pp}$.

For α scattering δ^{0n} and δ^{0p} are the amplitude-enhancement parameters for valence neutrons and protons, respectively, whereas in the procedure of Ref. 1 it is e_0 for both. Anticipation of these differences was one of the principal motivations discussed in Sec. II for considering the lack of purity of the giant-resonance excitations in isospin.

V. SUMMARY AND CONCLUSIONS

We have shown that the lack of purity of isoscalar and isovector giant states mixed into the shell-model states gives rise to complications in the connection between polarization effects in inelastic scattering and electromagnetic transitions. The terms isoscalar and isovector become ambiguous; what may, for example, be an isoscalar transition for the nucleus as a whole will be both isoscalar and isovector in the shell-model space. As a result one obtains a different enhancement factor for α scattering from valence protons than from valence neutrons, as expected from intuitive arguments. We have tabulated strength parameters including the core-polarization effects for electromagnetic transitions and for inelastic scattering of various projectiles in terms of a 2×2 polarization matrix with elements $\epsilon_{\tau'\tau}$. The connection is made between these four parameters and the terms in Bohr-Mottelson electromagnetic polarization charges,[3] which also take into account the lack of purity of the giant states. A further approximate formulation is presented which frees one polarization parameter ϵ_{00}, the off-diagonal elements being given in terms of ϵ_{00} and ϵ_{11}. In principle any one experimental transition rate can then be used to determine all four parameters and, therefore, all polarization and effective-charge parameters. Numerical results are presented for ^{207}Pb and ^{118}Sn.

It is well known that effective charges are dependent on j_1 and j_2 of a single-particle transition, whereas the separable interaction gives j-independent values. This deficiency of our calculation is compensated for somewhat by the use of an empirical effective charge, since the j dependence of inelastic scattering amplitudes is expected to follow that of electromagnetic amplitudes to some extent. Nevertheless, the results we have presented are not expected to give detailed agreement with experiment, but rather to show expected trends in the connection between inelastic scattering and electromagnetic transitions as a function of N and Z.

The use of the factor γ^t to correct for the deficiency of the schematic model in describing the expected surface and volume character[3] of the isoscalar and isovector giant resonances, respectively, has been slightly inconsistent. It has been used in Sec. II for the polarization parameter $\sigma_{\tau'\tau}$ and in Sec. III for calculating the energy shift, in both cases giving increased interaction in the isovector mode. It has not been used to correct the strength parameters of Table II for interaction of the core with the external field. This is because the external field interaction is expected to emphasize the nuclear surface region, diminishing the difference between its interaction with the two modes.

In a separate publication the results of this paper for effective polarization parameters will be applied to studying the possible differences of the deformation parameter β in (p,p'), (n,n'), and electromagnetic transitions.

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APPENDIX A. DETAILS OF THE SCHEMATIC MODEL

We include here some of the details of the schematic model discussed in Sec. III. Equations (7), (20), (21), and (22) are used for the development below.

A. Energies of the giant resonances

The collective core-excited states are obtained by diagonalizing the Hamiltonian in neutron and proton particle-hole states (TDA). The creation operator for core state t is

$$\mathcal{O}_t^\dagger = \sum_a \left[C_a^n B_a^{\dagger n} + C_a^p B_a^{\dagger p} \right] , \tag{A1}$$

where for example

$$B_a^{n\dagger} = [a_{m1/2}^\dagger b_{i-1/2}^\dagger]_\lambda , \tag{A2}$$

and the index "a" designates a particular particle-hole state, $a = mi$. These particle-hole excitations can have isospin $T=0$ or 1 or a mixture of both. The Schrödinger equation then yields

$$(E-\epsilon_a^n)C_a^n = D_a^n \left[\alpha \sum_b D_b^n C_b^n + \beta \sum_b D_b^p C_b^p \right] , \tag{A3a}$$

$$(E-\epsilon_a^p)C_a^p = D_a^p \left[\beta \sum_b D_b^n C_b^n + \alpha \sum_b D_b^p C_b^p \right] , \tag{A3b}$$

where ϵ_a^n and ϵ_a^p are particle-hole energies, the D's are given by Eq. (20), and use has been made of Eqs. (21) and (22). Solving for the coefficients gives the result

$$C_a^n = D_a^n \frac{\alpha K^n + \beta K^p}{E-\epsilon_a^n} , \quad C_a^p = D_a^p \frac{\beta K^n + \alpha K^p}{E-\epsilon_a^p} , \tag{A4}$$

where

$$K^n = \sum_b D_b^n C_b^n , \quad K^p = \sum_b D_b^p C_b^p . \tag{A5}$$

Putting in the solutions for C_a we have

$$K^n = \sum_b (D_b^n)^2 \frac{\alpha K^n + \beta K^p}{E-\epsilon_b^n} ,$$

$$K^p = \sum_b (D_b^p)^2 \frac{\beta K^n + \alpha K^p}{E-\epsilon_a^p} . \tag{A6}$$

Solving for K^n and K^p yields the results

$$K^n = \frac{\beta x_n K^p}{1-\alpha x_n} = \frac{1-\alpha x_p}{\beta x_p} K^p , \tag{A7}$$

where

$$x_n = \sum_b \frac{D_b^{n\,2}}{E-\epsilon_b^n} , \quad x_p = \sum_b \frac{D_b^{p\,2}}{E-\epsilon_b^p} \tag{A8}$$

are neutron and proton strength parameters. Compatibility of the two forms (A7) gives a condition on the energies

$$(1-\alpha x_n)(1-\alpha x_p) - \beta^2 x_p x_n = 0 . \tag{A9}$$

From here on we consider the degenerate case $\epsilon_b^n = \epsilon_b^p = \epsilon$ for all particle-hole pairs b because it gives simple solutions which will permit us to study systematics. Multiplying by $(E-\epsilon)^2$ gives two solutions with

$$d \equiv E-\epsilon = \frac{y_n+y_p}{2}\alpha \pm \tfrac{1}{2}[(y_n-y_p)^2\alpha^2 + 4\beta^2 y_p y_n]^{1/2} , \tag{A10}$$

where

$$y_n = x_n d = \sum_b D_b^{n2} ,$$

$$y_p = x_p d = \sum_b D_b^{p2} . \tag{A11}$$

All the other solutions remain degenerate[2] with $E=\epsilon$, which from Eqs. (A6) require $K^n = K^p = 0$. Since for realistic nuclear forces α, $\beta<0$, $|\alpha|<|\beta|$, one of the solutions, Eq. (A10), is positive and one is negative. A pure isoscalar interaction V_0 in Eq. (22) gives $\alpha=\beta$, and we have $d=(y_n+y_p)\alpha < 0$ from the lower sign and $E=\epsilon$ from the upper sign of Eq. (A10). Thus the upper and lower signs are to be identified as the collective isovector and isoscalar solutions.

B. Particle-hole amplitudes

Next Eq. (A3) is solved for the coefficients corresponding to the energy solutions just obtained. For the degenerate solutions $E=\epsilon$ and

$$\sum_b D_b^n C_b^n = \sum_b D_b^p C_b^p = 0 . \tag{A12}$$

The collective solutions from Eq. (A4) have

$$\frac{C_a^n}{D_a^n} \equiv c_n^t = \text{constant} , \tag{A13}$$

$$\frac{C_a^p}{D_a^p} \equiv c_p^t = \text{constant} , \tag{A14}$$

and with $y_\pm = y_n \pm y_p$

$$c_p^t \approx \frac{1}{2\beta y_p} [(-1)^{t+1}|\beta| y_+ - \alpha y_-] c_n^t , \tag{A15a}$$

$$(c_n^t)^2 \approx \frac{1}{2y_n} \frac{1}{1+(-1)^t(\alpha/|\beta|)(y_-/y_+)} , \tag{A15b}$$

where $t=1,0$ correspond to the upper and lower signs in Eq. (A10). If $y_n=y_p$, then $c_n^t=\mp c_p^t$ (remember that $\beta<0$) are the coefficients for the solutions tentatively identified as isovector and isoscalar, respectively. If there is a large excess of neutron particle-hole strength, for example take $y_n=2y_p$, then there results

$$\frac{c_n^t}{c_p^t}=\begin{cases}-0.57 & t=1\\ 0.80 & t=0\end{cases} \tag{A16}$$

for typical values $\beta=2\alpha$ of neutron and proton interaction parameters. This is closely analogous to the coupled pendulum problem. For equal masses the normal modes have equal or opposite amplitudes. The low frequency mode (isoscalar like) has equal amplitudes for unequal masses, but for the high frequency mode they are opposite in sign and inversely proportional to the masses. A qualitatively similar effect is seen in Eqs. (A15) and (A16), the strength parameters y_n and y_p playing the role of masses.

C. Transition rates to collective states

Unambiguous identification of the nature of the solutions Eq. (A10) is obtained by calculating isoscalar and isovector transition rates. We obtain pure isoscalar and isovector transition amplitudes from Eq. (7) and Eq. (16)

$$S_\tau^t=\sum_a C_a^n D_a^n+(-1)^\tau\sum_a C_a^p D_a^p\ . \tag{A17}$$

We see that the degenerate solutions, Eq. (A12), have zero transition rates. For the collective states Eq. (A17) gives

$$\begin{aligned}S_\tau^t&=c_n^t y_n+(-1)^\tau c_p^t y_p\\ &\approx c_n^t\left\{\tfrac{1}{2}y_+[1+(-1)^{t+\tau}]\right.\\ &\quad\left.+\tfrac{1}{2}y_-[1-(-1)^\tau(\alpha/\beta)]\right\}\ ,\end{aligned} \tag{A18}$$

where

$$c_n^t=\left[(y^+)^{-1}\left(1-\frac{y_-}{y_+}+(-1)^t\,\frac{\alpha}{\beta}\,\frac{y_-}{y_+}\right)\right]^{1/2}\ . \tag{A19}$$

We see from Eq. (A18) that our identification was essentially correct, but that the collective core modes are impure in isospin for $y_n\neq y_p$. The transition amplitude S_τ^t is zero for $t\neq\tau$ *only* when $y_n=y_p$, which would be expected to be the case for $N=Z$ nuclei. This result has been obtained in numerical calculations by Veje.[7]

The results of this Appendix can be summarized by stating that in the case of a nucleus with a large neutron excess the individual particle-hole excitations are totally mixed in isospin (pure neutron or pure proton), but the nuclear interaction puts together collective excitations consisting of linear combinations of the states having nearly pure isospin $\tau=0$ or 1.[17] A large neutron excess is compensated by a large proton amplitude in the isovector-type state, and the transitions to the collective states depart from being pure isoscalar or isovector by a term of order $(y_n-y_p)/(y_n+y_p)$, where y_n and y_p are neutron and proton strength parameters. This qualitative feature has been anticipated by Bohr and Mottelson.[3]

APPENDIX B. SUM RULE FOR QUADRUPOLE STRENGTHS

The parameters y_n, defined in Eq. (23), can be written as

$$y_n=\sum_{mi}(D_{mi}^n)^2=\sum_{mi}\langle[mi]_{\lambda\mu}|Q_{\lambda\mu}|0\rangle^2\ , \tag{B1}$$

which, according to the Wigner-Eckart theorem, is independent of μ. Equation (B1) can be rewritten as

$$\begin{aligned}y_n&=\sum_{j_m j_i}\langle j_m\|Q_\lambda\|j_i\rangle^2\hat{\lambda}^{-2}\\ &=\sum_{j_m m_m j_i m_i}\langle j_m m_m|Q_{\lambda\mu}|j_i m_i\rangle^2\ .\end{aligned} \tag{B2}$$

At this point we use a harmonic-oscillator model and assume all states below this Fermi level are filled. Then closure can be used to help carry out the sums. For $\lambda=1$, the only single-particle states i which can contribute substantially to the particle-hole pair are those within $\hbar\omega$ below the top of the Fermi sea; the $j_i m_i$ sum is therefore confined to those values. Correspondingly the only particle states that can contribute are those within $\hbar\omega$ above the Fermi sea. If, instead of using the coordinate x_i in $Q_{1\mu}$ we use $(\hbar/2\mu\omega)^{1/2}C_i^\dagger$ where $C_i^\dagger$ creates a harmonic-oscillator quantum then, to the extent that a harmonic-oscillator picture is accurate, only the upward transitions from states of energies $E_F-\hbar\omega<E<E_F$ to states of energy $E_F<E<E_F+\hbar\omega$ can take place even when the sum over $j_m m_m$ is extended over all single-particle states, where E_F is the Fermi energy.

For quadrupole transitions, to which the following development is confined, the same arguments apply except that two major shells above and below the Fermi sea contribute. Since Eq. (B1) is independent of μ, consider the case $\mu=1$. The quadrupole operator is

$$Q_2^1=-\left(\frac{15}{8\pi}\right)^{1/2}(x+iy)z\ , \tag{B3}$$

which will be replaced by the form

$$Q_2^1 = -\left(\frac{15}{8\pi}\right)^{1/2} \frac{\hbar}{2\mu\omega} (C_x^\dagger + iC_y^\dagger)C_z^\dagger \ . \tag{B4}$$

The operator Eq. (B4) has only upward transitions of $2\hbar\omega$. In terms of the C operators Eq. (B1) can be written

$$y_n = \frac{15}{8\pi}\left(\frac{\hbar}{2\mu\omega}\right)^2 \times \sum_{j_i m_i} \langle j_i m_i | C_z (C_x - iC_y)(C_x^\dagger + iC_y^\dagger)C_z^\dagger | j_i m_i \rangle \ , \tag{B5}$$

the $j_m m_m$ sum having been done using closure. The $j_i m_i$ sum is now restricted to two major shells below the Fermi level. For calculation simplicity we go to a rectangular representation. It is assumed that the sum over $j_i m_i$ represents a spherically symmetric ground state, as in the case of closed shells, so all rectangular quantum numbers take on the same sets of values in the sum. Evaluation of Eq. (B5) gives

$$\begin{aligned} y_n &= \frac{15}{8\pi}\left(\frac{\hbar}{2\mu\omega}\right)^2 \sum\sum_{n_x n_y n_z} (n_z+1)(n_x+1+n_y+1) \\ &= \frac{15}{8\pi}\left(\frac{\hbar}{2\mu\omega}\right)^2 \sum_{\eta'=\eta-1}^{\eta} \tfrac{1}{6}(\eta'+3)(\eta'+4) \\ &= \frac{1}{4\pi}\left(\frac{\hbar}{2\mu\omega}\right)^2 \tfrac{5}{2}(\eta+1)(\eta+2)^2(\eta+3) \ , \end{aligned} \tag{B6}$$

where η is the principle harmonic-oscillator quantum number of the last shell, $E_\eta = \hbar\omega(\eta + \frac{3}{2})$. Although strictly speaking, Eq. (B6) applies only for a closed-shell nucleus, we will apply the results to other cases allowing η to be a fraction determined by the number of nucleons.

Since differences $y_n - y_p$ play an important role in the transition strengths S_τ^t in Eq. (24), it is important that they be treated carefully. It is not appropriate to take the same $\hbar\omega$ for neutrons and protons, since these would give considerable differences in the neutron and proton radii. The rms radius for the harmonic oscillator can be calculated as follows:

$$\begin{aligned} R_{\rm rms}^2 = \langle r^2 \rangle &= N^{-1} \left\langle GS \left| \sum_i r_i^2 \right| GS \right\rangle = \frac{N^{-1}}{\mu\omega^2} \sum_k E_k \\ &= \frac{\hbar}{N\mu\omega} \sum_{\eta'=0}^{\eta} (\eta' + \tfrac{5}{2})(\eta'+1)(\eta'+2) \\ &= \frac{\hbar}{N\mu\omega} \tfrac{1}{4}(\eta+1)(\eta+2)^2(\eta+3) \ . \end{aligned} \tag{B7}$$

Furthermore, the total number of particles is

$$N = \sum_{\eta'=0}^{\eta} (\eta'+1)(\eta'+2) = \frac{(\eta+1)(\eta+2)(\eta+3)}{3} \ . \tag{B8}$$

These three equations involve y_n, $R_{\rm rms}$, N, η, $\hbar\omega$; from them η and $\hbar\omega$ can be eliminated to give, for example,

$$\begin{aligned} y_n &= \frac{1}{4\pi}\,\frac{5}{2}\left(\frac{\hbar}{\mu\omega}\right) N\langle r^2 \rangle = \frac{10}{4\pi} R_{\rm rms}^4 \left(\frac{N}{9}\right)^{2/3} \\ &= \frac{10}{4\pi} r_{\rm rms}^4 \left(\frac{NA^2}{9}\right)^{2/3} \ . \end{aligned} \tag{B9}$$

Similarly y_p can be calculated using Z instead of N. In the last form of Eq. (B9) we can easily impose the constraint that neutron and proton rms radii are the same. By comparison, a single collective state at $2\hbar\omega$ having all the isoscalar quadrupole sum strength has an *energy-weighted* sum S, from which it follows that

$$y_n + y_p = y_+ = \frac{S}{2\hbar\omega} = \frac{1}{2\hbar\omega}\,\frac{\hbar^2 A}{4\pi\mu}\,5\langle r^2 \rangle \ . \tag{B10}$$

Equation (B10) agrees exactly with Eq. (B9) summed over both neutrons and protons, assuming common $\hbar\omega$. Of course the use of Eq. (B10) gives up the equality of neutron and proton radii for equal $\hbar\omega$. Equation (B10) applied to the case of a double closed shell, for which the only 2^+ states are particle-hole states, provides a check of Eq. (B9), which was calculated just summing over particle-hole states. Generally the energy-weighted sum rule cannot be used because it includes a sum over all states; in our case we just want to include particle-hole states.

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