General formalism for pion production in nuclei: Application to $p\,d \rightarrow t\,\pi$[†]

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Motivated by the recent interest in (p, π) reactions which leave the nucleus in a definite final state, we have obtained a general formalism for such reactions and applied it to the reaction $p\,d \rightarrow t\,\pi$, for which there is recent medium energy data. The approach is essentially impulse approximation based on the assumption, originally used by Ruderman, that the reaction mechanism is the same as in $p\,p \rightarrow d\,\pi$. It differs from previous such calculations in that we have included distortion effects and have derived and used an expression for the form factor which does not require the crude approximations for the wave functions used before. We apply this formula to $p\,d \rightarrow t\,\pi$ and obtain zero parameter fits which are a significant improvement over previous calculations and which reproduce the shape and normalization of the data quite well at 470 and 590 MeV and qualitatively at 340 MeV, except that as in other calculations the backward peak at 470 MeV is not reproduced. Distortion effects are important but purely absorptive and serve, as do most other theoretical uncertainties, simply to adjust the normalization.

[NUCLEAR REACTIONS (p, π) derived general formalism $\sigma(\theta)$ at medium energies; distorted-wave impulse approx. Applied $d(p, \pi)t$; $E = 340$, 470, 590 MeV.]

I. INTRODUCTION

There has recently been a great deal of interest in reactions of the form $p + A \rightarrow \pi + (A+1)$, i.e., in those (p, π) reactions on nuclei which leave the final nucleus in some definite state. This interest was originally generated in large part as a result of the very beautiful and detailed measurements of the Uppsala group[1] of the near threshold production of pions from ^{12}C and ^{9}Be. These results include pion energy spectra and separate pion angular distributions for reactions leaving the final nucleus in the ground state or in one of the first few excited states. Earlier higher energy data on the 0° pion production cross section[2] had also served to stimulate interest in these reactions.

More recently an experiment on $p + d \rightarrow \pi + t$, one of the simplest reactions of this type, has been completed.[3] These data, while not so detailed as the Uppsala data, were taken at two different energies and thus, together with some earlier data,[4] provide a survey of pion angular distributions over the range of incident proton energies 300–600 MeV. In this range the dominant feature is a resonance at a proton energy of 400–500 MeV which is presumably related to the $\Delta(1236)$. The existence of data for this reaction over a range of energies thus puts a new, and hopefully important, constraint on the theories which can be used to describe such (p, π) reactions.

A number of different theoretical approaches have been used so far in an attempt to understand these reactions. For the most part recent work has concentrated on the ^{12}C and ^{9}Be data. The simplest approach, used by several groups[5–11] so far, assumes the so-called single-nucleon mechanism (SNM) for the interaction. That is, the pion is assumed to be emitted directly from the incoming proton (leading to a neutron which is subsequently captured) via the usual $\gamma_5\pi$-N interaction, as in Fig. 1. One makes a nonrelativistic reduction of the field theory amplitude for this interaction and so obtains an effective Hamiltonian. The result one obtains is not unique[12] but is usually chosen in a conventional Galilean invariant form.[5] This Hamiltonian can then be used directly in the standard types of distorted-wave Born-approximation (DWBA) calculations familiar from low energy nuclear physics. The main advantages of this approach are its simplicity and compatibility with existing DWBA codes. It also does not lead to π^- production, which provides a natural explanation for the strong suppression of π^-'s observed experimentally.[1,2] This mechanism has the serious disadvantage, however, of requiring that the entire momentum transfer, which is quite large, be absorbed by a single nucleon. Thus the results become very sensitive to the large momentum components of the wave functions, which are in general known very poorly, and thus also to the details of the potentials used to generate the distorted waves. This approach includes pion rescattering and effects of the resonance, which seem to dominate the reaction at energies above threshold (at least for $p + d \rightarrow t + \pi$, the only case which has been measured at these higher energies) only in an average

way via the optical potential.

A number of groups have performed calculations of this type,[5-11] differing in various details. Early calculations gave results in disagreement with the over-all normalization of the data by one to three orders of magnitude. The extreme sensitivity to the various parameters allows a great deal of freedom, however, and later calculations have achieved much better results by varying the parameters of the wave functions or distorting potentials. The angular distributions obtained in these calculations generally follow the trend of the data but do not reproduce the details and may differ from the data by more than an order of magnitude at some angles. Generally all such calculations do agree, however, on the importance of distortion effects.

A second approach which has been used assumes that a two-nucleon mechanism (TNM) is the important interaction. An example of one contribution to such an interaction is given in Fig. 2. Here a pion emitted from one nucleon scatters from another before being emitted from the nucleus. In this approach resonance effects can be incorporated directly by including appropriate diagrams. Also, since two nucleons are involved the momentum transfer can be shared among the nucleons and thus the result is not quite so sensitive to the unknown large momentum components of the wave functions. A disadvantage is that the mechanism does not give suppression of π^-'s in an obvious fashion. Also, one requires as input some information about, or a model of, off-mass-shell pion-nucleon scattering.

Several groups have used variations of this approach.[13-15] For example, Reitan[13] considered 0° pions from protons on ^{12}C and, using experimental on-shell information for the π-nucleon scattering amplitude, found a result too small by an order of magnitude. Rescattering corrections were important and the TNM could be larger or smaller than the SNM depending on the wave functions chosen. A quite detailed calculation of this type was recently performed by Grossman, Lenz, and Locher.[15] They cast the TNM into an effective interaction form, analogous to the SNM effective Hamiltonian, which includes resonance effects. However, the approximations required limit the result to near threshold production. Their results give angular distributions comparable to those obtained with the SNM and normalization uncertainties of the order of a factor of 5. In addition, they find that the SNM is inadequate even near threshold and agree with previous calculations that distortion effects are quite important. Yet another approach, but basically TNM, was used by Wienke,[16] who attempted a relativistic field theory calculation of these reactions.[17]

It thus appears that the two different mechanisms SNM and TNM can each individually be forced to give a qualitative fit to the data, at least at a single energy, where by qualitative one means an over-all normalization correct to a factor of 5 or so and angular distributions differing at some angles by an order of magnitude. The results with both approaches are extremely sensitive to various wave-function and distorting potential parameters, and it is this sensitivity which is at least partially responsible for the variation among the results of different groups and for the fact that comparable fits can be obtained using somewhat different approaches.

These methods have not been applied to data, e.g. for $p+d \to \pi+t$, in the resonance region and in fact, with a few exceptions, cannot be applied because of various restrictive approximations which limit the results to the threshold region. Thus to describe the new $p+d \to \pi+t$ data one must use a different method.

One possibility which has been used with some success is a method originally applied by Ruderman[20] to $p+d \to \pi+t$ which, in modern language, one would describe as plane-wave impulse approximation (PWIA). The basic assumption involved is that the mechanism for pion production in nuclei is the same as that in p-p collisions and, in particular, because of kinematic constraints, the same as in $p+p \to \pi+d$ (cf. Fig. 4). This approach has the definite advantage, in view of the discussion above, of not requiring one to distinguish between SNM and TNM, since both are included in the experimental cross section for $p+p \to \pi+d$. It does require this cross section off mass shell, however, and thus shares with the TNM approaches the necessity of a model or prescription for the off-mass-shell extrapolation.

Calculations of this type as applied to older data

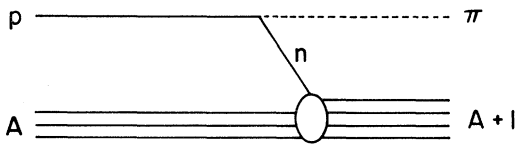

FIG. 1. Diagram for the single-nucleon mechanism of pion production.

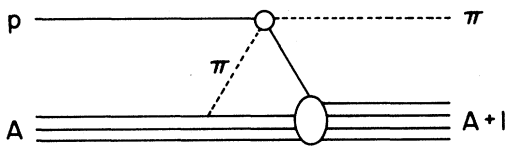

FIG. 2. A contribution to the two-nucleon mechanism of pion production.

on $p+d\to\pi+t$ have been carried out by Ingram *et al.*[21] and by Barry[18] with fairly encouraging results. In its simplest form this approach seems to give qualitatively correct results both for normalization and angular distribution (except for a failure to predict the backward peak seen in later data[3] at some energies). The results are nearly comparable to those obtained for protons on ^{12}C by more detailed calculations using SNM or TNM.

This suggestion of possible success indicates that it would be worthwhile to make a more refined calculation of this type, particularly one which includes the distortion effects which have been shown to be important in the various SNM and TNM calculations and which improves upon as many of the approximations of Ref. 18 and 21 as possible. Such a calculation would also appear to be important since this approach is almost the only one so far which is directly applicable to the $p+d\to\pi+t$ data in the resonance region or to the mass of medium energy (p,π) data which will become available as a result of the new meson facilities. This approach has an additional advantage in that it tends to separate questions of the correct interaction from those dealing with distortion effects or with the choice of wave functions. Thus it may be a very useful first step in a problem which seems to be extremely sensitive to so many poorly known ingredients.

Thus our calculation is in the spirit of Refs. 18, 20 and 21 but with a number of important refinements. In the first place, we include effects of distortion or multiple scattering which have not been previously included in impulse approximation calculations and which we find to be relatively important. We also have made a number of technical improvements over previous calculations, the major one being a better treatment of the overlap of initial and final wave functions. This leads to a new formula for the effective form factor which involves fewer approximations than previous results. We have also found and corrected a number of what appear to be errors in previous calculations.[18,21]

The outline of this paper is as follows. In the next section we derive the general formulas for (p,π) reactions on nuclei in what would conventionally be called a distorted-wave impulse approximation (DWIA). We particularly emphasize the physical assumptions used and the ways in which our calculation differs from Refs. 18 and 21, and we attempt to obtain results in a form which is easily applicable to arbitrary nuclei. The next section is devoted to an application of our general results to the reaction $p+d\to\pi+t$, for which there is recent new data available.[3] The final section is devoted to a discussion of our numerical results for this particular case, their sensitivity to various input parameters, and to a discussion of some general conclusions.

II. CALCULATION OF THE CROSS SECTION

In this section we want to derive the general result for pion production on a nucleus, Eq. (19) below, which we subsequently apply to a specific example of interest here. After defining our notation and the basic assumption required for the interaction, we describe the techniques for obtaining the final results. Included also is a description of the way we put in distortion effects and some discussion of the physical import of our assumptions. The reader interested primarily in the results should note Eqs. (16), (17), and (19) and the discussion following Eq. (19).

We consider the process $p+A\to(A+1)+\pi$ where A and $(A+1)$ correspond to nuclei of A and $A+1$ nucleons, respectively. Let the corresponding masses, energies, and four-momenta be m_p, E_p, p_p, etc.[22] The scattering matrix then becomes in the usual way

$$\begin{aligned}\langle f|S|i\rangle &= \delta_{fi} - 2\pi i\delta(E_f - E_i)\langle f|\mathcal{T}|i\rangle \\ &= \delta_{fi} - 2\pi i\delta^4(p_f - p_i)\langle f|T|i\rangle \ . \end{aligned} \tag{1}$$

With this normalization of $\mathcal{T}$ the cross section becomes[23]

$$\begin{aligned} d\sigma = &\left[\frac{2(2\pi)^5 dt}{\lambda(s_{pA}, m_p^2, m_A^2)}\right] \\ &\times\left[\overline{\sum_{\text{spins}}} E_p E_A E_\pi E_{A+1}\left|\langle f|T|i\rangle\right|^2\right], \end{aligned} \tag{2}$$

where we have defined the usual invariant energy and momentum transfer as $s_{pA}=(p_p+p_A)^2$ and $t=(p_p-p_\pi)^2$. The bracketed terms are separately Lorentz invariants and $\overline{\sum}$ indicates a sum on all spins and average over initial spins. In the $p-A$ center-of-mass system dt is related to the pion solid angle $d\Omega$ by

$$\begin{aligned} dt &= \frac{p_p p_\pi}{\pi}\, d\Omega \\ &= \frac{\lambda^{1/2}(s_{pA}, m_p^2, m_A^2)\lambda^{1/2}(s_{pA}, m_\pi^2, m_{A+1}^2)}{4\pi\, s_{pA}}\, d\Omega \ . \end{aligned} \tag{3}$$

Next define the coordinates $r_0,\ldots r_A$ of the initial $A+1$ nucleons, $r_0',\ldots,r_A'$ of the final nucleons, and r_π of the pion. The $\mathcal{T}$ matrix for the interaction is in general a function of all of these coordinates. The basic physical assumption of the calculation is that the interaction responsible for nuclear pion production can be treated as a sum of *at most* two nucleon interactions, with the other

nucleons acting as spectators. Thus we write

$$\mathcal{T}(r_\pi, r_0', \ldots, r_A', r_0, \ldots, r_A) = \sum_{i<j} \mathcal{T}_{ij}(r_\pi, r_i', r_j', r_i, r_j) \prod_{k \neq j,i} \delta^3(r_k' - r_k), \tag{4}$$

where $\mathcal{T}_{ij}$ is the $\mathcal{T}$ matrix for production of a pion in an interaction between nucleons i and j. With this approach we avoid the controversy of single-nucleon mechanism versus two-nucleon mechanism since $\mathcal{T}_{ij}$ includes a contribution from both mechanisms, e.g. from both diagrams Fig. 1 and Fig. 2 as well as possibly others. It does not, however, include any three-nucleon interactions as, for example, that of Fig. 3, though the generalization of this approach to include such interactions is obvious.

We take for the $\pi - (A+1)$ wave function $\Psi_f(r_\pi, \alpha_\pi; r_0', \ldots, r_A', \alpha_{A+1})$, which is understood to be antisymmetric in the coordinates $r_0', \ldots, r_A'$. The parameters α_π and α_{A+1} stand for all other quantum numbers, in particular spin and isospin, which are not explicit. Similarly, let $\Psi_i(r_0, \alpha_p; r_1, \ldots, r_A, \alpha_A)$ be the wave function for the incoming nucleon (r_0, α_p) and a completely antisymmetrized nucleus of A nucleons. The initial wave function is then obtained by antisymmetrizing on the incoming nucleon, i.e.,

$$\frac{1}{(A+1)^{1/2}} \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \Psi_i(r_0, \alpha_p; r_1, \ldots, r_A, \alpha_A), \tag{5}$$

where $\mathcal{P}$ is the set of permutations $r_0 \rightleftarrows r_i$. Using these wave functions one obtains in a straightforward way the nuclear matrix element in terms of the matrix element $\langle \Psi_f | \mathcal{T}_{01} | \Psi_i \rangle$.

We next wish to relate this matrix element to that for the two-body reaction $NN \to NN\pi$. To do this we simply insert complete sets of two-nucleon and two-nucleon-plus-pion states on either side of $\mathcal{T}_{01}$. In general these states will be full scattering states and not necessarily plane waves. Thus let $\psi_i(r_0, r_1, \beta_i)$ be an antisymmetrized two-nucleon state for which the nucleons have momenta p_p' and q. Again, β_i corresponds to all other quantum numbers. Similarly, let $\psi_f(r_\pi, \beta_\pi; r_0', r_1', \beta_d)$ be an antisymmetrized two-nucleon–plus–pion state with pion momentum p_π' and total nucleon momentum p_d. Here β_π and β_d include the other quantum numbers plus, in general, the relative momentum of the two nucleons. We eventually expect to neglect all of the complete set but the pion-deuteron state and so have chosen notation accordingly, using d to refer to the final two-nucleon state, though we should emphasize that so far the results are completely general.

Putting all of this together we obtain for the nuclear matrix element

$$\langle \pi A+1 | \mathcal{T} | pA \rangle = A(A+1)^{1/2} \sum_{\beta_\pi \beta_d \beta_i} \int d^3r_\pi \prod_i d^3r_i \prod_i d^3r_i' \, d^3p_p' \, d^3q \, d^3p_d \, d^3p_\pi' \langle \pi(p_\pi', \beta_\pi) d(p_d, \beta_d) | \mathcal{T}_{01} | pp(p_p', q, \beta_i) \rangle$$
$$\times \prod_{k \neq 0,1} \delta^3(r_k' - r_k) \Psi_f^*(r_\pi, \alpha_\pi; r_0', \ldots, r_A', \alpha_{A+1}) \psi_f(r_\pi, \beta_\pi; r_0', r_1', \beta_d)$$
$$\times \psi_i^*(r_0, r_1, \beta_i) \Psi_i(r_0, \ldots, r_A, \alpha_A). \tag{6}$$

We now have explicitly, albeit under the integral sign, the matrix element $\langle \pi d | \mathcal{T}_{01} | pp \rangle$ for the reaction $NN \to NN\pi$. Since this matrix element involves full two-nucleon scattering states rather than plane waves, we can use experimental information on the two-body process, without further approximation, to evaluate this matrix element and thus obtain the matrix element for the nuclear process, provided we can calculate the appropriate wave function overlaps.

To calculate these overlaps we must specify the wave functions in more detail. Formally the wave functions are (to first order in the interaction producing free pions) solutions of the complete scattering problem. They thus in general contain information about both absorption and multiple scattering of the pion or proton as it traverses the nucleus. In the usual notation one would write each wave function as a product of a plane wave for the center-of-mass motion and a relative wave

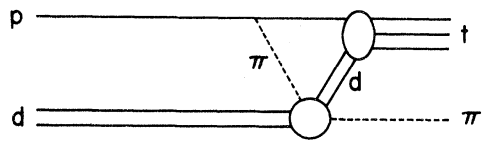

FIG. 3. A possible three-nucleon mechanism suggested in Refs. 18 and 19 as contributing to a backward peak in $pd \to t\pi$.

function χ.

Previous impulse approximation calculations have simply replaced χ by a plane wave and neglected all distortion effects. This does not seem to be sufficient, however, and we want to improve upon it here. Thus we use for χ the usual plane wave times a factor F into which we absorb all distortion effects. In general F will be a function of the coordinates of the projectile and of all of the target nucleons and of the various relative momenta. To reduce the number of integrations we are essentially forced to approximate F as a function of the projectile-target center of mass relative coordinate and velocity only. We expect to use the usual Glauber approximation for F, as discussed in detail at the end of this section, but need not make a specific choice yet.

Thus we can now write the wave functions necessary to evaluate the overlaps in the matrix element. We make an additional refinement apparently not normally made in that we put in explicitly the center of mass constraint on the nuclear coordinates. Thus, define R_A and R'_{A+1}, respectively, as the center of mass coordinates of the initial and final nuclei and let $x_\pi = r_\pi - R'_{A+1}$, $x'_i = r'_i - R'_{A+1}$, $x_i = r_i - R_A$.

We then have

$$\Psi_i = \frac{|\alpha_p\rangle}{(2\pi)^3 A^{3/2}} e^{ip_p \cdot r_0} \int d^3R_A e^{ip_A \cdot R_A} \phi_A(x_1, \ldots, x_A, \alpha_A) F_{pA}(x_0, v_{pA}) \delta^3\left(R_A - \frac{1}{A}\sum_{i=1}^{A} r_i\right) \tag{7}$$

and

$$\Psi_f = \frac{|\alpha_\pi\rangle}{(2\pi)^3 (A+1)^{3/2}} e^{ip_\pi \cdot r_\pi} \int d^3R'_{A+1} e^{ip_{A+1} \cdot R'_{A+1}} \phi_{A+1}(x'_0, \ldots, x'_A, \alpha_{A+1}) F_{\pi A+1}(x_\pi, v_{\pi A+1})$$

$$\times \delta^3\left(R'_{A+1} - \frac{1}{A+1}\sum_{i=0}^{A} r'_i\right), \tag{8}$$

where $|\alpha_p\rangle$ and $|\alpha_\pi\rangle$ are the spin-isospin wave functions for the initial proton and outgoing pion and where we have put appropriate subscripts on the F's and the relative velocities v. The functions ϕ_A and ϕ_{A+1} are the completely antisymmetric nuclear wave functions normalized according to, for example,

$$\int d^3x_1 \cdots d^3x_A \phi_A(x_1, \ldots, x_A, \alpha_A)$$

$$\phi_A^*(x_1, \ldots, x_A, \alpha'_A) \delta^3\left(\sum_{i=1}^{A} x_i\right) = \delta_{\alpha_A \alpha'_A}, \tag{9}$$

as required by the over-all normalization condition on Ψ_i and Ψ_f. Similarly, we take for the two-body wave functions

$$\psi_f = \frac{|\beta_\pi\rangle}{(2\pi)^3} e^{ip'_\pi \cdot r_\pi} e^{(i/2)p_d \cdot (r'_0 + r'_1)} \phi_d(r'_0 - r'_1, \beta_d)$$

$$\times F_{\pi d}(r_\pi - \tfrac{1}{2}(r'_0 + r'_1), v_{\pi d}) \tag{10}$$

and

$$\psi_i = \frac{e^{(i/2)(p'_p + q) \cdot (r_0 + r_1)}}{(2\pi)^{3/2}} |\beta_0\beta_1\rangle F_{pp}(r_0 - r_1, v_{pp}), \tag{11}$$

where $|\beta_\pi\rangle$ is the isospin wave function for the pion and $|\beta_0\beta_1\rangle$ is the antisymmetric spin-isospin wave function for two nucleons. The reduction of ψ_i from the general form to the symmetric space state x antisymmetric spin-isospin state form given above actually requires some further approximations to be made below, but is done now to simplify the algebra. The two-nucleon wave function ϕ_d is normalized as usual to

$$\int d^3r \, \phi_d(r, \beta_d) \phi_d^*(r, \beta'_d) = \delta_{\beta_d \beta'_d}. \tag{12}$$

Note that we calculate members of the complete set of two-nucleon scattering states ψ_i and ψ_f in exactly the same distorted-wave approximation used for the exterior states Ψ_i and Ψ_f, hence the presence of the two additional distortion functions F_{pp} and $F_{\pi d}$.

Using these wave functions we can evaluate the matrix element of Eq. (6). We define $y'_i = r'_i - R_{A-1}$ and $y_i = r_i - R_{A-1}$, where R_{A-1} is the center of mass of the $A-1$ spectator nucleons, and transform integration variables to the set x_0, x_π, R_A, R'_{A+1}, $y_1 \cdots y_A$, $y'_0 \cdots y'_A$ for which the Jacobian is $(A-1)^6/[A(A+1)]^3$. The integrations on R_A and R'_{A+1} can be done, giving a δ function of over-all momentum conservation which cancels that extracted from $\mathcal{T}$. A δ function extracted from $\mathcal{T}_{01}$ can be used to do the integration on $p_d = p'_p + q - p'_\pi$

and we are left with the following result:

$$\langle \pi A+1|\, T|\, pA\rangle = \frac{(A-1)^3}{A^{1/2}(A+1)(2\pi)^9}$$

$$\times \sum_{\substack{\beta_\pi \beta_d \\ \beta_0 \beta_1}} \int d^3x_\pi d^3x_0 d^3y_0' d^3y_1' d^3y_1 \cdots d^3y_A d^3p_p' d^3p_\pi' d^3q\, \delta^3\left(\sum_{i=2}^{A} y_i\right) \langle \pi d|\, T_{01}|\, pp\rangle \langle \alpha_\pi|\, \beta_\pi\rangle$$

$$\times \phi^*_{A+1}(y_0', y_1', y_2 \cdots y_A, \alpha_{A+1}) \phi_d(y_0' - y_1', \beta_d) \langle \beta_0 \beta_1|\, \alpha_p\rangle\, \phi_A(y_1 \cdots y_A, \alpha_A)$$

$$\times F^*_{\pi A+1}(x_\pi, v_{\pi A+1}) F_{\pi d}\left(x_\pi - \frac{A-1}{A+1}\frac{1}{2}(y_0'+y_1'), v_{\pi d}\right) F^*_{pp}\left(x_0 - \frac{A-1}{A}\, y_1, v_{pp}\right)$$

$$\times F_{pA}(x_0, v_{pA}) e^{iz}\ , \tag{13}$$

where

$$z = (p_\pi' - p_\pi)\left(x_\pi - \frac{A-1}{A+1}\,\frac{y_0'+y_1'}{2}\right) + (p_p' - p_p)\left(\frac{y_0'+y_1'}{2} - x_0 - \frac{1}{A}\, y_1\right) + \frac{y_0'+y_1'}{2}\left[\Delta + q - \frac{1}{A}\, p_A\right] + y_1\left[\frac{1}{A}\, p_A - q\right],$$

and where we have defined an effective momentum transfer Δ by

$$\Delta = \frac{A-1}{A+1}\left[p_p - \frac{1}{A} p_A - p_\pi\right]. \tag{14}$$

Note that the wave function $\phi_A(y_i)$, and similarly $\phi_{A+1}(y_i')$, are obtained from the original definitions $\phi_A(x_i)$ by substituting for each x_i the appropriate function of the y_i.

So far it has been necessary to make very few approximations, but the result involves a $3(A+7)$ dimensional integration and so clearly will require further simplifications. We observe first that the integrand consists of a part—containing the matrix element and wave functions—which may be considered as a relatively smoothly varying function of the variables, and a part—containing exponentials—which is rapidly varying. Next note that we are basically interested in describing the nuclear process in the resonance region, where both proton and pion have fairly high energies. Thus it would seem reasonable to use an eikonal type approximation. The physical picture is thus that the proton and pion traverse the nucleus with a number of small scatterings and perhaps some absorption but essentially no change in momentum. Hence we expect $p_p \simeq p_p'$ and $p_\pi \simeq p_\pi'$ and will make that approximation in the slowly varying portions of the integrand, i.e., in the matrix element and the F's. This allows us to do the integrations on p_p' and p_π' and to obtain the δ functions

$$\delta^3\left(x_\pi - \frac{A-1}{A+1}\,\frac{y_0'+y_1'}{2}\right)$$

and

$$\delta^3\left(\frac{y_0'+y_1'}{2} - x_0 - \frac{1}{A}\, y_1\right).$$

We then assume that q, the remaining variable in the two-body matrix element, can be evaluated at an average value $\bar{q}$ and the matrix element removed from the integrand. The question of how to evaluate $\bar{q}$ will be discussed later. Thus, after a change of variables $q - (1/A)p_A \to q$, the q integration gives a further δ function

$$\delta^3[y_1 - \tfrac{1}{2}(y_0'+y_1')]\ .$$

We can understand the significance of these approximations in another way if we note that, expressed in terms of the original coordinates, the arguments of the δ functions can be written as $r_\pi - \frac{1}{2}(r_0'+r_1')$, $r_0 - \frac{1}{2}(r_0'+r_1')$, and $r_1 - r_0$. Thus our approximations require that the incident and target protons interact at a point and that the pion and center of mass of the final two nucleons appear at that same point. These approximations, which we obtained from an eikonal point of view, are thus equivalent to the zero range approximation commonly used in low energy DWBA or DWIA calculations. These are the approximations which lead to the special form for ψ_i given in Eq. (11). Since $r_0 = r_1$, the antisymmetric space part of the wave function vanishes, and one is left with the result we used.

At this point we want to limit the sum on the intermediate states β_d to the deuteron. Thus β_d now represents only the possible spin projections for the deuteron. Such limitation is necessary since we do not have sufficiently complete experi-

mental information on the two-body amplitude for $NN \to NN\pi$ for all possible states of the final two nucleons, whereas information on $pp \to \pi d$ is fairly complete.[24] We do expect, however, that neglect of these other states is in fact a good approximation. The arguments for this are primarily kinematic and have been given by Ruderman[20] and Ingram *et al.*[21] Some support for this assumption can also be inferred from the experimental observation that π^- production is strongly suppressed[1,2] (by factors of up to 100) since π^- cannot be produced in either pn or pp interactions resulting in a deuteron but requires one of the isospin amplitudes we are neglecting. Conversely, one could say that this assumption "explains" the observed absence of π^-.

We also assume that ϕ_A, ϕ_{A+1}, and ϕ_d can be written as the product of a space wave function and a spin-isospin wave function and so extract explicitly the spin-isospin wave functions $|\alpha_A\rangle$, $|\alpha_{A+1}\rangle$, and $|\beta_d\rangle$. Putting these things together we obtain for the nuclear matrix element

$$\langle \pi, A+1|\, T|\, p, A\rangle = \sum_{\substack{\beta_d \\ \beta_0 \beta_1}} \langle \pi(p_\pi, \alpha_\pi) d(p_d, \beta_d)|\, T_{01}|\, pp(p_p, \bar{q}, \beta_0, \beta_1)\rangle \langle\langle\, \alpha_{A+1}|\, \beta_d\rangle \langle\, \beta_0\beta_1|\, \alpha_p\, \alpha_A\rangle\rangle\, F(\Delta)\,, \tag{15}$$

where we define a form factor $F(\Delta)$ by

$$F(\Delta) = \frac{(A-1)^3}{(A+1)(A)^{1/2}} \int d^3y_0' d^3y_1' d^3y_2 \cdots d^3y_A \delta^3\Big(\sum_{i=2}^{A} y_i\Big) e^{i\,\Delta\cdot y_1} F^*_{\pi A+1}(x_\pi, v_{\pi A+1}) F_{pA}(x_0, v_{pA}) F^*_{pp}(0, v_{pp})$$

$$\times F_{\pi d}(0, v_{\pi d})\, \phi^*_{A+1}(y_0', y_1', y_2, \ldots, y_A)\, \phi_d(y_0' - y_1')\, \phi_A(y_1, \ldots, y_A) \tag{16}$$

in which we evaluate, in accordance with the δ functions,

$$y_1 = \tfrac{1}{2}(y_0' + y_1'), \quad x_0 = \frac{1}{2}\frac{A-1}{A}(y_0' + y_1'),$$

and

$$x_\pi = \frac{1}{2}\frac{A-1}{A+1}(y_0' + y_1')\,.$$

The notation $\langle\langle\, \alpha_{A+1}|\, \beta_d\rangle \langle\, \beta_0\beta_1|\, \alpha_p\, \alpha_A\rangle\rangle$ for the spin-isospin matrix element is used to emphasize that $|\beta_d\rangle$ and $|\beta_0\beta_1\rangle$, which depend only on coordinates 0 and 1, must be projected onto the nuclear wave functions first, before taking the matrix element on the remaining coordinates.

To obtain the required cross section we simply square the matrix element above and sum on spins. Observe that in general the two-body T matrix appears in a complicated linear combination of various spin terms. To extract the required cross section we define a spin factor g as follows:

$$g = \frac{\sum_{\text{spins}} \Big| \sum_{\beta_d \beta_0 \beta_1} \langle \beta_d|\, T_{01}|\, \beta_0\beta_1\rangle \langle\langle\, \alpha_{A+1}|\, \beta_d\rangle \langle \beta_0\beta_1|\, \alpha_p\alpha_A\rangle\rangle \Big|^2}{\sum_{\text{spins}} \Big| \langle \beta_d|\, T_{01}|\, \beta_0\beta_1\rangle \Big|^2}\,. \tag{17}$$

Then

$$\overline{\sum_{\text{spins}}} |\langle \pi, A+1|\, T|\, p, A\rangle|^2 = g|\, F(\Delta)|^2 \overline{\sum_{\text{spins}}} |\langle \pi d|\, T_{01}|\, pp\rangle|^2\,, \tag{18}$$

where, as before, the bar on the summation sign indicates an average over initial spins. Finally, using Eq. (2) and Eq. (18) and, explicitly, the Lorentz invariance of the bracketed quantities in Eq. (2), we obtain

$$\Big[\frac{d\sigma}{d\Omega_\pi}(s_{pA}, t)\Big]^{pA\to\pi A+1}_{pA\ \text{c.m.}} = \frac{s_{pp}}{s_{pA}} \left[\frac{\lambda(s_{pA}, m_\pi^2, m_{A+1}^2)}{\lambda(s_{pA}, m_p^2, m_A^2)} \frac{\lambda(s_{pp}, m_p^2, m_p^2)}{\lambda(s_{pp}, m_\pi^2, m_d^2)}\right]^{1/2}$$

$$\times \frac{E_A E_{A+1}}{E_q E_d}\, g|\, F(\Delta)|^2 \Big[\frac{d\sigma}{d\Omega_\pi}(s_{pp}, t)\Big]^{pp\to\pi d}_{pp\ \text{c.m.}}\,, \tag{19}$$

where all factors not explicitly invariant are to be evaluated in the p-A center of mass system and where $s_{pp}=(p_p+\bar{q})^2$ is the energy, determined by $\bar{q}$, at which we evaluate the two-body amplitude. Here

$$E_q=(\bar{q}^2+m_p^2)^{1/2} \text{ and } E_d=E_p+E_q-E_\pi \ .$$

The results above, Eq. (16) and Eq. (19), allow us to evaluate the cross section for the nuclear pion production reaction in terms of the experimental cross section for $pp\to\pi d$ and information about the nuclear wave functions. Before proceeding to such an evaluation for a special case, however, we need to discuss the two questions deferred in the derivation above, namely what should one use for the distortion functions F and how does one evaluate the average momentum $\bar{q}$ and handle the off-mass-shell extrapolation necessary to evaluate the $pp\to\pi d$ cross section. We also want to discuss in some detail how our results, particulary those for the form factor $F(\Delta)$, differ from previous work.

We consider first the question of the evaluation of the distortion functions F. Recall that these functions describe the way in which the complete scattering wave function deviates from a plane wave. They thus contain effects of elastic scattering of the pion or proton on other nucleons and effects of various inelastic reactions leading to absorption. We are primarily interested in the resonance region where both proton and pion have momenta of some hundreds of MeV/c. Thus it should be sufficient at these energies to use an eikonal approximation, as was used, for example, by Eisenberg *et al.*[8] This is computationally much simpler than the method, used at lower energies, of solving the Schrödinger equation for the wave functions. Thus, following Glauber,[25] we assume that the main effect of the distortion is to produce a phase change in the wave function without changing the projectile momentum, and that the phase function can by calculated by integrating the scattering potential along the path of the incoming or outgoing particle. Thus, we use, for incoming particles,

$$F(x,v)=\exp\left[-\frac{i}{v}\int_0^\infty V(x-\hat{v}\tau)d\tau\right], \tag{20}$$

where $\vec{v}$ is the relative velocity and V is the scattering potential. For outgoing particles we must take the complex conjugate of the exponent and change the unit vector $\hat{v}$ to $-\hat{v}$. There is some ambiguity in the choice of reference frame for the evaluation of $\vec{v}$. We have obtained $\vec{v}$ from the projectile momentum evaluated in the rest frame of the target, as it is in this frame that the potential is really defined.

We used for the potentials the relations, given for example by Glauber,[25] which express the potentials in terms of the nuclear density function $\rho(r)$ and the forward scattering amplitude. Thus we have for an incoming projectile on a target nucleus of Z protons and $A-Z$ neutrons

$$F(x,v)=\exp\left\{\tfrac{1}{2}i\left[Z\sigma_p(\eta_p+i)+(A-Z)\sigma_n(\eta_n+i)\right]\int_0^\infty \rho(x-\hat{v}\tau)d\tau\right\}, \tag{21}$$

where $\sigma_p\,(\sigma_n)$ and $\eta_p\,(\eta_n)$ are, respectively, the total cross section and the ratio of real to imaginary parts of the forward amplitude for scattering of the projectile on protons (neutrons) evaluated at an energy corresponding to the relative velocity. To get F for outgoing particles we again change $\hat{v}\to-\hat{v}$ and take the complex conjugate of the exponent. The density function was taken as the usual Woods-Saxon form

$$\rho(r)=\rho_0\left[1+e^{(r-R)/a}\right]^{-1} \tag{22}$$

with

$$\rho_0=\left[\frac{4\pi R^3}{3}\left(1+\frac{\pi^2a^2}{R^2}\right)\right]^{-1} .$$

Here a is the skin thickness parameter and R the radius parameter, related to the rms radius, which can be obtained from electron scattering by[26]

$$R^2\cong\frac{5}{3}\langle r^2\rangle_{\rm rms}\left(1-\frac{7}{5}\frac{\pi^2a^2}{\langle r^2\rangle_{\rm rms}}\right) . \tag{23}$$

Note that in principle we must include four different distortion functions in Eq. (16). F_{pA} and $F_{\pi A+1}$ come from the external states and F_{pp} and $F_{\pi d}$ come from the intermediate two-nucleon states inserted into the original matrix element which, recall, were calculated in the same distorted-wave approximation used for the external states. These latter two are in principle necessary because the experimental $pp\to\pi d$ cross section results from the interaction of the complete pp and πd scattering states and so it is these states rather than plane waves which must be inserted in the matrix element if we are to avoid the further approximation (which is, however, often made) of equating the plane-wave $pp\to d\pi$ amplitude with the physical one obtained from experiment. However, F_{pp} and $F_{\pi d}$ are eventually evaluated at $x=0$ and so lead, as can be seen from Eq. (21), to a simple over-all phase factor which makes no contribution, and to a real exponential factor. This real exponential

reflects the presence of absorption effects and simply modifies the over-all normalization of the result.

Having argued that *in principle* F_{pp} and $F_{\pi d}$ should be present, we now observe that *in practice* it may be best to neglect them. The reason is that our other approximations require that F_{pp} and $F_{\pi d}$ be evaluated at zero, i.e., at the center of the proton and deuteron, respectively. Since the distortion is a cumulative effect depending on the path of the projectile through the potential, this means that we obtain for F_{pp} and $F_{\pi d}$ the absolute maximum value possible corresponding to a complete overlap of target and projectile, rather than the more realistic situation of very little overlap. Similar arguments may apply to a lesser extent to F_{pA} and $F_{\pi A+1}$ when A is small. In any case, however, the result is that the other approximations of the calculation force us to evaluate F_{pp} and $F_{\pi d}$ in a way which may grossly overestimate the absorption they produce. Thus it may be a better approximation just to neglect them.

One may also question the use of the simple central potential approximation to obtain the distorted wave functions, particularly for such light nuclei. Clearly one can calculate exact distorted wave functions, especially for the pp system where the exact potential is known. For our purposes, however, the simple approximation we use would appear to be sufficient for the following reason. Observe that in the context of the eikonal approximation we need only the forward amplitude for projectile-nucleon scattering to calculate the distorting phase. This same information, however, allows us to calculate in the same approximation the forward amplitude for the *nuclear* elastic scattering. We can check the result against data for the total cross section and forward elastic cross section and thus obtain an important constraint on the distorting potentials. Parameters of the density function can then be adjusted if necessary to improve the agreement. For the pp system the check is purely one of consistency and we obtain acceptable results provided we increase the proton rms charge radius of 0.81 fm by 0.55 fm and the central density ρ_0 by 20%. This increase in radius is consistent with the physical picture of the strong potential being effective over a region somewhat larger than the charge distribution as a result of the range of the nuclear force. A similar adjustment was not required for the deuteron but was needed for the triton in the particular application considered below. Thus, by virtue of this constraint, the potentials we obtain can be expected to lead to reasonable distortion factors for our purposes, even though they may be poor for calculating elastic scattering at large angles. Similar arguments would indicate for the pion case that the use of fancier potentials is not necessary.

We next need to discuss the evaluation of the amplitude for $pp \to \pi d$ and in particular the choice of the average momentum $\bar{q}$. In general the amplitude required is off the mass shell and so we must first make an extrapolation to an on-mass-shell point. We do this in the usual way by simply taking the point corresponding to the momenta as though the particles were on the mass shell. It is also quite natural to evaluate the two-nucleon amplitude at the four-momentum transfer $t = (p_p - p_\pi)^2$ determined by the external momenta p_p and p_π. These restrictions are not sufficient, however, to determine the total energy s_{pp} (or equivalently $\bar{q}$) at which to evaluate the amplitude. They simply give an equation relating the magnitude of the target nucleon momentum $\bar{q}$ to its direction. To restrict $\bar{q}$ further recall that we originally removed the two-body amplitude from the integral on q by assuming that it was slowly varying compared to the exponential factors. The dominant contribution of the resulting integrand comes at small q and this thus suggests that we evaluate $\bar{q}$ at the minimum value of q allowed by our other restrictions. This gives in the $p-A$ center of mass system

$$\bar{q} = \frac{1}{2t}\{(t + m_p^2 - m_d^2)[(E_p - E_\pi)^2 - t]^{1/2} + (E_p - E_\pi)\lambda^{1/2}(t, m_p^2, m_d^2)\} \tag{24}$$

with the direction of $\bar{q}$ being $-(\vec{p}_p - \vec{p}_\pi)$. The square of the total energy s_{pp} is then determined from the definition $s_{pp} = (p_p + \bar{q})^2$ and the $pp \to \pi d$ amplitude is evaluated at s_{pp} and t.

This particular choice is apparently the one used by Ingram *et al.*[21] although the explicit formulas were not given. It differs, however, from that used by Barry[18] and it is worth discussing the reasons for the differences. To understand the procedure used by Barry[18] consider the diagram for the nuclear reaction given in Fig. 4. Barry[18] requires first that the $A-1$ spectator nucleons be on the mass shell and then makes one of two choices: either the proton or the deuteron is on the mass shell. Each of these choices, together with a nonrelativistic approximation, determines a value

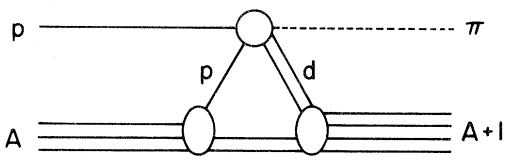

FIG. 4. The mechanism assumed in this calculation to dominate (p, π) reactions. The circle represents the complete $pp \to \pi d$ amplitude.

of s_{pp}, with the deuteron-on-mass-shell choice being preferred as it gives the correct resonance energy. Such an approach seems slightly inconsistent, however, since to evaluate the $pp \to \pi d$ amplitude from experiment both proton and deuteron must be put on the mass shell. Hence one ends up putting all three internal particles on the mass shell. This inconsistency, together with the inadequacy of certain of the nonrelativistic approximations, is reflected in the fact that near large and small pion angles the pair (s_{pp}, t) determined by Barry falls outside the physical region for $pp \to \pi d$, and in the case of the proton-on-shell choice, far outside. In contrast, the method we use puts both the proton and deuteron on the mass shell at the beginning, as is required to evaluate the $pp \to \pi d$ amplitude. The integration over the momentum of the $A-1$ nucleons never appears explicitly, but is effectively buried in the coordinate space integration over the nuclear wave functions. The resulting s_{pp} is more complicated than Barry's[18] preferred choice. It differs little from that choice over mid-range angles, but remains within the physical region at forward and backward angles. Thus it should be the preferable choice.

Our result, Eq. (19), also differs from the calculation of Barry[18] with regard to phase space factors. Our phase space factor agrees with that of Ingram *et al.*[21] provided one correctly transforms the $pp \to \pi d$ cross section from the pp center of mass system to the $p-A$ center of mass system. It differs from the factor used in Ref. 18, however, by the energy term $E_A E_{A+1}/E_q E_d$, which does not appear in that calculation. This difference arises apparently because Barry[18] assumes that the invariant amplitude

$$(E_\pi E_p E_A E_{A+1})^{1/2} \langle \pi A+1 \mid T \mid pA \rangle$$

for the nuclear process corresponds to the invariant amplitude

$$(E_p E_q E_\pi E_d)^{1/2} \langle \pi d \mid T_{01} \mid pp \rangle$$

for $pp \to \pi d$ instead of assuming that the T operators are the same. Thus the energy factor does not appear. However, in simple models, for example in Born approximation with T given by some two nucleon potential, the energy factors should be there. Thus we would conclude that Barry's results are incorrect by this factor and should be multiplied by $E_A E_{A+1}/E_q E_d \simeq \frac{1}{2}A(A+1)$, which will be a major correction for large A.

The major difference between this work and previous calculations is to be found in the more general expression for the form factor $F(\Delta)$ which we have obtained in Eq. (16). Before proceeding to an application of our formalism we want to discuss the additional approximations and simplifying assumptions which would be required if one wished to reduce our results to those used previously. As emphasized above, we have included distortion effects, which has not been done previously in this type of calculation. Thus to reduce our result we would first have to put the F's equal to one. In addition, the following approximations would be required in Eq. (16):

(a) Drop the δ function which enforces the center of mass constraint.

(b) Assume that the nuclear wave functions can be written as a product of a wave function ϕ_{A-1} for the $A-1$ spectator nucleons and either a single particle wave function $\tilde{\phi}_A$ in the case of ϕ_A or a two-particle wave function $\tilde{\phi}_2$ in the case of ϕ_{A+1}. Furthermore ϕ_{A-1} must be the same for initial and final nuclei, e.g. for harmonic oscillator wave functions these assumptions require that the oscillator parameter for initial and final wave functions be the same. These assumptions then allow the integration over $y_2, \ldots, y_A$ to be carried out.

(c) Assume that the two-particle part $\tilde{\phi}_2$ of ϕ_{A+1} can be written as the product of the deuteron wave function in the relative coordinate and a wave function $\tilde{\phi}_{A+1}$ in the coordinate of the two-nucleon center of mass relative to the $A-1$ nucleon core, i.e.

$$\tilde{\phi}_2 = \phi_d(y_0' - y_1')\tilde{\phi}_{A+1}(\tfrac{1}{2}(y_0' + y_1')).$$

This then allows an integration on the relative coordinate $y_0' - y_1'$ and elimination of the deuteron wave function. In previous applications $\tilde{\phi}_{A+1}$ has been obtained from some appropriate nuclear wave function simply evaluated at $y_0' = y_1'$.

With these rather special assumptions on the wave functions, Eq. (16) reduces to

$$F(\Delta) = C \int \tilde{\phi}_{A+1}(x)\tilde{\phi}_A(x) e^{i\Delta \cdot x} d^3x, \tag{25}$$

where we have replaced $x = \frac{1}{2}(y_0' + y_1')$ and where C is a constant containing the various factors involving A which come from proper antisymmetrization and normalization of the wave functions. Thus, up to this factor our result reduces to one of the same simplified form used before, provided that one makes enough extra assumptions. The difficulty is that these assumptions are quite restrictive. For example, in the particular case $pd \to t\pi$ to be discussed below the simple wave functions one normally uses for the triton do not satisfy assumption (c) and, in fact, with this assumption one ends up with a peculiar form for the wave function bearing little resemblance to the original one. Furthermore, there are ambiguities in the normalization of $\tilde{\phi}_2$. Usually it is divided by $\phi_d(0)$

so that $\tilde{\phi}_2$ agrees with the original wave function at $y_0' = y_1'$. However, it is then not a normalized wave function and hence the normalization of $F(\Delta)$ is incorrect, or at the least, ambiguous. In contrast such difficulties do not arise with the application of the full expression for $F(\Delta)$ as given in Eq. (16). The wave functions used there can be chosen arbitrarily and their normalizations, and hence the normalization of $F(\Delta)$, are precisely defined. Thus we would expect the full expression for $F(\Delta)$ which we have derived to be distinctly preferable to the simplified result of Eq. (25), which has been used in previous calculations.[20,21,18]

III. APPLICATION TO $p+d \to t+\pi$

The discussion above has purposely been kept quite general and the results presented in a form applicable to arbitrary nuclear targets since we eventually want to analyze the existing data[1] on ^{12}C and ^{9}Be as well as the higher energy data which will become available as the new high intensity medium energy accelerators such as TRIUMF and LAMPF begin operation. For the present, however, we limit our discussion to the reaction $p+d \to t+\pi$.

There are several reasons for choosing this particular reaction. In the first place it is the simplest of the pion production reactions of this type. Furthermore, the wave functions of both deuteron and triton are reasonably well known. This helps to separate the problem into known and unknown parts and tends to alleviate the difficulties which arise with heavier nuclei because the result is sensitive to so many different factors, all of which are imperfectly known. Thus hopefully we can obtain a better understanding of the strengths and weaknesses of this method of calculation. Finally, with the experiment of Dollhopf *et al.*,[3] reasonable pion angular distributions are available at several energies over the resonance region. Even though the data are not as detailed as the single energy data on ^{12}C and ^{9}Be, the fact that the model must provide a fit over a range of energies should provide a much more stringent test of the model, and perhaps shed some light on the difficulties and ambiguities which have plagued the single energy calculations.

To apply our results to the specific reaction $p+d \to t+\pi$ we simply must specify the input parameters, primarily the wave functions and parameters of the distorting potentials, and evaluate the general results of Eq. (16) and (19). For the wave function of the triton we tried several forms, all of which give reasonable fits to other data:

Irving-Gunn[27,28]:

$$\phi_3(x_0, x_1, x_2) = N_3 \frac{e^{-(1/2)\alpha u}}{u},$$

$$u = [|\vec{x}_0 - \vec{x}_1|^2 + |\vec{x}_1 - \vec{x}_2|^2 + |\vec{x}_0 - \vec{x}_2|^2]^{1/2}; \qquad (26)$$

$$N_3 = \frac{\alpha^2 3^{7/4}}{\pi^{3/2}\sqrt{2}}, \quad \alpha = 152 \text{ MeV};$$

Exponential[21]:

$$\phi_3(x_0, x_1, x_2) = N_3 e^{-(1/2)\alpha u},$$

$$u = |\vec{x}_0 - \vec{x}_1| + |\vec{x}_1 - \vec{x}_2| + |\vec{x}_0 - \vec{x}_2|; \qquad (27)$$

$$N = \frac{3\alpha^3}{\pi}\sqrt{\tfrac{6}{7}}, \quad \alpha = 178 \text{ MeV};$$

Gaussian[28]:

$$\phi_3(x_0, x_1, x_2) = N_3 e^{-(1/2)\alpha^2 u},$$

$$u = |\vec{x}_0 - \vec{x}_1|^2 + |\vec{x}_1 - \vec{x}_2|^2 + |\vec{x}_0 - \vec{x}_2|^2; \qquad (28)$$

$$N_3 = \frac{\alpha^3}{\pi^{3/2}} 3^{9/4}, \quad \alpha = 75.9 \text{ MeV}.$$

For each of these, to get $\phi(y_0, y_1, y_2)$, which is required, we eliminate the x_i using $x_i - x_j = y_i - y_j$. Note that the normalizations are obtained from Eq. (9), and are larger by a factor of $3\sqrt{3}$ than the usual normalizations,[28] since we have chosen to normalize in the individual variables x_i instead of the combinations usually used. This factor is eventually canceled by some of the A factors in $F(\Delta)$ which come from the consistent use of the same variables in the matrix element. Similarly, for the deuteron we use the usual Hulthén wave function:

Hulthén:

$$\phi_2(x_1, x_2) = N_2 \left[\frac{e^{-\beta|\vec{x}_1 - \vec{x}_2|} - e^{-\gamma|\vec{x}_1 - \vec{x}_2|}}{|\vec{x}_1 - \vec{x}_2|}\right], \qquad (29)$$

$$N_2 = \left[\frac{4\beta\gamma(\beta+\gamma)}{\pi(\beta-\gamma)^2}\right]^{1/2}, \quad \beta = 45 \text{ MeV}, \quad \gamma = 270 \text{ MeV}.$$

For $\phi_d(r)$, which is normalized differently, one must use the normalization factor $N_d = 2^{-3/2} N_2$. For both deuteron and triton the usual antisymmetric spin-isospin wave functions were used.[29]

The spin-isospin factor g can be evaluated in a straightforward manner as the spin-isospin wave functions lead to δ functions which allow one to express the numerator simply as a sum on spins of the square of the $pp \to \pi d$ matrix element. This sum includes only the spin-isospin states which give symmetric space states for the two protons, in accordance with the discussion above. There is evidence,[24] however, that these states, which lead

to p-wave pions, dominate the $pp \to \pi d$ reaction, so it should be a good approximation to simply cancel this sum against the complete sum appearing in the denominator. Thus, we obtain the result $g = 1/12$.

The distorting potentials were evaluated via Eq. (21) using data on the pp, pn, πp, πn cross sections taken from Refs. 30 and 31. The density function ρ was taken as a Woods-Saxon form, though for the most part we put the parameter $a = 0$, thus reducing the density to the usual uniform distribution. The radii parameters were determined from Eq. (23) starting with the rms radii $r_d = 2.11$ fm, $r_p = 0.81$ fm, and $r_t = 1.68$ fm, which were then adjusted to give potentials which reproduce the total cross sections. These adjustments involved an increase of 0.55 fm in r_p and of 0.65 fm in r_t, plus a 20% increase in the central density for the pp system. No adjustment seemed necessary for the πd system.

Finally, the experimental data for the $pp \to \pi d$ cross section was taken from the compilation of Ref. 24.

IV. RESULTS

In this section we compare the theroetical results with the data, particularly that of Ref. 3. In addition we want to explore in some detail the sensitivity of the results to the various aspects of the calculation, the aim being, as noted above, to gain some understanding of the strengths and weaknesses of this approach and some confidence in its application before using it in more complicated situations. In general it turns out that the results, particularly in over-all normalization, are fairly sensitive to input quantities. Thus it would be possible to vary some of these quantities as free parameters to obtain a best fit to the data. Such an approach does not seem to us to be particularly illuminating. We are more interested in learning whether the basic assumptions, e.g. impulse approximation and dominance of the $pp \to \pi d$ reaction, lead to a consistent physical picture of the process than in determining parameters of a phenomenological fit. Thus our philosophy has been that first one should evaluate the input parameters insofar as possible from other sources, and then see what results. Hence the curves we obtain depend on *no free parameters*. In particular, there has been no arbitrary normalization and no arbitrary adjustment depending on energy, so that the comparison of theory and experiment over a range of energies does provide a legitimate test of the theory. Clearly, however, the input parameters are not all precisely known, so as a second step we try some different values and/or assumptions to see just how sensitive the results are.

In Figs. 5 and 6 we show some of the theoretical curves plotted against the data at 470 and 590 MeV from Ref. 3, augmented by a few points from Refs. 2 and 4. Figure 7 shows a combination of older data taken at 340 and 325 MeV.[4] For these curves we have used the exponential form for the triton wave function and have simply taken square-well distorting potentials with radial parameters as given in the previous section. The curves shown correspond to: (a) no distortion, (b) distortion factors only for the p-A and π-$(A+1)$ systems in accordance with the discussion above, and (c) distortion factors for all four systems.

It is clear from these figures that distortion effects are fairly important, although only for the over-all normalization of the results. Furthermore, the general trend of the data is reproduced by the theoretical curves at all energies, though

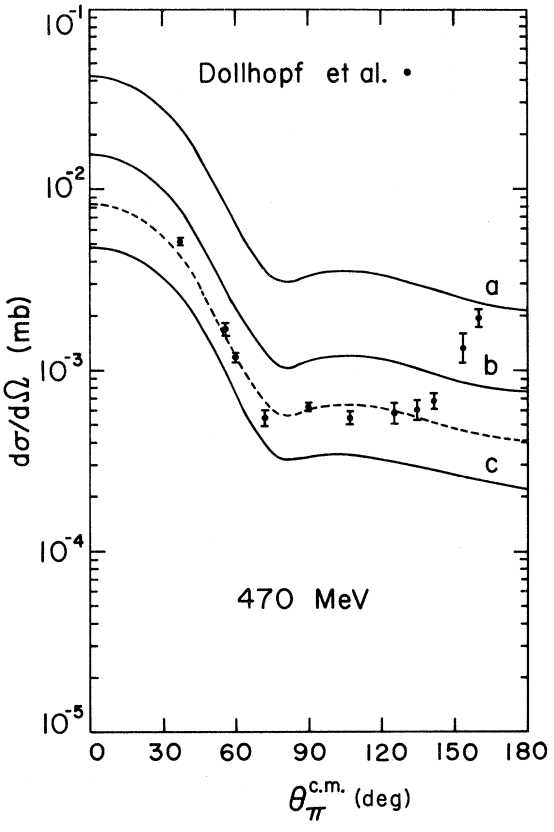

FIG. 5. Comparison with data from Ref. 3 for incident protons of 470 MeV. The curves correspond to (a) no distortion, (b) distortion factors only for the $p-A$ and $\pi-(A+1)$ systems, and (c) distortion factors for all four systems. Note that these results depend on no free parameters. The dotted curve is curve (b) shifted somewhat so as to aid the eye in comparing with the shape of the data.

as might be expected less well at 340 MeV, and at all but the backward angles. In some cases details are also reproduced.

Two aspects of this comparison need further discussion, the shape of the curves and their normalization. Consider first the 470 MeV results of Fig. 5, where we have included as a dotted line curve (b), shifted so as to aid the eye in comparing the shape of the curve with the data. Note that at forward angles the shape is reproduced well, as is the position of the dip and the beginning of the secondary rise. This is a distinct improvement over previous calculations (cf. Refs. 21 and 18) which put the dip at too large an angle. The correct shape results from the improved formula for $F(\Delta)$ rather than from the distortion effects. At backward angles, however, the theory does not reproduce the relatively sharp backward peak. At 590 MeV, we again get reasonable agreement with the shape of the curve, except for the older data point at 37° which falls below the curve. Again, however, the theory seems to fall a little too rapidly at large angles. Finally, at 340 MeV the shape of the curve is again qualitatively correct, though not in as good agreement as at the higher energies. One must be careful here, however, since the various high energy approximations in the theory would suggest that the theory will work less well at lower energies. Also, the data is fairly old and consists of two experiments at somewhat different energies which really only overlap at one angle, where the errors are large. Thus a relative normalization difference between the two experiments could lead to a large qualitative change in the shape of this angular distribution.

Next consider the question of absolute normalization. Recall that one of the difficulties with previous calculations, particularly those based on SNM, has been the failure to obtain reasonably correct normalizations. Early results were incorrect by factors of 50 to 100, although later calculations and those allowing free parameters have

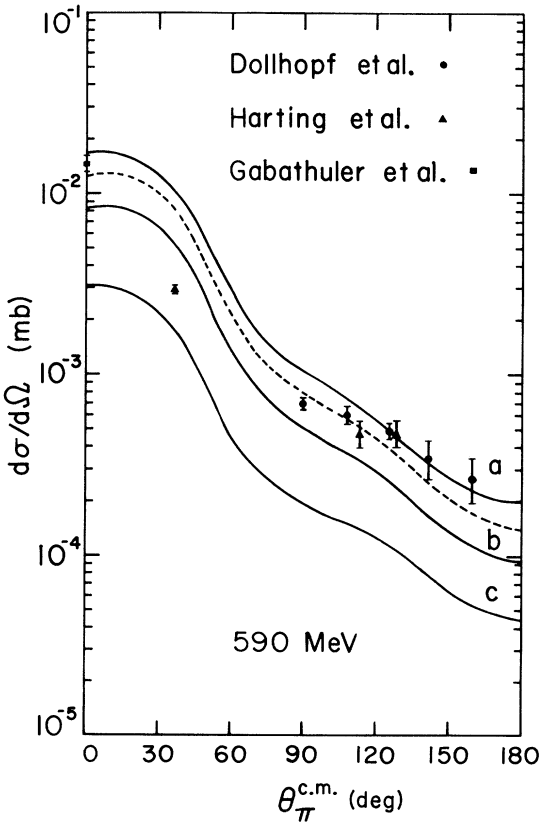

FIG. 6. Comparison with data from Refs. 2, 3, and 4 for incident protons of 590 MeV. The curves correspond to (a) no distortion, (b) distortion factors only for the $p-A$ and $\pi-(A+1)$ systems, and (c) distortion factors for all four systems. The dotted curve is curve (b) shifted somewhat so as to aid the eye in comparing with the shape of the data.

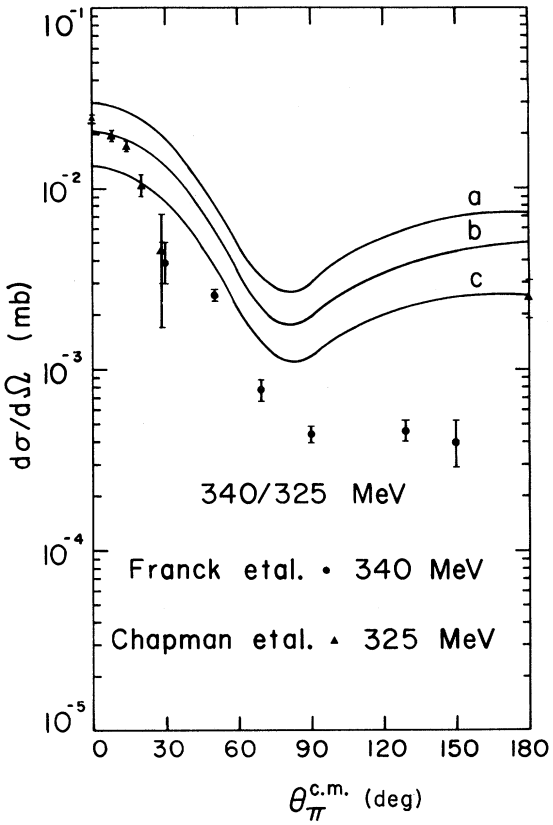

FIG. 7. Comparison with data at 340 and 325 MeV from Ref. 4 with theory at 340 MeV. The curves correspond to (a) no distortion, (b) distortion factors only for the $p-A$ and $\pi-(A+1)$ systems, and (c) distortion factors for all four systems.

done much better than this. If distortion is not included the present calculation gives results which for the most part tend to be too high. Distortion effects reduce the cross section significantly, however, by factors of 3 to 10. In fact, at 470 and 590 MeV the reduction is really too large if we include all four distortion functions F. Recall, however, our earlier argument which emphasized that because of the zero range assumptions which cause $F_{\pi d}$ and F_{pp} to be evaluated at their maximum values, including them may give an overestimate of distortion effects. It does appear that curve (b) corresponding to $F_{\pi d}=F_{pp}=1$, or perhaps something in between (b) and (c), gives a better normalization at 470 and 590 MeV than curve (c), though at 340 MeV all curves are somewhat too high. In general, however, we obtain normalizations which are correct at least at 470 and 590 MeV to within a factor of 2 or so, which is comparable to the best of the SNM and TNM analyses of the p-^{12}C data. It is perhaps worth noting that there may be a systematic trend in our results from being somewhat too high at 340 MeV to being slightly low at 590 MeV.

Another interesting feature evident from these results is the fact that the distortion is almost entirely a normalization effect and does not appreciably change the shape of the cross section curves. This result is obvious for $F_{\pi d}$ and F_{pp}, since they are evaluated at the origin and so are independent of coordinates. Thus the real part of the potentials appearing in their exponents give just an over-all phase, while the imaginary parts lead to a real exponential suppression factor, which simply reflects the absorption of the incoming proton or outgoing pion. It turns out, from the details of the calculation, that to a very good approximation $F_{\pi A+1}$ and F_{pA} are also independent of the pion angle. Thus again they lead to an irrelevant phase plus an over-all suppression factor.

This result has an important consequence in that it means that uncertainties in various parameters of the distorting potentials will be reflected primarily in normalization changes. For example, in Fig. 8 we see that the normalization of the results is changed significantly by non-zero values for the skin thickness parameters a or by a change in the radial parameters, while the shape is changed very little (Note, however, that the magnitude of the change is exaggerated in these curves because we have not imposed the constraint coming from the total cross sections which would require for each change in the radial parameters a partially compensating change in the skin thickness). Similarly, the choice of energy at which to evaluate the cross sections which determine the potentials or possible uncertainties in the values used for these cross sections affect primarily the over-all normalization. This means, from a practical point of view, that one may want to use an over-all normalization as a free parameter in fitting data. As long as this parameter remains within reasonable bounds, i.e. factors of 2 or 3, it can probably be considered not just as an arbitrary parameter but as a reasonable way of accounting for a number of legitimate theoretical uncertainties.

The next question one can ask is what, if anything, does affect the shape of the curve. As already noted, the improved formula for $F(\Delta)$ does put the dip in the right place in contrast to the older approximate result, so the formalism we have developed affects the shape. To a certain extent the wave functions determine the shape also. In Fig. 9 we have included results for the Irving-Gunn and Gaussian wave functions for the triton as

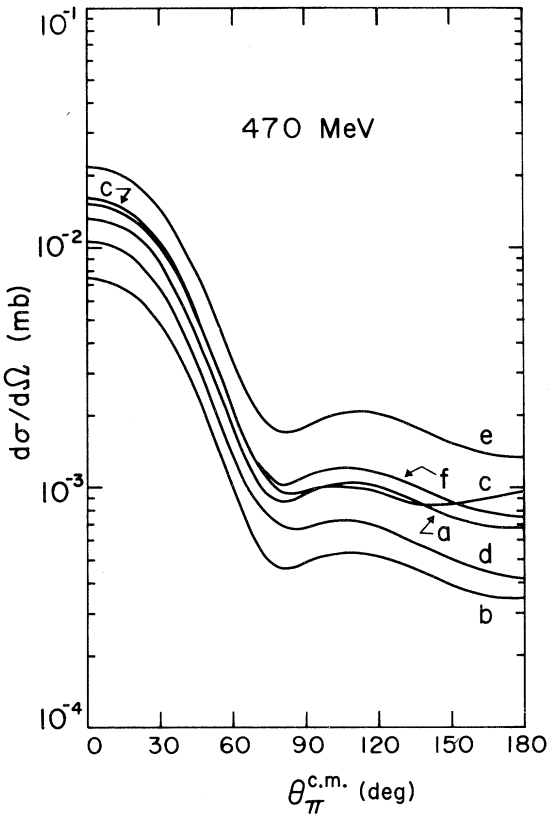

FIG. 8. Effect of variations in some parameters of the distorting potentials and in Δ. An exponential triton wave function has been used and $F_{\pi d}=F_{pp}=1$: (a) skin thickness parameters taken as $a_{\pi t}=0.30$ fm, $a_{pd}=0.30$ fm; (b) rms radius of triton changed to $r_t=1.68$ fm; (c) real part of πt and pd potentials multiplied by a factor of 10; (d) Δ increased by 10%; (e) Δ decreased by 10%; (f) standard result for comparison, same as curve (b) of Fig. 5.

well as an illustration of the effect of ±10% changes in the wave function parameters. One cannot distinguish between Irving-Gunn and exponential wave functions, except for normalization, but can see a small difference with the Gaussian wave-function which gives results which tend to fall more rapidly at large angles. The difficulty is that the triton wave function is so well known that reasonable variations are small, and produce relatively small changes in the results. For heavier nuclei one can probably still expect to obtain useful information about the wave functions, though perhaps not as much as originally hoped.

Finally, we show in Fig. 8 a few other more or less arbitrary changes to illustrate the stability of the calculation to various aspects of the theory. For curve (c) we have multiplied the real parts of the potentials by a factor of 10. The effect is small and indicates an almost total lack of dependence of the results on the choice of these real potentials. Curves (d) and (e) show the effect of a ±10% change in the magnitude of Δ, as might be produced by relaxing the eikonal assumptions. This results primarily in a normalization effect. Thus neither of these possible changes lead to significant changes in the shape of the curves.

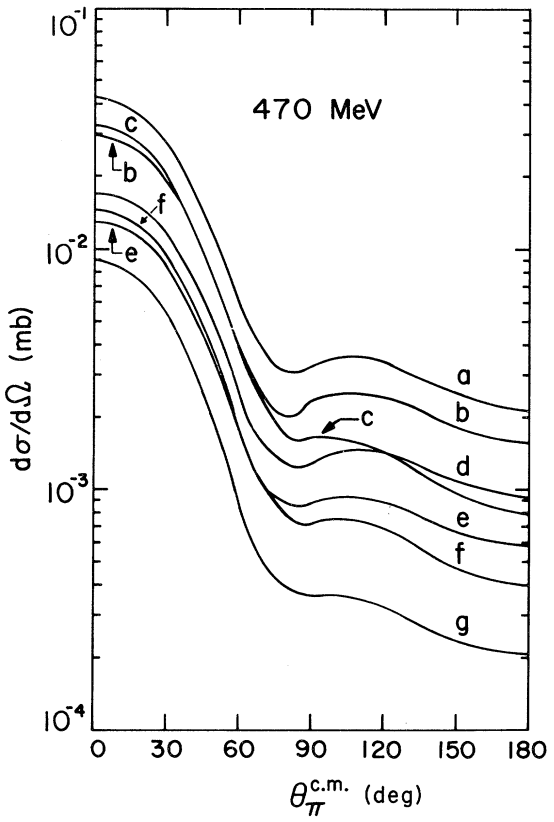

FIG. 9. Effect of variations in the wave functions and in the wave-function parameters: (a) exponential triton wave function, same as curve (a) of Fig. 5; (b) Irving-Gunn wave function; (c) Gaussian wave function; (d) and (e), exponential triton wave function with α respectively increased and decreased by 10%; (f) and (g), Gaussian wave function with α increased and decreased by 10%. Note that, to improve the clarity of the figure, for curves (a)–(c) distortion effects are not included, while for (d)–(g) they have been included for the pd and πt systems only.

This approach seems to be reasonably successful in describing the $pd \to t\pi$ data except for the backward peak at 470 MeV. A broad backward peak does appear at low energies simply as a result of the forward-backward symmetry of the $pp \to \pi d$ cross section. The form factor $F(\Delta)$ suppresses the cross section at larger angles, however, and by the time an energy of 470 MeV is reached the suppression is sufficient to eliminate completely the backward peak. A possible mechanism for a backward peak, based on the one pion exchange diagram of Fig. 3, was considered by Barry[18] and by Bhasin and Duck.[19] This mechanism, which in our language would be referred to as a three-nucleon mechanism, does seem to give a cross section in the backward direction somewhat larger than the results here. However, one obtains a broad enhancement rather than the sharp peak which is observed. Thus we can only conclude that the physical mechanism responsible for the backward peak has not yet been properly included in the calculations.

One final point is worth discussing. The alert reader will have noted that our results, with distortion effects neglected, do not agree very well with the analogous results of Ingram *et al.*[21] and Barry.[18] There are a number of reasons for this. In the first place, some over-all factors differ because of the energy factor discussed above which was neglected by Barry[18] and because of the various additional factors we included from spin and antisymmetrization. More importantly, as emphasized above, the more rigorous expression we have used for $F(\Delta)$ does make a difference. It changes both the shape, shifting the position of the dip and suppressing large angles somewhat, and the the normalization of the results. Some of this difference can be traced to the simple approximation for the wave function, given in (c) above Eq. (25), which has been used previously. For example, if one starts with a Gaussian wave function for the triton and following Refs. 21 and 18 uses this approximation, one gets a "Gaussian" wave function for use in the simple form for $F(\Delta)$. It is, however, quite different from the original Gaussian form [Eq. (28)] which is appropriate for use in the complete formula for $F(\Delta)$. Finally, as mentioned above, there is a further normalization

ambiguity in previous work brought about by the fact that the approximate wave function $\tilde{\phi}_2$ when divided by $\phi_d(0)$, as is usually done, is not completely normalized. In contrast, when using our complete formula [Eq. (16)] for $F(\Delta)$ there are no such ambiguities. No restrictive assumptions on the wave functions are required and the normalizations are uniquely defined via Eq. (9).

Finally, we summarize briefly our results. We have made a number of improvements in the impulse approximation calculation of (p, π) reactions on nuclei. These include a better treatment of the wave functions and the inclusion of distortion effects. For the simplest example of such reactions, $pd \to t\pi$, where wave functions are well known, we are able to obtain zero parameter predictions which reproduce in a reasonable way both the shape and normalization of the data over a range of energies in the resonance region, except for the failure to give the backward peak seen at 470 MeV. The proper treatment of the wave functions is important, as are distortion effects which are, however, almost entirely due to absorption and lead just to changes in the normalizations of the results.

Thus in general this method seems to work for this simple case fairly well and it would seem worthwhile to apply it to heavier nuclei, where one has more to learn, since wave functions are not as well known, and where one is more confident of some of the approximations going into the inclusion of distortion effects. Thus it would be extremely useful to have angular distribution data comparable to that of the Upsalla group for several nuclei over a range of energies in the resonance region. The data should cover at least central angles where the dip and shape of the curve may be more sensitive to the wave functions. It would also be useful to explore the backward angles to try to understand the backward peak. Finally, from a theoretical point of view, if this general approach proves as successful for heavier nuclei as for $pd \to t\pi$ one should perhaps look again at $pp \to \pi d$ to try to determine the appropriate interaction Hamiltonian for these $p\pi$ reactions.

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