Relativistic effects in phenomenological nucleon-nucleon potentials and nuclear matter*

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For any conventional phenomenological nucleon-nucleon potential, we derive a relativistic two-body Hamiltonian. With this two-body Hamiltonian we construct an approximately relativistic many-body Hamiltonian that is used to find the lowest order relativistic corrections to the Brueckner theory of nuclear matter. These corrections are explicitly presented as a sum of terms that are linear and quadratic functions of the nonrelativistic reaction matrix. Numerical results are obtained for the Reid soft core potential, the Ueda-Green potential, and several separable potentials. Due to extensive cancellations of the individual terms, the total correction is considerably smaller than that which would be expected from simple estimates. For the Reid potential the total correction to the energy per particle at saturation is only +0.22 MeV, and the change in the saturation density is negligible. The correction at saturation is also positive for all the other potentials considered.

I. INTRODUCTION

The motion of nucleons in nuclei is nonrelativistic to a good approximation. For a Fermi momentum $k_F = 1.36$ fm^{-1} the relativistic correction to the kinetic energy per particle in homogeneous nuclear matter is about -0.3 MeV or $1\frac{1}{2}\%$. The depth of a typical nuclear potential, the $^1S_0$ Reid Yukawa core potential, is about 100 MeV or about 10% of the nucleon rest energy mc^2. These figures give some indication of the order of magnitude one might expect for relativistic corrections. However, while the expansion parameter

$$\beta = k_F/mc\,, \tag{1.1}$$

is of order 0.3, momenta comparable to mc and larger are important in virtual intermediate states.

The relevant features of relativistic theories are (1) Lorentz invariance and relativistic kinematics, i.e., the relativistic relation between the momentum and the kinetic energy of individual nucleons; and (2) locality, i.e., the requirement that the interaction between nucleons be mediated by a local field or that the interaction be retarded. Historically these requirements have usually been considered as one indivisible entity. Breit's[1] phenomenological theory is based on this view and so are all derivations of potentials from local field theories, boson exchanges, and the Bethe-Salpeter equation. However, it is well known that Lorentz invariance and relativistic kinematics can be realized independently of locality of any of the familiar features of local field theories.[2-6] These requirements by themselves impose no significant restrictions on possible two-body potentials. On the other hand, the two-body interactions in the many-body Hamiltonian depend on the two-body center-of-mass momentum in a manner determined by Lorentz invariance,[7-9] and thus there are relativistic corrections to many-body calculations irrespective of any locality requirements.

Our purpose is to derive the corrections to the Brueckner theory of nuclear matter[10,11] required by Lorentz invariance and relativistic kinematics. Any phenomenological two-body potential fitted to scattering data is equally acceptable in this framework. For nucleons below the Fermi surface the ratios of both the kinetic and potential energies to the rest energy are treated as expansion parameters and corrections are obtained to first order in these parameters.

For potentials that are sufficiently soft so that the Hartree-Fock approximation is justified, relativistic effects can be obtained by formal expansion in inverse powers of the speed of light. For this case the theory has been developed by Bhakar,[8] but his approximations are not valid for the Tabakin potential, which he chose as a numerical example. For realistic potentials it is essential to keep in mind that a formal expansion in inverse powers of the speed of light is not legitimate. Neither can we assume that matrix elements of the potential are small compared to the nucleon rest energy. Lee and Tabakin[12] have carried out a relativistic Brueckner calculation, but they did not consider all terms of order β^2 and their nonrelativistic comparison problem is not the proper nonrelativistic limit of their relativistic calculation.

From the standpoint of field theory all physical particles are composites (dressed particles) and a nucleon-nucleon potential is an effective potential for such composites obtained after elimination of the internal (field) degrees of freedom. The important question of what constraints, if any, the

axioms of field theory impose on the potential is completely outside the scope of the present paper. Brown, Jackson, and Kuo,[13] and Richards, Haftel, and Tabakin[14] have made conjectures concerning "minimal" relativistic modifications of the potential. In their calculations a modified potential is refitted to the phase shifts and the resulting change in the binding energy per nucleon is computed in a standard nonrelativistic Brueckner calculation. Thus their results can be understood as special cases of approximately phase-shift equivalent potentials[15-17] and are not related to the effects considered in the present paper.

To implement our program we require a relativistically invariant relation between the phenomenological potential and the scattering cross section. Relativistic kinematics and the well-known relation between the relativistic S matrix and the cross sections[18] will be reviewed in Sec. II. The relativistic Lippmann-Schwinger equations relate the potentials to the S matrix. Various forms of these equations and their relations are discussed in Sec. III. Relativistic corrections to the many-body Hamiltonian and the Brueckner theory of nuclear matter are the subject of Sec. IV, in which the relativistic effects are displayed explicitly as corrections to the nonrelativistic binding energy.

Numerical results and the program used to produce them are discussed in Sec. V. Calculations have been done for both the Reid soft core (SC) potential and the Ueda-Green (UG) potential model III, as well as for several separable potentials. In all cases the saturation curves with and without relativistic effects have been computed. In the remaining sections of this paper we use units such that $\hbar = c = 1$.

II. RELATIVISTIC KINEMATICS

Let $\vec{p}_1$ and $\vec{p}_2$ be the momenta of two nucleons and m the nucleon mass. The components of the total momentum four-vector $\{\vec{P}, P^0\}$ are

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \tag{2.1}$$

and

$$P^0 = E = E_1 + E_2 , \tag{2.2}$$

where

$$E_i = (p_i^2 + m^2)^{1/2} . \tag{2.3}$$

Let $L(\vec{P}, P^0)$ be that Lorentz transformation which transforms $\{\vec{P}, P^0\}$ into $(0, 0, 0, \omega)$ where

$$\omega = (E^2 - P^2)^{1/2} , \tag{2.4}$$

and define the vector $\vec{k}$ by

$$L(\vec{P}, P^0)\{\vec{p}_1, E_1\} = \{\vec{k}, w\} , \tag{2.5}$$

where

$$w = (k^2 + m^2)^{1/2} . \tag{2.6}$$

The channel spin $\vec{S}$ of two spin-$\frac{1}{2}$ nucleons is related[19] to the individual spins $\vec{s}_1$ and $\vec{s}_2$ by the Wigner rotations $\mathcal{R}[\vec{p}_1, L(\vec{P}, P^0)]$ and $\mathcal{R}[\vec{p}_2, L(\vec{P}, P^0)]$

$$\vec{S} = \mathcal{R}[\vec{p}_1, L(\vec{P}, P^0)]\vec{s}_1 + \mathcal{R}[\vec{p}_2, L(\vec{P}, P^0)]\vec{s}_2 . \tag{2.7}$$

To second order in the nucleon velocities the Wigner rotations are represented by the operators

$$R[\vec{p}_i, L(\vec{P}, P^0)] = 1 - i\tfrac{1}{4}m^{-2}\vec{P}\cdot(\vec{s}_i \times \vec{p}_i) . \tag{2.8}$$

The Jacobian of the variable transformation $\{\vec{p}_1, \vec{p}_2\} \to \{\vec{k}, \vec{P}\}$ is

$$\frac{\partial(\vec{k}, \vec{P})}{\partial(\vec{p}_1, \vec{p}_2)} = \frac{Ew}{2E_1E_2} . \tag{2.9}$$

As a consequence of these definitions the following relations hold:

$$\omega = 2w , \tag{2.10}$$

$$E = (P^2 + \omega^2)^{1/2} , \tag{2.11}$$

$$L(\vec{P}, P^0)\{\vec{p}_2, E_2\} = \{-\vec{k}, w\} , \tag{2.12}$$

and

$$\vec{k} = \vec{p} + \frac{E_2 - E_1}{\omega + E}\vec{P} , \tag{2.13}$$

where

$$\vec{p} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) . \tag{2.14}$$

For small values of the center-of-mass velocities we have

$$E = \omega + \tfrac{1}{2}P^2/\omega - \tfrac{1}{8}P^4/\omega^3 + \cdots , \tag{2.15}$$

and

$$\vec{k} = \vec{p} - \tfrac{1}{2}\vec{P}(\vec{P}\cdot\vec{p})/\omega^2 + \cdots . \tag{2.16}$$

If both particle velocities are small we have to second order

$$\frac{\partial(\vec{k}, \vec{P})}{\partial(\vec{p}_1, \vec{p}_2)} \cong 1 - \frac{1}{2}\left(\frac{P}{2m}\right)^2 . \tag{2.17}$$

States are represented by square integrable functions $\psi(\vec{P}, \vec{k}, S, M_s)$ or $\tilde{\psi}(\vec{p}_1, \vec{p}_2, \mu_1, \mu_2)$, where the μ_i are the projections of the individual particle spins along the axis of quantization. The norm $\|\psi\|$ is given by

$$\begin{aligned} \|\psi\|^2 &= \sum_{S, M_s} \int d^3P \int d^3k\, |\psi(\vec{P}, \vec{k}, S, M_s)|^2 \\ &= \sum_{\mu_1, \mu_2} \int d^3p_1 \int d^3p_2\, |\tilde{\psi}(\vec{p}_1, \vec{p}_2, \mu_1, \mu_2)|^2 . \end{aligned} \tag{2.18}$$

These functions are related as follows:

$$\tilde{\psi}(\vec{p}_1, \vec{p}_2, \mu_1, \mu_2) = \sum_{S, M_s} \left[\frac{\partial(\vec{k}, \vec{P})}{\partial(\vec{p}_1, \vec{p}_2)}\right]^{1/2} \sum_{\mu_1', \mu_2'} (\mu_1 \mid R^{-1}[(\vec{p}_1, L(\vec{P}, P^0)] \mid \mu_1')(\mu_2 \mid R^{-1}[(\vec{p}_2, L(\vec{P}, P^0)] \mid \mu_2') \times (\tfrac{1}{2}, \tfrac{1}{2}; \mu_1', \mu_2' \mid S, M_s)\psi(\vec{P}, \vec{k}, S, M_s), \tag{2.19}$$

where it is understood that $\vec{P}$ and $\vec{k}$ are the functions of $\vec{p}_1$ and $\vec{p}_2$ given by Eqs. (2.1) and (2.13).

Poincaré transformations are represented by unitary operators acting on the functions $\psi(\vec{P}, \vec{k}, S, M_s)$. Let the operators $\vec{X}$ and $\vec{y}$ be defined by

$$\vec{X} = i\frac{\partial}{\partial \vec{P}} \tag{2.20}$$

and

$$\vec{y} = i\frac{\partial}{\partial \vec{k}}. \tag{2.21}$$

The generators of the Poincaré group are then represented by $\vec{P}$ and H_0, $\vec{J}$, and $\vec{K}_0$, where

$$H_0 = (P^2 + h_0^2)^{1/2}, \tag{2.22}$$

$$\vec{J} = \vec{X} \times \vec{P} + \vec{j}, \tag{2.23}$$

and

$$\vec{K}_0 = \tfrac{1}{2}(\vec{X}H_0 + H_0\vec{X}) - \vec{j} \times \vec{P}/(h_0 + H_0), \tag{2.24}$$

where

$$\vec{j} = \vec{y} \times \vec{k} + \vec{S} \tag{2.25}$$

and h_0 is the rest energy of the two-nucleon system

$$h_0 = \omega. \tag{2.26}$$

In Eqs. (2.22)–(2.24) the Poincaré generators are written as functions of the operators h_0, $\vec{X}$, and $\vec{j}$. These relations can be inverted. We have

$$h_0 = (H_0^2 - \vec{P}^2)^{1/2}, \tag{2.27}$$

$$\vec{X} = \tfrac{1}{2}(H_0^{-1}\vec{K}_0 + \vec{K}_0 H_0^{-1}) - [H_0 h_0 (H_0 + h_0)]^{-1}(\vec{P} \times \vec{W}), \tag{2.28}$$

and

$$\vec{j} = \vec{J} - \vec{X} \times \vec{P}, \tag{2.29}$$

where

$$\vec{W} = H_0\vec{J} + \vec{P} \times \vec{K}_0. \tag{2.30}$$

If the generators are defined abstractly, then Eqs. (2.27)–(2.30) may be considered definitions of h_0, $\vec{X}$, and $\vec{j}$.[20]

For the following discussion of scattering we assume spinless nucleons as a matter of convenience. Later, in the many-body problem, we must treat the spins explicitly. The requirement of relativistic invariance implies that the scattering operator S must commute with the generators $\vec{P}$, H_0, $\vec{J}$, and $\vec{K}_0$. This requirement is satisfied if we write the S matrix in the form

$$(\vec{P}', \vec{k}' \mid S \mid \vec{P}, \vec{k}) = \delta(\vec{P}' - \vec{P})\{\delta(\vec{k}' - \vec{k}) - 2\pi i \delta(\omega' - \omega)(\vec{k}' \mid \mathcal{T} \mid \vec{k})\}, \tag{2.31}$$

where

$$[\vec{j}, \mathcal{T}] = 0, \tag{2.32}$$

i.e., $\mathcal{T}$ is invariant under rotation. In Eq. (2.31) we have explicitly realized the requirement that S commute with $\vec{X}$, $\vec{P}$, $\vec{j}$, and h_0, which is both necessary and sufficient for relativistic invariance. From the unitarity condition

$$S^\dagger S = SS^\dagger = 1 \tag{2.33}$$

and rotational invariance, it follows that there are real phase shifts δ_L such that

$$(\vec{k}' \mid \mathcal{T} \mid \vec{k})_{|\vec{k}'|=|\vec{k}|} = -\sum_{L=0}^{\infty} \frac{2L+1}{4\pi^2 k^2}\frac{d\omega}{dk} e^{i\delta_L} \sin\delta_L P_L(\cos\theta), \tag{2.34}$$

where

$$\cos\theta = (\vec{k}' \cdot \vec{k})/k^2$$

and P_L is a Legendre polynomial. The differential cross section is related to the S matrix by

$$d\sigma = \frac{d\Omega}{k^2}\left|\frac{4\pi^2 k^2}{(d\omega/dk)}(\vec{k}' \mid \mathcal{T} \mid \vec{k})\right|^2_{|\vec{k}'|=|\vec{k}|}, \tag{2.35}$$

where $d\Omega$ is the differential solid angle. From Eqs. (2.34) and (2.35) it follows that

$$d\sigma = \frac{d\Omega}{k^2}\left|\sum_L (2L+1)e^{i\delta_L}\sin\delta_L P_L(\cos\theta)\right|^2. \tag{2.36}$$

This last expression does not involve the function $\omega(k)$; the relation between cross sections and phase shifts is the same for relativistic and nonrelativistic kinematics. If we let $\omega(k)$ in Eq. (2.35) be any other monotonic function of k this would merely imply a trivial change in the definition of $(\vec{k}' \mid \mathcal{T} \mid \vec{k})$.

III. LIPPMANN-SCHWINGER EQUATIONS

Following Bakamjian and Thomas[2] we introduce an interaction between the two nucleons by adding

an interaction term v to their rest energy

$$h = h_0 + v, \tag{3.1}$$

where v commutes with $\vec{X}$, $\vec{P}$, and $\vec{j}$. The wave operators are then given by[6]

$$\Omega_\pm = \lim_{t \to \pm\infty} e^{iht} e^{-ih_0 t}, \tag{3.2}$$

and the S operator is

$$S = \Omega_+^\dagger \Omega_-. \tag{3.3}$$

It follows that Eq. (2.31) holds with

$$\mathcal{T} = v\Omega_-. \tag{3.4}$$

The matrix $(\vec{k}' \mid \mathcal{T} \mid \vec{k})$ is therefore the solution of the Lippmann-Schwinger equation

$$(\vec{k}' \mid \mathcal{T} \mid \vec{k}) = (\vec{k}' \mid v \mid \vec{k}) - \lim_{\epsilon \to 0} \int d^3k'' \frac{(\vec{k}' \mid v \mid \vec{k}'')(\vec{k}'' \mid \mathcal{T} \mid \vec{k})}{\omega(k'') - \omega(k) - i\epsilon}. \tag{3.5}$$

The effects of relativistic kinematics show up in the energy denominator. This equation has been used by Schierholz[21] to fit a one-boson exchange potential to phase shifts.

On the other hand, according to Kato[22] we also have the operator relation

$$\Omega_\pm = \lim_{\tau \to \pm\infty} e^{i(1/4)h^2\tau} e^{-i(1/4)h_0^2\tau}, \tag{3.6}$$

and therefore

$$(\vec{P}', \vec{k}' \mid S \mid \vec{P}, \vec{k}) = \delta(\vec{P}' - \vec{P})\{\delta(\vec{k}' - \vec{k}) - 2\pi i m \delta(k'^2 - k^2)(\vec{k}' \mid T \mid \vec{k})\}, \tag{3.7}$$

where

$$T = V\Omega_- \tag{3.8}$$

with

$$V = \frac{1}{4m}(h^2 - h_0^2) = \frac{1}{4m}(h_0 v + v h_0 + v^2), \tag{3.9}$$

which in the nonrelativistic limit reduces to

$$V \cong v. \tag{3.10}$$

The kernel $(\vec{k}' \mid T \mid \vec{k})$ of the operator (3.8) is the solution of the Lippmann-Schwinger equation

$$(\vec{k}' \mid T \mid \vec{k}) = (\vec{k}' \mid V \mid \vec{k}) - \lim_{\epsilon \to 0} \int d^3k'' \frac{(\vec{k}' \mid V \mid \vec{k}'')(\vec{k}'' \mid T \mid \vec{k})}{k''^2/m - k^2/m - i\epsilon}. \tag{3.11}$$

This last equation has the same appearance as the nonrelativistic equation, but Eq. (3.7), with (3.11), satisfies all the requirements of relativistic invariance and unitarity for any Hermitian rotationally invariant V. The operators T and $\mathcal{T}$ are related by

$$T = \tfrac{1}{4}m^{-1}\{h_0 \mathcal{T} + \mathcal{T} h_0\} \tag{3.12}$$

or

$$(\vec{k}' \mid T \mid \vec{k}) = \tfrac{1}{4}m^{-1}(\omega' + \omega)(\vec{k}' \mid \mathcal{T} \mid \vec{k}). \tag{3.13}$$

Equation (3.11) can be made to look like the Blankenbecler-Sugar equation[23] by a slight change of notation:

$$(\vec{k}' \mid \hat{T} \mid \vec{k}) = m^{-1}(k'^2 + m^2)^{1/4}(\vec{k}' \mid T \mid \vec{k})(k^2 + m^2)^{1/4}$$

$$(\vec{k}' \mid \hat{V} \mid \vec{k}) = m^{-1}(k'^2 + m^2)^{1/4}(\vec{k}' \mid V \mid \vec{k})(k^2 + m^2)^{1/4}.$$

However, a careful inspection of the derivation of the Blankenbecler-Sugar equation from field theory makes it clear that the amplitude $\hat{T}$ defined here is *not* the Blankenbecler-Sugar amplitude except in the trivial static limit.

Conventional nonrelativistic procedures that fit phenomenological potentials to cross sections or phase shifts use Eq. (3.11) or the equivalent, together with the relation[24]

$$d\sigma = d\Omega \tfrac{1}{4}(2\pi)^4 m^2 \mid (\vec{k}' \mid T \mid \vec{k}) \mid^2_{|\vec{k}'| = |\vec{k}|}. \tag{3.14}$$

On the other hand, Eq. (3.14) may be derived from Eq. (2.35) with the use of Eqs. (3.13), (2.10), and (2.6). Thus for any conventional phenomenological potential V there is a relativistic two-body Hamiltonian

$$H = (P^2 + h^2)^{1/2} \tag{3.15}$$

implied by Eq. (3.9).

IV. MANY-BODY HAMILTONIAN AND THE BINDING ENERGY OF NUCLEAR MATTER

A fully relativistic many-body Hamiltonian within the present framework is not available; it is also not needed. Construction of the many-body Hamiltonian by formal expansion in inverse powers of the speed of light[7-9] is not justified unless all velocities are small compared to the speed of light and the norm of the interaction operator is small compared to m. For any two-body cluster under consideration the relative velocity may be quite high and the potential is in general unbounded. Our approximations will be based on the fact that both the kinetic energy and potential energy of a nucleon below the Fermi level are small compared to the rest energy of the nucleon. For any two-body cluster we can expand in powers of the total momentum of that cluster. The Hamiltonian is then

$$H = (P^2 + h^2)^{1/2} \cong h + P^2/2h - P^4/8h^3 + \cdots, \tag{4.1}$$

and the two-body potential $\mathcal{V}$ is defined by

$$\mathcal{V} = H - H_0 . \tag{4.2}$$

Since the Hamiltonian commutes with $\vec{P}$ the kernel of the operator $\mathcal{V}$ has the form

$$(\vec{P}', \vec{k}' \mid \mathcal{V} \mid \vec{P}, \vec{k}) = \delta(\vec{P}' - \vec{P})(\vec{k}' \mid \mathcal{V}(\vec{P}) \mid \vec{k}). \tag{4.3}$$

After expansion in powers of $\vec{P}$ to second order we have

$$(\vec{k}' \mid \mathcal{V}(\vec{P}) \mid \vec{k}) = (\vec{k}' \mid v \mid \vec{k}) - \tfrac{1}{2} P^2 (\vec{k}' \mid h_0^{-1} v h^{-1} \mid \vec{k}). \tag{4.4}$$

The spin variables have been suppressed for the sake of simplicity. The transformation to individual particle variables is given by Eq. (2.19).

Let us proceed to construct the many-body Hamiltonian. In addition to the invariance requirements, we must now satisfy the cluster separability condition first recognized by Foldy.[25,6,9] We may assume that the total momentum of the many-body system vanishes. If particles 1 and 2 interact with each other but not with the other particles in the system then the total energy has the form

$$h_{\text{tot}} = [P^2 + h(12)^2]^{1/2} + [P^2 + \bar{h}^2]^{1/2}, \tag{4.5}$$

where $\bar{h}$ is the rest energy of all the other particles. The two-body interaction term in the many-body Hamiltonian is thus given by

$$[P^2 + h(12)^2]^{1/2} - [P^2 + h_0(12)^2]^{1/2} = \mathcal{V}(\vec{P}), \tag{4.6}$$

independently of the presence of other particles at large distances. The many-body Hamiltonian will necessarily have three-body interactions, four-body interactions, etc.[9] For nuclear matter at normal densities the effect of these many-body forces should be small. At any rate it does not make sense to retain such three-body forces unless all three-body correlation effects are considered. In the following we are concerned only with the relativistic corrections to the lowest order Brueckner approximation. The calculations will therefore be based on a many-body Hamiltonian in which only the two-body interactions are retained:

$$H_{\text{tot}} = \int d^3p\, c^\dagger(\vec{p})[(p^2 + m^2)^{1/2} - m]c(\vec{p}) + \tfrac{1}{2} \int d^3p_1' \int d^3p_2' \int d^3p_1 \int d^3p_2 c^\dagger(\vec{p}_1') c^\dagger(\vec{p}_2')(\vec{p}_1', \vec{p}_2' \mid \mathcal{V} \mid \vec{p}_2, \vec{p}_1) c(\vec{p}_2) c(\vec{p}_1), \tag{4.7}$$

where

$$(\vec{p}_1', \vec{p}_2' \mid \mathcal{V} \mid \vec{p}_2, \vec{p}_1) = \delta(\vec{P}' - \vec{P}) \left\{ \left[\frac{\partial(\vec{k}', \vec{P}')}{\partial(\vec{p}_1', \vec{p}_2')} \right]^{1/2} (\vec{k}' \mid \mathcal{V}(\vec{P}) \mid \vec{k}) \left[\frac{\partial(\vec{k}, \vec{P})}{\partial(\vec{p}_1, \vec{p}_2)} \right]^{1/2} \right\}, \tag{4.8}$$

and the momenta $\vec{k}$, $\vec{P}$ are related to $\vec{p}_1$, $\vec{p}_2$ by Eqs. (2.13) and (2.1). Here $c(\vec{p})$ and $c^\dagger(\vec{p})$ are, respectively, destruction and creation operators for a nucleon of momentum $\vec{p}$.

With this many-body Hamiltonian we obtain the potential energy per particle $\mathcal{E}$ in homogeneous nuclear matter in the Brueckner approximation.[26] The different momentum dependence of the kinetic energy and the P dependence of the potential do not require any qualitative changes in the procedure. The result is

$$\mathcal{E} = \frac{3}{16\pi k_F^3} \int_{p_1 \le k_F} d^3p_1 \int_{p_2 \le k_F} d^3p_2 \sum_{S, M_S} \sum_{T} (2T+1)(\vec{k}, M_s \mid g_{S,T}(\vec{P}) \mid \vec{k}, M_s) \frac{\partial(\vec{k}, \vec{P})}{\partial(\vec{p}_1, \vec{p}_2)}, \tag{4.9}$$

where g is defined as the solution of the equation

$$(\vec{k}' \mid g(\vec{P}) \mid \vec{k}) = (\vec{k}' \mid \mathcal{V}(\vec{P}) \mid \vec{k}) - \int d^3k''(\vec{k}' \mid \mathcal{V}(\vec{P}) \mid k'')(E'' - E_\gamma)^{-1} Q(\vec{k}'', \vec{P})(\vec{k}'' \mid g(\vec{P}) \mid \vec{k}). \tag{4.10}$$

[Since the Wigner rotation R in Eq. (2.19) is a unitary matrix, it contributes no correction terms to the trace in Eq. (4.9).] The projection operator $Q(\vec{k}'', \vec{P})$ is unity if both $p_1(\vec{k}'', \vec{P})$ and $p_2(\vec{k}'', \vec{P})$ are above the Fermi level, zero otherwise. The energies E'' and E_γ are defined by

$$E'' \equiv [p_1''^2 + m^2]^{1/2} + [p_2''^2 + m^2]^{1/2} = [P^2 + 4(k''^2 + m^2)]^{1/2} \tag{4.11}$$

and

$$E_\gamma \equiv E + u(\vec{p}_1) + u(\vec{p}_2), \tag{4.12}$$

where

$$E \equiv (p_1^2 + m^2)^{1/2} + (p_2^2 + m^2)^{1/2} = [P^2 + 4(k^2 + m^2)]^{1/2}. \tag{4.13}$$

The single-particle potential energy below the Fermi level is

$$u(\vec{p}_1) = \tfrac{1}{2} \int_{p_2 \le k_F} d^3p_2 \sum_{S, M_S} \sum_{T} (2T+1) \times (\vec{k}, M_s \mid g(\vec{P}) \mid \vec{k}, M_s) \frac{\partial(\vec{k}, \vec{P})}{\partial(\vec{p}_1, \vec{p}_2)}. \tag{4.14}$$

It is convenient to define the quantity γ^2 such that

$$E_\gamma = [P^2 + 4(m^2 - \gamma^2)]^{1/2}. \tag{4.15}$$

It follows that

$$\gamma^2 = -\{k^2 + \tfrac{1}{2}[u(\vec{p}_1) + u(\vec{p}_2)]E + \tfrac{1}{4}[u(\vec{p}_1) + u(\vec{p}_2)]^2\}. \tag{4.16}$$

For weak soft potentials we have

$$(\vec{k}' \mid g \mid \vec{k}) \cong (\vec{k}' \mid \mathfrak{v} \mid \vec{k}) \cong (\vec{k}' \mid v \mid \vec{k})\left\{1 - \frac{1}{2}\left(\frac{P}{2m}\right)^2\right\}. \tag{4.17}$$

The potential energy per particle in the Hartree-Fock approximation is then

$$\mathcal{E} = \frac{3}{16\pi k_F^{\,3}} \int_{p_1 \le k_F} d^3p_1 \times \int_{p_2 \le k_F} d^3p_2 \sum_{S,T} (2S+1)(2T+1)[1 - (P/2m)^2] \times (\vec{k}' \mid v_{ST} \mid \vec{k}). \tag{4.18}$$

Since $\vec{p}_1$ and $\vec{p}_2$ are below the Fermi level we may expand in powers of p/m. From Eq. (2.16) it follows that

$$(\vec{k}' \mid v \mid \vec{k}) = (\vec{p}' \mid v \mid \vec{p}) - \tfrac{1}{8}m^{-2}(\vec{P}\cdot\vec{p}')(\vec{P}\cdot\nabla_{p'})(\vec{p}' \mid v \mid \vec{p}) - \tfrac{1}{8}m^{-2}(\vec{P}\cdot\vec{p})(\vec{P}\cdot\nabla_{p})(\vec{p}' \mid v \mid \vec{p}). \tag{4.19}$$

Bhakar's[8] result obtains if we insert Eq. (4.19) into Eq. (4.18). He found that the contribution of the correction terms in Eq. (4.19) is small compared to the term $-(P/2m)^2 v$.

Brueckner calculations with Eqs. (4.9) and (4.10) have been done by Lee and Tabakin.[12] They ignore the difference between $\mathfrak{v}$ and v [Eq. (4.4)] as well as the difference between $\vec{k}$ and $\vec{p}$ [Eq. (2.16)]. The results are compared to purely nonrelativistic computations done with the same potential v.

Our purpose is to obtain relativistic corrections to the Brueckner theory for a given empirical potential V. To that end we recast Eqs. (4.9) and (4.10) in a form that exhibits the dominant nonrelativistic contribution and the relativistic corrections explicitly. Let the operator $\tilde{G}$ be defined by

$$\tilde{G} = \tfrac{1}{4}m^{-1}(H_0 + E_\gamma)g, \tag{4.20}$$

and remember that the operators V and $\mathfrak{v}$ are related by

$$V = \tfrac{1}{4}m^{-1}(H_0\mathfrak{v} + \mathfrak{v}H_0 + \mathfrak{v}^2), \tag{4.21}$$

since

$$h^2 - h_0^{\,2} = H^2 - H_0^{\,2}. \tag{4.22}$$

Equation (4.10) may be written in operator form as follows:

$$g = \mathfrak{v} - \mathfrak{v}(H_0 - E_\gamma)^{-1}Qg. \tag{4.23}$$

With the definition

$$e = \tfrac{1}{4}(H_0^{\,2} - E_\gamma^{\,2})/m, \tag{4.24}$$

it follows from Eqs. (4.23), (4.20), and (4.24) that

$$\tilde{G} = V - V(Q/e)\tilde{G} - \mathfrak{v}\{\tfrac{1}{4}(H_0 - E_\gamma)/m + (H_0 + E_\gamma)^{-1}(1-Q)\tilde{G}\}. \tag{4.25}$$

With

$$G = [1 + V(Q/e)]^{-1}V \tag{4.26}$$

we find

$$\tilde{G} = G - [1 + V(Q/e)]^{-1}\mathfrak{v}D, \tag{4.27}$$

where

$$D = (H_0 + E_\gamma)^{-1}[e + (1-Q)G]. \tag{4.28}$$

The reaction matrix G is the leading term and the second term is a correction in which we have replaced $\tilde{G}$ by G.

In matrix elements of $\tilde{G}$ in which all nucleons are below the Fermi level we may expand in powers of P/m, k/m, and γ/m, which are all of order β. We neglect terms of order higher than β^2. For such matrix elements $H_0 + E_\gamma \cong 4m$ and

$$(\vec{k}' \mid D \mid \vec{k}) \cong [(k^2 + \gamma^2)/4m^2]\delta(\vec{k}' - \vec{k}) + (\vec{k}' \mid (1-Q)G \mid \vec{k})/4m. \tag{4.29}$$

It follows that

$$(\vec{k} \mid \tilde{G} \mid \vec{k}) = (\vec{k} \mid G(1-D) \mid \vec{k}) - (\vec{k} \mid [1 + V(Q/e)]^{-1}(\mathfrak{v} - V)D \mid \vec{k}). \tag{4.30}$$

That the last term is usually negligible may be seen as follows. From Eq. (4.21) it follows that

$$\mathfrak{v} = [4mV - (H - E_\gamma)\mathfrak{v}](H_0 + E_\gamma)^{-1}. \tag{4.31}$$

Since

$$\{1 + V(Q/e)\}^{-1}(H - E_\gamma) = 4m[e + G(1-Q)](H + E_\gamma)^{-1}, \tag{4.32}$$

it follows that

$$(\vec{k} \mid [1 + V(Q/e)]^{-1}(\mathfrak{v} - V)D \mid \vec{k}) \approx (\vec{k} \mid \{G[4m/(H_0 + E_\gamma) - 1] - 4mD^\dagger\mathfrak{v}(H + E_\gamma)^{-1}\}D \mid \vec{k}). \tag{4.33}$$

According to Eq. (4.29), the relevant matrix elements of D are of order β^2. The term (4.33) is therefore of order β^4. Whether or not the term is negligible compared to other corrections depends

on the size of the matrix elements of $4m(H+E_\gamma)^{-1}\mathfrak{V}$. If the potential $\mathfrak{V}$ is bounded and $\|\mathfrak{V}\| \ll m$, then the correction due to the term (4.33) is obviously negligible. For strong short-range potentials the norm $\|4m(H+E_\gamma)^{-1}\mathfrak{V}\|$ can be as large as $4m$; nevertheless it is easy to verify that the contributions to the term (4.33) remain small. In the following the second term of Eq. (4.30) will be dropped. Numerical tests of that approximation will be described in the next section.

From Eqs. (4.9), (4.14), (2.17), (4.20), (4.29), and (4.30) it follows that the potential energy per particle is

$$\mathcal{E} = \frac{3}{8\pi k_F^{\,3}} \int_{p_1 \le k_F} d^3p_1 u(\vec{p}_1), \tag{4.34}$$

where the single-particle potential energy is

$$u(p_1) = \tfrac{1}{2} \int_{p_2 \le k_F} d^3p_2 \sum_{S,M_S} \sum_{T} (2T+1)\{(\vec{k}, M_s \mid G_{S,T}(\vec{P}) \mid \vec{k}, M_s)\,[1 - (\vec{P}/2m)^2 - \tfrac{1}{2}(k/m)^2] - \tfrac{1}{4}m^{-1}(\vec{k}, M_s \mid G(1-Q)G \mid \vec{k}, M_s)\}. \tag{4.35}$$

It should be noted that by making use of the result of Eq. (4.33), we have succeeded in expressing the relativistic correction completely in terms of the reaction matrix G which is determined from the conventional potential V by the linear integral equation (4.26). Thus for a given phenomenological potential V, it is *not* necessary to solve the nonlinear Eq. (3.9) for v to be able to compute the nuclear matter corrections to order β^2.

Equation (4.35) does not yet explicitly display all the relativistic effects as corrections to a nonrelativistic energy since G is not the nonrelativistic reaction matrix. Indeed it can be seen that the difference between $\vec{k}$ and $\vec{p}$ [Eq. (2.16)], the relativistic form of E_γ in e [Eqs. (4.15) and (4.24)] and the relativistic kinematics used in the Pauli operator Q will all affect the reaction matrix G. We wish to display these effects explicitly so that they can be individually computed and compared to each other and to the terms already contained in Eq. (4.35). For notational simplicity we will suppress the isospin and spin labels for this discussion.

The last three terms in Eq. (4.35) are of order β^2 and hence in these terms $(\vec{k}' \mid G \mid \vec{k})$ may be replaced by its nonrelativistic limit

$$(\vec{p}' \mid G^0 \mid \vec{p}) = (\vec{p}' \mid [1 + V(Q_0/e_0)]^{-1} V \mid \vec{p}), \tag{4.36}$$

where $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$ and

$$Q_0(\vec{p}'', \vec{P}) = \begin{cases} 1 \text{ if both } |\tfrac{1}{2}\vec{P} \pm \vec{p}''| \ge k_F, \\ 0 \text{ otherwise}. \end{cases} \tag{4.37}$$

According to Sec. III, the operator V is the nonrelativistic potential. The nonrelativistic energy denominator e_0 is defined by

$$e_0\psi(\vec{p}') = m^{-1}(p'^2 + \gamma_0^{\,2})\psi(\vec{p}') \tag{4.38}$$

with

$$\gamma_0^{\,2} = -\{p^2 + m[u_0(\vec{p}_1) + u_0(\vec{p}_2)]\}. \tag{4.39}$$

To order β^2 we may replace $(\vec{k} \mid G \mid \vec{k})$ in the first term on the right of Eq. (4.35) by

$$(\vec{k} \mid G \mid \vec{k}) \approx (\vec{p} \mid G^0 \mid \vec{p}) + [(\vec{k} \mid G^0 \mid \vec{k}) - (\vec{p} \mid G^0 \mid \vec{p})] + (\vec{p} \mid G - G^0 \mid \vec{p}). \tag{4.40}$$

The corrections due to the relativistic form of the reaction matrix (4.26) are contained in the last term of Eq. (4.40) and may be evaluated by using the relation

$$\begin{aligned} G - G^0 &= -G[Q/e - Q_0/e_0]G^0 \\ &\approx -G^0[(Q-Q_0)/e_0]G^0 \\ &\quad + m^{-1}(\gamma^2 - \gamma_0^{\,2})G^0(Q_0/e_0^{\,2})G^0. \end{aligned} \tag{4.41}$$

From Eqs. (4.16), (4.39), and (2.16) it follows that

$$\begin{aligned} m^{-1}(\gamma^2 - \gamma_0^{\,2}) = &-\{(\tfrac{1}{2}p^2 + \tfrac{1}{8}P^2)/m^2 + \tfrac{1}{4}[u_0(p_1) + u_0(p_2)]/m\} \\ &\times [u_0(p_1) + u_0(p_2)] + \tfrac{1}{16}(p_1^{\,2} - p_2^{\,2})^2/m^3 \\ &- [u(p_1) + u(p_2)] + [u_0(p_1) + u_0(p_2)]. \end{aligned} \tag{4.42}$$

The first two terms in Eq. (4.42) are due to the relativistic form of γ^2 and will be referred to as the γ correction. The correction due to the relativistic single-particle energy will be called the spectral correction and the first term of Eq. (4.41) will be called the Pauli correction.

We will refer to terms that arise from the difference

$$\vec{k}' = \vec{k} - \vec{p} = -\tfrac{1}{8}m^{-2}(\vec{P} \cdot \vec{p})\vec{P} \tag{4.43}$$

as the Lorentz correction:

$$u_{\mathrm{Lor}}(\vec{p}_1) = \tfrac{1}{2} \int_{p_2 \le k_F} d^3p_2 \{(\vec{k} \mid G^0(\gamma_0^{\,2}) \mid \vec{k}) - (\vec{p} \mid G^0(\gamma_0^{\,2}) \mid \vec{p})\}, \tag{4.44}$$

where $\vec{p}$, $\vec{k}$, and $\gamma_0^{\,2}$ are functions of $\vec{p}_1$ and $\vec{p}_2$ defined, respectively, by Eqs. (2.14), (2.16), and (4.39). We evaluate the first term of Eq. (4.44) by

changing the integration variable to

$$\vec{p}_2' = \vec{p}_2 - 2\vec{k}', \tag{4.45}$$

so that $\vec{p}_2$ is replaced by $\vec{p}_2' + 2\vec{k}'$. The Jacobian of this transformation is

$$\frac{\partial(\vec{p}_2)}{\partial(\vec{p}_2')} = 1 + \frac{1}{8m^2}|\vec{p}_1 + \vec{p}_2|^2 + \frac{1}{2m^2}(p_2^2 - p_1^2), \tag{4.46}$$

where to lowest order $\vec{p}_2$ may be replaced by $\vec{p}_2'$. Finally the integration variable $\vec{p}_2'$ is renamed to be $\vec{p}_2$. The limits of the integral are then

$$|\vec{p}_2 + 2\vec{k}'| \leq k_F, \tag{4.47}$$

and the value of the argument γ_0^2 is $\gamma_0^2(\vec{p}_1, \vec{p}_2 + 2\vec{k}')$. The effective mass approximation[27] yields

$$\gamma_0^2(\vec{p}_1, \vec{p}_2 + 2\vec{k}') = \gamma_0^2(\vec{p}_1, \vec{p}_2) - \tfrac{1}{16}(p_2^2 - p_1^2)^2/(mm^*). \tag{4.48}$$

Thus we obtain the following expression for the right-hand side of Eq. (4.43):

$$u_{\mathrm{Lor}}(\vec{p}_1) = -\tfrac{1}{2}\int_\Omega d^3p_2 (\vec{p}\,|\,G^0(\gamma_0^2)\,|\,\vec{p}) + \tfrac{1}{2}\int_{p_2 \leq k_F} d^3p_2 \Big\{\Big[\frac{1}{8m^2}|\vec{p}_1 + \vec{p}_2|^2 + \frac{1}{2m^2}(p_2^2 - p_1^2)\Big](\vec{p}\,|\,G^0(\gamma_0^2)\,|\,\vec{p}) + \frac{1}{16m^2m^*}(p_2^2 - p_1^2)^2 (\vec{p}\,|\,G^0 \frac{Q_0}{e_0^2} G^0\,|\,\vec{p})\Big\}, \tag{4.49}$$

where the integration volume Ω is specified by the inequalities

$$p_2 \leq k_F, \tag{4.50}$$

and

$$|\vec{p}_2 + 2\vec{k}'| \geq k_F. \tag{4.51}$$

To order β^2 the first integral in Eq. (4.49) is

$$\int_\Omega d^3p_2 (\vec{p}\,|\,G^0\,|\,\vec{p}) = \tfrac{1}{8}m^{-2}(k_F^2 - p_1^2)\int d^3p_2 \delta(p_2 - k_F) \times (k_F + p_1 z_{12})(\vec{p}\,|\,G^0\,|\,\vec{p}), \tag{4.52}$$

where

$$z_{12} = \vec{p}_1 \cdot \vec{p}_2/(p_1 p_2). \tag{4.53}$$

Equations (4.41), (4.42), (4.49), and (4.52) display the corrections that are not explicitly given in Eq. (4.35). If these terms are added to Eq. (4.35) we may then replace G by G^0 and thus express the single-particle potential energy as the nonrelativistic value plus correction terms. Equation (4.34) will then give the corresponding form for the potential energy per particle. Actual computations are greatly facilitated by the use of the angle averaged Pauli operator[28] defined by

$$\overline{Q}(p, P) = \tfrac{1}{2}\int_{-1}^{1} dz\, Q(\vec{p}, \vec{P}), \tag{4.54}$$

where

$$z = \vec{p} \cdot \vec{P}/(pP). \tag{4.55}$$

For the nonrelativistic case this is

$$\overline{Q}_0(p, P) = \begin{cases} 0 & p \leq p_A, \\ z_0 & p_A \leq p \leq p_B, \\ 1 & p_B \leq p, \end{cases} \tag{4.56}$$

where

$$p_A = (k_F^2 - \tfrac{1}{4}P^2)^{1/2}, \tag{4.57}$$

$$p_B = k_F + \tfrac{1}{2}P, \tag{4.58}$$

and

$$z_0 = (\tfrac{1}{4}P^2 + p^2 - k_F^2)/(Pp). \tag{4.59}$$

To lowest order the relativistic correction to $\overline{Q}_0$ is

$$\overline{Q}(p, P) - \overline{Q}_0(p, P) = \begin{cases} \tfrac{1}{4}[(k_F^2 - p^2)^2 - \tfrac{1}{16}P^4]/(pPm^2), \\ \qquad\qquad \text{for } p_A \leq p \leq p_B, \\ 0 \quad \text{otherwise}. \end{cases} \tag{4.60}$$

For fixed values of P, $\overline{Q}$ is a scalar two-body operator as is e_0. Hence for fixed values of γ_0 and P, G^0 is a scalar two-body operator as are products involving G^0, $\overline{Q}$, and e_0^{-1}. The matrix elements of any such operator $\mathcal{O}$ may be expressed as a partial wave sum:

$$(\vec{p}', M_s' \,|\, \mathcal{O}_{S,T}(\gamma_0^2, \vec{P})\,|\,\vec{p}, M_s) = \sum_{J, M_J}\sum_{L, L'} (\hat{p}', M_s' \,|\, \mathcal{Y}_S \,|\, L', J, M_J) \times (p', L' \,|\, \mathcal{O}_{J,S,T}(\gamma_0^2, P)\,|\,p, L) \times (L, J, M_J \,|\, \mathcal{Y}_S^\dagger \,|\, \hat{p}, M_s), \tag{4.61}$$

where $\hat{p}$ is a unit vector in the direction of $\vec{p}$ and the transformation matrix $\mathcal{Y}$ is given in terms of Clebsch-Gordan coefficients and spherical harmonics by

$$(\hat{p}, M_s \,|\, \mathcal{Y}_S \,|\, L, J, M_J) = \sum_{M_L} (L, S; M_L, M_s \,|\, J, M_J) Y_{L, M_L}(\hat{p}). \tag{4.62}$$

It follows that

$$\sum_{M_S}(\vec{p}, M_S \mid \mathcal{O}_{S,T}(\gamma_0^2, \vec{P}) \mid \vec{p}, M_S) = \frac{1}{4\pi}\sum_{J,L}(2J+1)(p, L \mid \mathcal{O}_{J,S,T}(\gamma_0^2, P) \mid p, L). \tag{4.63}$$

Using this identity we write the single-particle potential energy as a partial wave sum and summarize the explicit expressions for all the relativistic corrections. Equations (4.34) and (4.35) take the form

$$\mathcal{E} = \tfrac{3}{2}k_F^{-3}\int_0^{k_F} dp_1 p_1^2 u_0(p_1), \tag{4.64}$$

and

$$u_0(p_1) = \tfrac{1}{4}\sum_{J,S,T,L}(2T+1)(2J+1)\int_0^{k_F} dp_2 p_2^2 \int_{-1}^{1} dz_{12} \times\{(p, L \mid G^0_{J,S,T} \mid p, L) + \sum_\alpha B_\alpha\}. \tag{4.65}$$

Here the B_α represent the various relativistic correction terms. The subscript α is symbolic; we will label the B's with names.

The first two correction terms of Eq. (4.35) will be referred to as the "main" correction since numerically they are of the largest magnitude:

$$B_{\text{main}} = -[(P/2m)^2 + \tfrac{1}{2}(k/m)^2](p, L \mid G^0_{J,S,T} \mid p, L). \tag{4.66}$$

The last term will be called the $1-Q$ correction:

$$B_{1-Q} = -\tfrac{1}{4}m^{-1}(p, L \mid G^0_{J,S,T}(1-\overline{Q}_0)G^0_{J,S,T} \mid p, L). \tag{4.67}$$

The Pauli correction [first term of Eq. (4.41)] is

$$B_{\text{Pauli}} = -(p, L \mid G^0_{J,S,T}[(\overline{Q}-\overline{Q}_0)/e_0]\, G^0_{J,S,T} \mid p, L). \tag{4.68}$$

From Eq. (4.42) and Eq. (4.39) the γ and spectral corrections are

$$B_\gamma = \tfrac{1}{4}m^{-3}[(\gamma_0^2+p^2)(p^2+\tfrac{1}{2}P^2-\gamma_0^2)+\tfrac{1}{4}(p_1^2-p_2^2)^2] \times(p, L \mid G^0_{J,S,T}[\overline{Q}_0/e_0^2]G^0_{J,S,T} \mid p, L), \tag{4.69}$$

and

$$B_{\text{spec}} = -[u(p_1)+u(p_2)-u_0(p_1)-u_0(p_2)] \times(p, L \mid G^0_{J,S,T}[\overline{Q}_0/e_0^2]G^0_{J,S,T} \mid p, L), \tag{4.70}$$

where $u(p_i)$ contains all the other relativistic corrections. From Eqs. (4.49) and (4.52) the Lorentz correction is

$$B_{\text{Lor}} = \tfrac{1}{8}m^{-2}\{[-\delta(p_2-k_F)(k_F^2-p_1^2)(k_F+p_1 z_{12}) + P^2 + 4(p_2^2-p_1^2)](p, L \mid G^0_{J,S,T} \mid p, L) + \frac{(p_2^2-p_1^2)}{2m^*}(p, L \mid G^0_{J,S,T}\frac{\overline{Q}_0}{e_0^2}\, G^0_{J,S,T} \mid p, L)\}. \tag{4.71}$$

To compute the binding energy we need the kinetic energy per particle:

$$K = \frac{4}{(16/3)\pi k_F^3}\int_{p_1 \le k_F} d^3p_1[(p_1^2+m^2)^{1/2}-m] \tag{4.72}$$

$$\approx (\tfrac{3}{10}-\tfrac{3}{56}\beta^2)k_F^2/m\,. \tag{4.73}$$

V. NUMERICAL RESULTS

A. Method of calculation

The nonrelativistic part of the program has been used previously[15,29] but the techniques employed were not described in detail. The first part of this section will be devoted to a brief description of the nonrelativistic calculation and of the accuracy of the numerical results. Momentum space integrals are approximated by Gaussian quadratures. Estimates of the errors so introduced were obtained by varying the number of mesh points. For multiple integrals we varied the mesh in each variable separately. The accuracies quoted in this section were determined using the $^1S_0$, $^3S_1 - {}^3D_1$, and $^3D_2$ partial waves of the Reid SC potential at $k_F = 1.36$ fm^{-1}. Checks using other potentials at other densities give comparable results.

Matrix elements of the reaction matrix for states beneath the Fermi level are obtained from

$$G^0 = V + V\mathcal{S}, \tag{5.1}$$

where

$$\mathcal{S} = -\frac{Q_0}{e_0}G^0 = -\left(1+\frac{Q_0}{e_0}V\right)^{-1}\frac{Q_0}{e_0}V. \tag{5.2}$$

The first term of Eq. (5.1) is independent of P and its contribution to $\mathcal{E}_0$ and $u_0(p_1)$ can therefore be reduced to one-dimensional integrals over p. We found that 8 Gauss points for the computation of $\mathcal{E}_0$ and 16 (8 in each of two regions of the integral) for $u_0(p_1)$ gave a precision of at least 10^{-4} MeV.

A three-dimensional grid of the variables p_1, p_2, and z_{12} was used to compute the contribution of the term $V\mathcal{S}$ in Eq. (5.1). Due to the symmetry under interchange of p_1 and p_2 only matrix elements for $p_2 \le p_1$ need to be computed. Since the integrals over p_1 and p_2 involve the weight factor p_i^2, Gauss-Jacobi quadratures are particularly efficient. Two

mesh points each for p_1 and p_2 gave values of $\mathcal{E}_0$ accurate to 10^{-4} MeV. The Gauss-Legendre quadrature over z_{12} required 4 points for an accuracy of 10^{-3} MeV. This high accuracy for a small number of grid points results from the fact that $(p, L \mid V\mathcal{S} \mid p, L)/p^{2L}$ is approximately constant for $p_1, p_2 \leq k_F$. Had we used the same quadrature on the first term in Eq. (5.1), the errors would have been much larger.

For each point in the (p_1, p_2, z_{12}) mesh, Eq. (5.2) was used to find $(p', L' \mid \mathcal{S}_{J,S,T}(\gamma_0) \mid p, L)$ for $p_A \leq p' < \infty$. The two regions indicated in Eq. (4.56) were considered separately in the p' integral. Linearly scaled Gauss-Legendre points were used for the interval $p_A \leq p' \leq p_B$. The interval $p_B \leq p' < \infty$ was mapped onto the interval $[-1, +1]$ by a rational function and Gauss-Legendre quadrature was used for the image interval. Varying the number of mesh points independently in the two regions, we found that 4 points in the first region and 12 points in the second region left errors of 10^{-3} MeV and 5×10^{-3} MeV, respectively.

The initial values of $u_0(p_i)$ are computed using the effective mass approximation.[27] Thereafter the $u_0(p_i)$ resulting from a given iteration are directly used as input to the next iteration. The iterations were continued until $\mathcal{E}_0$ had converged to better than 0.01 MeV which is comparable to the total error introduced by the numerical integrations.

The computation of $V\mathcal{S}$ requires the potential matrix elements $(p', L' \mid V_{J,S,T} \mid p'', L'')$ and $(p, L \mid V_{J,S,T} \mid p', L')$ for $p', p'' \geq p_A$. These are computed and stored at the beginning of the calculations. Note that the p' and p'' intervals defined by Eqs. (4.56)–(4.58) are different for each point on the (p_1, p_2, z_{12}) mesh and thus many matrices of potential elements must be computed and stored. This is the principal disadvantage of a calculation based on the (p_1, p_2, z_{12}) grid; the main advantage is that the $u_0(p_i)$ required for the next iteration

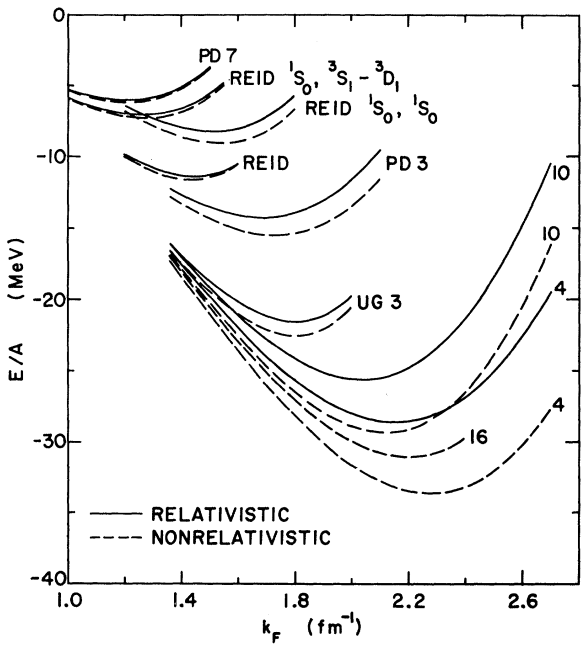

FIG. 1. Binding energy per particle for the potentials considered in this article. The dashed lines are the nonrelativistic case, the solid lines contain the relativistic corrections. The curves are labeled with the designations given in Table I.

are computed directly. The matrix elements $(p', L' \mid \mathcal{S}_{J,S,T} \mid p, L)$ from the final iteration are saved. Thus most of the input required for the relativistic corrections is available at the end of the nonrelativistic computation and need not be recomputed.

The $1 - Q$ term [Eq. (4.67)] requires matrix elements of G that are not available from the above computations. These are computed using Eq. (5.2) and most of the time spent on the relativistic corrections is used to find the new potential matrix

TABLE I. Summary of the potentials used in the calculations.

Designation	Ref.	Partial wave content	Rank per partial wave	Comments
Reid	30	All $J \leq 2$	Local	Reid soft core
UG3	31	All $J \leq 2$	Local	Ueda Green model III
PD 3	32	$^1S_0, ^3S_1$-$^3D_1$	2	ACS 3% deuteron D state; Mongan II $^1S_0$.
PD 7	32	$^1S_0, ^3S_1$-$^3D_1$	2	ACS 7% D state; Mongan II $^1S_0$.
^1S_0A		$^1S_0, ^3S_1$	1	Same v in both $^1S_0$ and $^3S_1$; fitted to Reid $^1S_0$.
^1S_0C		$^1S_0, ^3S_1$	3	Same v in both $^1S_0$ and $^3S_1$; fitted to Reid $^1S_0$.
4		$^1S_0, ^3S_1$	2	v fitted to $^1S_0$ and $^3S_1$ phases; repulsive term inverse range = 4 fm^{-1}.
10		$^1S_0, ^3S_1$	2	v fitted to $^1S_0$ and $^3S_1$ phases; repulsive term inverse range = 10 fm^{-1}.
16		$^1S_0, ^3S_1$	2	v fitted to $^1S_0$ and $^3S_1$ phases; repulsive term inverse range = 16 fm^{-1}.

elements required for this part of the calculation. Also the first term of the Lorentz correction [Eq. (4.71)] requires matrix elements of G^0 for $p_2 = k_F$ and p_1 and z_{12} on the grid. These matrix elements are not available from the nonrelativistic calculation. We computed the required values of $(p, L \mid V_{J,S,T} \mid p, L)$ exactly and obtained those for $(p, L \mid VS \mid p, L)$ by extrapolation from the (p_1, p_2, z_{12}) grid.

The stability of the relativistic corrections against changes in the numbers of Gauss points employed has also been investigated. The grids used in the nonrelativistic calculation result in errors of less than 5×10^{-4} MeV for all of the corrections except the Lorentz correction. The Lorentz correction has an error of nearly 0.01 MeV which is mainly due to the inadequacy of the z_{12} grid.

B. Numerical results

The main purpose of the numerical calculations is to exhibit the quantitative features of the relativistic correction terms and their dependence on various properties of the phenomenological potential. The features of the potentials used are summarized in Table I. The properties[30-32] and nonrelativistic nuclear matter saturation curves[16,17,32,33] of the first four potentials listed are known. The Ueda-Green model III potential was chosen as a representative of one-boson exchange potentials with defect integral κ significantly smaller than that of the Reid SC potential. The ACS potentials PD3 and PD7 exhibit the effects of varying the strength of the tensor force.

We have shown in Sec. IV that to order β^2 the relativistic corrections are functions of the reac-

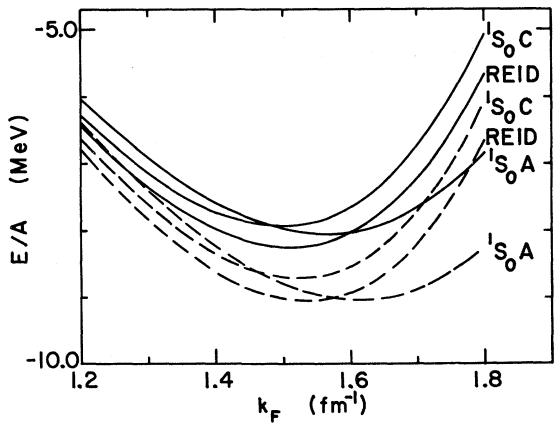

FIG. 2. Binding energy per particle for three S-wave potentials. The curves are as in Fig. 1. For each of these three cases, the same potential was used in the $^1S_0$ and $^3S_1$ channels.

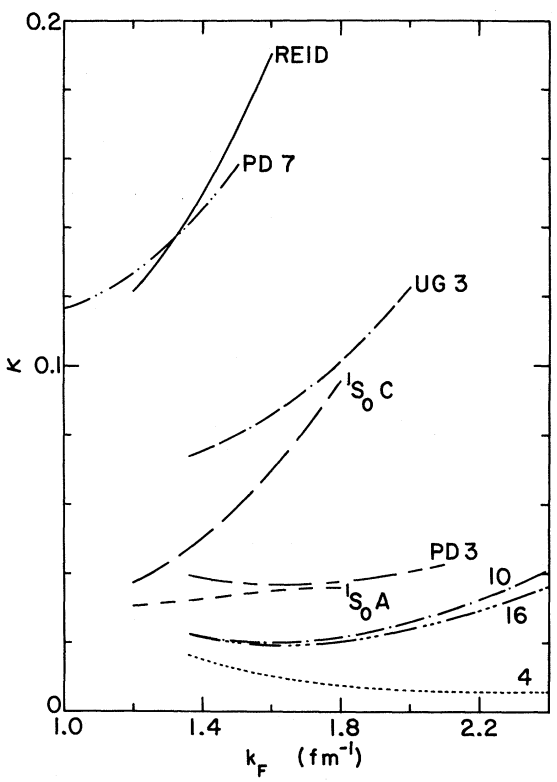

FIG. 3. The defect integral κ for the potentials in Table I.

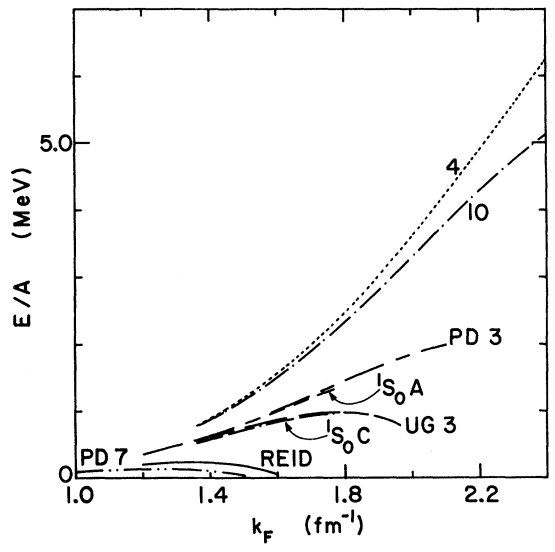

FIG. 4. The total relativistic correction for the potentials in Table I. The curves in this figure are the sum of the corresponding curves in Figs. 5–10, and (where applicable) Fig. 15, plus the kinetic energy correction in Eq. (4.73).

tion matrix G^0. Explicit dependence on the potential v appears only in the higher order term (4.33). Nevertheless it seems worthwhile to gain quantitative information about this term. The problem of solving Eq. (3.9) numerically for v presents significant practical difficulties, which we have not overcome. We have, therefore, created the last five potentials in Table I by fitting several separable forms for v to phase shifts. The corresponding conventional potentials V were then obtained from Eq. (3.9). Potentials A and C are, respectively, rank-1 and rank-3 potentials designed to mimic properties of the $^1S_0$ Reid potential. Models 4, 10, and 16 are rank-2 separable potentials designed to show the effect of widely varying the strength of the repulsive core. For model 4 the Hartree-Fock approximation is good, i.e., the second term in Eq. (5.1) is small compared to the first term. For the model 10 the two terms are of the same order of magnitude as their sum, while for model 16 the terms separately are about 200 times larger than G^0. In spite of the drastic differences in these three potentials the defect integrals and the saturation curves differ very little. The specifications of these potentials are given in Appendix A.

The nuclear matter saturation curves of the various potentials are shown in Figs. 1 and 2. In these figures we also show the saturation curves obtained using just the $^1S_0$ and $^3S_1 - {}^3D_1$ channels of the Reid SC and using the Reid $^1S_0$ in both the $^1S_0$ and $^3S_1$ channel. These curves may be compared to the separable potential curves with the same partial wave content. We do not show these two cases in the remaining graphs since they are, respectively, very similar to the PD7 and ^1S_0C curves. Since we did not obtain reliable values of

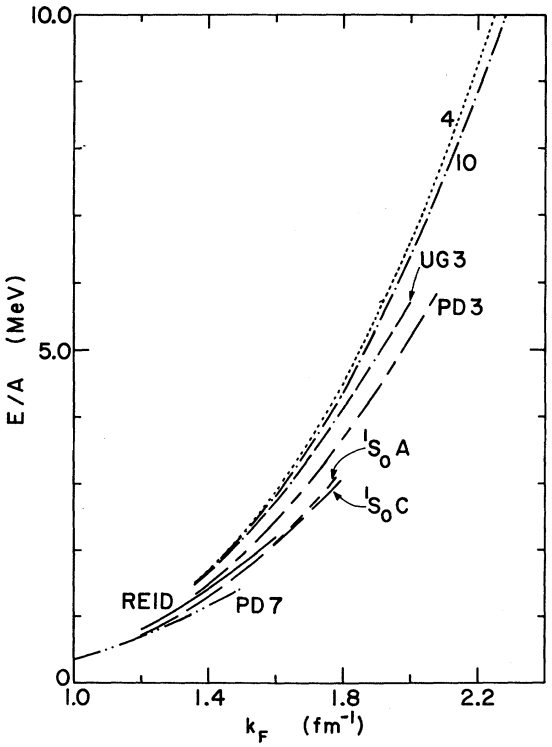

FIG. 5. The "main" correction for the potentials in Table I.

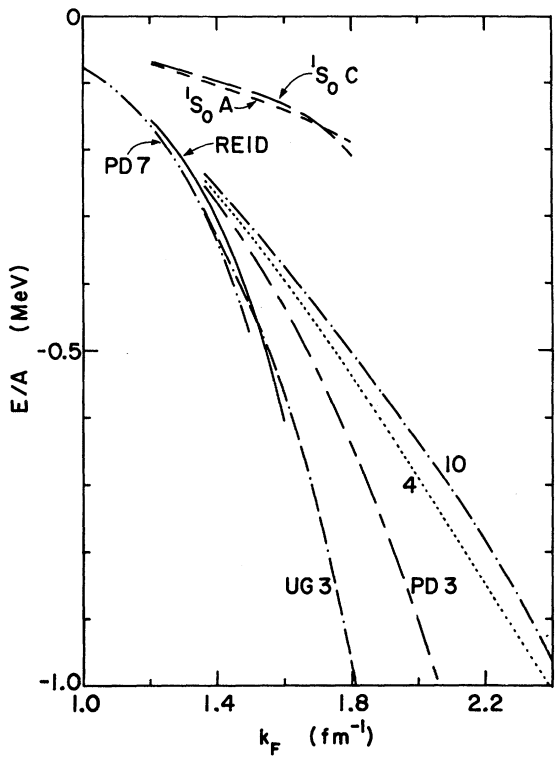

FIG. 6. The 1-Q correction for the potentials in Table I.

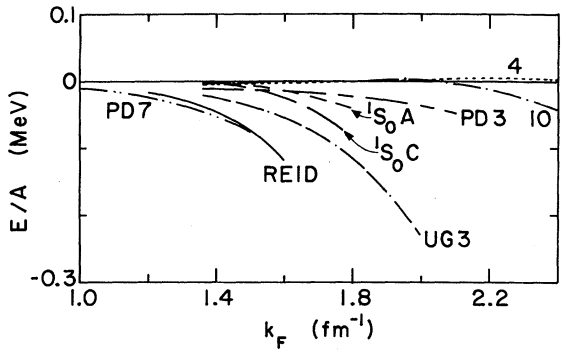

FIG. 7. The Pauli correction for the potentials in Table I.

the Lorentz correction for Model 16 its relativistic saturation curve is not included in Fig. 1. This potential was primarily intended for the study of the $\mathcal{U} - V$ correction.

The defect integral κ defined by[34]

$$\kappa = \frac{3}{4k_F^3}\int_0^{k_F} dp_1 p_1^2 \int_0^{k_F} dp_2 p_2^2 \times \int_{-1}^{1} dz_{12} \sum_{J,S,T,L} (2T+1)(2J+1) \times (p, L \mid G^0_{J,S,T}[\bar{Q}_0/e_0^2]G^0_{J,S,T} \mid p, L) \qquad (5.3)$$

is shown in Fig. 3 for these potentials. Comparison of Figs. 1–3 shows that we have considered potentials with a wide range of nuclear matter properties.

Figure 4 shows the total relativistic correction as a function of k_F while Figs. 5–10 show the individual correction terms defined in Sec. IV. Figure 11 shows the various corrections for the Reid potential and demonstrates that a large amount of cancellation occurs in computing the total relativistic correction.

The "main" correction is the largest and, as can be seen from Eq. (4.66) is proportional to the nonrelativistic potential energy. In fact, substitution of simple functions of P and p for $(p, L \mid G^0_{J,S,T} \mid p, L)$ in Eq. (4.66) shows that

$$\mathcal{E}_{\text{main}} \approx -\tfrac{9}{20}\beta^2 \mathcal{E}_0 \qquad (5.4)$$

for reasonable behavior of G^0. This simple relation holds true for all the curves in Fig. 5 to an accuracy of a few percent.

The integrands of the $1-Q$, Pauli, γ, and spectral corrections are all quadratic expressions in the reaction matrix that are similar to the integrand that appears in the definition of the defect integral κ. Thus we would expect these terms to increase with increasing κ. Furthermore, the $1-Q$ correction is manifestly negative while the spectral correction will have a sign opposite to that

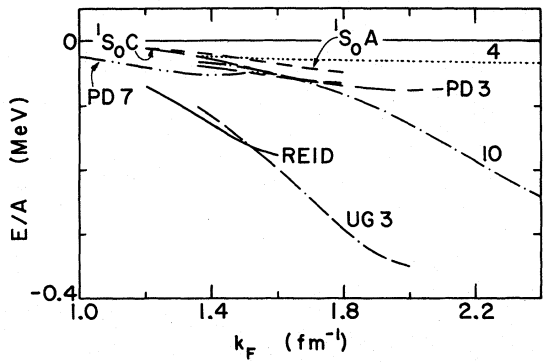

FIG. 8. The γ correction for the potentials in Table I.

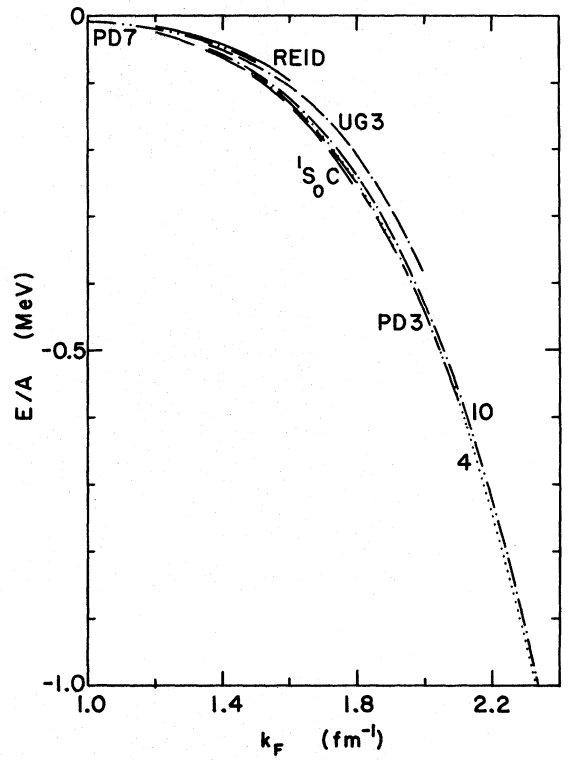

FIG. 9. The Lorentz correction for the potentials in Table I.

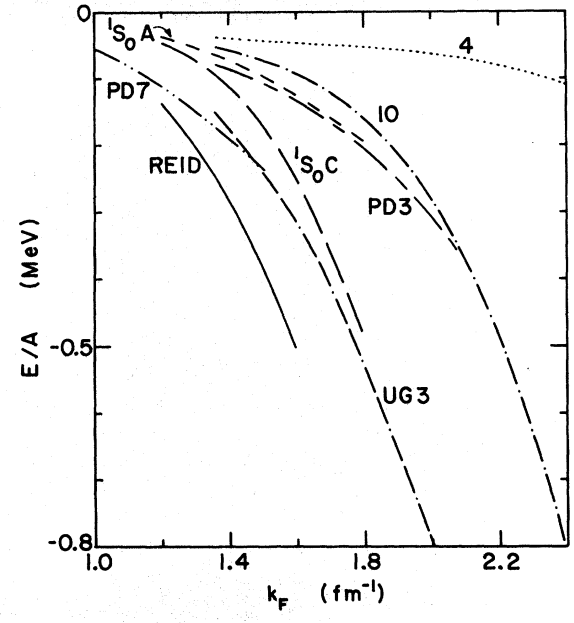

FIG. 10. The spectral correction for the potentials in Table I.

of the sum of the remaining terms. The integrands of the γ and Pauli corrections are also generally negative. Thus, as is clearly shown in Fig. 11, despite the fact that the "main" correction is considerably larger in magnitude than any one of the remaining corrections, the sum of the remaining corrections is comparable to it and the total correction is much smaller than the "main" correction. Since most of these remaining corrections are larger for larger κ, this cancellation is most effective for potentials with large defect integrals. This effect can be seen by comparing Figs. 3 and 4 which show that the potentials with a stronger tensor force (and hence a larger κ) produce a smaller relativistic correction.

The contributions of the individual partial waves of the potential to the corrections for the case of the Reid potential are shown in Figs. 12–14. For S waves we see again the cancellation of the "main" correction against all the others. The effect is particularly striking for the $^3S_1$ partial wave which,

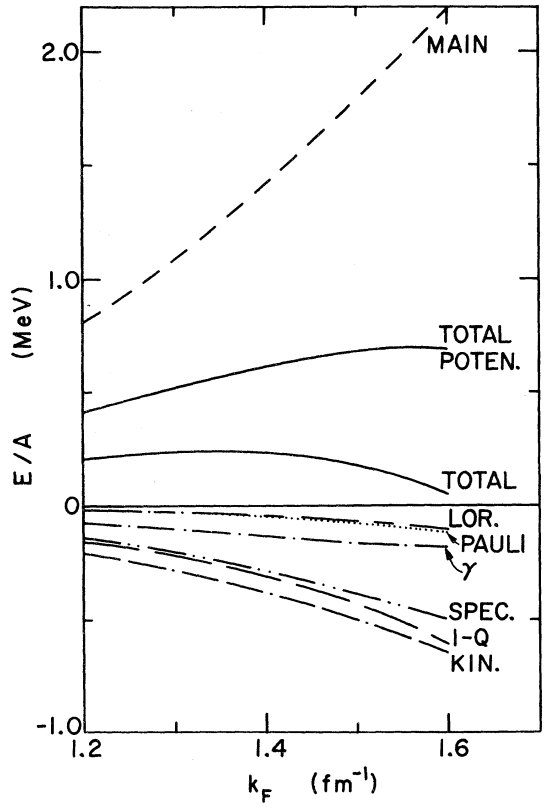

FIG. 11. The relativistic correction terms for the Reid SC potential. The curves are marked with the names of the corrections; "Kin." refers to the kinetic energy correction of Eq. (4.73). "Total Poten." is the sum of the potential energy corrections while "Total" is the sum of "Total Poten." and "Kin."

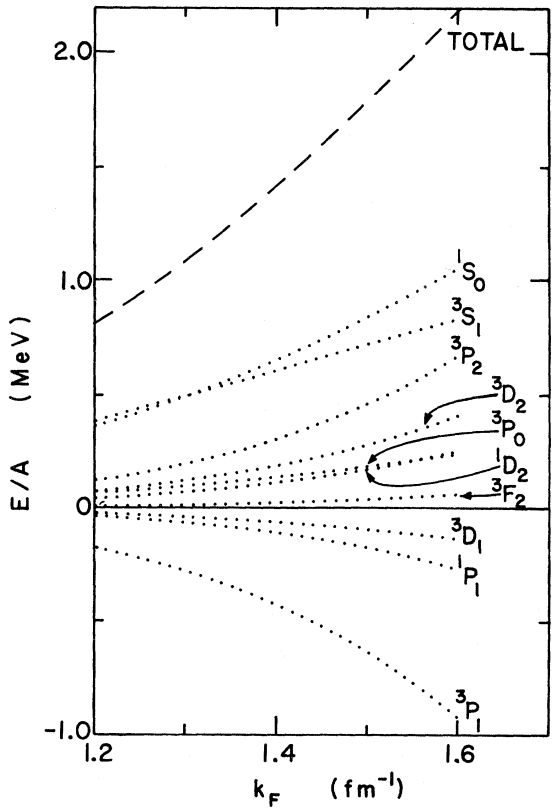

FIG. 12. Partial wave contributions to the "main" correction term for the Reid SC potential.

due to the tensor force, has a large partial wave defect integral. For the repulsive partial waves ($^1P_1$, $^3P_1$, and $^3D_1$) the main correction is itself negative as is the sum of the remaining correc-

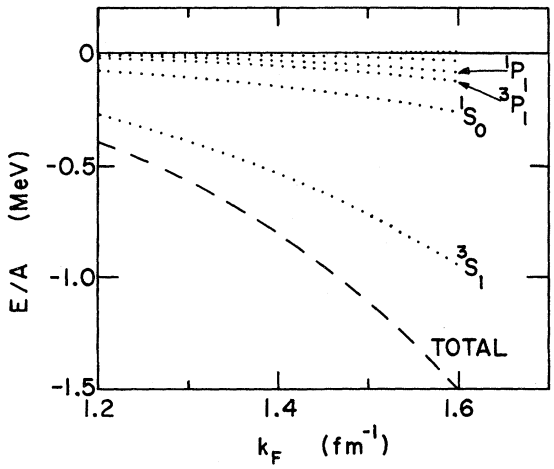

FIG. 13. Partial wave contributions to the sum of all the potential corrections except the "main" correction for the Reid SC potential.

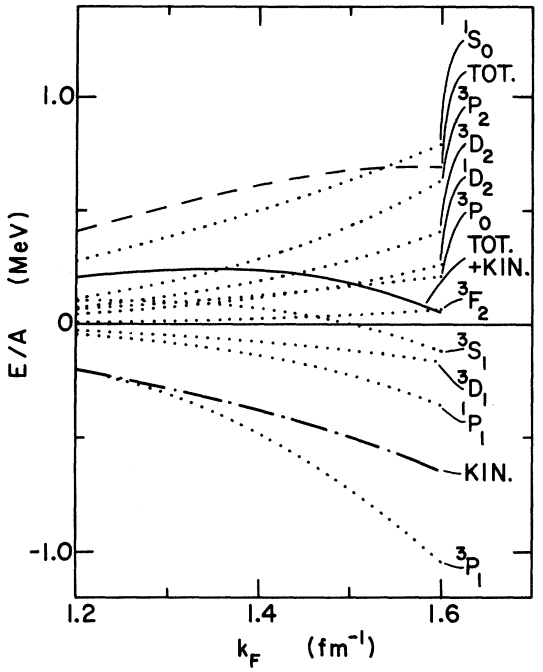

FIG. 14. Partial wave contributions to the total relativistic potential energy correction for the Reid SC potential. Also shown are the kinetic energy correction (dash-dot line) and the total correction (solid line).

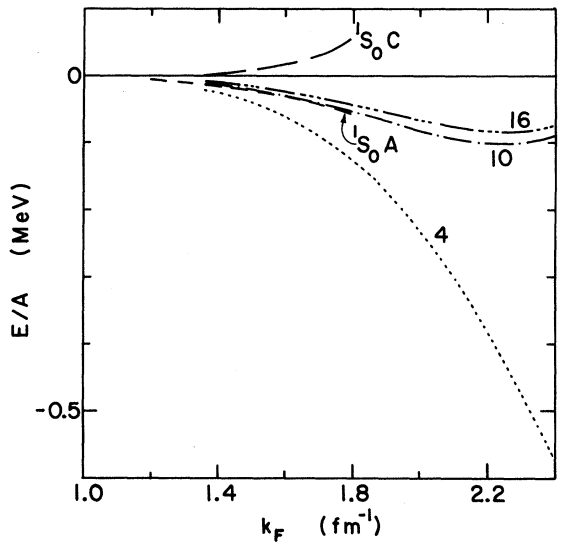

FIG. 15. The $\mathcal{U}$-V correction for the last five potentials of Table I.

tions. In these partial waves we thus have a significant attractive correction that contributes to the cancellation in the partial wave sum. For all other partial waves the "main" correction is dominant and the sum of all the others is negligible. This is to be expected since κ is quite small for these partial waves. We can, therefore, estimate the total relativistic corrections for all partial waves of $J>2$ by using Eq. (5.4) and published values for the nonrelativistic partial wave binding energies.[17,34] [Equation (5.4) is accurate to a few percent for all the partial wave states in Fig. 12.]

For $k_F=1.36$ the relativistic correction due to the higher partial waves is about -0.03 to -0.05 MeV.

In Table II we show the saturation properties of the various potentials used in this article. These properties were found by using several different schemes to interpolate to the minimum of the saturation curves. Based on the different results obtained with the different interpolations we estimate that the values of k_F and E/A are each good to at least $\pm 0.2\%$. The incompressibility,

$$\chi = k_F^2 \frac{d^2E}{dk_F^2}, \tag{5.5}$$

was much harder to find for the Reid and UG3 potentials and for these two cases has an error of about $\pm 10\%$. For the remaining potentials the error in χ is $\pm 1\%$.

For the last five potentials in Table I, the $\mathcal{U} - V$

TABLE II. Saturation properties. The Fermi momentum, binding energy per particle, and incompressibility at the saturation density for the various potentials used in this article. Shown are both the relativistic and nonrelativistic values. The nonrelativistic defect integrals are also shown.

	k_F (fm^{-1})		E/A (MeV)		χ (MeV)		κ
Potential	Rel	Nonrel	Rel	Nonrel	Rel	Nonrel	Nonrel
Reid	1.446	1.439	−11.42	−11.64	150	150	0.156
UG 3	1.798	1.798	−21.60	−22.59	255	285	0.101
PD 3	1.682	1.728	−14.28	−15.53	147	130.5	0.037
PD 7	1.205	1.208	−6.04	−6.17	63	66	0.127
^1S_0A	1.563	1.612	−8.05	−9.04	95	106	0.035
^1S_0C	1.496	1.522	−7.94	−8.71	110	126	0.061
4	2.150	2.271	−28.65	−33.62	248	300	0.0057
10	2.034	2.116	−25.66	−29.33	240	287	0.030

correction [the second term of Eq. (4.30)] has also been evaluated and the results are shown in Fig. 15. As expected, this term is small although there are some terms of order β^2 (Pauli and γ) that are equally small. Note that the very strong short-range repulsion in model 16 does not result in a large $\upsilon - V$ term.

VI. CONCLUSIONS

In the framework of relativistic two-particle quantum mechanics, the difference of the squares of the center-of-mass energies with and without an interaction has all the properties of a conventional phenomenological potential. Lorentz invariance then dictates the functional dependence of the two-body Hamiltonian on this potential and the total momentum of the two particles. Thus any conventional phenomenological potential implies a relativistic dynamical description of the two particles. For many-body systems the requirement of cluster separability as well as Lorentz invariance must be satisfied and this necessitates the introduction of many-body forces. In the low-density approximation applicable to nuclear matter, many-body interactions can be neglected in lowest order and expansion in powers of the velocities of the centers of mass of two-body clusters is legitimate. Relative velocities in such clusters will, in general, not be small.

On this basis, an approximately relativistic many-body Hamiltonian can be constructed for any conventional nucleon-nucleon potential. We have cast the Brueckner calculation in such a form that the resulting lowest order correction to the nonrelativistic binding energy per particle computed with the *same* potential is explicitly exhibited as a sum of several terms. An important practical aspect of our formulation is that all of the correction terms are functions of the nonrelativistic reaction matrix and other easily computed quantities.

Due to cancellation of the various correction terms the net effect is considerably smaller than the largest term and, by the same token, smaller than might be expected from crude estimates. This suppression of the correction is more effective for potentials with larger defect integrals κ. For the Reid potential the total correction to the energy per particle at saturation is only +0.22 MeV. The correction at saturation density is also positive for all the other potentials we considered. The changes in the saturation density were negligible for the realistic potentials. It thus appears that the relativistic corrections considered here do not help in explaining the discrepancy between present theoretical calculations[16,17,33] and the experimental values[11] of the nuclear matter saturation density and binding energy.

APPENDIX A: SEPARABLE REPRESENTATIONS OF THE INTERACTION $(k|v|k')$

Study of the $\upsilon - V$ corrections discussed in Sec. V B requires an explicit knowledge of the interaction v introduced in Sec. III. To this end, we have constructed two classes of central separable S-wave potentials.

The first class of potentials was made using a construction described by Ernst, Shakin, and Thaler.[36] This construction has the property that the resulting separable potential exactly reproduces the half-off-shell scattering amplitude of a given (nonseparable) potential at a number of selected energies. The Reid soft core $^1S_0$ potential[30] was used in Eq. (3.11) to find the half-off-shell amplitudes $(k|T|k')$. Equation (3.13) was then employed to generate the matrix elements of $(k|\tau|k')$ which in turn were used in the construction of separable representations of $(k|v|k')$. Two such potentials were constructed: a rank-one potential designated ^1S_0A that reproduces the Reid SC $^1S_0$ half-off-shell amplitude at 5 MeV and a rank-three potential designated ^1S_0C that fits the half-shell amplitudes at 3, 100, and 1500 MeV. In nuclear matter calculations with the interactions ^1S_0A and ^1S_0C the same potential was used in both the $^1S_0$ and $^3S_1$ channels.

The second class of potentials was made by assuming an explicit separable form for $(k|v|k')$ and then adjusting the interaction parameters until a

TABLE III. Parameters for the separable interactions v defined in Eqs. (A1)–(A3). The last two columns give some nuclear matter properties of these potentials at normal density ($k_F = 1.36$).

	$^1S_0$ parameters				$^3S_1$ parameters				Nuclear matter at $k_F = 1.36$	
Model	C_A (fm^{-3})	a_A (fm^{-1})	C_R (fm^{-3})	a_R (fm^{-1})	C_A (fm^{-3})	a_A (fm^{-1})	C_R (fm^{-3})	a_R (fm^{-1})	$\mathcal{E}_{HF}/\mathcal{E}$	κ
4	−6.8962	1.4712	107.22	4	−18.4728	1.8095	114.025	4	0.95	0.016
10	−6.1141	1.4336	11 889.9	10	−14.8540	1.7352	4 982.55	10	−1.7	0.022
16	−4.3287	1.3496	400 259	16	−12.8983	1.6791	269 678	16	−220	0.022

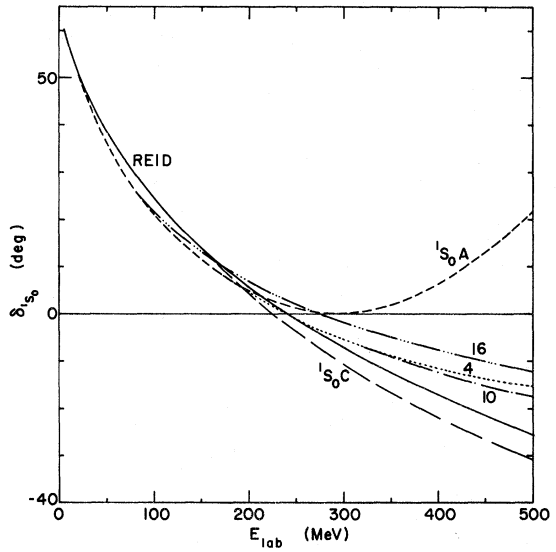

FIG. 16. The $^1S_0$ phase shifts for the last five potentials of Table I.

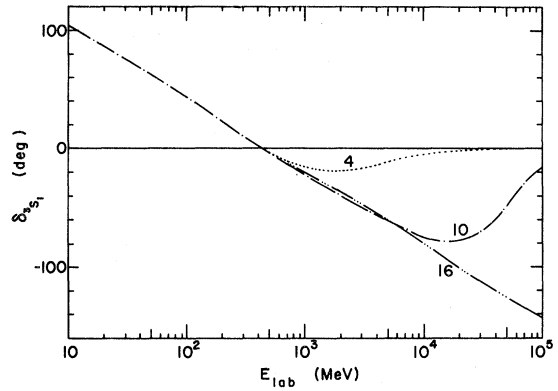

FIG. 17. The $^3S_1$ phase shifts for the last three potentials of Table I. The $^3S_1$ phase shifts of the $^1S_0$ A and $^1S_0$ C potentials are the same as their $^1S_0$ phase shifts.

reasonable fit to the experimental S-wave phase shifts was achieved. These central potentials are of rank two in each of the $^1S_0$ and $^3S_1$ channels and are of the form

$$(k \mid v \mid k') = g_A(k) C_A g_A(k') + g_R(k) C_R g_R(k'), \quad \text{(A1)}$$

where the form factors are

$$g_i(k) = [m/w(k)]^{1/2} (k^2 + a_i^2)^{-1}, \quad i = A, R \quad \text{(A2)}$$

and

$$w(k) = (k^2 + m^2)^{1/2}. \quad \text{(A3)}$$

Scattering amplitudes for these potentials are determined by solving Eq. (3.5). There are three potentials in this class, Models 4, 10, and 16; the model number derives from the repulsive range parameter a_R of each potential. These potentials were qualitatively fitted to experimental S-wave phase shifts up to 320 MeV but more importantly, yield, among themselves, nearly the same on-shell scattering amplitudes in this energy range. The potential parameters are presented in Table III. Increasing the repulsive range a_R leads to harder potentials in the sense that the Hartree-Fock approximation in nuclear matter becomes poorer. The last two columns of Table III show $\mathcal{E}_{III}/\mathcal{E}$, the ratio of the Hartree-Fock term to the potential energy in nuclear matter, and the defect integral κ. Despite the small variations in κ, there are large changes in the $\mathcal{E}_{III}/\mathcal{E}$ ratio for these potentials.

The phase shifts generated from the various S-wave separable potentials are shown in Figs. 16 and 17. Figure 16 shows the $^1S_0$ phase shifts up to a laboratory energy of 500 MeV for Models 4, 10, and 16 as well as the Reid $^1S_0$ and potentials ^1S_0A and ^1S_0C. The systematic increase in the short-range repulsion of the potential Models 4, 10, and 16 is graphically displayed in Fig. 17 where we show the $^3S_1$ phase shifts on a logarithmic energy scale up to a laboratory energy of 10^5 MeV (this corresponds to a relative momentum $k \approx 35$ fm^{-1}).

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