BCS variation method and universal behavior of trapped neutron systems at unitary limit

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In this work we study unitary neutron systems confined in a harmonic-oscillator trap, based on the BCS theory and the quasiparticle ring-diagram method. The low-momentum interaction $V_{\text{low-}k}$ renormalized from a fine-tuned CD-Bonn potential with the $^1S_0$ scattering length approaching the infinity is adopted. The energy ratios of trapped unitary neutron systems with various particle numbers, both those with magic numbers and those with nonmagic ones, to corresponding noninteracting systems, are shown to all flow to a constant value. The BCS wave functions for these trapped unitary systems, with different trap parameters and different particle numbers, manifest themselves with universal patterns: the effect of harmonic-oscillator shells is significant, and moreover, the orbits of the lowest valence shell are occupied with a uniform probability. We also discuss the self-energy given by the BCS calculation, and present a weighted sum rule satisfied by the monopole components of the unitary neutron-neutron interaction. In terms of these, the constant energy ratio independent of the trap parameter and the particle number can be interpreted.

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I. INTRODUCTION

The unitary limit of a fermionic system refers to a special scenario where the interfermion interaction has its scattering length approaching infinity. This scenario was originally formulated by Bertsch, with the question what will be the ground-state properties of such a system [1]. Universal behaviors of a Fermi gas at the unitary limit, such as those regarding the ground state, collective excitations, and nonequilibrium aspects, are expected and have attracted intensive attention; see, e.g., Refs. [2–21]. Meanwhile, a unitary Fermi system with finite particle number confined in a trap is attracting increasing attention; see, e.g., Refs. [22–28].

With the advances of cold-atom experimental technique, unitary Fermi systems became experimental accessible at the atomic level by way of the Feshbach resonance. From another perspective, the unitary limit naturally plays a crucial role in low-energy properties of nuclear systems, such as those of low-density neutron matters [29–31] and those of extremely-neutron-rich nuclei [32]. This is because the scattering lengths of two S channels, in particular the $^1S_0$ channel, of a realistic NN potential are both large. Recently, the effect of the unitary limit was shown to be significant in binding energies of nuclear few-body systems [33] and in fundamental properties of nuclear matters and neutron stars [34].

It is thus of much interest and importance to study neutron systems interacting via a unitary $^1S_0$ interaction, as well as nuclear systems interacting via unitary $^1S_0$ and $^3S_1$ interactions. In Refs. [11,12], unitary neutron matters were studied within the framework of the particle-particle-hole-hole (pphh) ring-diagram method [35,36], and these neutron matters, different in various aspects but all at the unitary limit, were shown to have a universal nature. In a previous work [28] we studied trapped unitary neutron systems with magic particle numbers, and energy ratios of these trapped unitary systems to corresponding noninteracting systems, were shown to be remarkably invariant with the variation of the trap parameter, as well as the change of the type of the unitary interaction and the variation of the decimation momentum of the adopted renormalization procedure.

Yet, the question whether the universality of trapped unitary neutron systems survives with respect to different choices of the system type, i.e., being a system of a magic number or a system of a nonmagic one, as well as the variation of the particle number, remains unanswered. In this work we generalize our study regarding trapped unitary neutron systems, based on the BCS theory and the quasiparticle ring-diagram method, in which a system of a magic particle number and a system of a nonmagic one are treated on the same footing. We check the aforementioned question, and study regularities emerging in trapped neutron systems at the unitary limit. The paper is organized as follows: In Sec. II we present a brief introduction to the theoretical framework of this work. In Sec. III we present and discuss our results for trapped unitary neutron systems. In Sec. IV we summarize the work.

II. THEORETICAL FRAMEWORK

The pairing correlation plays a crucial role in atomic nuclei and nuclear matter; see, e.g., Refs. [37,38]. Significant efforts have been devoted to various theories to treat pairing correlation, such as the well-known BCS theory [39–42], the

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seniority scheme [43] and its generalization [44,45], the interacting boson model [46], the fermion dynamical symmetry model [47,48], the nucleon-pair approximation of the shell model [49–52], etc. In Ref. [53] the formulation to solve a spherical BCS wave function considering a general Hamiltonian was presented, and BCS calculations in association with effective interactions derived from realistic *NN* potentials using renormalization methods, such as the *G*-matrix method [54], were further carried out [55–58].

In this work we study trapped and strongly coupled neutron systems at the unitary point, based on the BCS theory in association with a low-momentum interaction $V_{\text{low-}k}$ [59–65] renormalized from a unitary CD-Bonn interaction [11,66]. The formulas to carry out such a BCS calculation are given as below, which follow closely those considering a realistic effective interaction [53,55–58]. The spherical BCS wave function reads

$$|\Phi_{\rm BCS}\rangle = \prod_{j,m>0}[u_j + (-1)^{j-m} v_j a^\dagger_{jm} a^\dagger_{j\bar{m}}]|0\rangle. \quad (1)$$

Here we use $|j\bar{m}\rangle$ to denote $|j,-m\rangle$ for brevity; v_j^2 is the occupation probability of the single-particle state $|jm\rangle$ for all possible m values, and $u_j^2 = 1 - v_j^2$. With the single-particle energy for neutrons in a harmonic-oscillator trap $\epsilon_j \equiv \epsilon_{nlj} = (2n + l + 3/2)\hbar\omega$ and the normalized two-body matrix element of a unitary neutron-neutron interaction $V(j_a j_b j_c j_d, J)$, the energy of the BCS state is expressed as below:

$$\begin{aligned} E_{\rm BCS} &= \langle\Phi_{\rm BCS}|\hat{H}|\Phi_{\rm BCS}\rangle \\ &= \sum_j (2j+1)\left[\left(\epsilon_j - \frac{1}{2}\mu_j\right)v_j^2 - \frac{1}{2}\Delta_j u_j v_j\right]. \end{aligned} \quad (2)$$

Here μ is the self-energy given by

$$\begin{aligned} \mu_{j_a} &= 2(2j_a+1)^{-1}\sum_{j_b}\left[\sum_J (2J+1)G(j_a j_b j_a j_b, J)\right]v_{j_b}^2 \\ &= 2(2j_a+1)^{-1/2}\sum_{j_b}(2j_b+1)^{1/2}F(j_a j_a j_b j_b, 0)v_{j_b}^2, \end{aligned} \quad (3)$$

and Δ is half the pairing gap given by

$$\Delta_{j_a} = (2j_a+1)^{-1/2}\sum_{j_b}(2j_b+1)^{1/2}G(j_a j_a j_b j_b, 0)u_{j_b}v_{j_b}, \quad (4)$$

with

$$\begin{aligned} G(j_a j_b j_c j_d, J) &= -\sigma(j_a j_b)\sigma(j_c j_d)V(j_a j_b j_c j_d, J), \\ \sigma(j_a j_b) &= \begin{cases} 1 & \text{if } j_a = j_b \\ 1/\sqrt{2} & \text{otherwise,} \end{cases} \end{aligned} \quad (5)$$

and

$$\begin{aligned} &F(j_a j_c j_d j_b, J') \\ &= \sum_J (-)^{J+j_c+j_d}\begin{Bmatrix} j_a & j_b & J \\ j_d & j_c & J' \end{Bmatrix}(2J+1)G(j_a j_b j_c j_d, J). \end{aligned} \quad (6)$$

As is well known, the occupation probabilities v_j^2 are determined variationally to minimize the energy expectation under the constraint of the particle-number condition, i.e., by solving the following variational equation:

$$\frac{\delta}{\delta v_j}\langle\Phi_{\rm BCS}|(\hat{H} - \lambda\hat{N})|\Phi_{\rm BCS}\rangle = 0, \quad (7)$$

under the constraint $\langle\Phi_{\rm BCS}|\hat{N}|\Phi_{\rm BCS}\rangle = A$, where A is the particle number of the system. When one considers the energy expectation in the form of Eq. (2), the above variation equations correspond to the following equations:

$$\begin{aligned} 2u_j v_j &= \frac{\Delta_j}{\sqrt{(\epsilon_j - \lambda - \mu_j)^2 + \Delta_j^2}}, \\ u_j^2 - v_j^2 &= \frac{\epsilon_j - \lambda - \mu_j}{\sqrt{(\epsilon_j - \lambda - \mu_j)^2 + \Delta_j^2}}, \\ u_j^2 + v_j^2 &= 1, \quad \sum_j (2j+1)v_j^2 = A. \end{aligned} \quad (8)$$

With the solution of the BCS equations, one can define quasiparticles based on the Bogolyubov-Valatin transformation:

$$\alpha^\dagger_{jm} = u_j a^\dagger_{jm} - (-1)^{j-m} v_j a_{j\bar{m}}. \quad (9)$$

Then $|\Phi_{\rm BCS}\rangle$ in the particle representation is the vacuum state $|\tilde{0}\rangle$ in the quasiparticle representation, and the Hamiltonian $(\hat{H} - \lambda\hat{N})$ is transformed into a quasiparticle Hamiltonian $\hat{H}_{\rm q.p.}(\lambda)$:

$$\hat{H}_{\rm q.p.}(\lambda) = \hat{H}^{00}_{\rm q.p.} + \hat{H}^{11}_{\rm q.p.} + \hat{H}^{22}_{\rm q.p.} + \hat{H}^{13}_{\rm q.p.} + \hat{H}^{31}_{\rm q.p.} + \hat{H}^{04}_{\rm q.p.} + \hat{H}^{40}_{\rm q.p.},$$

where $\hat{H}^{mn}_{\rm q.p.}$ denotes a normal-ordered operator cluster consisting of m (n) quasiparticle creation (destruction) operators. Note that $\hat{H}^{00}_{\rm q.p.} = E_{\rm BCS} - \lambda A$ and $\hat{H}^{20}_{\rm q.p.} = \hat{H}^{02}_{\rm q.p.} = 0$.

Apparently, the physical ground state of the system should be composed of not only the vacuum state $|\tilde{0}\rangle$ but also components of quasiparticle states. We then also calculate the ground-state energy (denoted $E_{\rm 2nd}$) corrected by the second-order quasiparticle ring diagram generated by $\hat{H}^{40}_{\rm q.p.}$ and $\hat{H}^{04}_{\rm q.p.}$ vertices, to take into account the lowest-order contribution of four quasiparticle components in the ground-state energy. Note that there is no contribution from the first-order quasiparticle ring diagram since $\langle\tilde{0}|\hat{H}^{22}_{\rm q.p.}|\tilde{0}\rangle = 0$.

We have the energy correction given by the second-order quasiparticle ring diagram as below:

$$D_2 = -\sum_J (2J+1)\sum_{j_a\leqslant j_b, j_c\leqslant j_d}\frac{[H^{40}(j_a j_b j_c j_d, J)]^2}{E_{j_a} + E_{j_b} + E_{j_c} + E_{j_d}}. \quad (10)$$

Here E_j is the quasiparticle energy given by

$$E_j = \left[(\epsilon_j - \mu_j - \lambda)^2 + \Delta_j^2\right]^{1/2}. \quad (11)$$

And the matrix element for the vertex $\hat{H}^{40}_{\rm q.p.}$ is given by

$$\begin{aligned} H^{40}(j_aj_bj_cj_d,J) &= \langle\tilde{0}|A_{\rm q.p.}(j_cj_dJ\bar{M})A_{\rm q.p.}(j_aj_bJM)\hat{H}^{40}_{\rm q.p.}|\tilde{0}\rangle \\ &= [R(j_aj_bj_cj_d,J)-S(j_aj_bj_cj_d,J)] \\ &\quad\times[(1+\delta_{j_a,j_b})(1+\delta_{j_c,j_d})]^{-1/2}, \end{aligned} \tag{12}$$

where $A_{\rm q.p.}(j_aj_bJM)$ is the normalized quasiparticle pair destruction operator, and

$$\begin{aligned} R(j_aj_bj_cj_d,J) &= (u_{j_a}u_{j_b}+v_{j_a}v_{j_b})(u_{j_c}u_{j_d}+v_{j_c}v_{j_d}) \\ &\quad\times G(j_aj_bj_cj_d,J)+(u_{j_a}v_{j_b}-v_{j_a}u_{j_b}) \\ &\quad\times(u_{j_c}v_{j_d}-v_{j_c}u_{j_d})[F(j_aj_bj_cj_d,J) \\ &\quad+\theta(j_cj_dJ)F(j_aj_bj_dj_c,J)], \end{aligned} \tag{13}$$

$$\begin{aligned} S(j_aj_bj_cj_d,J) &= (u_{j_a}u_{j_b}-v_{j_a}v_{j_b})(u_{j_c}u_{j_d}-v_{j_c}v_{j_d}) \\ &\quad\times G(j_aj_bj_cj_d,J)+(u_{j_a}v_{j_b}+v_{j_a}u_{j_b}) \\ &\quad\times(u_{j_c}v_{j_d}+v_{j_c}u_{j_d})[F(j_aj_bj_cj_d,J) \\ &\quad-\theta(j_cj_dJ)F(j_aj_bj_dj_c,J)], \end{aligned} \tag{14}$$

with $\theta(j_cj_dJ)=(-)^{J+j_c+j_d}$ and G, F defined in Eqs. (5) and (6).

In this work we calculate $E_{\rm BCS}$ and $E_{\rm 2nd}$ as aforementioned and study the resulting ratios

$$R_{\rm BCS}=\frac{E_{\rm BCS}}{E_{\rm free}},\quad R_{\rm 2nd}=\frac{E_{\rm 2nd}}{E_{\rm free}} \tag{15}$$

for trapped unitary neutron systems. Here $E_{\rm free}$ denotes the ground-state energy of the corresponding noninteracting system.

III. RESULTS AND DISCUSSIONS

In this work we study trapped unitary neutron systems. We adopt a unitary neutron-neutron interaction renormalized from a fine-tuned CD-Bonn potential [11,66] via the $V_{\text{low-}k}$ procedure [59–65]. From the perspective of the Brown-Rho scaling mechanism [67–70], this fine-tuned CD-Bonn potential [11] was constructed by decreasing the σ-meson mass by 2.4% in comparison with the original one [66]. Its $^1S_0$ channel has the scattering length $a_s=-12\,070$ fm and the effective range $r_{\rm e}=2.54$ fm. Next the renormalization procedure of $V_{\text{low-}k}$ is enacted, where the high-momentum components of the bare NN potential are integrated out. Note that because the half-on-shell T matrix is preserved in the renormalization, both the scattering length and the effective range are preserved. The decimation momentum is taken to be $\Lambda = 2.0\ {\rm fm}^{-1}$, the same as in a number of nuclear structure studies [61]. For the harmonic-oscillator trap parameter, a few $\hbar\omega$ values which are comparable with those for realistic nuclei are adopted. Apparently, the scattering length of the fine-tuned CD-Bonn potential is enormous, i.e., approaching to the infinity, compared with any other length scales in the systems.

In Table I we present the ratios given by the BCS energy and the one also including the energy shift arising from the second-order quasiparticle ring diagram, denoted as $R_{\rm BCS}$ and $R_{\rm 2nd}$, for trapped unitary neutron systems with the mass number $A=50,54,\ldots,102$, respectively. As shown in Table I, for all these trapped unitary systems, the $R_{\rm 2nd}$ value is very slightly decreased in comparison with corresponding $R_{\rm BCS}$. Besides, we have done initial calculations also including energy corrections given by higher-order quasiparticle ring diagrams generated by $\hat{H}^{04}_{\rm q.p.}$, $\hat{H}^{40}_{\rm q.p.}$, and $\hat{H}^{22}_{\rm q.p.}$ vertices. According to the initial calculations, the effect of higher-order quasiparticle ring diagrams are much smaller than that of the second-order one. These suggest that the quasiparticle ring-diagram expansion with respect to the BCS state provides a rapidly converging framework for trapped unitary systems.

TABLE I. The ratios $R_{\rm BCS}$ and $R_{\rm 2nd}$ for trapped unitary neutron systems with the mass number $A=50,54,\ldots,102$, respectively. The trap parameter $\hbar\omega=8.5$, 9.5, 10.5, 12.0 MeV, respectively, which is comparable with those for realistic nuclei, is adopted. For the unitary interaction, the low-momentum interaction $V_{\text{low-}k}$ [59–65] renormalized from the fine-tuned CD-Bonn potential [11,66] with the $^1S_0$ scattering length $a_s=-12\,070$ fm is adopted.

A	$\hbar\omega$	Interaction	$R_{\rm BCS}$	$R_{\rm 2nd}$
50	8.5	CD-Bonn	0.757	0.754
54	8.5	CD-Bonn	0.756	0.752
58	8.5	CD-Bonn	0.755	0.751
	9.5	CD-Bonn	0.755	0.751
	10.5	CD-Bonn	0.757	0.752
	12.0	CD-Bonn	0.760	0.754
62	8.5	CD-Bonn	0.754	0.751
66	8.5	CD-Bonn	0.753	0.751
70	8.5	CD-Bonn	0.752	0.750
74	8.5	CD-Bonn	0.752	0.750
78	8.5	CD-Bonn	0.752	0.750
82	8.5	CD-Bonn	0.752	0.749
86	8.5	CD-Bonn	0.752	0.749
	9.5	CD-Bonn	0.753	0.750
	10.5	CD-Bonn	0.755	0.752
	12.0	CD-Bonn	0.759	0.755
90	8.5	CD-Bonn	0.752	0.749
94	8.5	CD-Bonn	0.751	0.749
98	8.5	CD-Bonn	0.750	0.749
102	8.5	CD-Bonn	0.750	0.749

One sees in Table I that, very interestingly, the energy ratios for all the trapped unitary systems, both the system with a magic number $A=70$ and the systems with nonmagic numbers, are remarkably close to a constant value of 0.75. This is a manifestation of *a universal nature* for trapped unitary systems, surviving *despite of the change of the system type, as well as the variation of the particle number*. As shown in Table I, the ratio for the trapped unitary system with a nonmagic number is also invariant with the variation of the adopted trap parameter $\hbar\omega$. This indicates the ground-state energy of the trapped unitary system is proportional to $\hbar\omega$, and we exemplify this point in Fig. 1 using the systems with the nonmagic particle number $A=58,66,74,86,98$, respectively. Plotted is the energy per neutron versus the $\hbar\omega$ value, as well as the corresponding linear-fitting result. One sees all lines fit the data very well, and they all converge to the origin (0,0). For such a linear scaling relation (i.e., $E_0=\alpha\hbar\omega$, with α

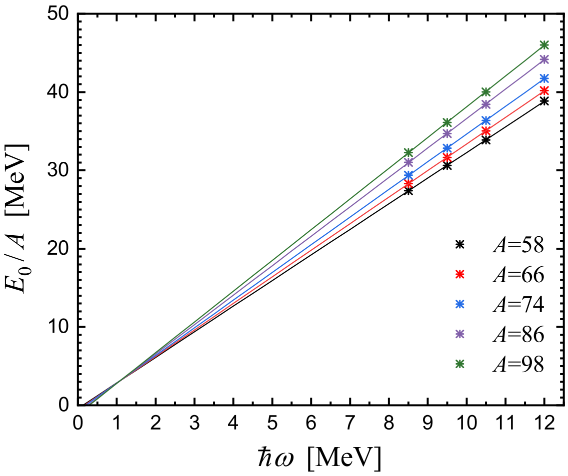

FIG. 1. The energy per neutron (denoted E_0/A) given by the BCS energy and the energy shift arising from the second-order quasiparticle ring diagram, versus the trap parameter $\hbar\omega$, for the unitary systems with the nonmagic particle number $A = 58, 66, 74, 86, 98$, respectively.

a constant independent of $\hbar\omega$), it has been analytically proved in our previous work [28] to be equivalent to the unitary-limit virial theorem [23] (i.e., $E_0 = 2\langle\Psi_0|U_{\rm osc.}|\Psi_0\rangle$, with $U_{\rm osc.}$ the harmonic-oscillator potential and Ψ_0 the ground-state wave function for the trapped unitary system).

Next we focus on the BCS results for trapped unitary neutron systems. In the BCS calculations, we consider $28\,j$ orbits of seven harmonic-oscillator major shells with $2n + l = 0, 1, \ldots, 6$, respectively. According to our calculations, for a given particle number, the distribution of the BCS occupation probabilities is robust with the variation of the trap parameter. Furthermore, the distributions of the occupation probabilities, for systems with various particle numbers, have a universal shell structure. In Fig. 2 we exemplify the distribution of the occupation probabilities, using the unitary systems of $A = 40, 50, 58, 66$ confined in the harmonic-oscillator trap with $\hbar\omega = 10.5$ MeV in Fig. 2(a), and using the systems of $A = 70, 78, 86, 94, 102$ in the trap with $\hbar\omega = 8.5$ MeV in Fig. 2(b).

As shown in Fig. 2, for both systems with magic numbers and those with nonmagic numbers, the effect of the harmonic-oscillator shells is significant and is further characterized by a special step-like distribution of the occupation probabilities. For systems with magic numbers, for example, for the system with $A = 70$, one can see in Fig. 2(b) that the distribution has a "one-step" structure: $v_{j_k}^2 = 1$ for $k = 1, \ldots, 15$ and $v_{j_k}^2 = 0$ for $k = 16, \ldots, 28$. For systems with nonmagic numbers, for example, for the system with $A = 86$, one can see in Fig. 2(b) that the distribution has a "two-step" structure: $v_{j_k}^2 = 1$ for $k = 1, \ldots, 15$, $v_{j_k}^2 \approx 16/42$ for $k = 16, \ldots, 21$, $v_{j_k}^2 = 0$ for $k = 22, \ldots, 28$. In conclusion, the BCS occupation probabilities for the trapped unitary neutron system can be described approximately as follows:

$$v_j^2 = \begin{cases} 1 & \text{for orbits in bottom shells} \\ A_{\rm val.}/\sum_{j'\in P}(2j'+1) & \text{for orbits in the lowest valence shell} \\ 0 & \text{for orbits in higher valence shells.} \end{cases} \tag{16}$$

Here $A_{\rm val.}$ is the number of valence neutrons (e.g., equals to zero for $A = 70$ and 16 for $A = 86$) and P denotes the lowest valence shell (e.g., is the $0h1f2p$ shell both for $A = 70$ and for $A = 86$). Note that for a given particle number, this special distribution of the occupation probabilities also gives the lowest energy to corresponding noninteracting system. In other words, the BCS wave function of the trapped unitary system is nearly identical to that of corresponding noninteracting one.

Let us discuss a little further about this special distribution of the occupation probabilities. For a system with a magic number, the BCS state defined with the occupation probabilities of Eq. (16), is equivalent to a Hartree-Fock state, i.e., a determinantal wave function with bottom shells fully occupied. Correspondingly the ground-state energy is given by the Hartree-Fock energy. Note that for such a system, the degree of freedom for excitations of one pair of neutrons across the closed shell is taken into account both in the BCS calculation here and in our previous work [28] using the particle-particle-hole-hole ring-diagram method. The two studies using different methods suggest the same structure for the ground state of the trapped unitary system with a magic number.

For a system with a nonmagic number, for example, for the system with $A = 86$, the occupation probabilities of Eq. (16) indicate that the system can be described to be 70 neutrons forming an inert core and 16 neutrons occupying the orbits of the lowest valence shell with a uniform probability. In other words, Fig. 2 suggests that for the trapped unitary system with a nonmagic number, the BCS state is well approximated to be the product of the determinantal wave function of an inert core and the wave function of valence particles described by

$$|\Phi_{\rm val.}\rangle = u^{\Omega}\exp\left[\frac{v}{u}S^{\dagger}\right]|0\rangle. \tag{17}$$

Here $v^2 = A_{\rm val.}/\sum_{j\in P}(2j+1)$ and $u^2 = 1 - v^2$, which are the uniform occupation probability and corresponding emptiness probability for the lowest valence major shell; $\Omega = \sum_{j\in P}\Omega_j$ with $\Omega_j = (2j+1)/2$ is the pair capacity of the lowest valence shell; $S^{\dagger} = \sum_{j\in P}S_j^{\dagger}$ with $S_j^{\dagger} = \sum_{m>0}(-)^{j-m}a_{jm}^{\dagger}a_{j\bar{m}}^{\dagger}$ is the quasispin raising operator of the lowest valence shell. It is interesting to note that the above BCS wave function of valence particles has a definite quasispin quantum number $\Omega/2$.

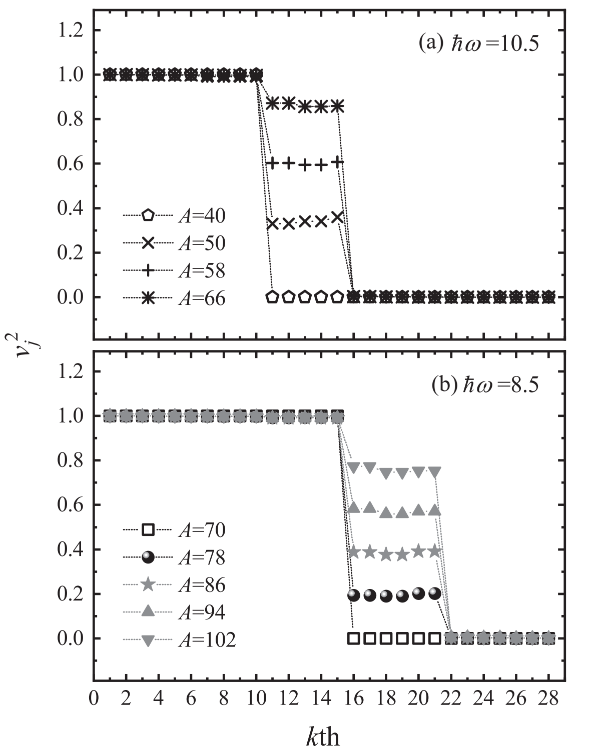

FIG. 2. Shell structure of the BCS occupation probabilities v_j^2's, for the trapped neutron system at the unitary limit. Panel (a) is for the systems of $A = 40, 50, 58, 66$ confined in the harmonic-oscillator trap with $\hbar\omega = 10.5$ MeV; Panel (b) is for the systems of $A = 70, 78, 86, 94, 102$ in the trap with $\hbar\omega = 8.5$ MeV. The model space for the BCS calculations is taken to be consisting of $28j$ orbits, i.e., seven harmonic-oscillator major shells with $2n + l = 0, 1, \ldots, 6$, respectively.

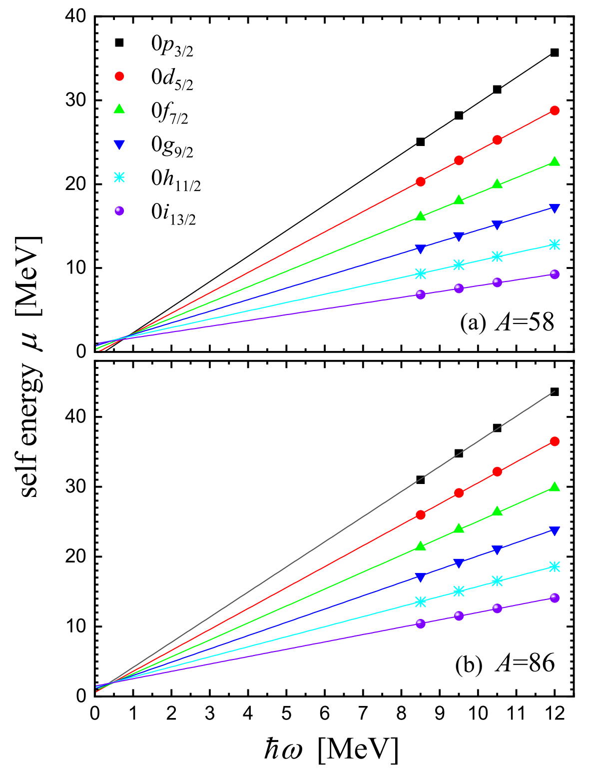

FIG. 3. Self-energies μ's [defined in Eq. (3) and in units of MeV] for the $0p_{3/2}$, $0d_{5/2}$, $0f_{7/2}$, $0g_{9/2}$, $0h_{11/2}$, $0i_{13/2}$ orbits, versus the adopted trap parameter $\hbar\omega$, for the trapped unitary systems with (a) $A = 58$ and (b) $A = 86$.

In our calculations for trapped unitary systems, the contribution of the pairing energy, i.e., the $-\frac{1}{2}\sum_j(2j+1)\Delta_j u_j v_j$ term, in the BCS energy is small, compared with the left part which can be expressed to be

$$E_{\rm BCS}^{\rm (m)} = \sum_j (2j+1)\left[\left(\epsilon_j - \frac{1}{2}\mu_j\right)v_j^2\right] = \sum_j (2j+1)\epsilon_j v_j^2 + \sum_{j_a \leqslant j_b}\sum_J (2J+1) \times V(j_a j_b j_a j_b, J) v_{j_a}^2 v_{j_b}^2. \tag{18}$$

Here the self-energy μ is defined in Eq. (3). In Fig. 3 we present the μ values for various j orbits versus the trap parameter $\hbar\omega$ adopted in the calculation, for the trapped unitary systems with $A = 58$ and 86, respectively. As shown in Fig. 3, the self-energy μ is approximately proportional to the trap parameter $\hbar\omega$. This, as well as the robust distribution of the occupation probabilities independent of the $\hbar\omega$ value, will give rise to a linear scaling relation where $E_{\rm BCS}^{\rm (m)}$ is proportional to $\hbar\omega$, according to Eq. (18). As aforementioned the $E_{\rm BCS}^{\rm (m)}$ is closely equal to $E_{\rm BCS}$, and as shown in Table I $E_{\rm BCS}$ is very slightly shifted by the second-order quasiparticle ring diagram. Thus one expects a linear scaling relation that $E_{\rm 2nd}$ is proportional to $\hbar\omega$, which is indeed the case as shown in Fig. 1.

In Fig. 4 we present the single-particle energy (SPE) corrected by the self-energy, $\varepsilon_j = \epsilon_j - \mu_j$, for the trapped unitary systems. Such shifted SPE's, which are obtained self-consistently in the BCS calculation, were found in Ref. [58] to play a crucial role in reproducing a few transition matrix elements of two-neutrino double beta decays using the QRPA together with realistic effective interactions. Noting ε_j can be also expressed as

$$\varepsilon_j = \epsilon_j + \frac{1}{2j+1}\sum_{j'}\sum_J (1+\delta_{jj'})(2J+1)V(jj'jj', J)v_{j'}^2, \tag{19}$$

one sees that the shifted SPE can be interpreted to be the effective single-particle energy (ESPE) based on the BCS occupation numbers. In Fig. 4(a) we present the distribution of the shifted SPE's for the trapped unitary systems with $A = 62$, 78 and 94, respectively. One sees that, interestingly, the distribution of the shifted SPE's for a trapped unitary neutron system, exhibits a special shell structure, which is the same as that in the distribution of the SPE's for corresponding noninteracting system. In Fig. 4(b) we further present the average value of the shifted SPE's over orbits of one major shell, denoted $\overline{\varepsilon}$, versus the quantum number $(2n + l)$ of the shell. One sees that the average of shifted SPE's of each major shell for trapped unitary systems follows remarkably the straight

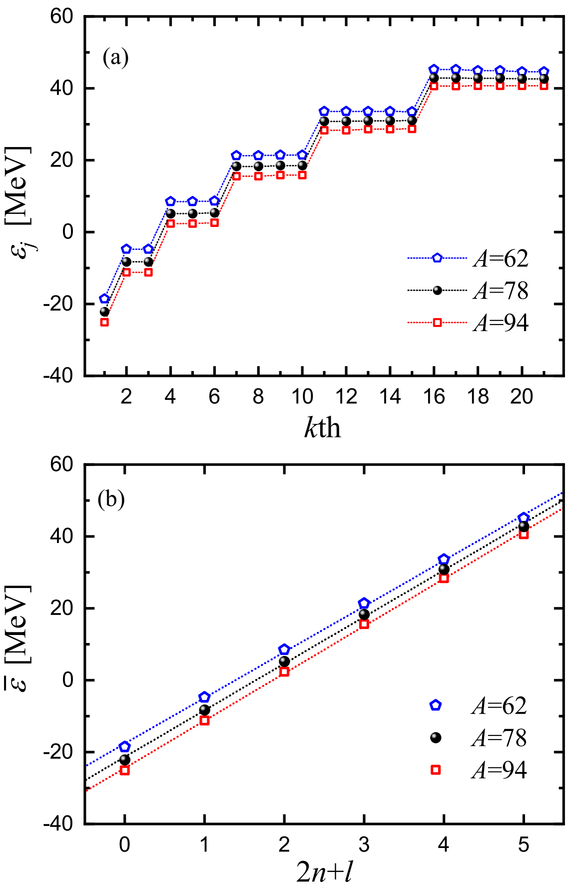

FIG. 4. (a) Shell structure of the single-particle energy (SPE) corrected by the self-energy, $\varepsilon_j = \epsilon_j - \mu_j$, for the trapped unitary systems with $A = 62$, 78 and 94, respectively. (b) The average value of the shifted SPE's over orbits of one major shell, denoted as $\bar{\varepsilon}$, versus the quantum number $(2n + l)$ of the shell, for the systems with $A = 62$, 78 and 94, respectively. The trap parameter $\hbar\omega = 8.5$ MeV is adopted in these calculations.

line described by $[C_1(2n + l) + C_0]$, the same as the SPE of each major shell for noninteracting systems.

At last we come back to the constant energy ratio independent of the particle number, i.e., the ratios for various particle numbers all flowing to the value of 3/4. Recall that for a trapped unitary neutron system, we have $E_{\mathrm{BCS}}^{(\mathrm{m})}$ close to E_{2nd}. At the same time, recall that we have a special distribution of the BCS occupation probabilities for the trapped unitary system, which is described by Eq. (16) and is identical to that of corresponding noninteracting system. Then one can see that $E_{\mathrm{2nd}} \cong \frac{3}{4} E_{\mathrm{free}}$ arises from a weighted sum rule for the monopole components of the unitary neutron-neutron interaction as follows.

$$\sum_{j \leqslant j'} V_{\mathrm{mono.}}^{(j,j')} (2j+1)(2j'+1) v_j^2 v_{j'}^2 \cong -\frac{1}{4} E_{\mathrm{free}}. \qquad (20)$$

Here $V_{\mathrm{mono.}}$ denotes the monopole components of the unitary interaction and is defined by $V_{\mathrm{mono.}}^{(j,j')} = \sum_J (2J + 1) V(jj'jj', J)/[(2j + 1)(2j' + 1)]$; and for a given particle number, the weight for the component $V_{\mathrm{mono.}}^{(j,j')}$ is the product of the occupation number in the j orbit $[(2j + 1)v_j^2]$ and that in the j' orbit $[(2j' + 1)v_{j'}^2]$ with v_j^2, $v_{j'}^2$ given by Eq. (16); and E_{free} is the energy of corresponding noninteracting system. In Fig. 5 we present the left-hand side of the above Eq. (20) versus the right-hand side, using the unitary CD-Bonn interaction combined with $\hbar\omega = 8.5$, 9.5, 10.5 MeV, respectively, for the systems of $A = 50, 54, \ldots, 102$. One sees that the results in Fig. 5 are in satisfactory agreement with the weighted sum rule of Eq. (20).

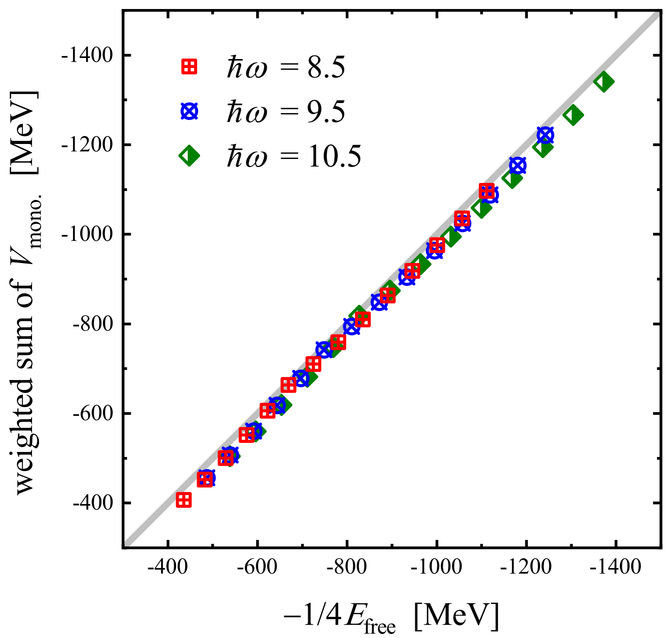

FIG. 5. The weighted sum of the monopole components of the unitary CD-Bonn interaction, $\sum_{j \leqslant j'} V_{\mathrm{mono.}}^{(j,j')} (2j + 1)(2j' + 1) v_j^2 v_{j'}^2$, versus the value of $-\frac{1}{4} E_{\mathrm{free}}$, using $\hbar\omega = 8.5$, 9.5, 10.5 MeV, respectively, for the systems of $A = 50, 54, \ldots, 102$. Here the monopole component of the neutron-neutron interaction is defined by $V_{\mathrm{mono.}}^{(j,j')} = \sum_J (2J + 1) V(jj'jj', J)/[(2j + 1)(2j' + 1)]$; and for a given particle number, the weight for the component $V_{\mathrm{mono.}}^{(j,j')}$ is the product of the occupation number in the j orbit $[(2j + 1)v_j^2]$ and that in the j' orbit $[(2j' + 1)v_{j'}^2]$ with v_j^2, $v_{j'}^2$ given by Eq. (16); and E_{free} is the energy of corresponding noninteracting system.

IV. SUMMARY

In this work we study unitary neutron systems in a harmonic-oscillator trap, based on the BCS theory and the quasiparticle ring-diagram method. The low-momentum interaction $V_{\mathrm{low}\text{-}k}$ renormalized from the fine-tuned CD-Bonn potential which has its $^1S_0$ scattering length approaching the infinity, is adopted.

For the energy ratio of the trapped unitary neutron system to corresponding noninteracting system, our results support a universal value, despite of different choices of the system type, i.e., being the system of a magic particle number or the one of a nonmagic number, and the variation of the particle number. For the BCS wave function, our results suggest universal patterns in the distribution of the occupation probabilities, despite of the variation of the trap parameter and the variation of the particle number: the wave function for the system with a magic number is approximately equivalent to a determinantal wave function with bottom shells fully occupied; and the wave function for the system with a nonmagic number is closely equal to a product of an inert core with bottom shells fully occupied and valence particles occupying the orbits of the lowest valence shell with a uniform probability.

We also discuss the self-energy and the single-particle energy (SPE) corrected by the self-energy, for the trapped unitary neutron system. The self-energy is shown to be proportional to the trap parameter. This, together with the robust distribution of the BCS occupation probabilities independent of the trap parameter, gives rise to the linear scaling relation of the ground-state energy versus the trap parameter. The shifted SPE's for the trapped unitary system are shown to have the same characters as those of the SPE's for the noninteracting system. At last we present a weighted sum rule satisfied by the monopole components of the unitary neutron-neutron interaction. This sum rule, as well as the special distribution of the BCS occupation probabilities for the trapped unitary system, gives rise to the constant ground-state energy ratio independent of the particle number.

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