Multiparticle integral and differential correlation functions

Claude Pruneau [1,*] Victor Gonzalez [1,†] Ana Marin [2,‡] and Sumit Basu [3,§]
[1] *Department of Physics and Astronomy, Wayne State University, Detroit, Michigan 48201, USA*
[2] *GSI Helmholtzzentrum für Schwerionenforschung GmbH, Research Division and ExtreMe Matter Institute EMMI, Planckstrasse 1, 64291 Darmstadt, Germany*
[3] *Department of Physics, Lund University, Division of Particle Physics, Box 118, SE-221 00 Lund, Sweden*

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This paper formalizes the use of integral and differential cumulants for measurements of multiparticle event-by-event transverse momentum fluctuations, rapidity fluctuations, as well as net-charge fluctuations. This enables the introduction of multiparticle balance functions, defined based on differential correlation functions (factorial cumulants), that suppress two- and three-prong resonance decays effects and enable measurements of underlying long-range correlations obeying quantum number conservation constraints. These multiparticle balance functions satisfy simple sum rules determined by quantum number conservation. It is additionally shown that these multiparticle balance functions arise as an intrinsic component of high-order net-charge cumulants. This implies that the magnitude of these cumulants, measured in a specific experimental acceptance, is strictly constrained by charge conservation and primarily determined by the rapidity and momentum width of these balance functions. The paper also presents techniques to reduce the computation time of differential correlation functions up to order $n = 10$ based on the methods of moments.

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I. INTRODUCTION

A variety of two-particle integral and differential correlation functions have been developed and deployed towards the analysis of particle production in heavy-ion collisions [1–4]. Correlation functions formulated as functions of rapidity and azimuth angle differences have enabled, in particular, the discovery of away-side jet suppression in large collision systems [5,6]. Subsequent measurements of long-range particle correlations in both large and small collision systems have additionally enabled detailed studies of the collision dynamics. Of particular interest are two-particle number correlations, R_2, based on normalized two-particle differential cumulants [3,4], transverse momentum (p_T) correlation functions, P_2, designed to study p_T fluctuations [7], and G_2, designed for the study of viscous effects based on their longitudinal broadening in A–A collisions [2,8,9]. Differential two-particle correlation functions have been studied for charge-inclusive (CI) particle pairs as well as charge-dependent (CD) pairs. The latter are of particular interest because they relate to charge balance functions, which provide a tool to study the correlation length, in momentum space, of charge balancing particles. Balance functions were initially developed to investigate the presence of isentropic expansion and delayed hadronization in large collision systems [10–12]. In that context, they were defined according to

$$B(p_2|p_1) \equiv \tfrac{1}{2}[\rho(b, p_2|a, p_1) - \rho(b, p_2|b, p_1) + \rho(a, p_2|b, p_1) - \rho(a, p_2|a, p_1)], \quad (1)$$

where $\rho(b, p_2|a, p_1)$ is the conditional density of particles of type b in momentum bin p_2 given the existence of a particle of type a in a bin p_1. The labels a and b were then meant to refer to the charge of (specific) hadrons. For instance, b could refer to positive kaons whereas a identifies negative kaons. Balance functions were later shown to also be sensitive to quark susceptibilities [13] as well as the diffusivity of light quarks [14,15]. Recent works have shown they are best measured in terms of differential two-particle cumulants and feature simple sum-rule that might be instrumental in the study of thermal hadron production [16–18].

Two-particle correlation functions are by construction dominated by contributions from two particle correlated production processes such as hadronic decays and jet production. They are also nominally sensitive to higher-order particle correlations even though they cannot specifically discriminate such higher-order processes. In many instances, it is these higher-order correlation processes that are of interest to investigate particle production and properties of the matter produced in A–A collisions. As a first example, consider measurements of transverse momentum fluctuation deviates. Correlators of transverse momentum deviates were initially invoked as a proxy to study temperature fluctuations in A–A

*claude.pruneau@wayne.edu
†victor.gonzalez@cern.ch
‡a.marin@gsi.de
§sumit.basu@cern.ch

collisions [19]. However, it may be argued that fluctuations observed in A–A collisions are strictly driven by initial state conditions [20]. There is thus interest in establishing the magnitude of such collective fluctuations. The use of the multiple particle $p_{\rm T}$ integral correlator has then been advocated to study the effects of initial fluctuations [21,22]. However note that ongoing studies may suffer from short-range correlations (a.k.a. nonflow) and it might thus be desirable to develop differential analysis techniques that enable rapidity gaps designed to suppress short-range correlations. As a second example of interest, consider charge, strangeness, and baryon balance functions. In this case also, one expects the correlations to receive sizable contributions from two- and three-prong particle decays as well as short-range correlation from jets. There is thus an interest in obtaining higher-order correlation functions that are sensitive to charge (strangeness, baryon) balance but suppress contributions from decays and jets and it is the primary purpose of this work to develop such multiparticle balance functions. As in studies of anisoptropic flow measurements, multiparticle cumulants, of order $n = 4, 6, \ldots$, shall be used to suppress lower order correlations and obtain multiparticle balance functions. Unfortunately, measurements and calculations of higher-order cumulants nominally require several nested loops to include contributions from all n-tuplets of interest on an event-by-event basis. Although such calculations are conceptually simple and remain practical for collisions and experimental acceptances featuring a modest particle multiplicity, they become prohibitively CPU intensive for large multiplicities. Fortunately, a number of techniques have been developed, particularly in the context of anisotropic flow analyses, to reduce the computational challenge of handling large multiplicities and high-order cumulants [23]. One such application successfully deployed in the context of anisotropic flow studies and towards the computation of higher moment deviates relies of the method of moments [21]. A second purpose of this paper is to further develop this method towards measurements of differential multiparticle correlations of inclusive as well as identified particle species.

This paper is organized as follows. First, Sec. II presents a discussion detailing the need for higher-order integral and differential correlators based on several examples. Higher-order correlators are then introduced in Sec. III in terms of integral and differential cumulants of arbitrary order as well as expectation values of the form $\langle\langle q_1 q_2 \cdots q_n\rangle\rangle$ and $\langle\langle \Delta q_1 \Delta q_2 \cdots \Delta q_n\rangle\rangle$, where q_i, $i = 1,\ldots,n$ represent particle variables of interest (e.g., transverse momentum, charge, etc.) and $\Delta q_i = q_i - \langle\langle q\rangle\rangle$, $i = 1,\ldots,n$, are their deviates relative to their event ensemble average $\langle\langle q\rangle\rangle$. Techniques to compute these expectation values based on the method of moments are presented in Secs. III C and III D. Equipped with these different correlators and computing tools, the notion of multiparticle balance function is then introduced in Sec. IV. Finally, multiparticle balance functions are considered in Sec. V in the context of measurements of net-charge cumulants and it is shown that net-charge cumulants of all orders are explicitly constrained by charge conservation. The paper is summarized in Sec. VI. Given much of the calculations performed in this work are somewhat tedious and lengthy, details of these calculations are presented in several appendices. Appendix A presents calculation methods and formula for moments, cumulants, factorial moments, and factorial cumulants for single and multivariable systems, as well as for the net charge of particles measured in a specific acceptance Ω. Appendix B extends these formula to differential correlations. General formula for the computations of deviates of the form $\langle\langle \Delta q_1 \Delta q_2 \cdots \Delta q_m\rangle\rangle$ are derived in Appendix C while equations for the calculation of correlators of the form $\langle\langle q_1 q_2 \cdots q_m p_1 \cdots p_n\rangle\rangle$ are listed in Appendix D. Finally, Appendix E lists definitions of multiparticle balance functions up to order $n = 10$.

II. MOTIVATIONS

Let us consider single particle densities of particles of type α, denoted $\rho_1^{\alpha}(\vec{p})$, and n-particle densities of mixed species $\alpha_1,\ldots,\alpha_n$, denoted $\rho_n^{\alpha_1\cdots\alpha_n}(\vec{p}_1,\ldots,\vec{p}_n)$. In general, mixed n-particle densities, $\rho_n^{\alpha_1\cdots\alpha_n}(\vec{p}_1,\ldots,\vec{p}_n)$, correspond to the yield (per event) of n-tuplets of particles of types $\alpha_1,\alpha_2,\ldots,\alpha_n$ at momenta $\vec{p}_1,\ldots,\vec{p}_n$. Such n-tuplets may arise from a single process yielding n correlated (mixed) particles, or combinations of processes jointly yielding n-particles. In general, the particle categories α and β may be either identical, distinct, or partially overlapping. This evidently impacts the calculation of integral factorial moments and integral cumulants as discussed in further details in Sec. III A. To focus a study on correlated particles exclusively, one commonly relies on the notions of integral and differential correlation function cumulants. Indeed recall that, by construction, integration of $\rho_1^{\alpha}(\vec{p})$ over a specific kinematic range Ω yields the average number of particles of type α in this acceptance, whereas integration of two-, three-, or n-mixed-particle densities yield the average number of pairs, triplets, and more generally n-tuplets of such groupings of particles. These integrals do not discriminate correlated from uncorrelated particles.

Differential cumulants and their integrals are of particular interest because they identically vanish in the *absence* of n or more particle correlations. They are thus an essential tool for the study of particle production. They can also be straightforwardly corrected for uncorrelated particle losses (detection efficiency). This feature is exploited in the formulation of number correlation function ratios such as

$$R_2^{\alpha\beta}(\vec{p}_1,\vec{p}_2) \equiv \frac{\rho_2^{\alpha\beta}(\vec{p}_1,\vec{p}_2) - \rho_1^{\alpha}(\vec{p}_1)\rho_1^{\beta}(\vec{p}_2)}{\rho_1^{\alpha}(\vec{p}_1)\rho_1^{\beta}(\vec{p}_2)} \equiv \frac{C_2^{\alpha\beta}(\vec{p}_1,\vec{p}_2)}{\rho_1^{\alpha}(\vec{p}_1)\rho_1^{\beta}(\vec{p}_2)} \equiv \frac{F_2^{\alpha\beta}}{F_1^{\alpha}F_1^{\beta}}, \tag{2}$$

with C_2, the second-order cumulant, and F_1 and F_2, the first and second-order factorial cumulants, respectively. The ratios R_2 are said to be robust against particle losses, i.e., independent of efficiencies provided these are approximately constant within the acceptance of a measurement [24]. Differential correlation functions $C_n(\vec{p}_1,\ldots,\vec{p}_n)$ and their integrals F_n are formally defined in Sec. III. Ratios of factorial cumulants, similarly formulated, have also been used in recent studies [2,4]. They too feature the property of robustness against particle (efficiency) losses. Particular combinations of differential and integral ratios $R_2^{\alpha\beta}$ have also been used in the context of relative yield fluctuation studies and balance functions [25]. Integral and differential correlation functions have also been used to study fluctuations of specific kinematic variables. Fluctuations of event-wise total transverse momentum, in particular, have been proposed to study temperature and energy fluctuations in the initial stage of heavy-ion collisions. Voloshin *et al.* [19] showed fluctuations measures are best formulated in terms of transverse momentum deviates $\Delta p_{\rm T,1}\Delta p_{\rm T,2}$ according to

$$\langle\langle \Delta p_{\rm T,1}\Delta p_{\rm T,2}\rangle\rangle \equiv \frac{\int_\Omega \Delta p_{\rm T,1}\Delta p_{\rm T,2}\;\rho_2(\vec{p}_1,\vec{p}_2)\,d\vec{p}_1\,d\vec{p}_2}{\int_\Omega \rho_2(\vec{p}_1,\vec{p}_2)\,d\vec{p}_1\,d\vec{p}_2}. \quad (3)$$

This integral correlator is commonly reported in terms of a dimensionless ratio $\langle\langle \Delta p_{\rm T,1}\Delta p_{\rm T,2}\rangle\rangle/\langle\langle p_{\rm T}\rangle\rangle^2$ [26,27]. A differential version of this dimensionless correlator was also used [2,4,7,28] and can be written according to

$$P_2(y_1,\varphi_1,y_2,\varphi_2) \equiv \frac{1}{\langle\langle p_{\rm T}\rangle\rangle^2}\frac{\int_\Omega \Delta p_{\rm T,1}\Delta p_{\rm T,2}\;\rho_2(p_{\rm T,1},y_1,\varphi_1,p_{\rm T,2}y_2,\varphi_2)\,dp_{\rm T,1}dp_{\rm T,2}}{\int_\Omega \rho_2(p_{\rm T,1},y_1,\varphi_1,p_{\rm T,2},y_2,\varphi_2)\,dp_{\rm T,1}dp_{\rm T,2}}. \quad (4)$$

A generalization of $\langle\langle \Delta p_{\rm T,1}\Delta p_{\rm T,2}\rangle\rangle$ to four particle correlations was first proposed by Voloshin [26] towards the study of temperature and energy fluctuations. The study of third and fourth $p_{\rm T}$ moments, defined according to

$$\langle\langle \Delta p_{\rm T,1}\Delta p_{\rm T,2}\Delta p_{\rm T,3}\rangle\rangle = \frac{1}{\langle\langle p_{\rm T}\rangle\rangle^3}\frac{\int_\Omega \Delta p_{\rm T,1}\Delta p_{\rm T,2}\Delta p_{\rm T,3}\;\rho_3(\vec{p}_1,\vec{p}_2,\vec{p}_3)\,d\vec{p}_1\,d\vec{p}_2\,d\vec{p}_3}{\int_\Omega \rho_3(\vec{p}_1,\vec{p}_2,\vec{p}_3)\,d\vec{p}_1\,d\vec{p}_2\,d\vec{p}_3}, \quad (5)$$

$$\langle\langle \Delta p_{\rm T,1}\Delta p_{\rm T,2}\Delta p_{\rm T,3}\Delta p_{\rm T,4}\rangle\rangle = \frac{1}{\langle\langle p_{\rm T}\rangle\rangle^4}\frac{\int_\Omega \Delta p_{\rm T,1}\Delta p_{\rm T,2}\Delta p_{\rm T,3}\Delta p_{\rm T,4}\;\rho_4(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)\,d\vec{p}_1\,d\vec{p}_2\,d\vec{p}_3\,d\vec{p}_4}{\int_\Omega \rho_4(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)\,d\vec{p}_1\,d\vec{p}_2\,d\vec{p}_3\,d\vec{p}_4}, \quad (6)$$

was proposed to probe initial stage fluctuations [21]. It is good to note that Eqs. (3)–(6) can in principle be simplified by replacing the number densities $\rho_n(\vec{p}_1,\ldots,\vec{p}_n)$ by probability densities $p_n(\vec{p}_1,\ldots,\vec{p}_n)$ defined according to

$$p_n(\vec{p}_1,\ldots,\vec{p}_n) \equiv \frac{\rho_n(\vec{p}_1,\ldots,\vec{p}_n)}{\int_\Omega \rho_n(\vec{p}_1,\ldots,\vec{p}_n)d\vec{p}_1\cdots d\vec{p}_n}. \quad (7)$$

In practice, however, such a substitution is not particularly useful because, experimentally, one does not measure ρ_n directly but rather obtain "estimators" (statistics) of such densities based on sums over the many particles of measured events. For instance, for pairs, triplets, and quadruplets, one could write inclusive estimator according to

$$\langle\langle \Delta p_{\rm T}\Delta p_{\rm T}(y_1,y_2)\rangle\rangle = \frac{\sum_{i\neq j}\Delta p_{\rm T,i}\Delta p_{\rm T,i}}{\langle\langle N_{\rm pairs}(y_1,y_2)\rangle\rangle}, \quad (8)$$

$$\langle\langle \Delta p_{\rm T}\Delta p_{\rm T}\Delta p_{\rm T}(y_1,y_2,y_3)\rangle\rangle = \frac{\sum_{i\neq j\neq k}\Delta p_{\rm T,i}\Delta p_{\rm T,i}\Delta p_{\rm T,k}}{\langle\langle N_{\rm triplets}(y_1,y_2,y_3)\rangle\rangle}, \quad (9)$$

$$\langle\langle \Delta p_{\rm T}\Delta p_{\rm T}\Delta p_{\rm T}\Delta p_{\rm T}(y_1,y_2,y_3,y_4)\rangle\rangle = \frac{\sum_{i\neq j\neq k\neq l}\Delta p_{\rm T,i}\Delta p_{\rm T,i}\Delta p_{\rm T,k}\Delta p_{\rm T,l}}{\langle\langle N_{\rm quads}(y_1,y_2,y_3,y_4)\rangle\rangle}, \quad (10)$$

where the sum in the numerators proceeds over particles in bins at rapidities $y_1,\ldots,y_n$, whereas $N_{\rm pairs}(y_1,y_2)$, $N_{\rm triplets}(y_1,y_2,y_3)$, $N_{\rm quads}(y_1,y_2,y_3,y_4)$ denote the number of pairs, triplets, and quadruplets of particles in rapidity bins at $y_1,\ldots,y_n$ in a given event.

Measurements of the correlators (5) and (6) were recently reported by the ALICE collaboration [22]. Clearly, it is trivial to also consider differential versions of these two correlators and such generalizations of $P_2(y_1,\varphi_1,y_2,\varphi_2)$ might be useful to carry out higher moment analyses with finite rapidity gaps. Additionally, as we discuss in the next sections, extension to particle correlators of this form to $n > 4$ particles are readily accessible based on the methods of moments presented in Sec. III C.

The integral and differential correlation functions, Eqs. (3)–(6), may also be applicable to the study of other types of fluctuations. For instance, replacing $\Delta p_{\rm T,i}$ by rapidity deviates $\Delta y_i \equiv y_i - \langle\langle y\rangle\rangle$, where y_i are the rapidities of particles $i = 1,\ldots,N$ of an event, it becomes possible to study event-by-event fluctuations in the rapidity of particles. Such fluctuations might provide an alternative way to probe the longitudinal correlation length (rapidity) of produced particles.

The multiparticle correlators and the set of tools for their extraction presented in Sec. III provide a basis for the extension of former techniques used for measurements of transverse momentum correlations, rapidity correlations, as well as charge correlations (including baryon and strangeness numbers correlations) heretofore completed mostly at low orders $n \leqslant 4$. These techniques also connect to measurements of anisotropic flow and correlations between flow and other variables discussed elsewhere [23].

The techniques, whether used with a single or several variables, are nominally very powerful because they enable joint measurements involving many particles simultaneously. However, it is also clear that the complexity of such measurements can quickly grow out of hand. For instance, assuming an interest in the rapidity, transverse momentum, and azimuth of particles, one would nominally get differential observables $\langle\langle \Delta q_1 \Delta q_2 \cdots \Delta q_n \rangle\rangle$ featuring $3 \times n$ degrees of freedom. Measuring such features would then require collecting data in as many as $3 \times n$ dimensions. Clearly, a considerable reduction of this "feature" space is required to enable the feasibility of measurements both in terms of data volumes (i.e., capturing sufficiently many n-tuples of particles to cover all partitions of the feature space with meaningful values) and in terms of its representation and interpretation. While we do not wish to preclude or dismiss possibilities of complex multidimensional analyses, we focus the discussion, as a kind of extended motivation, on some basic applications of the formalism and methods presented later in this work towards some specific physics analyses.

On general grounds, one can classify analyses of potential interest based on the number of kinematic partitions being used (i.e., partition of the $3 \times n$ momentum space), the number of observables of interest (e.g., transverse momentum, $p_{\rm T}$, rapidity, y, charge, anisotropic flow coefficients, etc.) and the number of particle types or species being considered (e.g., inclusive charged particles, positively versus negatively charged particles, specific species such as pions, kaons, etc.). We thus organize the discussion in terms of few use cases, beginning with the simplest case involving a single variable q, and next considering progressively more and more complex use cases involving two variables: q, p, as well as several variables.

Analyses based on a single variable q are already quite popular and have featured studies of fluctuations of transverse momentum, net charge, etc. However, most prior analyses have been limited to two particles [2,7,8,28] and only few recent works have undertaken higher number of particles [22,26]. Notable exceptions evidently include measurements of anisotropic flow based on multiparticle cumulants [29], multiparticle correlations between arbitrary numbers of particles of interest selected for their strangeness, heavy flavor, and conserved charges [30] as well as older works [31–33].

Measurements of $p_{\rm T}$ (alternatively y or q, etc.) correlations $\langle\langle \Delta p_{{\rm T},1} \Delta p_{{\rm T},2} \cdots \Delta p_{{\rm T},n} \rangle\rangle$ involving $n \geqslant 4$ particles of a given type of particle in a specific acceptance can be readily undertaken based on the methods discussed in Sec. III. Various types of scaling are evidently possible to obtain dimensionless observables and assess the evolution of n-order momentum correlators as a function of the A–A collision centrality or produced particle multiplicity. An obvious choice is the inclusive momentum average $\langle\langle p_{\rm T} \rangle\rangle$ already used in several studies [2,7,8] but other choices of scaling have also been discussed [21]. Differential measurements can then be achieved, for instance, by studying the magnitude of $\langle\langle \Delta p_{{\rm T},1} \Delta p_{{\rm T},2} \cdots \Delta p_{{\rm T},n} \rangle\rangle$ versus the width of the acceptance in rapidity.

Analyses involving two variables, q and p, are of interest, for instance, towards the study of some specific observable (e.g., anisotropic flow, transverse momentum fluctuations, or net-charge fluctuations) in two kinematic partitions separated by a finite-size rapidity gap.

Considering examples of $p_{\rm T}$ fluctuation studies, let q_i and p_i represent the transverse momentum of particles measured in two distinct rapidity acceptance ranges Ω_A and Ω_B of equal widths separated by a finite rapidity gap $\Delta\eta$, as schematically illustrated in Fig. 1. One can then measure correlators of the form $\langle\langle \Delta q_1 \Delta p_1 \rangle\rangle$, $\langle\langle \Delta q_1 \Delta q_2 \Delta p_1 \Delta p_2 \rangle\rangle$, etc., at any order to determine the strength of $n = 2, 4$, etc., transverse momentum correlations as a function of the width of the rapidity gap. Measurements of transverse momentum fluctuations and correlations have been thus far mostly limited to two-particle studies and implemented with a single acceptance bin [34–37] or in a fully differential manner [2,8]. The methods discussed in Sec. III, however, enable differential measurements involving multiple $n \geqslant 4$ particles. It then becomes possible to study momentum correlations arising from initial state fluctuations [21] while suppressing the influence of short-range correlations (a.k.a. nonflow) associated with hadron decays and jet fragmentation. Letting q_i, $i = 1, \ldots$, represent the $p_{\rm T}$ of particles in partition A, and p_i, $i = 1, \ldots$ represent the $p_{\rm T}$ of particles in partition B, the described methods allow to obtain $\langle\langle q_1 q_2 \cdots q_n p_1 p_2 \cdots p_n \rangle\rangle$ and the n-order deviates $\langle\langle \Delta q_1 \Delta q_2 \cdots \Delta q_n \Delta p_1 \Delta p_2 \cdots \Delta p_n \rangle\rangle$ corresponding to $p_{\rm T}$ correlators involving n particles from partition A and n particles from partition B. As in flow studies, one expects that it becomes possible to progressively suppress resonance and jet contributions by increasing the rapidity gap $\Delta\eta$ between partitions A an B. The analysis can also be made more differential by also using bins in azimuth as illustrated in Figs. 1(c) and 1(d), thereby enabling the suppression of or focus on back-to-back jet contributions.

The methodology can readily be adopted also for measurements of charge correlations and, as it will be introduced, multiparticle balance functions. In this case, q_i and p_i represent the charge of particles in partitions A and B. Then generic charge correlators $\langle\langle q_1 q_2 \cdots q_n p_1 p_2 \cdots p_n \rangle\rangle$ and their deviates $\langle\langle \Delta q_1 \Delta q_2 \cdots \Delta q_n \Delta p_1 \Delta p_2 \cdots \Delta p_n \rangle\rangle$ can be obtained and, it will be seen, these can then be related to multiparticle balance functions.

Two other use cases based on two variables q_i and p_i are worth mentioning. One involves the study of two distinct physics observables (e.g., charge, $p_{\rm T}$, rapidity, etc.) in a single kinematic partition whereas the other involves the measurement of a specific particle observable, e.g., the $p_{\rm T}$, for two types of particle species. In the first case, the variable q_i and p_i represent the two observables of interest whereas in the second they tag the species of interest. These latter use cases enable multiparticle correlations with specific species or between identical particles in two distinct $p_{\rm T}$ ranges.

The examples discussed in the previous paragraphs are readily extended towards the computation of correlation functions involving three or more kinematic partitions and particle types. Of particular interest is the determination of multiple particle balance functions. Although it may not be practical to conduct analyses involving explicit computation of more than three or four kinematic partitions or species, it remains possible to consider balance functions involving large number of particles towards the study of long-range multi-

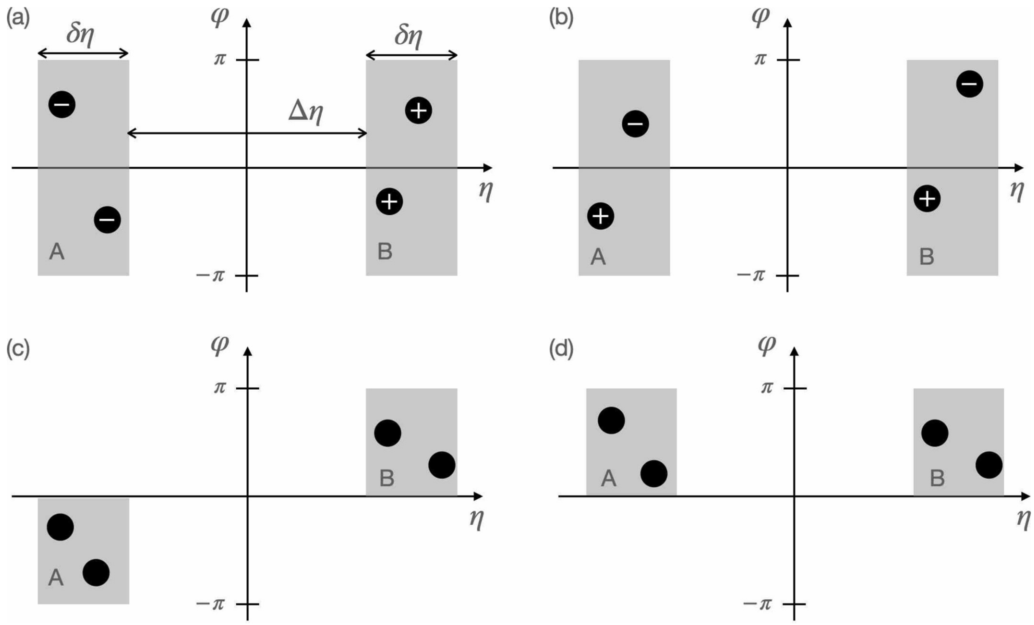

FIG. 1. Schematic illustrations of four-particle correlation analyses involving a finite rapidity gap to suppress resonance decays and other short-range correlations. The horizontal and vertical axes η and φ, respectively, denote the pseudorapidity and azimuthal coordinates of measured particles. Gray areas schematically indicate the azimuthal (φ) and pseudorapidity (η) acceptance of measurements. Panels (a, b) illustrate measurement involving negatively and positively charged particles in valid unlike sign combinations into two distinct ranges of pseudorapidity (full azimuthal coverage) separated by a rapidity gap $\Delta\eta$. Panels (c, d) illustrate four-particle inclusive correlation measurements involving a finite rapidity gap and a specific coverage in azimuth, which enable acceptance configurations sensitive to (c) back-to-back particle production or (d) out-of-back-to-back emission.

particle correlations constrained by charge conservation (or other quantum number conservation laws). Let us first consider measurements of four-particle balance functions based on two kinematic partitions A and B separated by a finite rapidity gap, as illustrated in Figs. 1(a) and 1(b). The partitions A and B could be azimuthally symmetric (i.e., with full azimuth coverage $0 \leqslant \varphi < 2\pi$), or feature partial coverage to focus or suppress contributions from back-to-back jets, as schematically illustrated in Figs. 1(c) and 1(d), respectively, of the same figure. Figure 1(a) illustrates a measurement involving two positively charged particles in B and two negatively particles in A. A measurement of the multiparticle balance function shall then be sensitive to the strength (or probability) of processes featuring four correlated particles separated by a finite rapidity gap. Since four-prong resonance decays are relatively few, this would reveal the likelihood of long-range correlations determined by stringlike fragmentation processes. In contrast, the analysis illustrated in Fig. 1(b) would focus on correlated quartets featuring two nearby pairs of unlike sign particles. These could be produced by stringlike fragmentation processes yielding four or more correlated particles, but they could also result from string fragmentation producing two neutral objects, each decaying into pairs of +ve and −ve particles. An explicitly selection of the charge states to be measured in partitions A and B might thus enable a discriminating study of the relative yields of distinct processes. Indeed, an analysis of the dependence of the relative strengths of processes depicted in Figs. 1(a) and 1(b) could shed additional light on particle production process in elementary collisions. An analysis of the correlation strength performed as a function of the rapidity gap might then provide better sensitivity to the correlation length of string break up processes. Additionally, such analyses conducted as a function of collision centrality and beam energy in large systems (A–A), and comparisons to dependencies observed in small systems (e.g., pp and p–A), might then reveal whether this correlation length evolves with energy density, system size, collision energy, etc. Clearly, the position and size of measurement bins can be varied. Figure 1(c) illustrates a measurement geometry emphasizing back-to-back jet emission whereas Fig. 1(d) suppresses such processes and thus enables the study of long-range nonjet and not resonance decay processes such as longitudinal string fragmentation. Obviously, a wide variety of other detection geometries can be implemented to explicitly favor or inhibit specific particle processes.

Analyses probing correlations of three or four particles of different charge, strangeness, and baryon number are also of interest and are possible with the framework presented in this paper. For illustrative purposes, consider the two scenarios displayed in Fig. 2. Figure 2(a) illustrates a measurement involving a neutral $\overline{\Lambda}(\overline{\text{uds}})$ antibaryon and a proton (uud) in rapidity partition A, observed jointly with a negative Ξ^-(dss) baryon and a neutral $\overline{\Lambda}(\overline{\text{uds}})$ antibaryon in rapidity partition B, which simultaneously probes baryon and strangeness balancing. Similarly, Fig. 2(b) shows a measurement involving a neutral $\overline{\Lambda}(\overline{\text{uds}})$ baryon and an proton (uud) in partition A measured jointly with a negative Ω^-(sss) baryon and a positive $\overline{\Xi}^+(\overline{\text{dss}})$ antibaryon in partition B, which probes charge, strangeness, and baryon number balancing all at once.

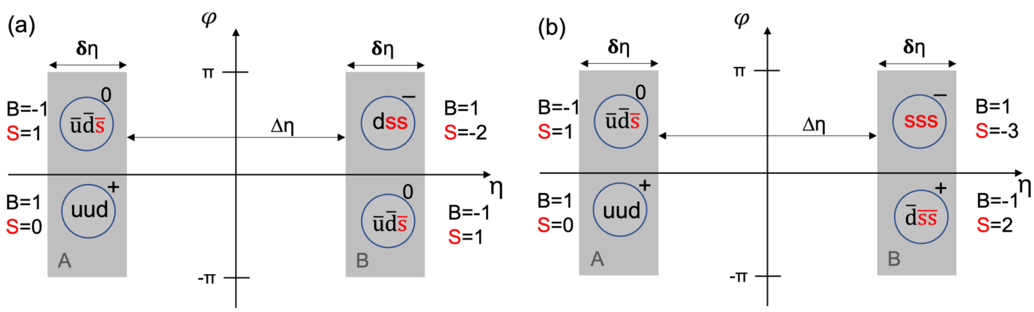

FIG. 2. Examples of measurements involving a $\overline{\Lambda}$ antibaryon and a proton in rapidity partition A with (a) a Ξ^- baryon and a neutral $\overline{\Lambda}$ antibaryon or (b) an Ω^- baryon and a $\overline{\Xi}^+$ antibaryon in partition B. The horizontal and vertical axes η and ϕ, respectively, denote the pseudorapidity and azimuthal coordinates of measured particles. Gray areas schematically indicate the azimuthal (φ) and pseudorapidity (η) acceptance of measurements.

In general, a large number of such particle species combinations could be established to study charge (Q), strangeness (S), baryon number (B), and isospin I_3 balancing. The tools could also be applied to charmness (C) and/or bottomness (B) balancing.

III. MULTIPARTICLE CORRELATION FUNCTIONS

A. Cumulants and factorial cumulants

As already mentioned, to focus a study on correlated particles exclusively, one relies on the notions of integral and differential correlation function cumulants. Differential cumulants have been discussed in details elsewhere [23,29]. In the context of this work, it suffices to remember they can be "reverse engineered" by listing all cluster decompositions of n-tuple densities. As such, single-, two-, and three-particle mixed-density cumulants may be obtained by writing [25]

$$C_1^{\alpha}(\vec{p}_1) \equiv \rho_1^{\alpha}(\vec{p}_1), \tag{11}$$

$$\begin{aligned} C_2^{\alpha_1\alpha_2}(\vec{p}_1,\vec{p}_2) &\equiv \rho_2^{\alpha_1\alpha_2}(\vec{p}_1,\vec{p}_2) - C_1^{\alpha_1}(\vec{p}_1)C_1^{\alpha_2}(\vec{p}_2) \\ &= \rho_2^{\alpha_1\alpha_2}(\vec{p}_1,\vec{p}_2) - \rho_1^{\alpha_1}(\vec{p}_1)\rho_1^{\alpha_2}(\vec{p}_2), \end{aligned} \tag{12}$$

$$\begin{aligned} C_3^{\alpha_1\alpha_2\alpha_3}(\vec{p}_1,\vec{p}_2,\vec{p}_3) &\equiv \rho_3^{\alpha_1\alpha_2\alpha_3}(\vec{p}_1,\vec{p}_2,\vec{p}_3) \\ &\quad - \sum_{(3)} C_2^{\alpha_1\alpha_2}(\vec{p}_1,\vec{p}_2)C_1^{\alpha_3}(\vec{p}_3) \\ &\quad - C_1^{\alpha_1}(\vec{p}_1)C_1^{\alpha_2}(\vec{p}_2)C_1^{\alpha_3}(\vec{p}_3), \end{aligned} \tag{13}$$

where the notation $\sum_{(k)}$ stands for a sum over k-ordered permutations of the labels α_1, α_2, and α_3, as well as the corresponding momentum vectors $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$. For $n = 3$, substitution of first and second-order cumulants and expansion of the permutations yield

$$\begin{aligned} &C_3^{\alpha_1\alpha_2\alpha_3}(\vec{p}_1,\vec{p}_2,\vec{p}_3) \\ &\quad = \rho_3^{\alpha_1\alpha_2\alpha_3}(\vec{p}_1,\vec{p}_2,\vec{p}_3) - \rho_2^{\alpha_1\alpha_2}(\vec{p}_1,\vec{p}_2)\rho_1^{\alpha_3}(\vec{p}_3) \\ &\qquad - \rho_2^{\alpha_1\alpha_3}(\vec{p}_1,\vec{p}_3)\rho_1^{\alpha_2}(\vec{p}_2) - \rho_2^{\alpha_2\alpha_3}(\vec{p}_2,\vec{p}_3)\rho_1^{\alpha_1}(\vec{p}_1) \\ &\qquad + 2\rho_1^{\alpha_1}(\vec{p}_1)\rho_1^{\alpha_2}(\vec{p}_2)\rho_1^{\alpha_3}(\vec{p}_3). \end{aligned} \tag{14}$$

Higher-order cumulants are computed and listed in Appendix A.

Mixed cumulants $C_n^{\alpha_1\cdots\alpha_n}(\vec{p}_1,\ldots,\vec{p}_n)$ nominally feature $3\times n$ degrees of freedom and are, as such, challenging to measure and visually represent. The dimensionality, and thus the number of degrees of freedom, can fortunately be reduced by integrating over several coordinates. Considering the densities, when all coordinates are integrated within acceptances $\Omega_k, k = 1, 2, \ldots, n$, one obtains mixed factorial moments. For instance, the integration of $\rho_1^{\alpha}(\vec{p})$ over a specific kinematic range Ω yields the average number of particles of type α in this acceptance, whereas integration of two-, three-, or n-mixed-particle densities yield the average number of pairs, triplets, and more generally n-tuplets of such groupings of particles. These averages are known as mixed factorial moments, herewith denoted $f_n^{\alpha_1\cdots\alpha_n}$, and computed according to

$$f_1^{\alpha} \equiv \int_{\Omega}\rho_1^{\alpha}(\vec{p})d\vec{p} = \langle N^{\alpha}\rangle, \tag{15}$$

$$\begin{aligned} f_2^{\alpha_1\alpha_2} &\equiv \int_{\Omega_{\alpha_1}}\int_{\Omega_{\alpha_2}}\rho_2^{\alpha_1\alpha_2}(\vec{p}_1,\vec{p}_2)d\vec{p}_1d\vec{p}_2 \\ &= \langle N^{\alpha_1}(N^{\alpha_2}-\delta_{\alpha_1\alpha_2})\rangle, \end{aligned} \tag{16}$$

$$\begin{aligned} f_3^{\alpha_1\alpha_2\alpha_3} &\equiv \int_{\Omega_{\alpha_1}}\cdots\int_{\Omega_{\alpha_3}}\rho_3^{\alpha_1\alpha_2\alpha_3}(\vec{p}_1,\vec{p}_2,\vec{p}_3)d\vec{p}_1d\vec{p}_2d\vec{p}_3 \\ &= \langle N^{\alpha_1}(N^{\alpha_2}-\delta_{\alpha_2\alpha_1})(N^{\alpha_3}-\delta_{\alpha_3\alpha_1}-\delta_{\alpha_3\alpha_2})\rangle, \end{aligned} \tag{17}$$

$$\begin{aligned} f_n^{\alpha_1\cdots\alpha_n} &\equiv \int_{\Omega_{\alpha_1}}\cdots\int_{\Omega_{\alpha_3}}\rho_n^{\alpha_1\cdots\alpha_n}(\vec{p}_1,\ldots,\vec{p}_n)d\vec{p}_1\cdots d\vec{p}_n \\ &= \langle N^{\alpha_1}(N^{\alpha_2}-\delta_{\alpha_2\alpha_1})\cdots(N^{\alpha_n}-\delta_{\alpha_n\alpha_1}-\delta_{\alpha_n\alpha_2}-\cdots \\ &\quad -\delta_{\alpha_n\alpha_{n-1}})\rangle, \end{aligned} \tag{18}$$

where $\langle N^{\alpha}\rangle$, $\langle N^{\alpha_1}(N^{\alpha_2}-\delta_{\alpha_1\alpha_2})\rangle$, and so on, denote the ensemble average of the number of mixed tuplets of the corresponding order. The terms $\delta_{\alpha\beta}$ are unity for identical species or particle classes, i.e., $\alpha = \beta$, and null otherwise. For instance, for a single species α, the average number of pairs and triplets would be $\langle N_{\alpha}(N_{\alpha}-1)\rangle$ and $\langle N_{\alpha}(N_{\alpha}-1)(N_{\alpha}-2)\rangle$, respectively. If two distinct species α and β are considered, then the average number of pairs is $\langle N_{\alpha}N_{\beta}\rangle$. For triplets (and higher mixed orders), one must specify the type of triplets being considered. For triplets consisting of two particles of type α and one particle of type β, the average is $\langle N_\alpha(N_\alpha - 1)N_\beta\rangle$. Extensions to higher-order factorial moments proceed in a similar manner. A slight complication arises when categories α and β partially overlap. This can be the case, for instance, when partially overlapping momentum or rapidity ranges are used in measurements of correlation functions. In such cases, the number of particles N_α and N_β must be split to explicitly indicate the subset of particles of both types that are, event by event, in the overlapping region. Denoting the splits $N_\alpha = N'_\alpha + N_s$ and $N_\beta = N'_\beta + N_s$ where the primed quantities represent multiplicities in the non overlapping ranges and N_s represent the multiplicity in the overlapping range, the number of pairs is then $\langle N'_\alpha(N'_\alpha - 1)\rangle + \langle N'_\beta(N'_\beta - 1)\rangle + \langle N_s(N'_\alpha + N'_\beta)\rangle$. Extensions to higher-order moments have been discussed in the context of flow azimuthal harmonics computation [38] and will be addressed in the context of the observables considered in this work in a forthcoming paper.

Being integrals of densities, factorial moments do not discriminate correlated from uncorrelated particles and it is thus also convenient to consider mixed factorial moment cumulants, herein denoted $F_n^{\alpha_1\cdots\alpha_n}$. Mixed factorial moment cumulants, hereafter simply called factorial cumulants, are the integrals of the mixed cumulants $C_n^{\alpha_1\cdots\alpha_n}(\vec{p}_1, \ldots, \vec{p}_n)$ over acceptances Ω_k, $k = 1, \ldots, n$, and are readily expressed in term of the factorial moments but they can also be computed based on generating functions, as discussed in Appendix A. Lowest orders yield

$$F_1^\alpha = \int C_1^\alpha(\vec{p}_1)d\vec{p}_1 = \langle N^\alpha\rangle = f_1^\alpha, \tag{19}$$

$$F_2^{\alpha_1\alpha_2} = \int C_2^{\alpha_1\alpha_2}(\vec{p}_1, \vec{p}_2)d\vec{p}_1 d\vec{p}_2 = f_2^{\alpha_1\alpha_2} - f_1^{\alpha_1}f_1^{\alpha_2}, \tag{20}$$

$$F_3^{\alpha_1\alpha_2\alpha_3} = f_3^{\alpha_1\alpha_2\alpha_3} - f_2^{\alpha_1\alpha_2}f_1^{\alpha_3} - f_2^{\alpha_1\alpha_3}f_1^{\alpha_2} - f_2^{\alpha_2\alpha_3}f_1^{\alpha_1} + 2f_1^{\alpha_1}f_1^{\alpha_2}f_1^{\alpha_3}, \tag{21}$$

while formula for higher orders are also listed in Appendix A.

Moments, centered moments, cumulants, factorial moments, and factorial (moment) cumulants constitute distinct tools to characterize the particle production encountered in elementary and nuclear collisions. Moments and centered moments are used as basic characterizations (e.g., mean, standard deviation, skewness, etc.) of the underlying n-particle densities $\rho_n^{\alpha_1\cdots\alpha_n}$ whereas n-factorial moments provide useful measures of the average number of single, pair, triplet, etc., of particles produced and detected on an event-by-event basis in collisions.

Factorial (moment) cumulants (F_n), introduced in its mixed version in Eqs. (19)–(21) and defined in Eqs. (A6)–(A15), based on factorial moments (f_n), are of particular interest because they constitute a true measure of correlation, i.e., deviation from statistical independence or Poisson statistics [25]. They are thus ideal to investigate particle production and transport mechanisms. Particular combinations of mixed factorial moment cumulants, such as $\nu_{\rm dyn}$, defined in Eq. (63), are commonly used to study event-by-event fluctuations of particle yields measured in a specific experimental acceptance [24]. Measurements of relative yield fluctuations have been studied in several contexts, including search for the critical point of nuclear matter, proximity of second-order (or cross over) phase transition ([39] and references therein). Factorial moment cumulants are also readily extended to measurements of fluctuations of event by event transverse momentum and charge deviates discussed in Sec. III B.

Correlation tools are broadly divided into integral and differential correlation functions. The former group include integral cumulants and factorial cumulants whereas the latter are measured as explicit functions of one or many particle variables. Integral cumulants (κ_n), defined in Eq. (A3), are of particular interest because they are nominally related to charge (Q), baryon (B), and strangeness (S) susceptibilities, $\chi^{\rm Q}$, $\chi^{\rm B}$, and $\chi^{\rm S}$, respectively, of the matter formed in the collision when described in the context of Grand Canonical Ensemble (GCE) thermal models [40]. Mixed cumulants are likewise related to mixed, QB, QS, BS, or QBS, etc., susceptibilities [40]. Their connection to multiparticle balance functions is discussed in Sec. IV.

A wide variety of differential n-particle observables are commonly used in studies of elementary and nuclear collisions. Differential observables may be formulated based on n-particle densities but can also be weighted by various functions of kinematic variables such as the particles transverse momenta, their azimuthal angle of emission, and so on. Observables based on number densities include particle pair densities and more generally n-particle densities, n-particle densities normalized by the number of "trigger" particles, commonly called triggered two-particle correlations, as well as the n-particle differential cumulants discussed in this section. Observables weighted by functions of azimuthal angles evidently include all varieties of flow measurements observables, including flow cumulants and related functions [25]. In general, measurements of "true" correlations are best accomplished with differential correlation functions based on the differential cumulants, Eqs. (12) and (13), and the higher-order cumulants listed in Appendix B.

B. Joint moments of observables and their deviates

As we discussed in Secs. II and III A, a wide variety of multiparticle correlation functions can be formulated in terms of the expectation value of products of particle observables of the form $\langle\langle q_1 q_2 \ldots q_n\rangle\rangle$ or their deviates $\langle\langle \Delta q_1 \Delta q_2 \ldots \Delta q_n\rangle\rangle$. In this section, we first introduce such expectations values based on moments of the sum $\sum_{i=0}^N q_i$ computed for all selected particles of an event with all self-correlations removed. We then consider the expectation value of off-diagonal products of deviates of the form $\langle\langle \Delta q_1 \Delta q_2 \cdots \Delta q_n\rangle\rangle$. First note that these expressions are totally general and thus applicable for any type of particle observables, e.g., transverse momentum, rapidity, electric charge, or other quantum numbers. Together with formula introduced in Sec. III C and Appendix C, these correlators enable the formulation of both integral and differential measurements of multiple particle correlations of basic particle observables.

To obtain expressions sought for, first consider a particle observable of interest q. This observable could be the transverse momentum of the particle, its rapidity, its charge, some other quantum numbers, or simply unity (i.e., to count the particles). We will denote the value of this observable for a specific particle q_i, with the index i spanning all selected particles (i.e., satisfying specific kinematic and quality selection criteria) in a given event. We are interested in computing moments of q and its deviates $\Delta q_i \equiv q_i - \langle\langle q\rangle\rangle$, where $\langle\langle q\rangle\rangle$ is the inclusive event ensemble average of q for specific collision conditions (i.e., events satisfying specific selection criteria).

Inclusive event ensemble (joint) averages of products of qs are defined according to

$$\langle\langle q\rangle\rangle = \frac{1}{\langle N\rangle}\left\langle \sum_{i=1}^{N} q_i \right\rangle, \tag{22}$$

$$\langle\langle q_1 q_2\rangle\rangle = \frac{1}{\langle N(N-1)\rangle}\left\langle \sum_{i_1\neq i_2=1}^{N} q_{i_1} q_{i_2} \right\rangle, \tag{23}$$

$$\langle\langle q_1 q_2 \cdots q_n\rangle\rangle = \frac{1}{\langle N(N-1)\cdots(N-n+1)\rangle}\left\langle \sum_{i_1\neq i_2\neq\cdots\neq i_n=1}^{N} q_{i_1} q_{i_2}\cdots q_{i_n} \right\rangle, \tag{24}$$

where the sums run over all N selected particles in a given event. The notation $\sum_{i_1\neq i_2}$ indicates the sums are computed for distinct particles, i.e., distinct values of i_1, i_2, etc. As such, $\sum_{i=1}^{N} q_i$ represents the sum of the q_i in a given event, and $\langle\sum_{i=1}^{N} q_i\rangle$ is the event ensemble average of this sum across all events of a selected data sample. Similarly, $\langle\sum_{i\neq j=1}^{N} q_i q_j\rangle$, and higher orders, represent ensemble averages of products of the qs evaluated for n-tuplets of particles. However, the fact that sums proceed on $i \neq j$ implies auto-correlations are explicitly removed. Herewith, we will use inclusive averages (i.e., averages computed over an event ensemble) but it is trivial to change the definition to event-by-event averages [21]. See for instance Eqs. (D13) and (D14). Additionally, note that if it is possible to prescan the dataset of interest to determine the inclusive average $\langle\langle q\rangle\rangle$, then one can readily replace q_i by $\Delta q_i = q_i - \langle\langle q\rangle\rangle$ in Eqs. (22)–(24) instead of carrying out the factorization discussed in detail below.

We proceed to compute event ensemble averages of moments of Δq_i in terms of moments of q_i. The inclusive first moment of Δq is calculated according to

$$\langle\langle \Delta q\rangle\rangle = \frac{1}{\langle N\rangle}\left\langle \sum_{i=1}^{N} \Delta q_i \right\rangle = \frac{1}{\langle N\rangle}\sum_{i=1}^{N} q_i - \frac{\langle N\rangle}{\langle N\rangle}\langle\langle q\rangle\rangle = \langle\langle q\rangle\rangle - \langle\langle q\rangle\rangle = 0 \tag{25}$$

and vanishes by construction. To compute moments of order $n = 2$ and $n = 3$, we write

$$\Delta q_i \Delta q_j = q_i q_j - \langle\langle q\rangle\rangle(q_i + q_j) + \langle\langle q\rangle\rangle^2, \tag{26}$$

$$\Delta q_i \Delta q_j \Delta q_k = q_i q_j q_k - \langle\langle q\rangle\rangle(q_i q_j + q_i q_k + q_j q_k) + \langle\langle q\rangle\rangle^2(q_i + q_j + q_k) - \langle\langle q\rangle\rangle^3, \tag{27}$$

whereas higher-order products can be written

$$\begin{aligned}\Delta q_{i_1}\Delta q_{i_2}\cdots\Delta q_{i_n} = {}& q_{i_1} q_{i_2}\cdots q_{i_n} - \langle\langle q\rangle\rangle \sum_{\binom{n}{1}} q_{i_1} q_{i_2}\cdots q_{i_{n-1}} + \langle\langle q\rangle\rangle^2 \sum_{\binom{n}{2}} q_{i_1} q_{i_2}\cdots q_{i_{n-2}} \\ & - \langle\langle q\rangle\rangle^3 \sum_{\binom{n}{3}} q_{i_1} q_{i_2}\cdots q_{i_{n-3}} \cdots + (-1)^{n-1}\langle\langle q\rangle\rangle^{n-1} \sum_{\binom{n}{n-1}} q_{i_1} + (-)^n \langle\langle q\rangle\rangle^n,\end{aligned} \tag{28}$$

where the notation $\sum_{\binom{n}{k}}$ indicates a sum over all $\binom{n}{k}$ permutations of the indices i_1, i_2, ..., i_{n-k}. Ensemble averages of products of order $n = 2,\ 3,\ 4$ yield

$$\langle\langle \Delta q_i \Delta q_j\rangle\rangle = \frac{1}{\langle N(N-1)\rangle}\left\langle \sum_{i\neq j=1}^{N} \left(q_i q_j - \langle\langle q\rangle\rangle(q_i + q_j) + \langle\langle q\rangle\rangle^2\right) \right\rangle = \langle\langle q_i q_j\rangle\rangle - \langle\langle q\rangle\rangle^2, \tag{29}$$

$$\langle\langle \Delta q_i \Delta q_j \Delta q_k\rangle\rangle = \langle\langle q_i q_j q_k\rangle\rangle - 3\langle\langle q\rangle\rangle\langle\langle q_i q_j\rangle\rangle + 2\langle\langle q\rangle\rangle^3, \tag{30}$$

$$\langle\langle \Delta q_i \Delta q_j \Delta q_k \Delta q_l\rangle\rangle = \langle\langle q_i q_j q_k q_l\rangle\rangle - 4\langle\langle q\rangle\rangle\langle\langle q_i q_j q_k\rangle\rangle + 6\langle\langle q\rangle\rangle^2\langle\langle q_i q_j\rangle\rangle - 3\langle\langle q\rangle\rangle^4, \tag{31}$$

whereas higher orders can be computed according to

$$\begin{aligned}\langle\langle \Delta q_{i_1}\Delta q_{i_2}\cdots\Delta q_{i_n}\rangle\rangle = {}& \langle\langle q_{i_1} q_{i_2}\cdots q_{i_n}\rangle\rangle - \binom{n}{n-1}\langle\langle q\rangle\rangle\langle\langle q_{i_1} q_{i_2}\cdots q_{i_{n-1}}\rangle\rangle + \binom{n}{n-2}\langle\langle q\rangle\rangle^2\langle\langle q_{i_1} q_{i_2}\cdots q_{i_{n-2}}\rangle\rangle \\ & + \cdots (-1)^{n-2}\binom{n}{2}\langle\langle q\rangle\rangle^{n-2}\langle\langle q_{i_1} q_{i_2}\rangle\rangle + (-1)^{n-1}\binom{n}{1}\langle\langle q\rangle\rangle^{n-1}\langle\langle q_1\rangle\rangle + (-1)^n\binom{n}{0}\langle\langle q\rangle\rangle^n\end{aligned} \tag{32}$$

$$= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\langle\langle q\rangle\rangle^{n-k}\langle\langle q_{i_1}\cdots q_{i_k}\rangle\rangle, \tag{33}$$

where it is assumed that $\langle\langle q_{i_1}\cdots q_{i_k}\rangle\rangle = \langle\langle q\rangle\rangle$ for $k=1$ and $\langle\langle q_{i_1}\cdots q_{i_k}\rangle\rangle = 1$ for $k=0$.

The computation of moments of order n of qs amounts to sums of the form $\sum_{i\neq j=1}^N q_i q_j$, $\sum_{i\neq j\neq k=1}^N q_i q_j q_k$, etc., which nominally require up to n nested loops for each event considered. Though conceptually trivial, such calculations involve a computation time proportional to N^n that becomes quickly prohibitive for large values of the multiplicity N. Fortunately, the method of moments, which we discuss in the next two subsections, enables the computation of averages of these sums based on a single loop (per event), i.e., with a computation time proportional to N.

C. Method of moments

The method of moments is introduced to facilitate the computation of moments $\langle\langle q_1 q_2 \ldots q_n\rangle\rangle$ and deviates $\langle\langle \Delta q_1 \Delta q_2 \ldots \Delta q_n\rangle\rangle$ based on a single loop over all particles of an event rather than using nested loops.

1. Method of moments for a single variable

Let Q_n represent an event-wise sum of the nth power of the variable q_i of the N selected particles of an event

$$Q_n = \sum_{i=1}^N q_i^n. \tag{34}$$

The method of moments relies on the evaluation of ensemble averages of Q_n and products of the form Q_nQ_m, $Q_nQ_mQ_o$, etc. The (inclusive) ensemble average of Q_n is

$$\langle\langle Q_n\rangle\rangle = \frac{1}{\langle N\rangle}\langle\langle q^n\rangle\rangle, \tag{35}$$

and one obviously obtains

$$\langle\langle q^n\rangle\rangle = \langle N\rangle\langle\langle Q_n\rangle\rangle. \tag{36}$$

Computation of the ensemble average of products Q_nQ_m, $Q_nQ_mQ_o$, etc., requires one properly handles the expansions of the sums. For instance, product of two and three Qs yield

$$Q_{n_1}Q_{n_2} = \sum_{i_1,i_2=1}^N q_{i_1}^{n_1} q_{i_2}^{n_2} = \sum_{i=1}^N q_i^{n_1+n_2} + \sum_{i_1\neq i_2=1}^N q_{i_1}^{n_1} q_{i_2}^{n_2}, \tag{37}$$

$$\begin{aligned} Q_{n_1}Q_{n_2}Q_{n_3} &= \sum_{i=1}^N q_i^{n_1+n_2+n_3} + \sum_{i_1\neq i_2=1}^N q_{i_1}^{n_1+n_2} q_{i_2}^{n_3} \\ &\quad + \sum_{i_1\neq i_2=1}^N q_{i_1}^{n_1+n_3} q_{i_2}^{n_2} + \sum_{i_1\neq i_2=1}^N q_{i_1}^{n_1} q_{i_2}^{n_2+n_3} \\ &\quad + \sum_{i_1\neq i_2\neq i_3=1}^N q_{i_1}^{n_1} q_{i_2}^{n_2} q_{i_3}^{n_3}, \end{aligned} \tag{38}$$

$$\begin{aligned} &= \sum_{i=1}^N q_i^{n_1+n_2+n_3} + \sum_{(3)}\sum_{i_1\neq i_2=1}^N q_{i_1}^{n_1+n_2} q_{i_2}^{n_3} \\ &\quad + \sum_{i_1\neq i_2\neq i_3=1}^N q_{i_1}^{n_1} q_{i_2}^{n_2} q_{i_3}^{n_3}, \end{aligned} \tag{39}$$

where the notation $\sum_{(3)}$ represents a sum spanning all three ordered permutations of n_1, n_2, and n_3.

Clearly, calculation of the ensemble average of $Q_{n_1}Q_{n_2}$ yields

$$\langle\langle Q_{n_1}Q_{n_2}\rangle\rangle = \langle N\rangle\langle\langle q^{n_1+n_2}\rangle\rangle + \langle N(N-1)\rangle\langle\langle q_1^{n_1}q_2^{n_2}\rangle\rangle. \tag{40}$$

This expression contains a term of the form $\langle N\rangle\langle\langle q_{i_1}^{n_1+n_2}\rangle\rangle$ that can be readily replaced by $\langle\langle Q_{n_1+n_2}\rangle\rangle$ based on Eq. (35). Solving for $\langle\langle q_1^{n_1}q_2^{n_2}\rangle\rangle$, one then gets

$$\langle N(N-1)\rangle\langle\langle q_1^{n_1}q_2^{n_2}\rangle\rangle = \langle\langle Q_{n_1}Q_{n_2}\rangle\rangle - \langle\langle Q_{n_1+n_2}\rangle\rangle, \tag{41}$$

which for $n_1 = n_2 = 1$ evidently simplifies to

$$\langle N(N-1)\rangle\langle\langle q_1q_2\rangle\rangle = \langle\langle Q_1^2\rangle\rangle - \langle\langle Q_2\rangle\rangle. \tag{42}$$

Proceeding similarly for the ensemble average of $Q_{n_1}Q_{n_2}Q_{n_3}$, one gets

$$\begin{aligned} \langle\langle Q_{n_1}Q_{n_2}Q_{n_3}\rangle\rangle &= \langle N\rangle\langle\langle q_{i_1}^{n_1+n_2+n_3}\rangle\rangle + \langle N(N-1)\rangle\langle\langle q_{i_1}^{n_1+n_2}q_{i_2}^{n_3}\rangle\rangle \\ &\quad + \langle N(N-1)\rangle\langle\langle q_{i_1}^{n_1+n_3}q_{i_2}^{n_2}\rangle\rangle + \langle N(N-1)\rangle \\ &\quad \times \langle\langle q_{i_1}^{n_2+n_3}q_{i_2}^{n_1}\rangle\rangle + \langle N(N-1)(N-2)\rangle \\ &\quad \times \langle\langle q_{i_1}^{n_1}q_{i_2}^{n_2}q_{i_3}^{n_3}\rangle\rangle. \end{aligned} \tag{43}$$

The first term, $\langle N\rangle\langle\langle q_{i_1}^{n_1+n_2+n_3}\rangle\rangle$, is equal to $\langle\langle Q_{n_1+n_2+n_3}\rangle\rangle$, while the next three terms are of the form of Eq. (41). Substituting these terms and solving for $\langle\langle q_{i_1}^{n_1}q_{i_2}^{n_2}q_{i_3}^{n_3}\rangle\rangle$, one gets

$$\begin{aligned} &\langle N(N-1)(N-2)\rangle\langle\langle q_1^{n_1}q_1^{n_2}q_1^{n_3}\rangle\rangle \\ &\quad = \langle\langle Q_{n_1}Q_{n_2}Q_{n_3}\rangle\rangle - \langle\langle Q_{n_1+n_2}Q_{n_3}\rangle\rangle - \langle\langle Q_{n_1+n_3}Q_{n_2}\rangle\rangle \\ &\quad\quad - \langle\langle Q_{n_2+n_3}Q_{n_1}\rangle\rangle + 2\langle\langle Q_{n_1+n_2+n_3}\rangle\rangle, \end{aligned} \tag{44}$$

which, for $n_1 = n_2 = n_3 = 1$, reduces to

$$\langle N(N-1)(N-2)\rangle\langle\langle q_1q_2q_3\rangle\rangle = \langle\langle Q_1^3\rangle\rangle - 3\langle\langle Q_2Q_1\rangle\rangle + 2\langle\langle Q_3\rangle\rangle. \tag{45}$$

The above calculation can be repeated iteratively for higher-order products of qs and thus yield expressions for $\langle\langle q_1q_2\cdots q_n\rangle\rangle$ at arbitrarily high order n. In practice, such calculations become rather tedious for $n > 4$ and are best computed programmatically, as discussed in Appendix D.

2. Higher-order moments of mixed acceptance variates

In the previous sections, we considered calculations of higher moments $\langle\langle q_1q_2\cdots q_n\rangle\rangle$ computed for a single acceptance or particle species (i.e., for a single kinematic bin or for a specific species or both). To compute differential correlation functions involving several kinematic bins or species, we now proceed to compute expectation values of the form $\langle\langle q_1q_2\cdots q_np_1\cdots p_mr_1\cdots r_o\rangle\rangle$, where q, p, and r represent distinct kinematic bins or species. The discussion is here limited to three bins (or species) for simplicity's sake but it is trivially extended to an arbitrary number of bins and species. The particle multiplicities in each bin are denoted N_i, $i = 1,\ldots,3$. Deviates are denoted Δq_i, Δp_j, and Δr_k for particles in bins 1, 2, and 3, respectively. Moments for all particles detected in a single bin are given by expressions of the form of Eqs. (41)–(44) already considered in Sec. III C 1.

We thus need to consider mixed moments only in this section. Lowest mixed moments are given by expressions of the form

$$\langle\langle q_1 p_1 \rangle\rangle = \frac{1}{\langle N_1 N_2\rangle}\left\langle \sum_{i=1}^{N_1} q_i \sum_{j=1}^{N_2} p_j \right\rangle, \quad (46)$$

$$\langle\langle q_1 q_2 p_1 \rangle\rangle = \frac{1}{\langle N_1(N_1-1)N_2\rangle}\left\langle \sum_{i_1\neq i_2=1}^{N_1} q_{i_1} q_{i_2} \sum_{j=1}^{N_2} p_j \right\rangle, \quad (47)$$

$$\langle\langle q_1 p_1 r_1 \rangle\rangle = \frac{1}{\langle N_1 N_2 N_3\rangle}\left\langle \sum_{i=1}^{N_1} q_i \sum_{j=1}^{N_2} p_j \sum_{k=1}^{N_3} r_k \right\rangle. \quad (48)$$

More generally, considering moments of order m_1, m_2, and m_3 for bins 1, 2, and 3, one gets expressions of the form

$$\langle\langle q_1 \cdots q_{m_1} p_1 \cdots p_{m_2} r_1 \cdots r_{m_3} \rangle\rangle = \frac{1}{N_{m_1,m_2,m_3}}\left\langle \sum_{i_1\neq\cdots\neq i_{m_1}=1}^{N_1} q_{i_1}q_{i_2}\cdots q_{i_{m_1}} \sum_{j_1\neq\cdots\neq j_{m_2}=1}^{N_2} p_{j_1}p_{j_2}\cdots p_{j_{m_2}} \sum_{k_1\neq\cdots\neq k_{m_3}=1}^{N_3} r_{k_1}r_{k_2}\cdots r_{k_{m_3}} \right\rangle, \quad (49)$$

where the sums $\sum_{i_1\neq\cdots\neq i_{m_1}=1}^{N_1}$, $\sum_{j_1\neq\cdots\neq j_{m_2}=1}^{N_2}$ and $\sum_{k_1\neq\cdots\neq k_{m_3}=1}^{N_3}$ span tuplets of distinct particles in bins 1, 2, and 3, respectively. Also note that the normalization corresponds to the average number of such tuplets:

$$N_{m_1,m_2,m_3} \equiv \langle N_1(N_1-1)\cdots(N_1-m_1+1)N_2(N_2-1)\cdots(N_2-m_2+1)N_3(N_3-1)\cdots(N_3-m_3+1)\rangle. \quad (50)$$

Event ensembles of mixed moments of deviates are defined in a similar fashion. At lowest orders, one gets expressions of the form

$$\langle\langle \Delta q_1 \Delta p_1 \rangle\rangle = \langle\langle q_1 p_1 \rangle\rangle - \langle\langle q\rangle\rangle\langle\langle p\rangle\rangle, \quad (51)$$

$$\langle\langle \Delta q_1 \Delta p_1 \Delta r_1 \rangle\rangle = \langle\langle q_1 p_1 r_1 \rangle\rangle - \langle\langle p\rangle\rangle\langle\langle q_1 r_1\rangle\rangle - \langle\langle q\rangle\rangle\langle\langle p_1 r_1\rangle\rangle - \langle\langle r\rangle\rangle\langle\langle q_1 p_1\rangle\rangle + 2\langle\langle q\rangle\rangle\langle\langle p\rangle\rangle\langle\langle r\rangle\rangle, \quad (52)$$

and higher-order moments are discussed in Appendix C. The computation of mixed moments and their deviates nominally requires nested loops over all particles and bins of interest. But as for the single bin case discussed in the previous section, it is advantageous to introduce event-wise sums of the variables q_i, p_i, and r_i according to

$$Q_n = \sum_{i=1}^{N} q_i^n, \qquad P_n = \sum_{i=1}^{N} p_i^n, \qquad R_n = \sum_{i=1}^{N} r_i^n. \quad (53)$$

It then becomes possible, as discussed in Appendix C, to compute the event ensemble averages $\langle\langle q_1 \cdots q_{m_1}\ p_1 \cdots p_{m_2} r_1 \cdots r_{m_3}\rangle\rangle$ and their corresponding deviates $\langle\langle \Delta q_1 \cdots \Delta q_{m_1} \Delta p_1 \cdots \Delta p_{m_2} \Delta r_1 \cdots \Delta r_{m_3}\rangle\rangle$ based on recursive formula of the moments of Q_n, P_n, and R_n.

D. Differential measurements of multiple-particle correlations

The multiparticle correlators $\langle\langle q_1 q_2 \cdots q_n\rangle\rangle$ and $\langle\langle \Delta q_1 \Delta q_2 \cdots \Delta q_n\rangle\rangle$ presented in Sec. III A, equipped with the method of moments discussed in Sec. III C, provide a basis for the extension of former techniques used for measurements of transverse momentum correlations, rapidity correlations, as well as charge correlations (including baryon and strangeness numbers correlations).

The method of moments, whether used with a single or several variables, is nominally very powerful because it enables joint measurements involving many particles simultaneously. However, it is also clear that the complexity of such measurements can quickly grow out of hand.

As was described in Sec. II, one can classify analyses of potential interest based on the number of kinematic bins being used (i.e., partitions of the 3× momentum space), the number of observables of interest (e.g., transverse momentum p_T, rapidity y, charge q, anisotropic flow coefficients, etc.) and the number of particle types or species being considered (e.g., inclusive charged particles, positively versus negatively charged particles, specific species such as pions, kaons, etc.). We thus organize the discussion of this section in terms of few use cases, beginning with the simplest case involving a single variable q, with event-wise variable Q_n, and next considering progressively more complex use cases involving two variables: q, p with event-wise variables Q_n and P_n, as well as more complex analyses based on several variables.

1. One variable (q)

Measurements of p_T (alternatively y or q, etc.) correlations $\langle\langle \Delta p_{T,1} \Delta p_{T,2} \cdots \Delta p_{T,n}\rangle\rangle$ involving $n \geqslant 4$ particles of a given type of particle in a specific acceptance can be readily undertaken based on the method of moments discussed in Sec. III C. If it is possible or practical to carry out the analysis in two or more passes on the data, than one can use the first pass to determine $\langle\langle p_T\rangle\rangle$. In the second pass, one can then define and compute $Q_n = \sum_i^N (p_T - \langle\langle p_T\rangle\rangle)$ event by event and then use Eqs. (D3)–(D9) to obtain n-order moments $\langle\langle \Delta p_{T,1} \Delta p_{T,2} \cdots \Delta p_{T,n}\rangle\rangle$. If the determination of $\langle\langle p_T\rangle\rangle$ in a first pass is not practical, then one can define and compute $Q_n = \sum_i^N p_T$ event by event, use Eqs. (D3)–(D9) to obtain $\langle\langle p_{T,1} p_{T,2} \cdots p_{T,n}\rangle\rangle$ and Eqs. (C1)–(C5) or Eq. (C6) to obtain the n-order deviates $\langle\langle \Delta p_{T,1} \Delta p_{T,2} \cdots \Delta p_{T,n}\rangle\rangle$.

2. Two variables (q and p)

We first discuss examples of p_T fluctuation studies. Let q_i and p_i represent the transverse momentum of particles measured in two distinct rapidity acceptance ranges Ω_A and

Ω_B of equal widths separated by a finite rapidity gap $\Delta\eta$, as schematically illustrated in Fig. 1. One can then measure correlators of the form $\langle\langle \Delta q_1 \Delta p_1 \rangle\rangle$, $\langle\langle \Delta q_1 \Delta q_2 \Delta p_1 \Delta p_2 \rangle\rangle$, etc., at any order to determine the strength of $n = 2, 4$, etc., transverse momentum correlations as a function of the width of the rapidity gap. The method of moments, however, enables differential measurements involving multiple $n \geqslant 4$ particles. Let q_i, $i = 1, \ldots$, represent the $p_{\rm T}$ of particles in bin A, and p_i, $i = 1, \ldots$ represent the $p_{\rm T}$ of particles in bin B. One then defines event-wise variables $Q_n = \sum_i^N p_{{\rm T},i}^n$ (from bin A) and $P_n = \sum_i^N p_{{\rm T},i}^n$ (from bin B). Equations (D3)–(D9) are then used to obtain $\langle\langle q_1 q_2 \cdots q_n p_1 p_2 \cdots p_n \rangle\rangle$ and Eqs. (C1)–(C5) or Eq. (C6) are used to obtain the n-order deviates $\langle\langle \Delta q_1 \Delta q_2 \cdots \Delta q_n \Delta p_1 \Delta p_2 \cdots \Delta p_n \rangle\rangle$ corresponding to $p_{\rm T}$ correlators involving n particles from bin A and n particles from bin B.

The method described above can readily be adopted also for measurements of charge correlations and multiparticle balance functions. In this case, q_i and p_i represent the charge of particles in bins A and B. Applications of Eqs. (D3)–(D9) then yield generic charge correlators $\langle\langle q_1 q_2 \cdots q_n p_1 p_2 \cdots p_n \rangle\rangle$ and Eqs. (C1)–(C5) or Eq. (C6) can be used to compute deviates.

Two additional use cases based on two variables q_i and p_i (with the corresponding event-wise variables Q_n and P_n) are worth mentioning. One involves the study of two distinct physics observables (e.g., charge, $p_{\rm T}$, rapidity, etc.) in a single rapidity bin whereas the other involves the measurement of a specific particle observable, e.g., the $p_{\rm T}$, for two types of particle species. In the first case, the variable q_i and p_i represent the two observables of interest whereas in the second they tag the species of interest. The determination of correlators between two observables or two types of particle species then proceeds in the manner already described. First get $\langle\langle q_1 q_2 \cdots q_n p_1 p_2 \cdots p_n \rangle\rangle$ with Eqs. (D3)–(D9) and next used Eqs. (C1)–(C5) to compute the deviates of interest.

3. Three or more variables (q, p, r,...)

The examples discussed in the previous paragraph are readily extended towards the computation of factorial cumulants or correlation functions involving three or more kinematic bins and particle types. Of particular interest is the determination of multiple particle balance functions. Although it may not be practical to conduct analyses involving explicit computation of more than three or four kinematic bins or species, it remains possible to consider balance functions involving large number of particles towards the study of long-range multiparticle correlations constrained by charge conservation (or other quantum number conservation laws).

As mentioned in Sec. II, we consider here the study of four-particle balance functions using two kinematic bins A and B separated by a finite rapidity gap, as was illustrated in Figs. 1(a) and 1(b). The bins A and B could be azimuthally symmetric (i.e., with full azimuth coverage $0 \leqslant \varphi < 2\pi$), or feature partial coverage to suppress contributions from back-to-back jets, as shown in Figs. 1(c) and 1(d) of the same figure. Figure 1(a) illustrates a measurement involving two positively charged particles in A and two negatively particles in B. A measurement of B_4^{2+2-} shall then be sensitive to the strength (or probability) of processes featuring four correlated particles with two +ve and two −ve particles separated by a finite rapidity gap. By contrast, the analysis illustrated in Fig. 1(b) would focus on correlated quartets featuring two nearby pairs of +ve and −ve particles. These could be produced by string-like fragmentation processes yielding four or more correlated particles, but it could also result from string fragmentation producing two neutral objects, each decaying into pairs of +ve and −ve particles.

It is also worth mentioning prior discussions and exemplars of multiparticle correlation functions and their cumulants [30–33].

IV. MULTIPARTICLE BALANCE FUNCTIONS

Yet another application of integral and differential correlation functions of the form of Eqs. (3)–(6), involves the study of net-charge (and other quantum numbers) fluctuations. We first show how Eq. (3) can be used to measure net-charge fluctuations and how it connects to measurements of balance functions [10,17,18]. We also remind the reader how second moments of the charge are connected to the $\nu_{\rm dyn}$ observable [24] and differential charge balance functions [17,18]. This then provides a convenient mechanism to introduce higher-order balance functions.

To express net-charge fluctuations, we rewrite Eq. (3) by replacing the transverse momentum $p_{\rm T}$ by the charge q_i of particles,

$$\langle\langle \Delta q_1 \Delta q_2 \rangle\rangle \equiv \frac{1}{\langle N(N-1)\rangle} \int_\Omega \Delta q_1 \Delta q_2 \rho_2(\vec{p}_1, \vec{p}_2) d\vec{p}_1 d\vec{p}_2, \quad (54)$$

where deviates are defined as $\Delta q_i \equiv q_i - \langle\langle q \rangle\rangle$. The variables q_i are considered implicit functions of the momentum of the particles and thus cannot be factorized out of the integral. To build this point, let us consider the expression of the average charge in the acceptance Ω of interest

$$\langle\langle q \rangle\rangle \equiv \frac{1}{\langle N \rangle} \int_\Omega \tilde{\rho}_1(\vec{p}) d\vec{p}, \quad (55)$$

where $\tilde{\rho}_1(\vec{p})$ represents a *charge density*, i.e., not the number density $\rho_1(\vec{p}_1)$. Equation (55) may be computed based on number densities if $\tilde{\rho}_1(\vec{p})$ is replaced (symbolically) by $q_\alpha \rho_1^\alpha(\vec{p})$ where q_α is the charge of the particle species of interest, α. A similar development can be done for strangeness or baryon quantum numbers by considering strangeness and baryon densities instead of charge densities. Here for the sake of simplicity, let us restrict the discussion to three types of charged particles: positively charged, neutral, and negatively charged hadrons. The single particle density is then $\rho_1(\vec{p}) = \rho_1^{(+)}(\vec{p}) + \rho_1^{(0)}(\vec{p}) + \rho_1^{(-)}(\vec{p})$. Substituting this expression in Eq. (55) and inserting the values $q_i = +1, 0, -1$ for each of the three types, one gets

$$\langle\langle q\rangle\rangle \equiv \frac{1}{\langle N\rangle}\left[(+1)\times\int_{\Omega}\rho_1^{(+)}(\vec{p})d\vec{p} + (0)\times\int_{\Omega}\rho_1^{(0)}(\vec{p})d\vec{p} + (-1)\times\int_{\Omega}\rho_1^{(-)}(\vec{p})d\vec{p}\right] = \frac{1}{\langle N\rangle}[\langle N^+\rangle - \langle N^-\rangle], \quad (56)$$

where, in the second line, we applied Eq. (15). In the following, we assume neutral particles are not measured and consider the total multiplicity N as the sum of the multiplicities of positively and negatively charged particles, i.e., $N = N^{(+)} + N^{(-)}$.

The calculation of $\langle\langle \Delta q_1 \Delta q_2\rangle\rangle$, which corresponds to the covariance of the charges of two measured particles, proceeds in the same way. First expand $\Delta q_1 \Delta q_2$ and compute its ensemble average

$$\langle\langle \Delta q_1 \Delta q_2\rangle\rangle = \langle\langle q_1 q_2\rangle\rangle - \langle\langle q\rangle\rangle^2, \quad (57)$$

which may be expressed according to

$$\langle\langle \Delta q_1 \Delta q_2\rangle\rangle = \frac{1}{\langle N(N-1)\rangle}\iint_{\Omega}[q_1 q_2 \rho_2(\vec{p}_1,\vec{p}_2) - q_1\rho_1(\vec{p}_1)q_2\rho_1(\vec{p}_2)]d\vec{p}_1 d\vec{p}_2 \quad (58)$$

$$= \frac{1}{\langle N(N-1)\rangle}\iint_{\Omega} q_1 q_2 C_2(\vec{p}_1,\vec{p}_2)d\vec{p}_1 d\vec{p}_2, \quad (59)$$

where in the second line, we used the expression of the second cumulant C_2 given by Eq. (12). Expanding $q\rho_1$ as in Eq. (56) and $q_1q_2\rho_2$ according to

$$q_1 q_2 \rho_2 = \rho_2^{(++)} - \rho_2^{(+-)} - \rho_2^{(-+)} + \rho_2^{(--)}, \quad (60)$$

the integration yields

$$\langle\langle \Delta q_1 \Delta q_2\rangle\rangle = \frac{\langle N^+(N^+-1)\rangle + \langle N^-(N^- -1)\rangle - 2\langle N^+N^-\rangle - (\langle N^+\rangle - \langle N^-\rangle)^2}{\langle N(N-1)\rangle}. \quad (61)$$

At LHC, A–A collisions produces approximately equal multiplicities (and densities) of positively and negatively charged particle: $\langle N^+\rangle \approx \langle N^-\rangle$. The above expression for $\langle\langle \Delta q_1 \Delta q_2\rangle\rangle$ can thus be written

$$\langle\langle \Delta q_1 \Delta q_2\rangle\rangle \approx \frac{\langle N^+\rangle\langle N^-\rangle}{\langle N(N-1)\rangle}\left[\frac{\langle N^+(N^+-1)\rangle}{\langle N^+\rangle\langle N^+\rangle} + \frac{\langle N^-(N^- -1)\rangle}{\langle N^-\rangle\langle N^-\rangle} - 2\frac{\langle N^+N^-\rangle}{\langle N^+\rangle\langle N^-\rangle}\right] \quad (62)$$

within which one recognizes the expression of $\nu_{\rm dyn}^{+-}$ [24]

$$\nu_{\rm dyn}^{+-} \equiv \frac{\langle N^+(N^+-1)\rangle}{\langle N^+\rangle\langle N^+\rangle} + \frac{\langle N^-(N^- -1)\rangle}{\langle N^-\rangle\langle N^-\rangle} - 2\frac{\langle N^+N^-\rangle}{\langle N^+\rangle\langle N^-\rangle} = \frac{F_2^{++}}{F_1^+F_1^+} + \frac{F_2^{--}}{F_1^-F_1^-} - 2\frac{F_2^{+-}}{F_1^+F_1^-} = R_2^{++} + R_2^{--} - 2R_2^{+-}, \quad (63)$$

where in the second line we used the definition of factorial cumulants F_1^{α} and $F_2^{\alpha_1\alpha_2}$ and in the third line, we used the normalized cumulant ratios $R_2^{\alpha_1\alpha_2}$, defined in Eq. (2), with $\alpha_1 = +$ and $\alpha_2 = -$. One then obtains the useful result

$$\langle\langle \Delta q_1 \Delta q_2\rangle\rangle \approx \frac{\langle N\rangle^2}{4\langle N(N-1)\rangle}\nu_{\rm dyn}^{+-}. \quad (64)$$

A similar development can be carried out with a differential version of $\langle\langle \Delta q_1 \Delta q_2\rangle\rangle$ and one finds

$$\langle\langle \Delta q_1 \Delta q_2\rangle\rangle(\vec{p}_1,\vec{p}_2) = -\frac{\langle N\rangle}{4\langle N(N-1)\rangle} \times [B_2^{+-}(\vec{p}_1,\vec{p}_2) + B_2^{-+}(\vec{p}_1,\vec{p}_2)], \quad (65)$$

where $B_2^{+-}(\vec{p}_1,\vec{p}_2)$ and $B_2^{-+}(\vec{p}_1,\vec{p}_2)$ are bound unified balance functions [17] defined according to

$$B_2^{+-}(\vec{p}_1,\vec{p}_2) = \frac{1}{\langle N^-\rangle}[C_2^{+-}(\vec{p}_1,\vec{p}_2) - C_2^{--}(\vec{p}_1,\vec{p}_2)], \quad (66)$$

$$B_2^{-+}(\vec{p}_1,\vec{p}_2) = \frac{1}{\langle N^+\rangle}[C_2^{-+}(\vec{p}_1,\vec{p}_2) - C_2^{++}(\vec{p}_1,\vec{p}_2)]. \quad (67)$$

The functions $B_2^{+-}(\vec{p}_1,\vec{p}_2)$ and $B_2^{-+}(\vec{p}_1,\vec{p}_2)$ are constructed in such a way that their respective integral each converge to unity in the full acceptance limit [17].[1] The fluctuations $\langle\langle \Delta q_1 \Delta q_2\rangle\rangle(\vec{p}_1,\vec{p}_2)$ thus have an upper bound $\langle N\rangle/\langle N(N-1)\rangle$. And since $\langle N(N-1)\rangle \rightarrow \langle N\rangle^2$ in the large N (and Poisson) limit, one concludes that $\langle\langle \Delta q_1 \Delta q_2\rangle\rangle(\vec{p}_1,\vec{p}_2)$ should scale in inverse proportion of the system size and the produced particle multiplicity. This expectation is verified from a number of recent measurements of charge fluctuations [41–43].

It is natural to seek to extend Eq. (65) to higher moments by considering expressions of the form $\langle\langle \Delta q_1 \Delta q_2 \cdots \Delta q_n\rangle\rangle$. We begin, in this section, with a discussion of four-particle balance functions based on an expansion of $\langle\langle \Delta q_1 \Delta q_2 \Delta q_3 \Delta q_4\rangle\rangle$.

[1] For charge conserving processes.

Extensions to higher orders $n = 6, \dots, 10$ are presented in Appendix E.

To compute $\langle\langle \Delta q_1 \Delta q_2 \Delta q_3 \Delta q_4 \rangle\rangle$, we first expand the deviates, compute the event ensemble of the resulting expression and find

$$\begin{aligned}\langle\langle \Delta q_1 \Delta q_2 \Delta q_3 \Delta q_4 \rangle\rangle = \langle\langle q_1 q_2 q_3 q_4 \rangle\rangle - \langle\langle q \rangle\rangle \langle\langle q_1 q_2 q_3 \\ + q_1 q_2 q_4 + q_1 q_3 q_4 + q_2 q_3 q_4 \rangle\rangle \\ + \langle\langle q \rangle\rangle^2 \langle\langle q_1 q_2 + q_1 q_3 + q_1 q_4 \\ + q_2 q_3 + q_2 q_4 + q_3 q_4 \rangle\rangle - 3\langle\langle q \rangle\rangle^4.\end{aligned} \tag{68}$$

We observe the above expression nearly matches the fourth cumulant expansion, Eq. (A33), but misses a term of the form $3\langle\langle \Delta q_1 \Delta q_2 \rangle\rangle \langle\langle \Delta q_3 \Delta q_4 \rangle\rangle$. This additional term corresponds to the square of two-particle contributions and needs to be subtracted to eliminate such trivial contributions to the four particle correlator. Subtracting $3\langle\langle \Delta q_1 \Delta q_2 \rangle\rangle \langle\langle \Delta q_3 \Delta q_4 \rangle\rangle$, the integral of C_4, denoted I_4^{+-}, may then be written according to

$$I_4^{+-} \equiv \langle\langle \Delta q_1 \Delta q_2 \Delta q_3 \Delta q_4 \rangle\rangle - 3\langle\langle \Delta q_1 \Delta q_2 \rangle\rangle \langle\langle \Delta q_3 \Delta q_4 \rangle\rangle \tag{69}$$

$$= \frac{1}{\langle N(N-1)(N-2)(N-3) \rangle} \iint_\Omega q_1 q_2 q_3 q_4 \times C_4(\vec{p}_1, \dots, \vec{p}_4) d\vec{p}_1 \cdots d\vec{p}_4. \tag{70}$$

We thus proceed to use the integral I_4^{+-} and the differential cumulant $C_4(\vec{p}_1, \dots, \vec{p}_4)$ to introduce four-particle balance functions. To this end, we write the charged-particle four-tuplet decomposition of C_4 as follows:

$$\begin{aligned}C_4(\vec{p}_1, \dots, \vec{p}_4) = C_4^{++++}(\vec{p}_1, \dots, \vec{p}_4) + C_4^{----}(\vec{p}_1, \dots, \vec{p}_4) \\ - 4C_4^{+++-}(\vec{p}_1, \dots, \vec{p}_4) - 4C_4^{---+} \\ \times (\vec{p}_1, \dots, \vec{p}_4) + 6C_4^{++--}(\vec{p}_1, \dots, \vec{p}_4),\end{aligned} \tag{71}$$

where we assumed the densities are symmetric under permutations of indices, e.g., $\rho_4^{++--} = \rho_4^{-+-+} == \rho_4^{--++}$, to compute each of the coefficients. The integral, Eq. (69), becomes

$$\begin{aligned}I_4^{+-} = \frac{1}{N(N-1)(N-2)(N-3))} \bigg[\int_\Omega C_4^{++++}(\vec{p}_1, \dots, \vec{p}_4) d\vec{p}_1 \cdots d\vec{p}_4 + \int_\Omega C_4^{----}(\vec{p}_1, \dots, \vec{p}_4) d\vec{p}_1 \cdots d\vec{p}_4 \\ -4 \int_\Omega C_4^{-+++}(\vec{p}_1, \dots, \vec{p}_4) d\vec{p}_1 \cdots d\vec{p}_4 - 4 \int_\Omega C_4^{+---}(\vec{p}_1, \dots, \vec{p}_4) d\vec{p}_1 \cdots d\vec{p}_4 \\ +6 \int_\Omega C_4^{++--}(\vec{p}_1, \dots, \vec{p}_4) d\vec{p}_1 \cdots d\vec{p}_4 \bigg]\end{aligned} \tag{72}$$

$$= \frac{[F^{++++} - 4F^{+++-} + 6F^{++--} - 4F^{---+} + F^{----}]}{N(N-1)(N-2)(N-3)}, \tag{73}$$

where in the last line we used the definition, Eq. (A33), of the four-particle factorial cumulant. By analogy to Eq. (65), one can then introduce four-particle differential "balance functions" according to

$$\begin{aligned}I_4^{+-}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \equiv \frac{4!}{2 \cdot 2!} \frac{\langle N^-(N^- - 1) \rangle}{\langle N(N-1)(N-2)(N-3) \rangle} B_4^{+-}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \\ + \frac{4!}{2 \cdot 2!} \frac{\langle N^+(N^+ - 1) \rangle}{\langle N(N-1)(N-2)(N-3) \rangle} B_4^{-+}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4),\end{aligned} \tag{74}$$

where

$$B_4^{+-}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = \frac{2}{4!/2!} \frac{[3C_4^{++--}(\vec{p}_1, \dots, \vec{p}_4) - 4C_4^{+---}(\vec{p}_1, \dots, \vec{p}_4) + C_4^{----}(\vec{p}_1, \dots, \vec{p}_4)]}{\langle N^-(N^- - 1) \rangle}, \tag{75}$$

$$B_4^{-+}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = \frac{2}{4!/2!} \frac{[3C_4^{--++}(\vec{p}_1, \dots, \vec{p}_4) - 4C_4^{-+++}(\vec{p}_1, \dots, \vec{p}_4) + C_4^{++++}(\vec{p}_1, \dots, \vec{p}_4)]}{\langle N^+(N^+ - 1) \rangle}. \tag{76}$$

We use the notations B_4^{+-} and B_4^{-+}, to denote n particle balance functions involving, the balancing of negatively and positively charged particles by positively and negatively charged particles, respectively. Inclusion of the ratio of factorial coefficients $(4!/2!)$ insures the integral of $B_4^{\pm\mp}$ converge to unity in the limit of full acceptance. Similar coefficients are introduced, in Appendix E to achieve proper normalization of the higher-order balance functions. To indeed verify the integrals of B_4^{+-} and B_4^{-+} integrate to unity over the full acceptance, let $P(n)$ represent the probability of a process involving the production of n pairs of positively and negatively charged particles. Mixed factorial moments are thus trivially given by

$$f_1^{\pm} = \langle N^{\pm} \rangle = \sum_{n=0}^{\infty} nP(n) \equiv \langle n \rangle, \tag{77}$$

$$f_2^{\pm\pm} = \langle N^\pm (N^\pm - 1)\rangle = \langle n^2\rangle - \langle n\rangle, \quad (78)$$

$$f_2^{\pm\mp} = \langle N^\pm N^\mp\rangle = \langle n^2\rangle, \quad (79)$$

$$f_4^{\pm\pm\pm\pm} = \langle n^4\rangle - 6\langle n^3\rangle + 11\langle n^2\rangle - 6\langle n\rangle, \quad (80)$$

$$f_4^{\pm\pm\pm\mp} = \langle n^4\rangle - 3\langle n^3\rangle + 2\langle n^2\rangle, \quad (81)$$

$$f_4^{\pm\pm\mp\mp} = \langle n^4\rangle - 2\langle n^3\rangle + \langle n^2\rangle. \quad (82)$$

Assuming these are in fact fourth-order (or higher) correlations, the integral of $B_4^{+-}(\vec{p}_1, \ldots, \vec{p}_4)$ over the full momentum volume thus yields

$$\begin{aligned}\int_\Omega & B_4^{+-}(\vec{p}_1, \ldots, \vec{p}_4) d\vec{p}_1 \cdots d\vec{p}_4 \\ &= \frac{2}{4!/2!}\frac{1}{\langle n(n-1)\rangle}[3(\langle n^4\rangle - 2\langle n^3\rangle + \langle n^2\rangle) - 4(\langle n^4\rangle \\ &\quad - 3\langle n^3\rangle + 2\langle n^2\rangle) + (\langle n^4\rangle - 6\langle n^3\rangle + 11\langle n^2\rangle - 6\langle n\rangle)] \\ &= 1, \end{aligned} \quad (83)$$

which is indeed equal to unity. By symmetry, the integral of B_4^{-+} also converges to unity in a full acceptance measurement. We thus have two equivalent four particle balance functions to study charge balanced four particle correlations.

As for two-particle balance functions, the above expressions can also be generalized to cross species balance function and these shall feature simple sum rules similar to those satisfied by B_2 functions [17].

Given they are based on four-particle cumulants, the functions $B_4^{+-}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)$ and $B_4^{-+}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)$ shall suppress, by construction, contributions from four-tuplets of uncorrelated particles. Only four-tuplets featuring genuinely correlated particles would contribute to the strength of the correlators. Tuplets involving only two or three correlated particles would have vanishing contributions. As such the four-cumulant components of B_4 should indeed suppress contributions from resonance decays resulting into two or three correlated particles. Contributions from hadron resonance decays would then be limited to four prong decays. Considering these typically have small probabilities, one would expect the magnitude of B_4 is then determined by other processes such as "string fragmentation", jet fragmentation, and other multiparticle production processes. Contributions from jet fragmentation processes can be singled out by using a cone-shaped acceptance with, e.g., $\varphi \approx 1$ and $y \approx 1$. Conversely, jets can be suppressed by using at least two relatively narrow kinematic bins separated by a sizable rapidity gap. Removal of two- and three-prong decays, as well as contributions from jets, enables a direct study of the underlying processes, such as string fragmentation, that lead to particle production over extended ranges of rapidity. Long-range correlations have already been observed in the context of anisotropic flow measurements in pp, pA, and AA collisions. These measurements show that the long correlations are largely dominated by the geometry and fluctuations of the geometry of collisions. They however say little about the underlying nature of the correlations or the correlation length. Measurements of multiparticle charge (or baryon) balance functions would change the focus from the transverse geometry to the longitudinal structure of these correlations and might then shed light on the nature and origin of these correlations.

By construction, n-cumulants are nonvanishing only if particle correlations of n-particle are present in the system considered. Consequently, should observations yield vanishing four-cumulants, it would imply that correlation between balancing charges are only limited to second-order contributions. If the four-cumulants are nonvanishing, they would indicate more intricate production mechanisms. Either way, measurements of B_4 would provide new and valuable information on the structure of particle production dynamics.

Higher-order balance functions, $B_n^{\pm\mp}$, with $6 \leqslant n \leqslant 10$ can be constructed in a similar way as $B_4^{\pm\mp}$ and are listed in Appendix E. As for $B_4^{\pm\mp}$, higher-order balance functions would suppress contributions from lower order correlations. A systematic study of $B_n^{\pm\mp}$ for $n = 4, 6, 8$, etc., would then provide sensitivity to increasingly more complicated production processes featuring a growing number of correlated particles. Such measurements should then provide additional and powerful constrains on multiparticle production models.

V. RELATION TO NET-CHARGE CUMULANTS

Cumulants of the net-charge Q (as well as net baryon B and net strangeness S) of the particles measured in a specific acceptance Ω nominally provide a probe of the susceptibilities of the matter formed in nucleus-nucleus collisions [44–48]. Several measurements of lower order cumulants (as well as mixed cumulants) have been reported in the recent literature. Ratios of lower order cumulants have been studied, in particular, in the context of the beam energy scans recently performed at the Relativistic Heavy Ion Colliders (RHIC) to identify signatures of a critical point of nuclear matter [49,50]. Given the potential significance of such critical point, considerable theoretical and experimental efforts have been deployed to obtain relations between the properties of nuclear matter, net-quantum number cumulants, as well as robust techniques to measure these observables [47]. In this context, note that it was recently shown that a simple relation exists between the second net-charge cumulant, κ_2^Q and net-charge balance functions B [16] (also see discussions in Ref. [51]). This relation is of particular interest because it expresses the magnitude of the non trivial part (non-Poissonian) of κ_2^Q in terms of an integral of the charge balance function B across a specific experimental acceptance (i.e., a specific kinematic range). Given this integral converges to unity, by construction, in the full acceptance limit, it implies the magnitude of κ_2^Q is determined by the shape and width of the balance function relative to the width of the acceptance. This is critical for the beam energy scan because, although the acceptance can be kept fixed, the shape of the B is known to evolve with produced species, system size, nucleus-nucleus collision centrality, and beam energy [1,43,52–56]. As such, the magnitude of κ_2^Q thus constitutes a poorly defined reference in the search of a critical point of nuclear matter. That said, it is also of interest to consider how higher cumulants might be impacted by charge conservation, the size of the experimental acceptance of a measurement, and the dynamics of collisions.

We saw, in Sec. IV, that balance functions naturally arise in the calculation of moments $\langle\langle \Delta q_1 \Delta q_2 \rangle\rangle$ and $\langle\langle \Delta q_1 \Delta q_2 \Delta q_3 \Delta q_4 \rangle\rangle$ and yield expressions proportional to factorial cumulants of the particle multiplicities. We thus seek to determine relations between net-charge cumulants, net-charge factorial cumulants, and factorial cumulants of multiplicities of positively and negatively charge particles. Details of the derivations are presented in Appendix A. In this section, we summarize results of interest which indicate that highest order contributions of net-charge cumulants are identical to integrals of multiparticle balance functions of same order.

It is well known that moments, cumulants, factorial moments, and factorial cumulants are readily computed based on their respective generating functions which, herewith, we denote by $G_m(\theta_Q)$, $G_c(\theta_Q) \equiv \ln G_m(\theta_Q)$, $G_f(s_Q)$, and $G_F(s_Q) \equiv \ln G_f(s_Q)$, where the sub-indices m, c, f, and F indicate generating functions of moments, cumulants, factorial moments, and factorial cumulants, respectively. As discussed in Appendix A 3, moments m_n^Q are obtained by computing nth derivatives of G_m w.r.t. θ_Q, evaluated at $\theta_Q = 0$, while cumulants κ_n^Q are obtained from nth order derivatives of G_c w.r.t. θ_Q also evaluated at $\theta_Q = 0$. Given $G_c = \ln G_m$, it is then straightforward to compute κ_n^Q in terms of moments $m_{n'}^Q$, with $n' \leqslant n$ [see Eqs. (A41)–(A43)]. Similarly, factorial and factorial cumulants can be also obtained by nth order derivatives of G_f and G_F w.r.t. to s_Q. It is however more useful to express the generating functions in terms of multiplicities of positively and negatively charged particles N_+ and N_- and their associated dummy variables θ_+ and θ_- for moments and cumulants calculations and dummy variables s_+ and s_- for factorial moments and factorial cumulant calculations. It is then possible to obtain relations between net-charge cumulants and (mixed) cumulants of N_+ and N_- as well as with factorial cumulants of these multiplicities. As shown in detail in Appendix A 3, one finds even order n-cumulants are given by

$$\kappa_2^Q = F_1^+ + F_1^- + F_2^{++} - 2F_2^{+-} + F_2^{--}, \tag{84}$$

$$\kappa_4^Q = F_1^+ + F_1^- + \cdots + F_4^{4+} - 4F_4^{3+1-} + 6F_4^{2+2-} - 4F_4^{1+3-} + F_4^{4-}, \tag{85}$$

$$\kappa_6^Q = F_1^+ + F_1^- + \cdots + F_6^{6+} - 6F_6^{5+1-} + 15F_6^{4+2-} - 20F_6^{3+3-} + 15F_6^{2+4-} - 6F_6^{1+5-} + F_6^{6-}, \tag{86}$$

and so on for higher orders. Intermediate terms of order $1 < n' < n$ were omitted for the sake of clarity. Comparing the above expressions, as well as Eqs. (A61) and (A62), with Eqs. (E3)–(E7), we observe that the cumulants κ_n^Q feature a dependence on the mixed factorial moments $F_n^{k(+)n-k(-)}$ that exactly matches the expression of the balance functions of order $n \geqslant 4$. Indeed, as for κ_2^Q, we find that the nontrivial component of higher-order κ_n^Q are exactly proportional to integrals of balance functions B_n^{+-}, B_n^{-+} introduced in Sec. IV. Given these balance functions are governed by sum rules, i.e., their full acceptance integrals are entirely determined by charge conservation. We conclude that as for second-order cumulants κ_2^Q, the nontrivial components of higher cumulants, κ_2^Q, $n \geqslant 4$, are determined by integral of functions whose full acceptance limit is solely driven by charge conservation. As for basic balance functions, Eqs. (66) and (67), one expects that these higher-order balance function integrals feature a strong dependence on the rapidity and transverse momentum coverage of the measurements, as well as the details of the particle production processes at play in the collisions being studied [47]. Additionally, as for basic balance functions, it stands to reason that these higher balance function might feature some dependence on collision centrality and beam energy. Such dependencies might thus be better probed with differential balance functions. This suggests that rather than measuring cumulants κ_n^Q, which only feature information on the integrals of balance functions, it would be better advised to measure differential balance functions. Techniques to compute multiparticle balance functions without the drawbacks of multiple nested loops on particles of an event were discussed in Sec. III C whereas kinematic configurations of measurements of potential interest were presented in Sec. III D.

VI. SUMMARY

We first advocated, in Sec. II, for measurements of integral and differential of multiparticle correlation functions as tools to extract characteristics of heavy ion collisions and the matter they produce heretofore somewhat neglected and susceptible of enhancing the understanding of the physics of these complex systems. We next explicitly presented detailed formula of such multiparticle correlations as well as techniques to compute them in finite time (i.e., single loop on all particles of interest) based on the methods of moments. This set the stage for the development of what we called multiparticle balance functions. These higher-order balance functions were introduced based on expectation values of the form $\langle\langle \Delta q_1 \Delta q_2 \cdots \Delta q_n \rangle\rangle$ but are best computed in terms of combinations of nth order cumulants (or integral factorial cumulants). Much like the original balance functions B_2 introduced by Pratt *et al.* [10], these new balance functions are defined in such a way that they integrate to unity in full phase space (i.e., all rapidities and $p_{\rm T} \geqslant 0$). As such they too provide a measure of the fraction of charge (or other quantum number) balanced when measured in a finite acceptance. This fraction is expected to be rather sensitive to the details of the (charge conserving) particle production and transport. Indeed, given they are constructed based on n-particle cumulants, they should probe the particle production rapidity and momentum correlation length scales and the details of the particle production mechanisms.

We additionally showed these higher-order balance functions have integrals, formulated in terms of factorial cumulants, that are equal to the higher-order contributions of net-charge cumulants κ_n^Q. This is an important result that pertains to measurements of net-charge (as well as net strangeness and net baryon number) fluctuations based on cumulants κ_n^Q and their evolution with beam energy in the context of the beam energy scan (BES) at the Relativistic Heavy Ion Collider. The magnitude of these cumulants *cannot* be corrected for charge conservation given the balance functions integrate to unity in full acceptance. Indeed the magnitude of the integral of the balance functions $B_n^{\pm\mp}$ measured in a specific acceptance (in rapidity and transverse momentum) is determined by the details of the particle production and transport (e.g., presence of radial flow) and has thus relatively little to do with the intrinsic properties of the matter they originate from (i.e., the susceptibilities of the QGP).

The formalism developed in this work for the deployment of multiple particle correlations is in many ways similar to the techniques used in the context of measurements of anisotropic flow. It is thus likely that the correlation functions discussed in this work might be calculable, with minor or no adaptation, to existing generic frameworks of anisotropic flow measurements. Of particular interest, however, are practical implementations of $\vec{p}$ dependent acceptance and efficiency corrections at the single particle level. Also of interest are efficiency losses related to correlated detector effects that likely manifest themselves differently in the context of the correlation functions discussed in this work. The authors thus plan to follow up this work with additional studies of these practical effects.

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APPENDIX A: MOMENTS, CUMULANTS, FACTORIAL MOMENTS, AND FACTORIAL CUMULANTS

The calculation of moments, cumulants, factorial moments, and factorial cumulants based on their respective generating functions are discussed in Appendix A 1 for single variable systems, in Appendix A 2 for joint-measurements of multivariable systems, and in Appendix A 3 for the specific case of a collision system's net charge. Generalization of the notions of moments, cumulants, and their properties were discussed by Kubo [57]. Properties of cumulants were also discussed more recently in the context of flow observables [58]. In the following, we simply state the results needed elsewhere in this work.

1. Single variable systems

Recall that given a function $P(N)$ stipulating the probability of observing a value N, algebraic moments of N, denoted m_n, are defined as

$$m_n \equiv \langle N^n \rangle = \sum_{N=0}^{\infty} N^n P(N). \tag{A1}$$

Additionally, defining the moment generating function $G_m(\theta) = \langle e^{\theta N} \rangle$, one readily verifies the moments m_n can be computed according to

$$m_n = \partial_\theta^n G_m(\theta)|_{\theta=0}, \tag{A2}$$

where $\partial_\theta = \partial/\partial\theta$. Cumulants of N of order n, denoted κ_n, are similarly defined and computed with the introduction of a cumulant generating functions $G_c(\theta) \equiv \ln G_m(\theta)$ according to

$$\kappa_n = \partial_\theta^n G_c(\theta)|_{\theta=0} = \partial_\theta^n \ln G_m(\theta)|_{\theta=0}. \tag{A3}$$

Application of the right-hand side (r.h.s.) of the above expression readily yields the cumulants κ_n as linear combinations of the moments m_n. In the context of measurements of particle densities of order n, discussed in this work, it is also convenient to consider factorial moments and factorial cumulants. Factorial moments, f_n, are formally defined with the introduction of generating functions $G_f(\theta) = \langle s^N \rangle$, where $s = e^\theta$ and computed according to

$$f_n = \partial_s^n G_f(s)|_{s=1}, \tag{A4}$$

where $\partial_s = \partial/\partial s$. Likewise, factorial cumulants, F_n, are formally defined as derivatives of a factorial cumulant generating functions $G_F(s) \equiv \ln G_f(s)$

$$F_n = \partial_s^n G_F(s)|_{s=1} = \partial_s^n \ln G_f(s)|_{s=1}. \tag{A5}$$

Application of the r.h.s. of the above expression yields factorial cumulants F_n as combinations of the factorial moments f_n. Computation with Mathematica [59] yields the following expressions for the ten lowest orders

$$F_1 = f_1 \tag{A6}$$

$$F_2 = -f_1^2 + f_2 \tag{A7}$$

$$F_3 = 2f_1^3 - 3f_1f_2 + f_3 \tag{A8}$$

$$F_4 = -6f_1^4 - 12f_1^2f_2 - 3f_2^2 - 4f_1f_3 + f_4 \tag{A9}$$

$$\begin{aligned} F_5 = {} & 24f_1^5 - 60f_1^3f_2 + 30f_1f_2^2 + 20f_1^2f_3 - 10f_2f_3 \\ & -5f_1f_4 + f_5 \end{aligned} \tag{A10}$$

$$\begin{aligned} F_6 = {} & -120f_1^6 + 360f_1^4f_2 - 270f_1^2f_2^2 + 30f_2^3 - 120f_1^3f_3 \\ & +120f_1f_2f_3 - 10f_3^2 + 30f_1^2f_4 - 15f_2f_4 - 6f_1f_5 + f_6 \end{aligned} \tag{A11}$$

$$\begin{aligned} F_7 = {} & 720f_1^7 - 2520f_1^5f_2 + 2520f_1^3f_2^2 - 630f_1f_2^3 + 840f_1^4f_3 \\ & +1260f_1^2f_2f_3 + 210f_2^2f_3 + 140f_1f_3^2 - 210f_1^3f_4 \\ & +210f_1f_2f_4 - 35f_3f_4 + 42f_1^2f_5 - 21f_2f_5 \\ & -7f_1f_6 + f_7 \end{aligned} \tag{A12}$$

$$\begin{aligned} F_8 = {} & -5040f_1^8 + 20160f_1^6f_2 - 25200f_1^4f_2 + 10080f_1^2f_2^3 \\ & - 630f_2^4 - 6720f_1^5f_3 + 13440f_1^3f_2f_3 - 5040f_1f_2^2f_3 \\ & - 1680f_1^2f_3^2 + 560f_2f_3^2 + 1680f_1^4f_4 - 2520f_1^2f_4 \\ & + 420f_2^2f_4 + 560f_1f_3f_4 - 35f_4^2 - 336f_1^3f_5 \\ & + 336f_1f_2f_5 - 56f_3f_5 + 56f_1^2f_6 - 28f_2f_6 - 8f_1f_7 + f_8, \end{aligned} \tag{A13}$$

$$\begin{aligned} F_9 = {} & 40320f_1^9 - 181440f_1^7f_2 + 272160f_1^5f_2 - 151200f_1^3f_2^3 \\ & + 22680f_1f_2^4 + 60480f_1^6f_3 - 151200f_1f_2f_3^2 \\ & + 90720f_1^2f_2^2f_3 - 7560f_2^3f_3 + 20160f_1^3f_3 \\ & - 15120f_1f_2f_3^2 + 560f_3^3 - 15120f_1^5f_4 + 30240f_1^3f_2f_4 \end{aligned}$$

$$
\begin{aligned}
&- 11340 f_1 f_2^2 f_4 - 7560 f_1^2 f_3 f_4 + 2520 f_2 f_3 f_4 \\
&+ 630 f_1 f_4^2 + 3024 f_1^4 f_5 - 4536 f_1^2 f_2 f_5 + 756 f_2^2 f_5 \\
&+ 1008 f_1 f_3 f_5 - 126 f_4 f_5 - 504 f_1^3 f_6 + 756 f_2^2 f_5 \\
&+ 1008 f_1 f_3 f_5 - 126 f_4 f_5 - 504 f_1^3 f_6 + 504 f_1 f_2 f_6 \\
&- 84 f_3 f_6 + 72 f_1^2 - 36 f_2 f_7 - 9 f_1 f_8 + f_9, \qquad \text{(A14)}
\end{aligned}
$$

$$
\begin{aligned}
F_{10} = &- 362880 f_1^{10} + 1814400 f_1^8 f_2 - 3175200 f_1^6 f_2^2 \\
&+ 2268000 f_1^4 f_2^3 - 567000 f_1^2 f_2^4 + 22680 f_2^5 \\
&- 604800 f_1^7 f_3 + 1814400 f_1^5 f_2 f_3 - 1512000 f_1^3 f_2^2 f_3 \\
&+ 302400 f_1 f_2^3 f_3 - 252000 f_1^4 f_3^2 + 302400 f_1^2 f_2 f_3^2 \\
&- 37800 f_2^2 f_3^2 - 16800 f_1 f_3^3 + 151200 f_1^6 f_4 \\
&- 378000 f_1^4 f_2 f_4 + 226800 f_1^2 f_2^2 f_4 - 18900 f_2^3 f_4 \\
&+ 100800 f_1^3 f_3 f_4 - 75600 f_1 f_2 f_3 f_4 + 4200 f_3^2 f_4 \\
&- 9450 f_1^2 f_4^2 + 3150 f_2 f_4^2 - 30240 f_1^5 f_5 \\
&+ 60480 f_1^3 f_2 f_5 - 22680 f_1 f_2^2 f_5 - 15120 f_1^2 f_3 f_5 \\
&+ 5040 f_2 f_3 f_5 + 2520 f_1 f_4 f_5 - 126 f_5^2 + 5040 f_1^4 f_6 \\
&- 7500 f_1^2 f_2 f_6 + 1260 f_2^2 f_6 + 1680 f_1 f_3 f_6 \\
&- 210 f_4 f_6 - 720 f_1^3 f_7 + 720 f_1 f_2 f_7 - 120 f_3 f_7 \\
&+ 90 f_1^2 f_8 - 45 f_2 f_8 - 10 f_1 f_9 + f_{10} \qquad \text{(A15)}
\end{aligned}
$$

Factorial cumulants, F_n, are of particular interest in the context of measurements of particle production because they identically vanish in the absence of correlations of order n. Note that the relations between cumulants and moments are formally identical to the above given the definitions of cumulants and factorial cumulants in terms of log of their respective generating functions.

It is straightforward (and convenient) to express cumulants as combinations of factorial cumulants if one notices that $G_c(\theta) = G_F(s)$ given $s = e^{\theta}$. Taking n-order derivatives ∂_θ^n of the left-hand side (l.h.s.) yields cumulants κ_n, while derivatives on the r.h.s. are computed based on $\partial_\theta = (\partial s/\partial\theta)\partial_s = s\partial_s$ and yield expressions in terms of $F_{n'}$ with $n' \leqslant n$. The ten lowest orders are

$$\kappa_1 = \partial_\theta G_c|_{\theta=0} = s\partial_s G_F|_{s=1} = F_1, \qquad \text{(A16)}$$

$$\kappa_2 = \partial_\theta^2 G_c\big|_{\theta=0} = s\partial_s(s\partial_s G_F)|_{s=1} = F_1 + F_2, \qquad \text{(A17)}$$

$$\kappa_3 = F_1 + 3F_2 + F_3, \qquad \text{(A18)}$$

$$\kappa_4 = F_1 + 7F_2 + 6F_3 + F_4, \qquad \text{(A19)}$$

$$\kappa_5 = F_1 + 15F_2 + 25F_3 + 10F_4 + F_5, \qquad \text{(A20)}$$

$$\kappa_6 = F_1 + 31F_2 + 90F_3 + 65F_4 + 15F_5 + F_6, \qquad \text{(A21)}$$

$$\kappa_7 = F_1 + 63F_2 + 301F_3 + 350F_4 + 140F_5 + 21F_6 + F_7, \qquad \text{(A22)}$$

$$
\begin{aligned}
\kappa_8 = &F_1 + 127F_2 + 966F_3 + 1701F_4 + 1050F_5 + 266F_6 \\
&+28F_7 + F_8, \qquad \text{(A23)}
\end{aligned}
$$

$$
\begin{aligned}
\kappa_9 = &F_1 + 255F_2 + 3025F_3 + 7770F_4 + 6951F_5 \\
&+2646F_6 + 462F_7 + 36F_8 + F_9, \qquad \text{(A24)}
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{10} = &F_1 + 511F_2 + 9330F_3 + 34105F_4 + 45525F_5 \\
&+22827F_6 + 5880F_7 + 750F_8 + 45F_9 + F_{10}. \qquad \text{(A25)}
\end{aligned}
$$

First note that cumulants of a given order k feature a linear combination of all factorial cumulants of lower order $k' \leqslant k$. Second, remember that in the context of particle correlation measurements, one can conclude there are correlations of k or more particles only when a factorial cumulant F_k is nonvanishing. Consequently, if a factorial F_k is consistent with zero, within statistical uncertainties, there is no point in measuring κ_k or higher-order cumulants $\kappa_{k'}$, with $k' > k$ since these do not carry additional experimental information about the system under study. Indeed, in such cases, the magnitude of κ_k is primarily determined by factorial cumulants of the lowest orders involving few or, possibly, no particle correlations.

2. Multivariable systems

Given a function $P(\vec{N})$ stipulating the probability of jointly observing m variables $\vec{N} \equiv (N_1, N_2, \ldots, N_m)$ corresponding to categories $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, which in the context of this work corresponds to kinematic bins or particle species or both, mixed algebraic moments of $\vec{N}$, denoted $m_{\vec{n}}^{\vec{\alpha}}$, are defined as

$$m_{\vec{n}}^{\vec{\alpha}} \equiv \sum_{\vec{N}} \prod_{i=1}^{m} N_i^{n_i} P(\vec{N}), \qquad \text{(A26)}$$

and calculable based on a mixed moment generating functions $G_m(\vec{\theta}) \equiv \langle e^{\sum_{i=1}^m \theta_i N_i} \rangle$ according to

$$m_n^{\vec{\alpha}} = \left(\prod_{i=1}^{n} \partial_{\theta_i} \right) G_m(\vec{\theta}) \Bigg|_{\vec{\theta}=0}, \qquad \text{(A27)}$$

where $\vec{\alpha}$ represents all the categories for which moments are evaluated. For instance, a double derivative $\partial_{\theta_1}\partial_{\theta_1}$ would yield a second moment of N_1, whereas $\partial_{\theta_1}\partial_{\theta_2}$ would yield a mix moment of N_1 and N_2. Proceeding as for a single variable, one defines mixed cumulants according to

$$\kappa_n^{\vec{\alpha}} = \left(\prod_{i=1}^{n} \partial_{\theta_i} \right) \ln G_m(\vec{\theta}) \Bigg|_{\vec{\theta}=0}, \qquad \text{(A28)}$$

where $\vec{\theta} = 0$ specifies derivatives are evaluated with $\theta_i = 0$, for $i = 1, \ldots, m$. Similarly, mixed factorial moments, $f_n^{\vec{\alpha}}$, are defined based on mixed moment generating functions $G_f(\vec{s}) \equiv \langle \prod_{i=1}^m s^{N_i} \rangle$ according to

$$f_n^{\vec{\alpha}} = \left(\prod_{i=1}^{n} \partial_{s_i} \right) G_f(\vec{s}) \Bigg|_{\vec{s}=1}, \qquad \text{(A29)}$$

where $\vec{s} = 1$ specifies derivatives are evaluated with $s_i = 1$, for $i = 1, \ldots, m$. Factorial cumulants are defined as derivatives of the factorial cumulant generating functions $G_F(\vec{s}) \equiv \ln G_f(\vec{s})$ according to

$$F_n^{\vec{\alpha}} = \left(\prod_{i=1}^{n} \partial_{s_i} \right) \ln G_f(\vec{s}) \Bigg|_{\vec{s}=1}, \qquad \text{(A30)}$$

Mathematica [59] enables a speedy and reliable computation of $F_n^{\vec{\alpha}}$. The lowest orders are found to be

$$F_2^{\alpha_1\alpha_2} = f_2^{\alpha_1\alpha_2} - f_1^{\alpha_1} f_1^{\alpha_2}, \tag{A31}$$

$$F_3^{\alpha_1\cdots\alpha_3} = f_3^{\alpha_1\cdots\alpha_3} - \sum_{(3)} f_2^{\alpha_1\alpha_2} f_1^{\alpha_3} + 2 f_1^{\alpha_1} f_1^{\alpha_2} f_1^{\alpha_3}, \tag{A32}$$

$$F_4^{\alpha_1\cdots\alpha_4} = f_4^{\alpha_1\cdots\alpha_4} - \sum_{(4)} f_3^{\alpha_1\cdots\alpha_3} f_1^{\alpha_4} - \sum_{(3)} f_2^{\alpha_1\alpha_2} f_2^{\alpha_3\alpha_4} + 2\sum_{(6)} f_2^{\alpha_1\alpha_2} f_1^{\alpha_3} f_1^{\alpha_4} - 6 f_1^{\alpha_1} \times \cdots \times f_1^{\alpha_4}, \tag{A33}$$

$$\begin{aligned} F_5^{\alpha_1\cdots\alpha_5} = {} & f_5^{\alpha_1\cdots\alpha_5} - \sum_{(5)} f_4^{\alpha_1\cdots\alpha_4} f_1^{\alpha_5} - \sum_{(10)} f_3^{\alpha_1\cdots\alpha_3} f_2^{\alpha_4\alpha_5} - \sum_{(10)} f_3^{\alpha_1\cdots\alpha_3} f_1^{\alpha_4} f_1^{\alpha_5} + 2\sum_{(15)} f_2^{\alpha_1\alpha_2} f_2^{\alpha_3\alpha_4} f_1^{\alpha_5} \\ & - 6\sum_{(10)} f_2^{\alpha_1\alpha_2} f_1^{\alpha_3} f_1^{\alpha_4} f_1^{\alpha_5} + 24 f_1^{\alpha_1} \times \cdots \times f_1^{\alpha_5}, \end{aligned} \tag{A34}$$

$$\begin{aligned} F_6^{\alpha_1\cdots\alpha_6} = {} & f_6^{\alpha_1\cdots\alpha_6} - \sum_{(6)} f_5^{\alpha_1\cdots\alpha_5} f_1^{\alpha_6} - \sum_{(15)} f_4^{\alpha_1\cdots\alpha_4} f_2^{\alpha_5\alpha_6} - \sum_{(15)} f_4^{\alpha_1\cdots\alpha_4} f_1^{\alpha_5} f_1^{\alpha_6} \\ & - \sum_{(10)} f_3^{\alpha_1\alpha_2\alpha_3} f_3^{\alpha_4\alpha_5\alpha_6} + 2\sum_{(60)} f_3^{\alpha_1\alpha_2\alpha_3} f_2^{\alpha_4\alpha_5} f_1^{\alpha_6} - 6\sum_{(20)} f_3^{\alpha_1\alpha_2\alpha_3} f_1^{\alpha_4} f_1^{\alpha_5} f_1^{\alpha_6} \\ & + 2\sum_{(15)} f_2^{\alpha_1\alpha_2} f_2^{\alpha_3\alpha_4} f_2^{\alpha_5\alpha_6} - 6\sum_{(45)} f_2^{\alpha_1\alpha_2} f_2^{\alpha_3\alpha_4} f_1^{\alpha_5} f_1^{\alpha_6} + 24\sum_{(15)} f_2^{\alpha_1\alpha_2} f_1^{\alpha_3} f_1^{\alpha_4} f_1^{\alpha_5} f_1^{\alpha_6} \\ & - 120 f_1^{\alpha_1} \times \cdots \times f_1^{\alpha_6}, \end{aligned} \tag{A35}$$

where the notation $\sum_{(k)}$ stands for a sum over k (ordered) permutations of the labels α_1, α_2, and $\alpha_3, \ldots$.

As in the case of a single variable, one can express mixed cumulants $\kappa_n^{\vec{\alpha}}$ in terms of factorial cumulants $F_n^{\vec{\alpha}}$ based on the equality $G_c(\vec{\theta}) = G_F(\vec{s})$ by taking derivatives on l.h.s. relative to θ_i whereas derivative are taken relative to $\partial_{\theta_i} = (\partial s_i/\partial\theta_i)\partial_{s_i} = s_i\partial_{s_i}$ on the r.h.s..

3. Net-charge Q

Let $Q = N_+ - N_-$ and $N = N_+ + N_-$ define the net-charge and total charged-particle multiplicity, respectively, detected in a given event, with N_+ and N_-, respectively, representing the number of positively and negatively charged particles in that event. The number of positively and negatively charged particles are expected to fluctuate on an event-by-event basis both owing the stochastic nature of the particle production and variations in the processes yielding particles. Moments and cumulants of Q are of interest because they nominally relate to charge susceptibility of the matter formed in heavy-ion collisions [44–46,48]. Moments m_n^Q of the net charge are defined, as in Eq. (A1), according to

$$m_n^Q \equiv \langle Q^n\rangle = \sum_Q Q^n P(Q, N), \tag{A36}$$

where $P(Q,N)$ represents the probability of observing a net-charge Q and total multiplicity N in a particular event. Moments m_n^Q can evidently be computed based on a generating function of the form $G_m(N,Q) = \langle e^{N\theta_N + Q\theta_Q}\rangle$ but it is of greater interest to obtain the moments, the cumulants, and so on, in terms of moments of the multiplicities N_+ and N_-. Clearly, a simple change of variable enables the definition of $P(N_+, N_-)$ as the joint probability of observing events with N_+ and N_- positively and negatively charged particles. The moment generating functions of (mixed) moments of the multiplicities can then be written

$$G_m(\theta_+, \theta_-) = \langle e^{N_+\theta_+ + N_-\theta_-}\rangle \tag{A37}$$

and successive derivatives of G_m w.r.t. θ_+ and θ_-, evaluated at $\theta_+ = \theta_- = 0$ yield moments and mixed moments of N_+ and N_-. Introducing the notations $\partial_+ = \partial/\partial\theta_+$ and $\partial_- = \partial/\partial\theta_-$, one computes lowest-order mixed moments according to

$$m_1^{\pm} = \partial_\pm G_m(\theta_+, \theta_-)|_{\theta_+=\theta_-=0} = \langle N_\pm\rangle, \tag{A38}$$

$$m_2^{\pm\pm} = \partial_\pm\partial_\pm G_m(\theta_+, \theta_-)|_{\theta_+=\theta_-=0} = \langle N_\pm^2\rangle, \tag{A39}$$

$$m_2^{+-} = \partial_-\partial_+ G_m(\theta_+, \theta_-)|_{\theta_+=\theta_-=0} = \langle N_+N_-\rangle, \tag{A40}$$

and so on. Cumulants and mixed cumulants of the multiplicities N_+ and N_- are computed based on the cumulant generating function $G_c(\theta_+, \theta_-) \equiv \ln G_m(\theta_+, \theta_-)$ by taking successive derivatives w.r.t. θ_+ and θ_-. One for instance gets

$$\kappa_1^{\pm} = \partial_\pm G_c(\theta_+, \theta_-)|_{\theta_+=\theta_-=0} = G_m^{-1}\partial_\pm G_m|_{\theta_+=\theta_-=0} = \langle N_\pm\rangle, \tag{A41}$$

$$\kappa_2^{\pm\pm} = \partial_\pm\left(G_m^{-1}\partial_\pm G_m\right)\big|_{\theta_+=\theta_-=0} = \langle N_\pm^2\rangle - \langle N_\pm\rangle^2, \tag{A42}$$

$$\kappa_2^{+-} = \partial_-\left(G_m^{-1}\partial_+ G_m\right)\big|_{\theta_+=\theta_-=0} = \langle N_+N_-\rangle - \langle N_+\rangle\langle N_-\rangle, \tag{A43}$$

and similarly for higher orders. One recognizes $\kappa_2^{\pm\pm}$ and κ_2^{+-} as the variance and covariance of N_+ and N_- while higher-order κ_3 and κ_4 (not shown) are related to skewness and kurtosis of these multiplicities.

In the context of measurements of net-charge fluctuations, it is of interest to relate cumulants of the net-charge Q to factorial cumulants of the N_+ and N_-. First note that the factorial moments are calculable based on the factorial moment generating functions defined as $G_f(s_+, s_-) \equiv \langle s_+^{N_+} s_-^{N_-}\rangle$, where $s_+ = e^{\theta_+}$ and $s_- = e^{\theta_-}$. Introducing the notations $\partial_{s_+} \equiv \partial/\partial s_+$ and $\partial_{s_-} \equiv \partial/\partial s_-$, mixed factorial moments of N_+ and N_- are obtained by repeated evaluations of derivatives ∂_{s_+} and ∂_{s_-} evaluated at $s_+ = s_- = 1$:

$$f_1^{\pm} = \partial_{s_\pm} G_f(s_+, s_-)\big|_{s_+=s_-=1} = \langle N_\pm s_\pm^{N_\pm -1} s_\mp^{N_\mp}\rangle|_{s_+=s_-=1} = \langle N_\pm\rangle, \quad \text{(A44)}$$

$$f_2^{\pm\pm} = \partial_{s_\pm}\langle N_\pm s_\pm^{N_\pm-1} s_\mp^{N_\mp}\rangle|_{s_+=s_-=1} = \langle N_\pm(N_\pm - 1)\rangle, \quad \text{(A45)}$$

$$f_2^{+-} = \partial_{s_-}\langle N_+ s_+^{N_+-1} s_-^{N_-}\rangle|_{s_+=s_-=1} = \langle N_+ N_-\rangle, \quad \text{(A46)}$$

$$f_3^{\pm\pm\pm} = \langle N_\pm(N_\pm - 1)(N_\pm - 2)\rangle, \quad \text{(A47)}$$

$$f_3^{\pm\pm\mp} = \langle N_\pm(N_\pm - 1)N_\mp\rangle, \quad \text{(A48)}$$

and so on. To compute the relations between κ_n^Q and factorial cumulants of the multiplicities N_+ and N_-, we introduce ∂_{θ_Q}, with $\theta_Q = \theta_+ - \theta_-$, as a linear combination of ∂_{θ_+} and ∂_{θ_-} according to

$$\partial_{\theta_Q} \equiv \frac{\partial}{\partial\theta_Q} = \frac{\partial\theta_+}{\partial\theta_Q}\frac{\partial}{\partial\theta_+} + \frac{\partial\theta_-}{\partial\theta_Q}\frac{\partial}{\partial\theta_-} = \partial_{\theta_+} - \partial_{\theta_-}. \quad \text{(A49)}$$

Cumulants of Q of order n are obtained by computing n derivatives of $G_c(\theta_Q, \theta_N)$ w.r.t. θ_Q according to

$$\begin{aligned}\kappa_n^Q &\equiv \partial_{\theta_Q}^n G_c(\theta_Q, \theta_N) = (\partial_{\theta_+} - \partial_{\theta_-})^n G_c(\theta_+, \theta_-)\\ &= \sum_{k=0}^{n} (-)^{n-k}\binom{n}{k}\partial_{\theta_+}^k \partial_{\theta_-}^{n-k} G_c(\theta_+, \theta_-)\\ &= \sum_{k=0}^{n} (-)^{n-k}\binom{n}{k}\kappa_n^{(k)(n-k)},\end{aligned} \quad \text{(A50)}$$

where $\kappa_n^{(k)(n-k)}$ represent mixed cumulants of order k and $n-k$ in N_+ and N_-. One gets

$$\kappa_1^Q = \kappa_1^+ - \kappa_1^-, \quad \text{(A51)}$$

$$\kappa_2^Q = \kappa_2^{2+} - 2\kappa_2^{1+1-} + \kappa_2^{2-}, \quad \text{(A52)}$$

$$\kappa_3^Q = \kappa_3^{3+} - 3\kappa_3^{2+1-} + 3\kappa_3^{1+2-} - \kappa_3^{3-}, \quad \text{(A53)}$$

and so on. Experimentally, it is of greater interest to obtain the cumulants κ_n^Q in terms of factorial cumulants because these are easier to correct for (single) particle losses and vanish in the absence of correlations at order n. Evidently, it is only meaningful to report cumulants κ_n^Q if the corresponding factorial cumulant F_n are nonvanishing since only these provide new information not already included in cumulants of lower orders. Replacing derivatives ∂_{θ_i} by $s_i\partial_{s_i}$, and noting that $\partial s_i/\partial\theta_j = \delta_{ij} s_i \partial_{s_i}$, one gets

$$\kappa_n^Q = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}(s_+\partial_{s_+})^k (s_-\partial_{s_-})^{n-k} G_F(s_+, s_-). \quad \text{(A54)}$$

Lowest orders of interest for this work are found to be

$$\kappa_1^Q = F_1^+ - F_1^-, \quad \text{(A55)}$$

$$\kappa_2^Q = F_1^+ + F_1^- + F_2^{2+} - 2F_2^{+-} + F_2^{2-}, \quad \text{(A56)}$$

$$\begin{aligned}\kappa_3^Q &= F_1^+ - F_1^- - 3F_2^{2+} - 3F_2^{2-} + F_3^{3-} + 3F_3^{1+2-}\\ &\quad - + 3F_3^{2+1-} - F_3^{3+},\end{aligned} \quad \text{(A57)}$$

$$\begin{aligned}\kappa_4^Q &= F_1^+ + F_1^- + 7F_2^{2+} - 2F_2^{+-} + 7F_2^{2-} + 6F_3^{3+}\\ &\quad -6F_3^{1+2-} - 6F_3^{2+1-} + 6F_3^{3-} + F_4^{4+}\\ &\quad -4F_4^{3+1-} + 6F_4^{2+2-} - 4F_4^{1+3-} + F_4^{4-},\end{aligned} \quad \text{(A58)}$$

$$\begin{aligned}\kappa_5^Q &= F_1^+ - F_1^- + 15F_2^{2+} - 15F_2^{2-} + 25F_3^{3+} - 15F_3^{2+1-}\\ &\quad +15F_3^{1+2-} - 25F_3^{3-} + 10F_4^{4+} - 20F_4^{3+1-}\\ &\quad +20F_4^{1+3-} - 10F_4^{4+0-} + F_5^{5+} - 5F_5^{4+1-}\\ &\quad +10F_5^{3+2-} - 10F_5^{2+3-} + 5F_5^{1+4-} - F_5^{5-},\end{aligned} \quad \text{(A59)}$$

$$\begin{aligned}\kappa_6^Q &= F_1^+ + F_1^- + 31F_2^{2+} - 2F_2^{1+1-} + 31F_2^{2-}\\ &\quad +90F_3^{3+} - 30F_3^{2+1-} - 30F_3^{1+2-} + 90F_3^{3-}\\ &\quad +65F_4^{4+} - 80F_4^{3+1-} + 30F_4^{2+2-} - 80F_4^{1+3-}\\ &\quad +65F_4^{4-} + 15F_5^{5+} - 45F_5^{4+1-} + 30F_5^{3+2-}\\ &\quad +30F_5^{2+3-} - 45F_5^{1+4-} + 15F_5^{5-} + F_6^{6+} - 6F_6^{5+1-}\\ &\quad +15F_6^{4+2-} - 20F_6^{3+3-} + 15F_6^{2+4-} - 6F_6^{1+5-}\\ &\quad +F_6^{6-},\end{aligned} \quad \text{(A60)}$$

$$\begin{aligned}\kappa_8^Q &= F_1^+ + F_1^- + 127F_2^{2+} - 2F_2^{1+1-} + 127F_2^{2-}\\ &\quad +966F_3^{3+} - 126F_3^{1+2-} - 126F_3^{2+1-} + 966F_3^{3-}\\ &\quad +1701F_4^{4+} - 924F_4^{3+1-} + 126F_4^{2+2-} - 924F_4^{1+3-}\\ &\quad +1701F_4^{4-} + 1050F_5^{5+} - 1470F_5^{4+1-} + 420F_5^{3+2-}\\ &\quad +420F_5^{2+3-} - 1470F_5^{1+4-} + 1050F_5^{5-} + 266F_6^{6+}\\ &\quad +756F_6^{1+5-} + 630F_6^{2+4-} - 280F_6^{3+3-} + 266F_6^{6-}\\ &\quad +630F_6^{4+2-} - 756F_6^{5+1-} + 28F_7^{7+} - 140F_7^{6+1-}\\ &\quad +252F_7^{5+2-} - 140F_7^{4+3-} - 140F_7^{3+4-} + 252F_7^{2+5-}\\ &\quad -140F_7^{1+6-} + 28F_7^{7-} + F_8^{8+} - 8F_8^{7+1-}\\ &\quad +28F_8^{6+2-} - 56F_8^{5+3-} + 70F_8^{4+4-}\\ &\quad -56F_8^{3+5-} + 28F_8^{2+6-} - 8F_8^{1+7-} + F_8^{8-},\end{aligned} \quad \text{(A61)}$$

$$\begin{aligned}\kappa_{10}^Q = \quad & F_1^+ + F_1^- + \cdots +\\ & +F_{10}^{10+} - 10F_{10}^{9+1-} + 45F_{10}^{8+2-} - 120F_{10}^{7+3-}\\ & +210F_{10}^{6+4-} - 252F_{10}^{5+5-} + 210F_{10}^{4+6-}\\ & -120F_{10}^{3+7-} + 45F_{10}^{2+8-} + 10F_{10}^{1+9-} + F_{10}^{10-},\end{aligned} \quad \text{(A62)}$$

where, for κ_{10}^Q, we omitted terms of lesser interest. We first note that cumulants κ_n^Q of order n feature a dependence on factorial cumulants of all orders $n' \leqslant n$. Also recall, once again, that factorial cumulants of order n' are nonvanishing if and only if n' or more particles are correlated in the events of interest. This implies that high order cumulants κ_n^Q can be nonvanishing based on single particle or correlated pairs even in the absence of n particle correlations. Higher-order cumulants, $n \geqslant 3$, are thus nontrivial, i.e., carry new information relative to lower orders, only if factorial cumulants of same order are nonvanishing.

Additionally, note that κ_2^Q depends on $F_2^{++} - 2F_2^{+-} + F_2^{--}$ which amounts to the integral of the two-particle balance function B_2^{+-} across the acceptance of the measurement. Similarly, one observes that κ_4^Q depends on $F_4^{4+} - 4F_4^{3+1-} + 6F_4^{2+2-} - 4F_4^{1+3-} + F_4^{4-}$ which corresponds to the average of the four-particle balance functions, $\frac{1}{2}(B_4^{+-} + B_4^{+-})$ we introduced in Sec. IV. Additional inspection of the expressions for higher (even) order net-charge cumulants κ_n^Q reveal these also contain sums of mixed charged-particle cumulants corresponding to the higher-order balance functions defined in Appendix E. As shown in that Appendix, given the integrals of multiparticle balance functions are constrained by sum rules determined by charge conservation, we conclude that the magnitude of the cumulants κ_n^Q, for even values of n, are also entirely determined by effects associated to charge conservation and the widths of the measurement acceptance. A comprehensive study of the cumulants κ^Q with beam energy and system size thus requires a detailed understanding of the evolution of the factorial cumulants F_n with beam energy and system size. Given it is likely that multiparticle balance functions of produced particles have intricate dependencies on beam energy, and in particular the growing impact of nuclear stopping at lower energy, we advocate that differential measurements of balance functions provide better insight in the impact of effects associated to the collision dynamics that may otherwise impede the studies of the properties being sought for.

APPENDIX B: DIFFERENTIAL CORRELATIONS

Differential correlation functions of n particles, herein simply termed n-cumulants, may be obtained at any order n by listing all distinct ways to "cluster" n particles into smaller subsets (i.e., clusters) to obtain n-particle densities in terms of correlated clusters of particles (and thus cumulants) of lower order $n' \leqslant n$. Cumulants of order $n \leqslant 4$ can then be written

$$C_1^{\alpha}(\vec{p}) \equiv \rho_1^{\alpha}(\vec{p}), \tag{B1}$$

$$C_2^{\alpha_1\alpha_2}(\vec{p}_1, \vec{p}_2) \equiv \rho_2^{\alpha_1\alpha_2}(\vec{p}_1, \vec{p}_2) - C_1^{\alpha_1}(\vec{p}_1)C_1^{\alpha_2}(\vec{p}_2), \tag{B2}$$

$$\begin{aligned} C_3^{\alpha_1\cdots\alpha_3}(\vec{p}_1, \dots, \vec{p}_3) \equiv\ & \rho_3^{\alpha_1\cdots\alpha_3}(\vec{p}_1, \dots, \vec{p}_3) \\ & - \sum_{(3)} C_2^{\alpha_1\alpha_2}(\vec{p}_1, \vec{p}_2)C_1^{\alpha_3}(\vec{p}_3) \\ & - C_1^{\alpha_1}(\vec{p}_1)C_1^{\alpha_2}(\vec{p}_2)C_1^{\alpha_3}(\vec{p}_3), \end{aligned} \tag{B3}$$

$$\begin{aligned} C_4^{\alpha_1\cdots\alpha_4}(\vec{p}_1, \dots, \vec{p}_4) \equiv\ & \rho_4^{\alpha_1\cdots\alpha_4}(\vec{p}_1, \dots, \vec{p}_4) \\ & - \sum_{(4)} C_3^{\alpha_1\cdots\alpha_3}(\vec{p}_1, \vec{p}_2, \vec{p}_3)C_1^{\alpha_4}(\vec{p}_4) \\ & - \sum_{(3)} C_2^{\alpha_1\alpha_2}(\vec{p}_1, \vec{p}_2)C_2^{\alpha_3\alpha_4}(\vec{p}_3, \vec{p}_4) \\ & - \sum_{(6)} C_2^{\alpha_1\alpha_2}(\vec{p}_1, \vec{p}_2)C_1^{\alpha_3}(\vec{p}_3)C_1^{\alpha_4}(\vec{p}_4) \\ & - C_1^{\alpha_1}(\vec{p}_1)C_1^{\alpha_2}(\vec{p}_2)C_1^{\alpha_3}(\vec{p}_3)C_1^{\alpha_4}(\vec{p}_4), \end{aligned} \tag{B4}$$

where the symbols ρ_n and C_n, respectively, indicate n-particle densities and cumulants at momenta $\vec{p}_1, \dots, \vec{p}_n$ while the labels $\alpha_1, \dots, \alpha_n$ are species or kinematic bins identifiers. Clearly, a formula for $C_2^{\alpha_1\alpha_2}(\vec{p}_1, \vec{p}_2)$ in terms of densities is readily obtained by substituting expressions for C_1 by single densities $\rho_1^{\alpha}(\vec{p})$. Similarly, and recursively, cumulants of order $n \geqslant 2$ can be computed based on lower order cumulants $n' < n$. Alternatively, one can also formulate a generating function G_ρ according to [60]

$$\begin{aligned} & G_\rho[\theta_1(\vec{p}_{\rm T}), \dots, \theta_m(\vec{p}_{\rm T})] \\ & \quad \equiv \int \exp\left\{\int \prod_{i=1}^{m} \theta_i(\vec{p})n_i(\vec{p})\right\} P[n(\vec{p}_{\rm T})]Dn, \end{aligned} \tag{B5}$$

where $\theta_1, \dots, \theta_m$ identify m species or types of particles, $n_i(\vec{p})$ is the density of particles of type i, and $P[n(\vec{p}_{\rm T})]$ is the probability of having particles of type i with densities $n(\vec{p}_{\rm T})$ on an event-by-event basis, and $\int P[n(\vec{p}_{\rm T})]Dn = 1$. The moments of the densities n_i are then obtained in the usual way by computing derivatives w.r.t. $\theta_i(\vec{p})$ evaluated at $\theta_i(\vec{p}) = 0$:

$$\begin{aligned} & \langle n_{\alpha_1}(\vec{p}_1) \cdots n_{\alpha_n}(\vec{p}_n)\rangle \\ & \quad \equiv \int n_{\alpha_1}(\vec{p}_1) \cdots n_{\alpha_n}(\vec{p}_n)P[n(\vec{p}_{\rm T})]Dn \\ & \quad = \partial_{\theta_1(\vec{p}_1)} \cdots \partial_{\theta_n(\vec{p}_n)}G_\rho(\theta_1(\vec{p}_1), \dots, \theta_n(\vec{p}_1)\big|_{\theta_i=0}. \end{aligned} \tag{B6}$$

Of interest also are factorial moments and factorial cumulants corresponding to these moments. These are obtained by introducing continuous variables $s_i(\vec{p}_i) = \exp[\theta_i(\vec{p}_i)]$ and expressing G_ρ as a function of these variables

$$\begin{aligned} & G_\rho[s_1(\vec{p}_{\rm T}), \dots, s_m(\vec{p}_{\rm T})] \\ & \quad \equiv \int \exp\left\{\int \prod_{i=1}^{m} n_i(\vec{p}) \ln s_i(\vec{p})\right\} P[n(\vec{p}_{\rm T})]Dn. \end{aligned} \tag{B7}$$

Factorial cumulants, corresponding to connected parts (i.e., correlated) of the densities, are then obtained by taking functional derivatives of $\ln G_\rho[s_1(\vec{p}_{\rm T}), \dots, s_m(\vec{p}_{\rm T})]$ w.r.t. $s_i(\vec{p}_i)$. Structurally, expressions of n-cumulants in terms of densities ρ_n have the same dependence as cumulants F_n on factorial moments f_n, one can use the relations ((A6)–(A15)) and substitute densities ρ_k to moments f_k, with $k = 0, \dots, n$ to obtain the connected correlation functions of interest.

Additionally, note, as already stated in Sec. III A, that integrals of densities $\rho_n^{\alpha_1\cdots\alpha_n}(\vec{p}_1, \dots, \vec{p}_n)$ yield factorial moments

$$\begin{aligned} f_n^{\alpha_1\cdots\alpha_n} & \equiv \int \cdots \int_\Omega \rho_n^{\alpha_1\cdots\alpha_n}(\vec{p}_1, \dots, \vec{p}_n)d\vec{p}_1 \cdots d\vec{p}_n \\ & = \langle N(N-1) \cdots (N-n+1)\rangle, \end{aligned} \tag{B8}$$

whereas integrals of correlation functions $C_n^{\alpha_1\cdots\alpha_n}(\vec{p}_1, \dots, \vec{p}_n)$ yield factorial cumulants

$$F_n^{\alpha_1\cdots\alpha_n} \equiv \int \cdots \int_\Omega C_n^{\alpha_1\cdots\alpha_n}(\vec{p}_1, \dots, \vec{p}_n)d\vec{p}_1 \cdots d\vec{p}_n. \tag{B9}$$

There is indeed a one-to-one relation between densities $\rho_k(\vec{p}_1, \dots, \vec{p}_k)$ and factorial moments f_k as well as between functional cumulants (or cumulant densities) $C_k(\vec{p}_1, \dots, \vec{p}_k)$ and factorial cumulants F_k.

APPENDIX C: EVENT ENSEMBLE AVERAGE OF DEVIATES

Expressing event ensemble average of deviates $\langle\langle \Delta q_1 \cdots \Delta q_n \rangle\rangle$ in terms of sums averages $\langle\langle q_1 \cdots q_m \rangle\rangle$, $m \leqslant n$, is a trivial but somewhat tedious process. We created simple scripts to compute these for arbitrary orders n and show below expressions up to order 6:

$$\langle\langle \Delta q_1 \Delta q_2 \rangle\rangle = \langle\langle q_1 q_2 \rangle\rangle - \langle\langle q \rangle\rangle^2, \tag{C1}$$

$$\langle\langle \Delta q_1 \Delta q_2 \Delta q_3 \rangle\rangle = \langle\langle q_1 q_2 q_3 \rangle\rangle - 3\langle\langle q \rangle\rangle \langle\langle q_1 q_2 \rangle\rangle + 2\langle\langle q \rangle\rangle^3, \tag{C2}$$

$$\begin{aligned}\langle\langle \Delta q_1 \cdots \Delta q_4 \rangle\rangle = {} & \langle\langle q_1 q_2 q_3 q_4 \rangle\rangle - 4\langle\langle q \rangle\rangle \langle\langle q_1 q_2 q_3 \rangle\rangle \\ & +6\langle\langle q \rangle\rangle^2 \langle\langle q_1 q_2 \rangle\rangle - 3\langle\langle q \rangle\rangle^4,\end{aligned} \tag{C3}$$

$$\begin{aligned}\langle\langle \Delta q_1 \cdots \Delta q_5 \rangle\rangle = {} & \langle\langle q_1 \cdots q_5 \rangle\rangle - 5\langle\langle q \rangle\rangle \langle\langle q_1 q_2 q_3 q_4 \rangle\rangle \\ & +10\langle\langle q \rangle\rangle^2 \langle\langle q_1 q_2 q_3 \rangle\rangle \\ & -10\langle\langle q \rangle\rangle^3 \langle\langle q_1 q_2 \rangle\rangle + 4\langle\langle q \rangle\rangle^5,\end{aligned} \tag{C4}$$

$$\begin{aligned}\langle\langle \Delta q_1 \cdots \Delta q_6 \rangle\rangle = {} & +\langle\langle q_1 \cdots q_6 \rangle\rangle - 6\langle\langle q \rangle\rangle \langle\langle q_1 \cdots q_5 \rangle\rangle \\ & +15\langle\langle q \rangle\rangle^2 \langle\langle q_1 q_2 q_3 q_4 \rangle\rangle - 20\langle\langle q \rangle\rangle^3 \langle\langle q_1 q_2 q_3 \rangle\rangle \\ & +15\langle\langle q \rangle\rangle^4 \langle\langle q_1 q_2 \rangle\rangle - 5\langle\langle q \rangle\rangle^6.\end{aligned} \tag{C5}$$

Inspection of the above expressions reveal a simple pattern based on binomial coefficients as follows:

$$\langle\langle \Delta q_1 \cdots \Delta q_n \rangle\rangle = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \langle\langle q \rangle\rangle^{n-k} \langle\langle q_1 \cdots q_k \rangle\rangle, \tag{C6}$$

where we define $\langle\langle q_1 \cdots q_k \rangle\rangle \equiv \langle\langle q \rangle\rangle$ for $k = 1$ and $\langle\langle q_1 \cdots q_k \rangle\rangle \equiv 1$ for $k = 0$.

Moments of cross deviates of variable q and p are computed in a similar fashion. One gets at lowest orders

$$\langle\langle \Delta q_1 \Delta p_1 \rangle\rangle = \langle\langle q_1 p_1 \rangle\rangle - \langle\langle q \rangle\rangle \langle\langle p \rangle\rangle, \tag{C7}$$

$$\langle\langle \Delta q_1 \Delta q_2 \Delta p_1 \rangle\rangle = \langle\langle q_1 q_2 p_1 \rangle\rangle - \langle\langle p \rangle\rangle \langle\langle q_1 q_2 \rangle\rangle - 2\langle\langle q \rangle\rangle \langle\langle q_1 p_1 \rangle\rangle + 2\langle\langle p \rangle\rangle \langle\langle q \rangle\rangle^2, \tag{C8}$$

$$\langle\langle \Delta q_1 \Delta q_2 \Delta q_3 \Delta p_1 \rangle\rangle = \langle\langle q_1 q_2 q_3 p_1 \rangle\rangle - \langle\langle p \rangle\rangle \langle\langle q_1 q_2 q_3 \rangle\rangle - 3\langle\langle q \rangle\rangle \langle\langle q_1 q_2 p_1 \rangle\rangle + 3\langle\langle p \rangle\rangle \langle\langle q \rangle\rangle \langle\langle q_1 q_2 \rangle\rangle + 3\langle\langle q \rangle\rangle^2 \langle\langle q_1 p_1 \rangle\rangle - 3\langle\langle p \rangle\rangle \langle\langle q \rangle\rangle^3, \tag{C9}$$

$$\begin{aligned}\langle\langle \Delta q_1 \cdots \Delta q_4 \Delta p_1 \rangle\rangle = {} & \langle\langle q_1 \cdots q_4 p_1 \rangle\rangle - \langle\langle p \rangle\rangle \langle\langle q_1 \cdots q_4 \rangle\rangle - 4\langle\langle q \rangle\rangle \langle\langle q_1 q_2 q_3 p_1 \rangle\rangle + 4\langle\langle p \rangle\rangle \langle\langle q \rangle\rangle \langle\langle q_1 q_2 q_3 \rangle\rangle \\ & +6\langle\langle q \rangle\rangle^2 \langle\langle q_1 q_2 p_1 \rangle\rangle - 6\langle\langle p \rangle\rangle \langle\langle q \rangle\rangle^2 \langle\langle q_1 q_2 \rangle\rangle - 4\langle\langle q \rangle\rangle^3 \langle\langle q_1 p_1 \rangle\rangle + 4\langle\langle p \rangle\rangle \langle\langle q \rangle\rangle^4,\end{aligned} \tag{C10}$$

$$\begin{aligned}\langle\langle \Delta q_1 \Delta q_2 \Delta p_1 \Delta p_1 \rangle\rangle = {} & \langle\langle q_1 q_2 p_1 p_2 \rangle\rangle - 2\langle\langle p \rangle\rangle \langle\langle q_1 q_2 p_1 \rangle\rangle + \langle\langle p \rangle\rangle^2 \langle\langle q_1 q_2 \rangle\rangle - 2\langle\langle q \rangle\rangle \langle\langle q_1 p_1 p_2 \rangle\rangle \\ & +4\langle\langle p \rangle\rangle \langle\langle q \rangle\rangle \langle\langle q_1 p_1 \rangle\rangle - 3\langle\langle p \rangle\rangle^2 \langle\langle q \rangle\rangle^2 + \langle\langle q \rangle\rangle^2 \langle\langle p_1 p_2 \rangle\rangle,\end{aligned} \tag{C11}$$

$$\begin{aligned}\langle\langle \Delta q_1 \cdots \Delta q_3 \Delta p_1 \Delta p_2 \rangle\rangle = {} & \langle\langle q_1 q_2 q_3 p_1 p_2 \rangle\rangle - 2\langle\langle p \rangle\rangle \langle\langle q_1 q_2 q_3 p_1 \rangle\rangle + \langle\langle p \rangle\rangle^2 \langle\langle q_1 q_2 q_3 \rangle\rangle - 3\langle\langle q \rangle\rangle \langle\langle q_1 q_2 p_1 p_2 \rangle\rangle \\ & +6\langle\langle p \rangle\rangle \langle\langle q \rangle\rangle \langle\langle q_1 q_2 p_1 \rangle\rangle - 3\langle\langle p \rangle\rangle^2 \langle\langle q \rangle\rangle \langle\langle q_1 q_2 \rangle\rangle + 3\langle\langle q \rangle\rangle^2 \langle\langle q_1 p_1 p_2 \rangle\rangle \\ & -6\langle\langle p \rangle\rangle \langle\langle q \rangle\rangle^2 \langle\langle q_1 p_1 \rangle\rangle + 4\langle\langle p \rangle\rangle^2 \langle\langle q \rangle\rangle^3 - \langle\langle q \rangle\rangle^3 \langle\langle p_1 p_2 \rangle\rangle.\end{aligned} \tag{C12}$$

Once again, close inspection of the above expressions reveal straightforward patterns and one obtains a generic formula in terms of two binomial coefficients as follows:

$$\langle\langle \Delta q_1 \cdots \Delta q_n \Delta p_1 \cdots \Delta p_m \rangle\rangle = \sum_{k=0}^{n} \sum_{l=0}^{m} (-1)^{n-k} (-1)^{m-l} \binom{n}{k} \binom{m}{l} \langle\langle q \rangle\rangle^{n-k} \langle\langle p \rangle\rangle^{m-l} \langle\langle q_1 \cdots q_k \rangle\rangle, \langle\langle p_1 \cdots p_l \rangle\rangle. \tag{C13}$$

Similarly, products of Δqs, Δps, and Δrs yield at lowest orders

$$\langle\langle \Delta q_i \Delta p_j \Delta r_k \rangle\rangle = \langle\langle q_i p_j r_k \rangle\rangle - \langle\langle p \rangle\rangle \langle\langle q_i r_j \rangle\rangle - \langle\langle q \rangle\rangle \langle\langle p_i r_j \rangle\rangle - \langle\langle r \rangle\rangle \langle\langle q_i p_j \rangle\rangle + 2\langle\langle q \rangle\rangle \langle\langle p \rangle\rangle \langle\langle r \rangle\rangle, \tag{C14}$$

$$\begin{aligned}\langle\langle \Delta q_i \Delta q_j \Delta p_k \Delta r_l \rangle\rangle = {} & \langle\langle q_i q_j p_k r_l \rangle\rangle - 2\langle\langle q \rangle\rangle \langle\langle q_i p_j r_k \rangle\rangle + \langle\langle q \rangle\rangle^2 \langle\langle p_i r_j \rangle\rangle - \langle\langle p \rangle\rangle \langle\langle q_i q_j r_k \rangle\rangle + 2\langle\langle q \rangle\rangle \langle\langle p \rangle\rangle \langle\langle q_i r_j \rangle\rangle \\ & -3\langle\langle q \rangle\rangle^2 \langle\langle p \rangle\rangle \langle\langle r \rangle\rangle - \langle\langle r \rangle\rangle \langle\langle q_i q_j p_k \rangle\rangle + 2\langle\langle q \rangle\rangle \langle\langle r \rangle\rangle \langle\langle q_i p_j \rangle\rangle + \langle\langle p \rangle\rangle \langle\langle r \rangle\rangle \langle\langle q_i q_j \rangle\rangle,\end{aligned} \tag{C15}$$

$$\begin{aligned}\langle\langle \Delta q_i \Delta p_j \Delta r_k \Delta s_l \rangle\rangle = {} & \langle\langle q_i p_j r_k s_l \rangle\rangle - \langle\langle p \rangle\rangle \langle\langle q_i r_j s_k \rangle\rangle - \langle\langle q \rangle\rangle \langle\langle p_i r_j s_k \rangle\rangle - \langle\langle p \rangle\rangle \langle\langle q_i q_j r_k \rangle\rangle \\ & +\langle\langle q \rangle\rangle \langle\langle p \rangle\rangle \langle\langle r_i s_j \rangle\rangle - \langle\langle r \rangle\rangle \langle\langle q_i p_j s_l \rangle\rangle + \langle\langle p \rangle\rangle \langle\langle r \rangle\rangle \langle\langle q_i s_j \rangle\rangle + \langle\langle s \rangle\rangle \langle\langle q_i p_j r_k \rangle\rangle + \langle\langle p \rangle\rangle \langle\langle s \rangle\rangle \langle\langle q_i r_j \rangle\rangle \\ & +\langle\langle q \rangle\rangle \langle\langle s \rangle\rangle \langle\langle p_i r_j \rangle\rangle + \langle\langle r \rangle\rangle \langle\langle s \rangle\rangle \langle\langle q_i p_j \rangle\rangle + 3\langle\langle q \rangle\rangle \langle\langle p \rangle\rangle \langle\langle r \rangle\rangle \langle\langle s \rangle\rangle.\end{aligned} \tag{C16}$$

In this case, inspection of the above expressions reveals a formula involving three binomial coefficients

$$\begin{aligned}\langle\langle \Delta q_1 \cdots \Delta q_n \Delta p_1 \cdots \Delta p_m \Delta r_1 \cdots \Delta r_o \rangle\rangle = {} & \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{l} (-1)^{n-i} (-1)^{m-k} (-1)^{o-k} \binom{n}{i} \binom{m}{j} \binom{o}{k} \\ & \times \langle\langle q \rangle\rangle^{n-i} \langle\langle p \rangle\rangle^{m-j} \langle\langle p \rangle\rangle^{o-k} \times \langle\langle q_1 \cdots q_i \rangle\rangle \langle\langle p_1 \cdots p_j \rangle\rangle \langle\langle r_1 \cdots r_l \rangle\rangle.\end{aligned} \tag{C17}$$

Similar formula are readily obtained for four or more variables.

APPENDIX D: COMPUTATION OF CORRELATORS $\langle\langle q_1^{n_1} q_2^{n_2} \cdots q_m^{n_m} \rangle\rangle$

In Sec. III C, we derived expressions for $\langle\langle q_1^{n_1} q_2^{n_2} \rangle\rangle$ and $\langle\langle q_1^{n_1} q_2^{n_2} q_3^{n_3} \rangle\rangle$ in terms of functions of the event-wise sums Q_n defined by Eq. (35). The same approach can be used to define higher moments of the form $\langle\langle q_1^{n_1} \cdots q_m^{n_m} \rangle\rangle$, for arbitrary values of m. First note that the products of more than three Qs can be computed by straightforward expansion of the sums corresponding to each variable Q_n. At order m, one obtains expressions of the form

$$\begin{aligned} Q_{n_1} Q_{n_2} \cdots Q_{n_m} = & \sum_{i=1}^{N} q_i^{n_1+n_2+\cdots+n_m} + \sum_{\text{perms}} \sum_{i_1 \neq i_2=1}^{N} q_{i_1}^{n_1+n_2+\cdots+n_{m-1}} q_{i_2}^{n_m} \\ & + \sum_{\text{perms}} \sum_{i_1 \neq i_2=1}^{N} q_{i_1}^{n_1+n_2+\cdots+n_{m-2}} q_{i_2}^{n_{m-1}+n_m} + \sum_{\text{perms}} \sum_{i_1 \neq i_2 \neq i_3=1}^{N} q_{i_1}^{n_1+n_2+\cdots+n_{m-2}} q_{i_2}^{n_{m-1}} q_{i_3}^{n_m} \\ & + \sum_{\text{perms}} \sum_{i_1 \neq i_2=1}^{N} q_{i_1}^{n_1+n_2+\cdots+n_{m-3}} q_{i_2}^{n_{m-2}+n_{m-1}+n_m} + \sum_{\text{perms}} \sum_{i_1 \neq i_2 \neq i_3=1}^{N} q_{i_1}^{n_1+n_2+\cdots+n_{m-3}} q_{i_2}^{n_{m-2}+n_{m-1}} q_{i_3}^{n_m} \\ & + \sum_{\text{perms}} \sum_{(i_1,\cdots,i_m)=1}^{N} q_{i_1}^{n_1+n_2+\cdots+n_{m-3}} q_{i_2}^{n_{m-2}} q_{i_3}^{n_{m-1}} q_{i_4}^{n_m} + \cdots \\ & + \sum_{\text{perms}} \sum_{(i_1,\cdots,i_m)=1}^{N} q_{i_1}^{n_1+n_2} q_{i_2}^{n_3} \cdots q_{i_{m-2}}^{n_{m-1}} q_{i_{m-1}}^{n_m} + \sum_{(i_1,\cdots,i_m)=1}^{N} q_{i_1}^{n_1} q_{i_2}^{n_2} \cdots q_{i_m}^{n_m}, \end{aligned} \tag{D1}$$

where the notation $\sum_{\text{perms}}$ indicates a sum over all ordered permutations of the exponents n_i, $i = 1, \ldots, m$, whereas $\sum_{(i_1,\cdots,i_m)=1}^{N}$ represents a sum over all distinct m-tuples of values of the indices $i_1, i_2, \ldots, i_m$, i.e., $i_1 \neq i_2 \neq \cdots \neq i_m$. When an ensemble average is computed, each term of the above expression yields terms of the form $\sum_{\text{perms}} \langle N(N-1)\cdots(N-m+1)\rangle \langle\langle q_1^{n_1} \cdots q_m^{n_m} \rangle\rangle$. Clearly, terms of the form $\langle\langle q_1^{n_1} \cdots q_m^{n_m} \rangle\rangle$ can be computed iteratively based on sums of $\langle\langle q_1^{n_1} \cdots q_p^{n_p} \rangle\rangle$, with $p \leqslant m$. It is thus nominally possible to obtain expressions for $\langle\langle q_1^{n_1} \cdots q_m^{n_m} \rangle\rangle$ at any order m based on sums of lower order terms. In practice, one finds that the number of terms to be considered grows very rapidly as m increases. We thus opted to write scripts (in C + +) based on the TString root class [61]. The computation proceeds in four basic steps. In the first step, at given order m, one finds all the permutations of exponents n_1, n_2, etc., that yield terms that are products of two factors $q_1^a q_2^b$, three factors $q_1^a q_2^b q_3^c$, and so on. Once these permutations are listed, one proceeds to generate these terms by recursively calling functions that generate them from lower order products. This second step is then followed by an aggregation and simplification step in which identical terms are regrouped and rearranged to produce a latex output. Low orders were checked against the results of manual calculations and low order $m \leqslant 4$ expressions published elsewhere for $n_1 = n_2 = \cdots = n_m = 1$ [21,22]. As an example, we show below the expression for arbitrary integer values $n_1, \cdots, n_m$ obtained for $m = 4$:

$$\begin{aligned} \langle N(N-1)\cdots(N-3)\rangle \langle\langle q_1^{n_1} q_2^{n_2} q_3^{n_3} q_4^{n_4} \rangle\rangle = & -6\langle\langle Q_{n1+n2+n3+n4} \rangle\rangle + 2\langle\langle Q_{n1+n2+n3} Q_{n4} \rangle\rangle + 2\langle\langle Q_{n1+n2+n4} Q_{n3} \rangle\rangle \\ & + 2\langle\langle Q_{n1+n3+n4} Q_{n2} \rangle\rangle + 2\langle\langle Q_{n2+n3+n4} Q_{n1} \rangle\rangle + \langle\langle Q_{n1+n2} Q_{n3+n4} \rangle\rangle \\ & + \langle\langle Q_{n1+n3} Q_{n2+n4} \rangle\rangle + \langle\langle Q_{n1+n4} Q_{n2+n3} \rangle\rangle - \langle\langle Q_{n1+n2} Q_{n3} Q_{n4} \rangle\rangle \\ & - \langle\langle Q_{n1+n3} Q_{n2} Q_{n4} \rangle\rangle - \langle\langle Q_{n1+n4} Q_{n2} Q_{n3} \rangle\rangle - \langle\langle Q_{n2+n3} Q_{n1} Q_{n4} \rangle\rangle \\ & - \langle\langle Q_{n2+n4} Q_{n1} Q_{n3} \rangle\rangle - \langle\langle Q_{n3+n4} Q_{n1} Q_{n2} \rangle\rangle + \langle\langle Q_{n1} Q_{n2} Q_{n3} Q_{n4} \rangle\rangle \\ = & -6\langle\langle Q_{n_1+n_2+n_3+n_4} \rangle\rangle + 2\sum_{(4)} \langle\langle Q_{n_1+n_2+n_3} Q_{n_4} \rangle\rangle \\ & + \sum_{(3)} \langle\langle Q_{n_1+n_2} Q_{n_3+n_4} \rangle\rangle - \sum_{(6)} \langle\langle Q_{n_1+n_2} Q_{n_3} Q_{n_4} \rangle\rangle + \langle\langle Q_{n_1} Q_{n_2} Q_{n_3} Q_{n_4} \rangle\rangle, \end{aligned} \tag{D2}$$

where the notation $\sum_{(n)}$ indicate sums over all ordered permutations of the indices n_1, n_2, n_3, and n_4.[2]

The computation of event ensemble averages of deviates, Eqs. (C1)–(C6), require expressions for products of the form $\langle\langle q_1 \cdots q_m \rangle\rangle$. These are obtained by setting exponents $n_1 = n_2 = \cdots n_m = 1$ in the generic expressions $\langle\langle q_1^{n_1} \cdots q_n^{n_m} \rangle\rangle$. Although somewhat simpler, these remain fastidious to calculate by hand. We have extended our scripts to automatically set exponents n_i, $i = 1, \ldots, m$ to unity programmatically. Computation of the first eight orders yields

$$\langle N(N-1)\rangle \langle\langle q_1 \cdots q_2 \rangle\rangle = -\langle\langle Q_2 \rangle\rangle + \langle\langle Q_1^2 \rangle\rangle, \tag{D3}$$

[2]The code used to generate these and other expressions reported in this paper is available on Github [62].

$$\langle N(N-1)(N-2)\rangle\langle\langle q_1\cdots q_3\rangle\rangle = 2\langle\langle Q_3\rangle\rangle - 3\langle\langle Q_2Q_1\rangle\rangle + \langle\langle Q_1^3\rangle\rangle, \tag{D4}$$

$$\langle N(N-1)\cdots(N-3)\rangle\langle\langle q_1\cdots q_4\rangle\rangle = -6\langle\langle Q_4\rangle\rangle + 8\langle\langle Q_3Q_1\rangle\rangle + 3\langle\langle Q_2^2\rangle\rangle - 6\langle\langle Q_2Q_1^2\rangle\rangle + \langle\langle Q_1^4\rangle\rangle, \tag{D5}$$

$$\begin{aligned}\langle N(N-1)\cdots(N-4)\rangle\langle\langle q_1\cdots q_5\rangle\rangle = {} & 24\langle\langle Q_5\rangle\rangle - 30\langle\langle Q_4Q_1\rangle\rangle - 20\langle\langle Q_2Q_3\rangle\rangle + 20\langle\langle Q_3Q_1^2\rangle\rangle + 15\langle\langle Q_2^2Q_1\rangle\rangle \\ & -10\langle\langle Q_2Q_1^3\rangle\rangle + \langle\langle Q_1^5\rangle\rangle,\end{aligned} \tag{D6}$$

$$\begin{aligned}\langle N(N-1)\cdots(N-5)\rangle\langle\langle q_1\cdots q_6\rangle\rangle = {} & -120\langle\langle Q_6\rangle\rangle + 144\langle\langle Q_5Q_1\rangle\rangle + 90\langle\langle Q_4Q_2\rangle\rangle - 90\langle\langle Q_4Q_1^2\rangle\rangle + 40\langle\langle Q_3^2\rangle\rangle \\ & -120\langle\langle Q_3Q_2Q_1\rangle\rangle + 40\langle\langle Q_3Q_1^3\rangle\rangle - 15\langle\langle Q_2^3\rangle\rangle + 45\langle\langle Q_2^2Q_1^2\rangle\rangle - 15\langle\langle Q_2Q_1^4\rangle\rangle \\ & +\langle\langle Q_1^6\rangle\rangle,\end{aligned} \tag{D7}$$

$$\begin{aligned}\langle N(N-1)\cdots(N-6)\rangle\langle\langle q_1\cdots q_7\rangle\rangle = {} & +530\langle\langle Q_7\rangle\rangle - 850\langle\langle Q_6Q_1\rangle\rangle - 294\langle\langle Q_5Q_2\rangle\rangle + 504\langle\langle Q_5Q_1^2\rangle\rangle \\ & -335\langle\langle Q_4Q_3\rangle\rangle + 630\langle\langle Q_4Q_2Q_1\rangle\rangle - 210\langle\langle Q_4Q_1^3\rangle\rangle \\ & +290\langle\langle Q_3^2Q_1\rangle\rangle + 105\langle\langle Q_3Q_2^2\rangle\rangle - 420\langle\langle Q_3Q_2Q_1^2\rangle\rangle \\ & +70\langle\langle Q_3Q_1^4\rangle\rangle - 105\langle\langle Q_2^3Q_1\rangle\rangle + 105\langle\langle Q_2^2Q_1^3\rangle\rangle \\ & -21\langle\langle Q_2Q_1^5\rangle\rangle + \langle\langle Q_1^7\rangle\rangle,\end{aligned} \tag{D8}$$

$$\begin{aligned}\langle N(N-1)\cdots(N-7)\rangle\langle\langle q_1\cdots q_8\rangle\rangle = {} & -4760\langle\langle Q_8\rangle\rangle + 7720\langle\langle Q_7Q_1\rangle\rangle + 2450\langle\langle Q_6Q_2\rangle\rangle \\ & -3430\langle\langle Q_6Q_1^2\rangle\rangle + 3528\langle\langle Q_5Q_3\rangle\rangle - 5712\langle\langle Q_5Q_2Q_1\rangle\rangle \\ & +1344\langle\langle Q_5Q_1^3\rangle\rangle + 770\langle\langle Q_4^2\rangle\rangle - 4480\langle\langle Q_4Q_3Q_1\rangle\rangle \\ & -630\langle\langle Q_4Q_2^2\rangle\rangle + 2520\langle\langle Q_4Q_2Q_1^2\rangle\rangle - 420\langle\langle Q_4Q_1^4\rangle\rangle \\ & -1470\langle\langle Q_3^2Q_2\rangle\rangle + 1190\langle\langle Q_3^2Q_1^2\rangle\rangle + 2520\langle\langle Q_3Q_2^2Q_1\rangle\rangle \\ & -1120\langle\langle Q_3Q_2Q_1^3\rangle\rangle + 112\langle\langle Q_3Q_1^5\rangle\rangle + 105\langle\langle Q_2^4\rangle\rangle \\ & -420\langle\langle Q_2^3Q_1^2\rangle\rangle + 210\langle\langle Q_2^2Q_1^4\rangle\rangle - 28\langle\langle Q_2Q_1^6\rangle\rangle + \langle\langle Q_1^8\rangle\rangle.\end{aligned} \tag{D9}$$

Formula for the ensemble average of deviates of the form $\langle\langle \Delta q_1 \cdots \Delta q_m\rangle\rangle$, shown in Eqs. (C1)–(C6), are obtained by substitution of the expressions for $\langle\langle q_1 \cdots q_m\rangle\rangle$ listed above. The three lowest orders are

$$\langle\langle \Delta q_1\Delta q_2\rangle\rangle = \frac{\langle\langle Q_1^2\rangle\rangle - \langle\langle Q_2\rangle\rangle}{\langle N(N-1)\rangle} - \frac{\langle\langle Q_1\rangle\rangle^2}{\langle N\rangle^2}, \tag{D10}$$

$$\langle\langle \Delta q_1\Delta q_2\Delta q_3\rangle\rangle = \frac{\langle\langle Q_1^3\rangle\rangle - 3\langle\langle Q_2Q_1\rangle\rangle + 2\langle\langle Q_3\rangle\rangle}{\langle N(N-1)(N-2)\rangle} - 3\frac{\langle\langle Q_1\rangle\rangle}{\langle N\rangle}\frac{\left(\langle\langle Q_1^2\rangle\rangle - \langle\langle Q_2\rangle\rangle\right)}{\langle N(N-1)\rangle} + 2\frac{\langle\langle Q_1\rangle\rangle^3}{\langle N\rangle^3}, \tag{D11}$$

$$\begin{aligned}\langle\langle \Delta q_1\Delta q_2\Delta q_3\Delta q_4\rangle\rangle = {} & \frac{\langle\langle Q_1^4\rangle\rangle + 3\langle\langle Q_2^2\rangle\rangle - 6\langle\langle Q_2Q_1^2\rangle\rangle + 8\langle\langle Q_3Q_1\rangle\rangle - 6\langle\langle Q_4\rangle\rangle}{\langle N(N-1)(N-2)(N-3)\rangle} \\ & -4\frac{\langle\langle Q_1\rangle\rangle}{\langle N\rangle}\frac{\left(\langle\langle Q_1^3\rangle\rangle - 3\langle\langle Q_2Q_1\rangle\rangle + 2\langle\langle Q_3\rangle\rangle\right)}{\langle N(N-1)(N-2)\rangle} + 6\frac{\langle\langle Q_1\rangle\rangle^2}{\langle N\rangle^2}\frac{\left(\langle\langle Q_1^2\rangle\rangle - \langle\langle Q_2\rangle\rangle\right)}{\langle N(N-1)\rangle} - 3\frac{\langle\langle Q_1\rangle\rangle^4}{\langle N\rangle^4}.\end{aligned} \tag{D12}$$

The above expressions of event ensemble averages of products of deviates Δq_i involve inclusive averaging, i.e., computation of the average of products and powers of Qs separately. These are then divided by averages of the multiplicity $\langle N\rangle$ and average numbers of n-tuplets $\langle N(N-1)\ldots(N-n+1)\rangle$. One can readily switch to event-wise averaging, corresponding to calculations of the products and powers on an event-by-event basis by "moving" the double brackets to include the divisions by the number of n-tuples. For instance, the lowest two orders may be written [21]

$$\langle\langle \Delta q_1\Delta q_2\rangle\rangle = \left\langle\left\langle \frac{Q_1^2 - Q_2}{N(N-1)}\right\rangle\right\rangle - \left\langle\left\langle \frac{Q_1}{N}\right\rangle\right\rangle^2, \tag{D13}$$

$$\langle\langle \Delta q_1\Delta q_2\Delta q_3\rangle\rangle = \left\langle\left\langle \frac{Q_1^3 - 3Q_2Q_1 + 2Q_3}{N(N-1)(N-2)}\right\rangle\right\rangle - 3\left\langle\left\langle \frac{Q_1}{N}\right\rangle\right\rangle\left\langle\left\langle \frac{(Q_1^2 - Q_2)}{N(N-1)}\right\rangle\right\rangle + 2\left\langle\left\langle \frac{Q_1}{N}\right\rangle\right\rangle^3, \tag{D14}$$

where the notation $\langle\langle R\rangle\rangle$ denote ensemble averaging of ratios, R, of functions of Qs, calculating event-by-event, and the number of n-tuples formed by the N particles of a given event.

The computation of ensemble averages of mixed moment deviates based on event-wise sums of variables q_i, p_i, r_i, s_i, t_i, etc., proceeds in a similar fashion. One first defines event-wise sums Q_n, P_n, R_n, S_n, T_n according to

$$Q_n = \sum_{i=1}^N q_i^n, \quad P_n = \sum_{i=1}^N p_i^n, \quad R_n = \sum_{i=1}^N r_i^n, \quad S_n = \sum_{i=1}^N s_i^n, \quad T_n = \sum_{i=1}^N t_i^n, \quad \text{etc.} \tag{D15}$$

One next lists all required mixed products of these sums and finally proceed to evaluate their event ensemble averages.

We limit the discussion to three variables, q_i, p_j, r_k, corresponding to three distinct kinematic bins or species, but the technique is readily applicable to an arbitrary number of such variables. We thus seek to express cross moments of interest, $\langle\langle q_1 \cdots q_{m_1} p_1 \cdots p_{m_2} r_1 \cdots r_{m_3} \rangle\rangle$, in terms of ensemble averages of products of Q_n, P_n, and R_n, as appropriate. Given the moments defined in Eqs. ((C7)–(C17)), one expects to need ensemble averages of the form $\langle\langle Q_n\rangle\rangle$, $\langle\langle Q_nQ_m\rangle\rangle$, $\langle\langle Q_nQ_mQ_o\rangle\rangle$ already computed in Sec. III C (and in this Appendix) as well as cross moments of the form $\langle\langle Q_nP_m\rangle\rangle$, $\langle\langle Q_nQ_mP_o\rangle\rangle$, $\langle\langle Q_nQ_mQ_oP_p\rangle\rangle$, $\langle\langle Q_nQ_mP_oP_p\rangle\rangle$, etc., that we now proceed to compute. All other cross moments can be obtained by appropriate permutations of variable names and indices. Let N_q, N_p, and N_r represent the number of particles in bins corresponding to q, p, and r, respectively. Proceeding as in Sec. III C, the lowest order moments are found to be

$$\langle\langle Q_nP_m\rangle\rangle = \langle N_qN_p\rangle\langle\langle q_i^n p_j^m\rangle\rangle, \tag{D16}$$

$$\langle\langle Q_nQ_mPo\rangle\rangle = \langle N_qN_p\rangle\langle\langle q_i^{n+m} p_j^o\rangle\rangle + \langle N_q(N_q-1)N_p\rangle\langle\langle q_i^n q_j^m p_k^o\rangle\rangle, \tag{D17}$$

$$\begin{aligned}\langle\langle Q_nQ_mQ_oP_p\rangle\rangle = {}& \langle N_qN_p\rangle\langle\langle q_i^{n+m+o} p_j^p\rangle\rangle + \langle N_q(N_q-1)N_p\rangle\langle\langle q_i^{n+m} q_j^o p_k^p\rangle\rangle + \langle N_q(N_q-1)N_p\rangle\langle\langle q_i^{n+o} q_j^m p_k^p\rangle\rangle \\ & \langle N_q(N_q-1)N_p\rangle\langle\langle q_i^n q_j^{m+o} p_k^p\rangle\rangle + \langle N_q(N_q-1)(N_q-2)N_p\rangle\langle\langle q_i^n q_j^m q_k^o p_l^p\rangle\rangle,\end{aligned} \tag{D18}$$

$$\begin{aligned}\langle\langle Q_nQ_mP_oP_p\rangle\rangle = {}& \langle N_qN_p\rangle\langle\langle q_i^{n+m} p_j^{o+p}\rangle\rangle + \langle N_qN_p(N_p-1)\rangle\langle\langle q_i^{n+m} p_j^o p_k^p\rangle\rangle + \langle N_q(N_q-1)N_p\langle\langle q_i^n q_j^m p_k^{o+p}\rangle\rangle \\ & + \langle N_q(N_q-1)N_p(N_p-1\langle\langle q_i^n q_j^m p_k^o p_l^p\rangle\rangle,\end{aligned} \tag{D19}$$

$$\langle\langle Q_nP_mR_o\rangle\rangle = \langle N_qN_pN_r\rangle\langle\langle q_i^n p_j^m r_k^o\rangle\rangle, \tag{D20}$$

$$\langle\langle Q_nQ_mP_oR_p\rangle\rangle = \langle N_qN_pN_r\rangle\langle\langle q_i^{n+m} p_j^o r_l^p\rangle\rangle + \langle N_q(N_q-1)N_pN_r\rangle\langle\langle q_i^n q_j^m p_k^o r_l^p\rangle\rangle. \tag{D21}$$

Based on the above expressions, one can iteratively obtain expressions for the moments $\langle\langle q_i^n q_j^m p_k^o r_l^p\rangle\rangle$ in terms of moments of products of Qs, Ps, and Rs. Given the sum over q and p factorize, the moments become simple combinations of Qs and Ps, and one gets at lowest orders

$$\langle N_qN_p\rangle\langle\langle q_1^n p_1^m\rangle\rangle = \langle\langle Q_nP_m\rangle\rangle, \tag{D22}$$

$$\langle N_q(N_q-1)N_p\rangle\langle\langle q_1^{n_1} q_2^{n_2} p_1^m\rangle\rangle = \langle\langle Q_{n_1}Q_{n_2}P_m\rangle\rangle - \langle\langle Q_{n_1+n_2}P_m\rangle\rangle, \tag{D23}$$

$$\begin{aligned}\langle N_q(N_q-1)(N_q-2)N_p\rangle\langle\langle q_1^{n_1} q_2^{n_2} q_3^{n_3} p_1^m\rangle\rangle = {}& \langle\langle Q_{n_1}Q_{n_2}Q_{n_3}P_m\rangle\rangle - \sum_{(3)}\langle\langle Q_{n_1+n_2}Q_{n_3}P_m\rangle\rangle \\ & -2\langle\langle Q_{n_1+n_2+n_3}P_m\rangle\rangle,\end{aligned} \tag{D24}$$

$$\begin{aligned}\langle N_q\cdots(N_q-2)N_p(N_p-1)\rangle\langle\langle q_1^{n_1} q_2^{n_2} q_3^{n_3} p_1^{m_1} p_2^{m_2}\rangle\rangle = {}& -2\langle\langle Q_{n1+n2+n3}P_{m1+m2}\rangle\rangle + 2\langle\langle Q_{n1+n2+n3}P_{m1}P_{m2}\rangle\rangle \\ & +\langle\langle Q_{n1+n2}Q_{n3}P_{m1+m2}\rangle\rangle - \langle\langle Q_{n1+n2}Q_{n3}P_{m1}P_{m2}\rangle\rangle \\ & +\langle\langle Q_{n1+n3}Q_{n2}P_{m1+m2}\rangle\rangle - \langle\langle Q_{n1+n3}Q_{n2}P_{m1}P_{m2}\rangle\rangle \\ & +\langle\langle Q_{n2+n3}Q_{n1}P_{m1+m2}\rangle\rangle - \langle\langle Q_{n2+n3}Q_{n1}P_{m1}P_{m2}\rangle\rangle \\ & -\langle\langle Q_{n1}Q_{n2}Q_{n3}P_{m1+m2}\rangle\rangle + \langle\langle Q_{n1}Q_{n2}Q_{n3}P_{m1}P_{m2}\rangle\rangle.\end{aligned} \tag{D25}$$

Higher orders are increasingly tedious to compute for large values of n, m, and o. Fortunately, the scripts created for the computation of $\langle\langle q_1^{n_1} \cdots q_m^{n_m}\rangle\rangle$ are trivially extendable to multiple variables provided one appropriately considers all permutations of factors in q, p, and r. Finally, setting all exponents to unity one gets expressions of the form

$$\langle N_qN_p\rangle\langle\langle q_1p_1\rangle\rangle = \langle\langle Q_1P_1\rangle\rangle, \tag{D26}$$

$$\langle N_q(N_q-1)N_p\rangle\langle\langle q_1q_2p_1\rangle\rangle = -\langle\langle Q_2P_1\rangle\rangle + \langle\langle Q_1^2P_1\rangle\rangle, \tag{D27}$$

$$\langle N_q(N_q-1)(N_q-2)N_p\rangle\langle\langle q_1^{n_1} q_2^{n_2} q_3^{n_3} p_1^m\rangle\rangle = +2\langle\langle Q_3P_1\rangle\rangle - 3\langle\langle Q_2Q_1P_1\rangle\rangle + \langle\langle Q_1^3P_1\rangle\rangle, \tag{D28}$$

$$\begin{aligned}\langle N_q\cdots(N_q-2)N_p(N_p-1)\rangle\langle\langle q_1^{n_1} q_2^{n_2} q_3^{n_3} p_1^{m_1} p_2^{m_2}\rangle\rangle = {}& -2\langle\langle Q_3P_2\rangle\rangle + 2\langle\langle Q_3P_1^2\rangle\rangle \\ & +3\langle\langle Q_2Q_1P_2\rangle\rangle - 3\langle\langle Q_2Q_1P_1^2\rangle\rangle \\ & -\langle\langle Q_1^3P_2\rangle\rangle + \langle\langle Q_1^3P_1^2\rangle\rangle,\end{aligned} \tag{D29}$$

and so on.

APPENDIX E: COMPUTATION OF MULTIPARTICLE BALANCE FUNCTIONS

Balance function of arbitrary order n can be defined using the procedure introduced in Sec. IV based on differential correlators of the form $\langle \Delta q_1 \cdots \Delta q_n\rangle$ and corresponding n-order differential cumulant expansions. By construction, in the presence of

n-particle correlations, the defined balance functions must yield unity when integrated over all particle transverse momenta ($p_T > 0$), azimuths, and rapidity. As in the case of the second and fourth orders, n-cumulant can be expanded into their n-tuplet charge combinations. Such decompositions are straightforwardly obtained by considering all ways to cluster n particles into subgroups of $k \leqslant n$ positively and $n-k$ negatively charged particles. Such decompositions are herewith denoted $k(+)n-k(-)$ in which k and $n-k$, respectively, represent the number of positively and negatively charged particles in a decomposition of n particle. Given the order in which the +ve and −ve particles are listed is inconsequential, the number of equivalent permutations is given by binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{E1}$$

Additionally, the sign of each term evidently depends on the number of negative particles in a particular decomposition. Cumulants of order n can thus be written

$$C_n(\vec{p}_1,\ldots,\vec{p}_n) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}C_n^{k(+)n-k(-)}(\vec{p}_1,\ldots,\vec{p}_6). \tag{E2}$$

We split the cumulant decompositions to yield expressions of balance function corresponding to $m(+)$s given $m(-)$s and conversely, $m(-)$s given $m(+)$, for $m = n/2$. This requires an additional factor of two in the BF definitions. To avoid unnecessary repetitions of labels + and − as in Sec. IV, we introduce the notations B_n^{+-}, with n being even integers 2, 4, 6, etc., to indicate the balance functions of $n/2$ positively charged particles found at momenta $\vec{p}_1,\ldots,\vec{p}_{n/2}$ given $n/2$ negatively charged particles are detected at $\vec{p}_{n/2+1},\ldots,\vec{p}_n$, and conversely, B_n^{-+} shall indicate the BF of $n/2$ negatively charged particles found at momenta $\vec{p}_1,\ldots,\vec{p}_{n/2}$ given $n/2$ positively charged particles are detected at $\vec{p}_{n/2+1},\ldots,\vec{p}_n$. The five lowest orders are thus written

$$B_2^{+-}(\vec{p}_1,\vec{p}_2) = \frac{C_2^{+-}(\vec{p}_1,\vec{p}_2) - C_2^{--}(\vec{p}_1,\vec{p}_2)}{\langle N^-\rangle}, \tag{E3}$$

$$B_4^{+-}(\vec{p}_1,\ldots,\vec{p}_4) = \frac{1}{6}\times\frac{3C_4^{2+2-} - 4C_4^{1+3-} - C_4^{4-}}{\langle N^-(N^- - 1)\rangle}, \tag{E4}$$

$$B_6^{+-}(\vec{p}_1,\ldots,\vec{p}_6) = \frac{1}{60}\times\frac{10C_6^{3+3-} - 15C_4^{2+4-} + 6C_4^{1+5-} + C_4^{6-}}{\langle N^-(N^- - 1)(N^- - 2)\rangle}, \tag{E5}$$

$$B_8^{+-}(\vec{p}_1,\ldots,\vec{p}_8) = \frac{1}{840}\times\frac{35C_8^{4+4-} - 56C_8^{3+5-} + 25C_8^{2+6-} - 8C_8^{1+7-} + C_8^{8-}}{\langle N^-(N^- - 1)(N^- - 2)(N^- - 3)(N^- - 3)\rangle}, \tag{E6}$$

$$B_{10}^{+-}(\vec{p}_1,\ldots,\vec{p}_{10}) = \frac{1}{15120}\times\frac{126C_{10}^{5+5-} - 210C_{10}^{4+6-} + 120C_{10}^{3+7-} - 45C_{10}^{2+8-} + 10C_{10}^{1+9-} - C_{10}^{10-}}{\langle N^-(N^- - 1)(N^- - 2)(N^- - 3)(N^- - 3)(N^- - 4)\rangle}. \tag{E7}$$

Higher orders, $n > 10$, are readily obtained based on Eq. (E2) and can be written

$$B_n^{+-}(\vec{p}_1,\ldots,\vec{p}_n) = \frac{(-1)^{n/2}}{N(n)}\sum_{k=0}^{n}(-1)^{n-k}\left(\frac{1}{2}\right)^{\delta_{n,2k}}\binom{n}{k}C_n^{k(+)n-k(-)}, \tag{E8}$$

where the normalization coefficient $N(n)$ is calculated according to

$$N(n) = \frac{1}{2}\frac{n!}{(n/2)!}\langle N^-(N^- - 1)\cdots(N^- - n + 1)\rangle, \tag{E9}$$

and $\delta_{n,2k} = 1$ for $n = 2k$ but otherwise vanishes.

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