Recoil contributions in neutron β^- decay and their corrections to the correlation coefficients

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In this work, we evaluate the recoil corrections to the correlation coefficients in the neutron β^- decay with polarized neutron and polarized electron, up to order $O(m_n^{-2})$, where m_n represents the mass of neutron. In our calculations, we express the mass of proton m_p in terms of m_n and other small quantities to ensure that only one larger parameter, m_n, remains in the differential cross section. Subsequently, we expand the differential cross section on m_n^{-1}, and finally integrate the solid angles of the antineutrino to obtain the analytic expressions for the angular correlation coefficients. Comparing the analytic results to those in a previous study [Ivanov, Hollwieser, Troitskaya, Wellenzohn, and Berdnikov, Phys. Rev. C **95**, 055502 (2017); **104**, 069901(E) (2021)], we find that, at order $O(m_n^{-1})$, three of the results are consistent, while two differ slightly. In addition, the contributions at order $O(m_n^{-2})$ are presented when considering a general form of Wnp interactions. Numerical results show that contributions from order $O(m_n^{-2})$ and additional form factors of Wnp interactions are required to achieve a precision of 10^{-5}.

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I. INTRODUCTION

The β^- decay of a free neutron $n \rightarrow p e \bar{\nu}_e$ [1] provides a clean process to determine the elemental parameters in the standard model (SM). In the SM, a free neutron is unstable and its β^- decay is mainly governed by the weak interaction. In this process, the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{ud} and the weak coupling of Wnp play crucial roles. Due to its clean background, this process serves as a powerful laboratory for testing the universality of the quark mixing CKM matrix [2], validating the conserved-vector-current (CVC) hypothesis [3], and examining the absence of second-class currents (SCC) [4], among other things. In addition, the axial-vector coupling constant g_A plays a crucial role as an essential input in various fields such as nuclear physics, particle physics, cosmology, and astrophysics [5].

In the case of unpolarized β^- decay of the neutron, the experimental observables are the decay lifetime τ_n and the shape of the decay spectrum. However, when studying β^- decay involving a polarized electron and a polarized neutron, additional measurements can be made to determine the angular correlation coefficients $A(E_e)$, $G(E_e)$, $N(E_e)$, $Q(E_e)$, and $R(E_e)$ [6], where E_e represents the energy of the electron in the rest frame of the neutron. Recently, many high precise measurements have been carried out in the neutron β^- decay, including the nonpolarization case [7], the single neutron polarization [8], and both the electron and the neutron polarizations [9].

To extract the accurate value of V_{ud} and the coupling constants from the free neutron β^- decay, the theoretical estimations for the corresponding correlation coefficients should reach sufficient accuracy. For instance, in the case of unpolarized neutron β^- decay, the model-independent radiative corrections (MIRC) have been evaluated at different orders. The leading-order corrections $O(\alpha_e)$ were calculated in [10], the next-to-leading-order corrections $O(\alpha_e^2)$ were evaluated in [11], and the next-to-next-to-leading-order corrections $O(\alpha_e^3)$ were studied in [12], where α_e represents the fine structure constant. Additionally, model-dependent radiative corrections (MDRC) have been estimated using various methods. These include a renormalization group analysis method [13] and its improvement [14], the effective field theory method [15], and the dispersion relation method [16]. Furthermore, other contributions from proton recoil, finite proton radius, and lepton-nucleon convolution [17] are also significant and cannot be neglected in upcoming high-precision extractions [18].

In Refs. [19,20], the contributions to the correlation coefficients $A(E_e)$, $G(E_e)$, $N(E_e)$, $Q(E_e)$, and $R(E_e)$ at orders $O(\alpha_e)$ and $O(m_n^{-1})$ were calculated, where m_n represents the mass of the neutron. Furthermore, in Refs. [21,22], the contributions to the differential cross section at the level of 10^{-5} were estimated. In this study, we focus on calculating the contributions to the correlation coefficients $A(E_e)$, $G(E_e)$, $N(E_e)$, $Q(E_e)$, and $R(E_e)$ at order $O(m_n^{-2})$, considering a general form of Wnp interactions. Since the radiative corrections at orders $O(\alpha_e)$ and $O(\alpha_e m_n^{-1})$ can be combined directly with the recoil corrections to obtain the full corrections, and most of them have been extensively studied in previous references, we do not delve into them in detail. Instead, our main focus is solely on the recoil contributions at order $O(m_n^{-2})$ under the Born approximation.

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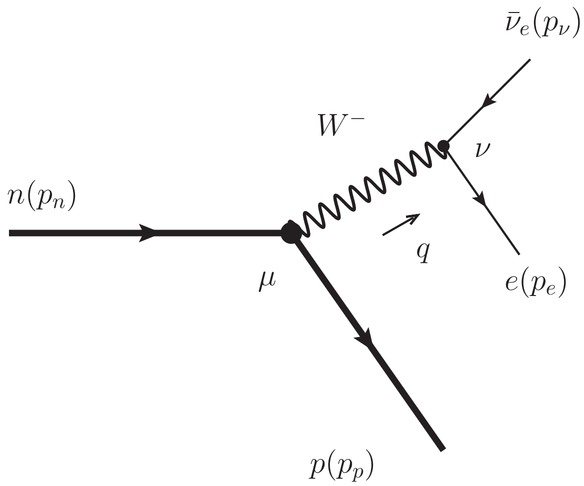

FIG. 1. Diagram for $n \to pe\bar{\nu}_e$ with one-W exchange.

The paper is structured as follows. In Sec. II, we provide the basic formula under the Born approximation. In Sec. III, we present the analytical expressions for the angular correlation coefficients up to order $O(m_n^{-2})$. In Sec. IV, we present numerical comparisons between our results and those reported in the literature. Additionally, we discuss the reasons for any observed differences.

II. BASIC FORMULA FOR $n \to pe\bar{\nu}_e$

Under the Born approximation for free neutron β^- decay, only the one-W-exchange diagram shown in Fig. 1 should be considered. The corresponding amplitude can be expressed as follows:

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}V_{ud}[\bar{u}(p_e, m_e)\gamma_\mu(g_{eV} - g_{eA}\gamma_5)u(p_\nu, m_\nu)] \times [\bar{u}(p_p, m_p)\Gamma^\mu_{Wnp}(q)u(p_n, m_n)], \quad (1)$$

where $G_F = \sqrt{2}g^2/8m_W^2$ is the Fermi weak constant [g is the $SU(2)$ gauge coupling constant], V_{ud} is the CKM matrix element [23], and $g_{eV,eA}$ are the coupling constants of $e\bar{\nu}_e W^-$. $\bar{u}(p_e, m_e)$, $u(p_\nu, m_\nu)$, $\bar{u}(p_p, m_p)$, and $u(p_n, m_n)$ are the spinors of the electron, antineutrino, proton, and neutron with the corresponding momentum and mass, respectively. The transfer momentum $q = p_n - p_p$. The most general form for the form factor Γ_{Wnp} in V-A theory [24] is given by

$$\Gamma^\mu_{Wnp}(q) = f_1(q^2)\gamma^\mu - i\frac{f_2(q^2)}{m_n + m_p}\sigma^{\mu\rho}q_\rho + \frac{f_3(q^2)}{m_n + m_p}q^\mu + \left[f_4(q^2)\gamma^\mu - i\frac{f_5(q^2)}{m_n + m_p}\sigma^{\mu\rho}q_\rho + \frac{f_6(q^2)}{m_n + m_p}q^\mu\right]\gamma_5, \quad (2)$$

where $\sigma^{\mu\rho} = \frac{i}{2}[\gamma^\mu, \gamma^\rho]$, $f_1(q^2)$, $f_2(q^2)$, $f_3(q^2)$, $f_4(q^2)$, $f_5(q^2)$, and $f_6(q^2)$ in Γ^μ_{Wnp} account for vector, weak magnetism, scalar, axial vector, weak electricity, and induced pseudoscalar contributions, respectively.

Under the Born approximation, the form factors $f_i(q^2)$ can be parametrized as functions of the low momentum transfer q^2 and expressed as

$$\begin{aligned} f_1(q^2) &= f_1 + \frac{q^2}{m_n^2}\lambda_{f_1}, \\ f_2(q^2) &= f_2, \\ f_3(q^2) &= f_3, \\ f_4(q^2) &= f_4 + \frac{q^2}{m_n^2}\lambda_{f_4}, \\ f_5(q^2) &= f_5, \\ f_6(q^2) &= f_6, \end{aligned} \quad (3)$$

where f_{1-6} and λ_{f_1,f_4} are the constants. For simplicity, we denote λ_{f_1} and λ_{f_4} as f_7 and f_8, respectively. In the literature, it is common to consider only the vector term f_1, the weak magnetism term f_2 and the axial vector term f_4 [20,21]. However, in our discussion, we retain all these terms for the sake of generality.

The differential cross section for the process involving a polarized neutron and a polarized electron can be expressed as

$$d\sigma_n = \frac{F(E_e, Z=1)}{2E_n}\prod_{i=e,\nu,p}\frac{d^3\vec{p}_i(2\pi)^4\delta^4(p_e + p_\nu + p_p - p_n)}{(2\pi)^3(2E_i)} \times \sum_{\text{helicity}}\mathcal{M}\mathcal{M}^*, \quad (4)$$

where E_i and $\vec{p}_i$ are the energy and three-momentum of the corresponding particle i in the rest frame of neutron, respectively. $F(E_e, Z=1)$ is the relativistic Fermi function [25] which describes the contribution of the Coulomb interaction between the final state electron and proton. After integrating the δ functions, one can obtain the following expression:

$$\frac{d^5\sigma_n(E_e, \vec{p}_e, \vec{\xi}_e, \vec{\xi}_n)}{dE_e d\Omega_e d\Omega_\nu} = F(E_e, Z=1)\beta\sum_{\text{helicity}}\mathcal{M}\,\mathcal{M}^*, \quad (5)$$

where $\vec{\xi}_n$ and $\vec{\xi}_e$ are the polarization vectors of the neutron and electron, respectively, $d\Omega_e$ and $d\Omega_\nu$ are the elements of the solid angles of the electron and the antineutrino, respectively. Here, the sums of the neutron's and electron's helicities are expressed by the corresponding polarization vectors. The expression for the three-body phase space factor, denoted as β, is given as follows:

$$\beta = \frac{1}{16m_n}\frac{1}{(2\pi)^5}\frac{\sqrt{E_e^2 - m_e^2}E_\nu}{E_p + E_\nu + \vec{n}_\nu\cdot\vec{p}_e}, \quad (6)$$

where the unit vector $\vec{n}_\nu$ is oriented along the direction of the antineutrino's three-momentum $\vec{p}_\nu$.

Experimentally, the direction of the antineutrino's three-momentum is not directly measured. Therefore, we need to integrate over $d\Omega_\nu$ to account for all possible directions. After integrating $d\Omega_\nu$, the form of the cross section can be expressed as [6,20]

$$\frac{d^3\sigma_n(E_e,\vec{p}_e,\vec{\xi}_e,\vec{\xi}_n)}{dE_e d\Omega_e} = \frac{G_F^2|V_{ud}|^2}{8\pi^4}(E_0-E_e)^2\sqrt{E_e^2-m_e^2}E_eF(E_e,Z=1)\bigg\{\zeta(E_e)+\bar{A}(E_e)\,\frac{\vec{\xi}_n\cdot\vec{p}_e}{E_e}+\bar{G}(E_e)\,\frac{\vec{\xi}_e\cdot\vec{p}_e}{E_e} + \bar{N}(E_e)\,\vec{\xi}_n\cdot\vec{\xi}_e+\bar{Q}(E_e)\,\frac{(\vec{\xi}_n\cdot\vec{p}_e)(\vec{\xi}_e\cdot\vec{p}_e)}{E_e(E_e+m_e)}+\bar{R}(E_e)\,\frac{\vec{\xi}_n\cdot(\vec{p}_e\times\vec{\xi}_e)}{E_e}\bigg\}, \tag{7}$$

where $E_0=(m_n^2-m_p^2+m_e^2)/2m_n$ represents the endpoint energy of the electron spectrum. In comparison with the expressions in Ref. [20], we introduce the following definitions: $\bar{A}(E_e)=A(E_e)\zeta(E_e)$, $\bar{G}(E_e)=G(E_e)\zeta(E_e)$, $\bar{N}(E_e)=N(E_e)\zeta(E_e)$, $\bar{Q}(E_e)=Q(E_e)\zeta(E_e)$, and $\bar{R}(E_e)=R(E_e)\zeta(E_e)$, respectively.

To calculate $\bar{A}(E_e)$, $\bar{G}(E_e)$, $\bar{N}(E_e)$, $\bar{Q}(E_e)$, and $\bar{R}(E_e)$, it is common to expand the phase space factor and the amplitude $\mathcal{M}$ separately on m_n^{-1} or m_N^{-1}, where $m_N=(m_n+m_p)/2$.

Various approximations of the phase space factor have been employed in the literature. For example, in Refs. [19,20], the phase space factor is approximated as follows:

$$\beta\to\beta_{\rm I}=\frac{1}{16m_n}\frac{1}{(2\pi)^5}\left[1+\frac{3}{(m_n+m_p)/2}(E_e-\vec{n}_\nu\cdot\vec{p}_e)\right]\times\frac{(E_0-E_e)^2\sqrt{E_e^2-m_e^2}}{m_nE_\nu}. \tag{8}$$

For comparison, if one expands the following variable Y_1 on m_n^{-1} up to order $O(m_n^{-2})$,

$$Y_1\equiv\frac{(E_0-E_e)^2}{E_p+E_\nu+\vec{n}_\nu\cdot\vec{p}_e}=16m_n(2\pi)^5\frac{(E_0-E_e)^2}{\sqrt{E_e^2-m_e^2}E_\nu}\beta, \tag{9}$$

then we have

$$\beta\to\beta_{\rm II}=\frac{1}{16m_n}\frac{1}{(2\pi)^5}\left[1+\frac{3}{m_n}(E_e-\vec{n}_\nu\cdot\vec{p}_e)\right]\times\frac{(E_0-E_e)^2\sqrt{E_e^2-m_e^2}}{m_nE_\nu}. \tag{10}$$

This expansion of Y_1 to order $O(m_n^{-2})$ introduces a factor difference compared to $\beta_{\rm I}$, and the difference is at higher order.

The authors of Ref. [20] extended $\beta_{\rm I}$ to higher orders of m_N^{-1} in Ref. [21] with the following expression:

$$\beta\to\beta_{\rm III}=\frac{1}{16m_n}\frac{1}{(2\pi)^5}\frac{(E_0-E_e)^2\sqrt{E_e^2-m_e^2}}{m_nE_\nu}\Phi_n(\vec{p}_e,\vec{p}_\nu) \tag{11}$$

with

$$\Phi_n(\vec{p}_e,\vec{p}_\nu)=1+3\frac{E_e}{m_N}\left(1-\frac{\vec{p}_e\cdot\vec{p}_\nu}{E_eE_\nu}\right)+6\frac{E_e^2}{m_N^2}\left(1-\frac{\vec{p}_e\cdot\vec{p}_\nu}{E_eE_\nu}\right)\left(1-\frac{\vec{p}_e\cdot\vec{p}_\nu}{E_eE_\nu}-\frac{1}{4}\frac{E_0}{E_e}\right). \tag{12}$$

In our calculation, we do not expand the phase space factor and the amplitude independently. Instead, we expand the entire expression of $\frac{d^5\sigma_n(E_e,\vec{p}_e,\vec{\xi}_e,\vec{\xi}n)}{dE_ed\Omega_ed\Omega\nu}$ on m_n^{-1}.

Since the entire expression $\frac{d^5\sigma_n(E_e,\vec{p}_e,\vec{\xi}_e,\vec{\xi}n)}{dE_ed\Omega_ed\Omega\nu}$ involves two large parameters, m_n and m_p, in order to perform a consistent expansion on m_n^{-1}, we apply the following three exact rules to the entire expression in Mathematica:

$$\begin{aligned} E_\nu &\to \frac{m_n(E_0-E_e)}{m_n-E_e+\vec{n}_\nu\cdot\vec{p}_e},\\ m_n+m_p &\to xm_n,\\ m_p &\to \sqrt{m_n^2+m_e^2-2E_0m_n}. \end{aligned} \tag{13}$$

After applying these replacements, the expression is left with only one large parameter, m_n, which can be safely expanded on m_n^{-1}. We would like to mention that this step differs significantly from the approach employed in Refs. [20,21]. Moreover, the resulting expression takes the form of a polynomial in terms of $\vec{n}_\nu$, allowing for straightforward integration over $d\Omega_\nu$.

To facilitate a comparison between the different expansions, we also separately utilize our method to expand the phase space factor. The result of this expansion is as follows:

$$\beta\to\beta_{\rm IV}=\frac{(E_0-E_e)\sqrt{(E_e^2-m_e^2)}}{512\pi^5m_n^4}\big[m_n^2+2m_nE_e+3E_e^2-2\sqrt{E_e^2-m_e^2}(3E_e+m_n)\vec{n}_\nu\cdot\vec{n}_e+3\big(E_e^2-m_e^2\big)(\vec{n}_\nu\cdot\vec{n}_e)^2\big]. \tag{14}$$

In Eq. (14), the resulting expression involves only one large parameter, m_n, and is a polynomial function of $\vec{n}_\nu\cdot\vec{n}_e$. On the other hand, in Eqs. (8), (10), and (11), a small variable E_ν, which is a function of $\vec{n}_\nu\cdot\vec{p}_e$, still appears in the denominator.

In Fig. 2, we present the numerical results for $1-\beta_X/\beta$ at $E_e=1$ MeV to compare the different phase space factors. The results depicted in Fig. 2 clearly demonstrate that the approximations used in Eqs. (8) and (10) are not small. This is expected since these two phase space factors are obtained through a Taylor expansion to a lower order of m_n^{-1} or m_N^{-1}. On the other hand, the results obtained using Eqs. (11) and (14) are significantly better and exhibit consistency.

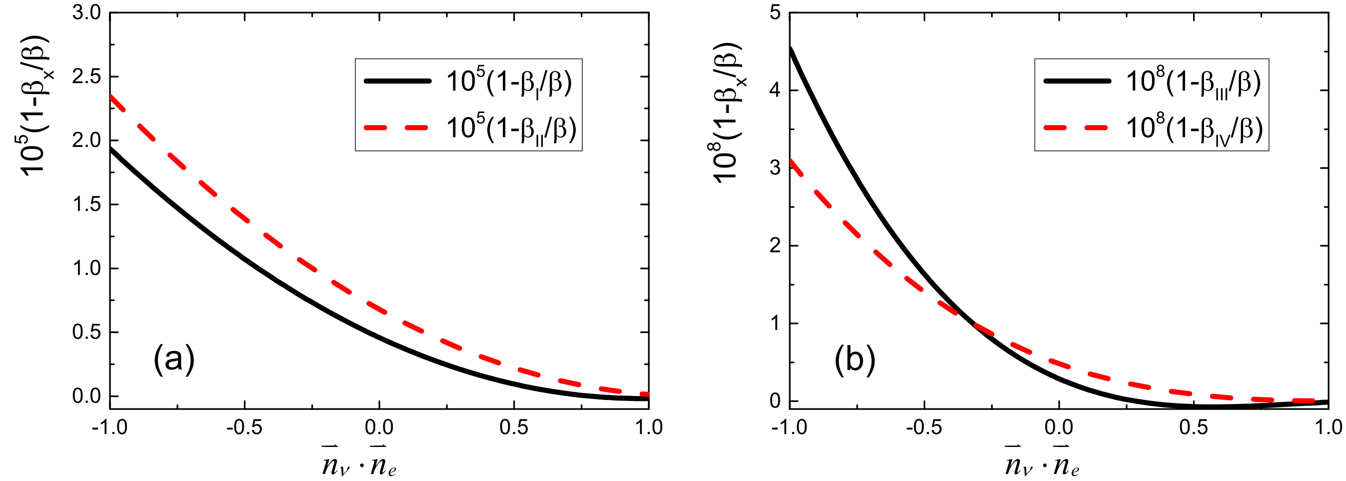

FIG. 2. Comparison of the phase space factors: $(1-\beta_X/\beta)$ *vs.* $\vec{n}_\nu \cdot \vec{n}_e$ with $E_e = 1$ MeV.

III. ANALYTIC EXPRESSIONS

The general recoil corrections to the angular correlation coefficients can be expressed as follows:

$$X(E_e) = \sum_{i=1}^{6}\sum_{j=i}^{8} \mathcal{C}_{ij}^X L_{ij}^X f_i f_j, \tag{15}$$

where X refers to $\zeta, \bar{A}, \bar{G}, \bar{N}, \bar{Q}$, $\mathcal{C}_{ij}^X$ refers to the contributions from the hadronic part, and L_{ij}^X refers to the factor from the leptonic part which can be either $g_{eV}^2 + g_{eA}^2$ or $g_{eV} g_{eA}$. In Eq. (15), we have expressed the contributions in a form where $i \leqslant j$, and we have ignored the contributions with $i = 7, 8$ as they are approximately of the order $O(m_n^{-4})$. Consequently, we have only taken into account the terms satisfying the

TABLE I. Analytical results for $\mathcal{C}_{ij}^\zeta$, where L_{ij}^ζ represents the factor from the lepton part. Certain contributions, such as $\mathcal{C}_{16}^\zeta$ and $\mathcal{C}_{23}^\zeta$, are zero and have been excluded from the table.

	L_{ij}^ζ	$O(1)$	$O(m_n^{-1})$	$O(m_n^{-2})$
$\mathcal{C}_{11}^\zeta$	$g_{eV}^2 + g_{eA}^2$	$\frac{1}{2}$	$\frac{E_e}{m_n}$	$\frac{(3E_0^2-8E_0E_e+32E_e^2)+(2E_0/E_e-11)m_e^2}{12m_n^2}$
$\mathcal{C}_{12}^\zeta$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{(6E_0^2-16E_0E_e+16E_e^2)+(7E_0/E_e-13)m_e^2}{6m_n(m_n+m_p)}$
$\mathcal{C}_{13}^\zeta$	$g_{eV}^2 + g_{eA}^2$	0	$\frac{m_e^2/E_e}{m_n+m_p}$	$\frac{(-E_0/E_e+5)m_e^2}{2m_n(m_n+m_p)}$
$\mathcal{C}_{14}^\zeta$	$g_{eV} g_{eA}$	0	$\frac{2(E_0-2E_e)+2m_e^2/E_e}{m_n}$	$\frac{8(4E_0E_e-7E_e^2)+8(-E_0/E_e+4)m_e^2}{3m_n^2}$
$\mathcal{C}_{15}^\zeta$	$g_{eV} g_{eA}$	0	0	$-\frac{2(E_0^2-2E_0E_e)+2E_0/E_e m_e^2}{m_n(m_n+m_p)}$
$\mathcal{C}_{17}^\zeta$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{4(E_0E_e-E_e^2)+(2E_0/E_e+1)m_e^2}{3m_n^2}$
$\mathcal{C}_{22}^\zeta$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{(3E_0^2-10E_0E_e+10E_e^2)+(4E_0/E_e-7)m_e^2}{3(m_n+m_p)^2}$
$\mathcal{C}_{24}^\zeta$	$g_{eV} g_{eA}$	0	$\frac{2(E_0-2E_e)+2m_e^2/E_e}{m_n}$	$\frac{8(4E_0E_e-7E_e^2)+8(-E_0/E_e+4)m_e^2}{3m_n^2}$
$\mathcal{C}_{25}^\zeta$	$g_{eV} g_{eA}$	0	0	$-\frac{2(E_0^2-2E_0E_e)+2E_0/E_e m_e^2}{m_n(m_n+m_p)}$
$\mathcal{C}_{33}^\zeta$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{m_e^2}{2(m_n+m_p)^2}$
$\mathcal{C}_{44}^\zeta$	$g_{eV}^2 + g_{eA}^2$	$\frac{3}{2}$	$-\frac{(E_0-5E_e)+m_e^2/E_e}{m_n}$	$-\frac{(3E_0^2+56E_0E_e-176E_e^2)+(-14E_0/E_e+77E_e)m_e^2}{12m_n^2}$
$\mathcal{C}_{45}^\zeta$	$g_{eV}^2 + g_{eA}^2$	0	$-\frac{2E_0+m_e^2/E_e}{m_n+m_p}$	$\frac{2(E_0^2-8E_0E_e)+(5E_0/E_e-3)m_e^2}{2m_n(m_n+m_p)}$
$\mathcal{C}_{46}^\zeta$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{(-E_0/E_e+1)m_e^2}{2m_n(m_n+m_p)}$
$\mathcal{C}_{48}^\zeta$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{20(E_0E_e-E_e^2)+(-2E_0/E_e+11)m_e^2}{3m_n^2}$
$\mathcal{C}_{55}^\zeta$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{(6E_0^2-4E_0E_e+4E_e^2)+(4E_0/E_e-1)m_e^2}{6(m_n+m_p)^2}$

TABLE II. Analytical results for $\mathcal{C}_{ij}^{\bar{A}}$, where $L_{ij}^{\bar{A}}$ represents the factor from the lepton part. Certain contributions, such as $\mathcal{C}_{13}^{\bar{A}}$ and $\mathcal{C}_{26}^{\bar{A}}$, are zero and have been excluded from the table.

	$L_{ij}^{\bar{A}}$	$O(1)$	$O(m_n^{-1})$	$O(m_n^{-2})$
$\mathcal{C}_{11}^{\bar{A}}$	$g_{eV}g_{eA}$	0	$\frac{-2(E_0-E_e)}{3m_n}$	$\frac{-(3E_0^2+8E_0E_e-8E_e^2)+3m_e^2}{6m_n^2}$
$\mathcal{C}_{12}^{\bar{A}}$	$g_{eV}g_{eA}$	0	$-\frac{4(E_0-E_e)}{3(m_n+m_p)}$	$\frac{-2(E_0^2+2E_0E_e)+6m_e^2}{3m_n(m_n+m_p)}$
$\mathcal{C}_{14}^{\bar{A}}$	$g_{eV}^2+g_{eA}^2$	-1	$-\frac{E_0+2E_e}{3m_n}$	$-\frac{8(E_0E_e-E_e^2)}{3m_n^2}$
$\mathcal{C}_{15}^{\bar{A}}$	$g_{eV}^2+g_{eA}^2$	0	$\frac{2(E_0-E_e)}{3(m_n+m_p)}$	$\frac{2(E_0^2+4E_0E_e-8E_e^2)-3m_e^2}{6m_n(m_n+m_p)}$
$\mathcal{C}_{16}^{\bar{A}}$	$g_{eV}^2+g_{eA}^2$	0	0	$\frac{m_e^2}{2m_n(m_n+m_p)}$
$\mathcal{C}_{18}^{\bar{A}}$	$g_{eV}^2+g_{eA}^2$	0	0	$\frac{-(4E_0E_e-4E_e^2+3m_e^2)}{3m_n^2}$
$\mathcal{C}_{22}^{\bar{A}}$	$g_{eV}g_{eA}$	0	0	$\frac{-(E_0^2-6E_0E_e+8E_e^2)+3m_e^2}{3m_n(m_n+m_p)}$
$\mathcal{C}_{24}^{\bar{A}}$	$g_{eV}^2+g_{eA}^2$	0	$\frac{-2(2E_0-5E_e)}{3(m_n+m_p)}$	$\frac{2(E_0^2-28E_0E_e+48E_e^2)-15m_e^2}{6m_n(m_n+m_p)}$
$\mathcal{C}_{25}^{\bar{A}}$	$g_{eV}^2+g_{eA}^2$	0	0	$\frac{2(E_0^2-2E_0E_e-2E_e^2)}{3(m_n+m_p)^2}$
$\mathcal{C}_{34}^{\bar{A}}$	$g_{eV}^2+g_{eA}^2$	0	0	$\frac{m_e^2}{2m_n(m_n+m_p)}$
$\mathcal{C}_{35}^{\bar{A}}$	$g_{eV}^2+g_{eA}^2$	0	0	$-\frac{m_e^2}{(m_n+m_p)^2}$
$\mathcal{C}_{44}^{\bar{A}}$	$g_{eV}g_{eA}$	-2	$\frac{4E_0-22E_e}{3m_n}$	$\frac{(3E_0^2+40E_0E_e-136E_e^2)+21m_e^2}{6m_n^2}$
$\mathcal{C}_{45}^{\bar{A}}$	$g_{eV}g_{eA}$	0	$\frac{4(2E_0+E_e)}{3(m_n+m_p)}$	$\frac{-2(E_0^2-14E_0E_e-8E_e^2)-3m_e^2}{3m_n(m_n+m_p)}$
$\mathcal{C}_{47}^{\bar{A}}$	$g_{eV}^2+g_{eA}^2$	0	0	$-\frac{(4E_0E_e-4E_e^2+3m_e^2)}{3m_n^2}$
$\mathcal{C}_{48}^{\bar{A}}$	$g_{eV}g_{eA}$	0	0	$-\frac{4(8E_0E_e-8E_e^2+3m_e^2)}{3m_n^2}$
$\mathcal{C}_{55}^{\bar{A}}$	$g_{eV}g_{eA}$	0	0	$-\frac{2E_0(E_0+2E_e)}{3(m_n+m_p)^2}$

conditions $1 \leqslant i \leqslant 6$ and $i \leqslant j \leqslant 8$. In practical calculations, the terms $\mathcal{C}_{i7}^X$ and $\mathcal{C}_{i8}^X$ with $i = 2, 3, 5, 6$ are also neglected as they are approximately of order $O(m_n^{-3})$. Furthermore, when considering only the one-boson-exchange contribution, it is found that $\bar{R}(E_e) = 0$, and our direct calculation also confirms this property.

All our calculations are performed by the Mathematica codes and the package FeynCalc [26] is employed to handle the trace of Dirac matrices in four dimensions. Following the above consistent expansion of the results on m_n^{-1} and integration over $d\Omega_\nu$, analytic expressions for $X(E_e)$ can be obtained.

The final expressions are presented in Tables I through V. The expressions of $\mathcal{C}_{ij}^X$ at orders $O(1)$, $O(m_n^{-1})$, $O(m_n^{-2})$ are presented in the third to the last columns of each table, respectively. Certain contributions, such as $\mathcal{C}_{16}^{\zeta}$ and $\mathcal{C}_{13}^{\bar{A}}$, which are zero, have been omitted from these tables. The analytical results show a general property that the contributions from f_1^2, f_1f_4, f_4^2 are dominant.

The coupling constants f_1 and f_4 correspond to the weak coupling constants g_V and g_A, respectively, where $f_1 \equiv g_V$ and $f_4 \equiv g_A$. The ratio $f_4/f_1 \equiv \lambda$ can be determined from Ref. [8]. Assuming $SU(2)$ symmetry and utilizing the CVC hypothesis, we have $f_1 = 1$ and $f_2 \equiv \kappa = \kappa_p - \kappa_n$, where κ_p and κ_n are the anomalous magnetic moments of the proton and neutron, respectively [23]. When there is no SCC [4], the scalar coupling constant f_3 and weak electric coupling constant f_5 are both set to zero. The pseudoscalar coupling f_6 is induced by strong interaction effects, and is determined by the partially conserved axial vector current (PCAC) hypothesis as $f_6 = \frac{4m_n^2 f_4}{m_\pi^2}$ [27].

At the order $O(1)$, when the conditions $f_1 = 1$, $f_2 = \kappa$, $f_4 = \lambda$, $f_3 = f_5 = f_6 = f_7 = f_8 = 0$, and $g_{eV} = g_{eA} = 1$ are imposed, the expressions in Tables I to V can reproduce the "bare" correlation coefficients given by Eq. (1) in Ref. [20]. It should be noted that in the "bare" results, a global factor of $1 + 3\lambda^2$ has been multiplied to ensure the same definition of $\zeta(E_e)$.

Moreover, at the order $O(m_n^{-1})$, when substituting the factor $1/m_n$ in Tables I–V with the factor $2/(m_n + m_p)$, the corresponding results $\zeta(E_e)$, $\bar{A}(E_e)$, and $\bar{G}(E_e)$ given by Eqs. (6), (7) in Ref. [20] can be fully reproduced. Since the difference between the two factors occurs at a higher order of m_n^{-1}, it means that they are consistent at the order $O(m_n^{-1})$. However, it should be noted that the results for $\bar{N}(E_e)$ and $\bar{Q}(E_e)$ as given by Eqs. (6) and (7) in Ref. [20] cannot be reproduced. After the above substitution,, the discrepancies for these two results can be expressed as follows:

$$\bar{N}_r(E_e) - \bar{N}(E_e) = \frac{2(E_0 - 2E_e)m_e\lambda}{E_e(m_n + m_p)},$$
$$\bar{Q}_r(E_e) - \bar{Q}(E_e) = \frac{2(E_0 - 2E_e - m_e)\lambda}{m_n + m_p}, \qquad (16)$$

TABLE III. Analytical results for $\mathcal{C}_{ij}^{\bar{G}}$, where $L_{ij}^{\bar{G}}$ represents the factor from the lepton part. Certain contributions, such as $\mathcal{C}_{16}^{\bar{G}}$ and $\mathcal{C}_{23}^{\bar{G}}$, are zero and have been excluded from the table.

	$L_{ij}^{\bar{G}}$	$O(1)$	$O(m_n^{-1})$	$O(m_n^{-2})$
$\mathcal{C}_{11}^{\bar{G}}$	$g_{eV}g_{eA}$	-1	$-\frac{2E_e}{m_n}$	$\frac{-(3E_0^2-8E_0E_e+32E_e^2)+9m_e^2}{6m_n^2}$
$\mathcal{C}_{12}^{\bar{G}}$	$g_{eV}g_{eA}$	0	0	$\frac{-2(3E_0^2-8E_0E_e+8E_e^2)+3m_e^2}{3m_n(m_n+m_p)}$
$\mathcal{C}_{13}^{\bar{G}}$	$g_{eV}g_{eA}$	0	0	$\frac{m_e^2}{m_n(m_n+m_p)}$
$\mathcal{C}_{14}^{\bar{G}}$	$g_{eV}^2+g_{eA}^2$	0	$-\frac{E_0-2E_e}{m_n}$	$-\frac{16E_0E_e-28E_e^2+3m_e^2}{3m_n^2}$
$\mathcal{C}_{15}^{\bar{G}}$	$g_{eV}^2+g_{eA}^2$	0	0	$\frac{E_0^2-2E_0E_e}{m_n(m_n+m_p)}$
$\mathcal{C}_{17}^{\bar{G}}$	$g_{eV}g_{eA}$	0	0	$-\frac{2(4E_0E_e-4E_e^2+3m_e^2)}{3m_n^2}$
$\mathcal{C}_{22}^{\bar{G}}$	$g_{eV}g_{eA}$	0	0	$\frac{-2(3E_0^2-10E_0E_e+10E_e^2)+6m_e^2}{3(m_n+m_p)^2}$
$\mathcal{C}_{24}^{\bar{G}}$	$g_{eV}^2+g_{eA}^2$	0	$-\frac{E_0-2E_e}{m_n}$	$-\frac{16E_0E_e-28E_e^2+3m_e^2}{3m_n^2}$
$\mathcal{C}_{25}^{\bar{G}}$	$g_{eV}^2+g_{eA}^2$	0	0	$\frac{E_0^2-2E_0E_e}{m_n(m_n+m_p)}$
$\mathcal{C}_{33}^{\bar{G}}$	$g_{eV}g_{eA}$	0	0	$\frac{m_e^2}{(m_n+m_p)^2}$
$\mathcal{C}_{44}^{\bar{G}}$	$g_{eV}g_{eA}$	-3	$\frac{2(E_0-5E_e)}{m_n}$	$\frac{(3E_0^2+56E_0E_e-176E_e^2)+27m_e^2}{6m_n^2}$
$\mathcal{C}_{45}^{\bar{G}}$	$g_{eV}g_{eA}$	0	$\frac{4E_0}{m_n+m_p}$	$-\frac{2E_0^2-16E_0E_e+3m_e^2}{m_n(m_n+m_p)}$
$\mathcal{C}_{46}^{\bar{G}}$	$g_{eV}g_{eA}$	0	0	$\frac{m_e^2}{m_n(m_n+m_p)}$
$\mathcal{C}_{48}^{\bar{G}}$	$g_{eV}g_{eA}$	0	0	$-\frac{2(20E_0E_e-20E_e^2+9m_e^2)}{3m_n^2}$
$\mathcal{C}_{55}^{\bar{G}}$	$g_{eV}g_{eA}$	0	0	$\frac{-2(3E_0^2-2E_0E_e+2E_e^2)+3m_e^2}{3(m_n+m_p)^2}$

where the terms with the subindex r refer to the results obtained from Tables I–V by replacing the factor $1/m_n$ with $2/(m_n+m_p)$, $\bar{X}(E_e)$ refer to the expressions given in Ref. [20], and only the contributions at the order $O(m_n^{-1})$ are considered in both cases.

Our calculation is based on the same amplitude and phase space factor as those used in Ref. [20]. In principle, these two calculations should yield identical results and any differences should only occur at order $O(m_n^{-2})$. Therefore, it is an intriguing question to investigate the underlying reasons for the observed discrepancy.

The results presented in Ref. [20] can be traced back to the original calculations provided in the Appendix of Ref. [19]. In that calculation, two-component Pauli spinors and Pauli matrix were employed to expand the hadronic part amplitude at first, and other two approximations are also used to express the final amplitude, which are shown in their Eqs. (A-7) to (A-13). In our calculation, we use the Lorentz covariant form to express the cross-section and expanded the entire result on m_n^{-1} after absorbing the variable m_p. All of our calculations are performed in Mathematica, and the five correlation functions are evaluated simultaneously.

To determine the cause of the discrepancy, we conducted practical calculations using Eq. (A-10) and Eq. (A-12) from Ref. [20] as inputs to calculate the correlation functions. Remarkably, we obtained consistent results with those obtained using the covariant form at order $O(m_n^{-1})$. As the analytic expressions for $\bar{G}(E_e)$, $\bar{N}(E_e)$, and $\bar{Q}(E_e)$ at this order are only reported by one research group, we believe that our results can serve as an independent verification or double-check of those expressions.

For convenience, we have included one of our Mathematica codes as Supplemental Material [28]. This code considers a simplified scenario where only f_1 and f_4 are nonzero. These codes provide a clear and transparent illustration of our calculations, which can help further clarify the discrepancy.

IV. NUMERICAL COMPARISON AND DISCUSSIONS

In the numerical comparison, we use the values $m_n = 939.56542$ MeV, $m_p = 938.27209$ MeV, $m_e = 0.51100$ MeV [23], and approximate the antineutrino mass as $m_\nu \approx 0$.

In Ref. [20], the phase space factor $\beta_{\rm I}$ is utilized, and the expansion of $\mathcal{M}$ is employed to derive the relevant correlation coefficients. To facilitate a direct comparison of the results, we also adopt $\beta_{\rm I}$ as an input and expand $d^5\sigma_n(E_e, \vec{p}_e, \vec{\xi}_e, \vec{\xi}_n)$ on m_n^{-1} to obtain the expressions for the correlation coefficients. We define the difference between these two sets of results as

$$\delta_1 X \equiv X_{\beta_{\rm I},A}^{\rm our} - X_{\beta_{\rm I},A}^{\rm ref}, \tag{17}$$

where $X_{\beta_{\rm I},A}^{\rm ref}$ refers to the results obtained from Eqs. (6), (7) in Ref. [20], $X_{\beta_{\rm I},A}^{\rm our}$ refers to the results obtained from our own calculations using $\beta_{\rm I}$ as input, and the subindex A refers to the choice of the parameters as specified in Ref. [20]:

$$f_1 = 1, \quad f_2 = 3.7058, \quad f_4 = -1.2767,$$
$$f_3 = f_5 = f_6 = f_7 = f_8 = 0, \quad g_{eV} = g_{eA} = 1. \tag{18}$$

TABLE IV. Analytical results for $\mathcal{C}_{ij}^{\bar{N}}$, where $L_{ij}^{\bar{N}}$ represents the factor from the lepton part. Certain contributions, such as $\mathcal{C}_{17}^{\bar{N}}$ and $\mathcal{C}_{26}^{\bar{N}}$, are zero and have been excluded from the table.

	$L_{ij}^{\bar{N}}$	$O(1)$	$O(m_n^{-1})$	$O(m_n^{-2})$
$\mathcal{C}_{11}^{\bar{N}}$	$\frac{(g_{eV}^2+g_{eA}^2)m_e}{E_e}$	0	$\frac{E_0-E_e}{3m_n}$	$\frac{3E_0^2+10E_0E_e-16E_e^2+3m_e^2}{12m_n^2}$
$\mathcal{C}_{12}^{\bar{N}}$	$\frac{(g_{eV}^2+g_{eA}^2)m_e}{E_e}$	0	$\frac{2(E_0-E_e)}{3(m_n+m_p)}$	$\frac{2E_0^2+10E_0E_e-15E_e^2+3m_e^2}{6m_n(m_n+m_p)}$
$\mathcal{C}_{13}^{\bar{N}}$	$\frac{(g_{eV}^2+g_{eA}^2)m_e}{E_e}$	0	0	$\frac{2E_0E_e-5E_e^2+3m_e^2}{6m_n(m_n+m_p)}$
$\mathcal{C}_{14}^{\bar{N}}$	$\frac{g_{eV}g_{eA}m_e}{E_e}$	2	$\frac{2(E_0+5E_e)}{3m_n}$	$\frac{2(4E_0E_e+8E_e^2-3m_e^2)}{3m_n^2}$
$\mathcal{C}_{15}^{\bar{N}}$	$\frac{g_{eV}g_{eA}m_e}{E_e}$	0	$-\frac{2(2E_0+E_e)}{3(m_n+m_p)}$	$-\frac{2E_0^2+9E_0E_e+E_e^2}{3m_n(m_n+m_p)}$
$\mathcal{C}_{16}^{\bar{N}}$	$\frac{g_{eV}g_{eA}m_e}{E_e}$	0	0	$\frac{-E_eE_0+E_e^2}{3m_n(m_n+m_p)}$
$\mathcal{C}_{18}^{\bar{N}}$	$\frac{g_{eV}g_{eA}m_e}{E_e}$	0	0	$\frac{2(2E_0E_e-2E_e^2+m_e^2)}{m_n^2}$
$\mathcal{C}_{22}^{\bar{N}}$	$\frac{g_{eV}g_{eA}m_e}{E_e}$	0	0	$\frac{(E_0-E_e)^2}{6m_n(m_n+m_p)}$
$\mathcal{C}_{23}^{\bar{N}}$	$\frac{(g_{eV}^2+g_{eA}^2)m_e}{E_e}$	0	0	$\frac{2E_0E_e-5E_e^2+3m_e^2}{6m_n(m_n+m_p)}$
$\mathcal{C}_{24}^{\bar{N}}$	$\frac{g_{eV}g_{eA}m_e}{E_e}$	0	$\frac{8(E_0-E_e)}{3(m_n+m_p)}$	$-\frac{2E_0^2-35E_0E_e+39E_e^2-6m_e^2}{3m_n(m_n+m_p)}$
$\mathcal{C}_{25}^{\bar{N}}$	$\frac{g_{eV}g_{eA}m_e}{E_e}$	0	0	$\frac{-2(2E_0^2-5E_e^2+3m_e^2)}{3(m_n+m_p)^2}$
$\mathcal{C}_{34}^{\bar{N}}$	$\frac{g_{eV}g_{eA}m_e}{E_e}$	0	$\frac{2E_e}{m_n+m_p}$	$\frac{-5E_0E_e+23E_e^2-6m_e^2}{3m_n(m_n+m_p)}$
$\mathcal{C}_{35}^{\bar{N}}$	$\frac{g_{eV}g_{eA}m_e}{E_e}$	0	0	$-\frac{2(2E_eE_0+E_e^2)}{3(m_n+m_p)^2}$
$\mathcal{C}_{44}^{\bar{N}}$	$\frac{(g_{eV}^2+g_{eA}^2)m_e}{E_e}$	1	$\frac{-2(E_0-4E_e)}{3m_n}$	$-\frac{3E_0^2+26E_0E_e-80E_e^2+15m_e^2}{12m_n^2}$
$\mathcal{C}_{45}^{\bar{N}}$	$\frac{(g_{eV}^2+g_{eA}^2)m_e}{E_e}$	0	$-\frac{2(2E_0+E_e)}{3(m_n+m_p)}$	$\frac{2E_0^2-18E_0E_e-17E_e^2+9m_e^2}{6m_n(m_n+m_p)}$
$\mathcal{C}_{46}^{\bar{N}}$	$\frac{(g_{eV}^2+g_{eA}^2)m_e}{E_e}$	0	0	$\frac{-2E_0E_e+5E_e^2-3m_e^2}{6m_n(m_n+m_p)}$
$\mathcal{C}_{47}^{\bar{N}}$	$\frac{g_{eV}g_{eA}m_e}{E_e}$	0	0	$\frac{2(2E_0E_e-2E_e^2+m_e^2)}{m_n^2}$
$\mathcal{C}_{48}^{\bar{N}}$	$\frac{(g_{eV}^2+g_{eA}^2)m_e}{E_e}$	0	0	$\frac{2(2E_0E_e-2E_e^2+m_e^2)}{m_n^2}$
$\mathcal{C}_{55}^{\bar{N}}$	$\frac{(g_{eV}^2+g_{eA}^2)m_e}{E_e}$	0	0	$\frac{E_0^2+2E_0E_e}{3(m_n+m_p)^2}$

The E_e dependencies of $\delta_1 X$ are shown in Fig. 3. In Fig. 3(a), the (orange) short dotted curve represents the results for $\delta_1\bar{Q}$, and the (green) dash-dotted curve represents the results for $\delta_1\bar{N}$. In Fig. 3(b), the (black) solid curve, the (red) dotted curve, and the (blue) dashed curve correspond to $\delta_1\zeta$, $\delta_1\bar{A}$, and $\delta_1\bar{G}$, respectively. The results clearly demonstrate that the differences $\delta_1\zeta$, $\delta_1\bar{A}$, and $\delta_1\bar{G}$ are of the order 10^{-5}, while the difference $\delta_1\bar{N}$ is of the order 10^{-4}, and the difference $\delta_1\bar{Q}$ even reaches the order 10^{-3} for large E_e. These differences arise from two main factors: (1) At the order $O(m_n^{-1})$, our analytical results for $\zeta(E_e)$, $\bar{A}(E_e)$, and $\bar{G}(E_e)$ are consistent with Eqs. (6), (7) in Ref. [20]. However, our expressions for $\bar{N}(E_e)$ and $\bar{Q}(E_e)$ differ slightly from those presented in Ref. [20]. (2) Our analytical results are derived to the order $O(m_n^{-2})$, whereas the analytical results from Ref. [20] are obtained to the order $O(m_n^{-1})$.

In our calculation, we prioritize the analytic expressions by considering the phase space factor β as an input. To illustrate the differences arising from the choice of the phase space factor, we define

$$\delta_2 X \equiv X_{\beta,\mathrm{A}}^{\mathrm{our}} - X_{\beta_{\mathrm{II}},\mathrm{A}}^{\mathrm{our}}. \tag{19}$$

Here, the index "our" refers to our calculation, "β, β_{II}" refer to the input phase space factors, and the index "A" refers to the choice of the parameters f_i and $g_{eV,eA}$ as given in Eq. (18).

The E_e dependence of $\delta_2 X$ is presented in Fig. 4, where the definitions of the curves are the same as those in Fig. 3. The results show that the absolute magnitudes of $\delta_2\zeta$, $\delta_2\bar{A}$, $\delta_2\bar{G}$, $\delta_2\bar{Q}$ increase as E_e increases and are on order 10^{-5}, while $\delta_2\bar{N}$ is not sensitive to E_e and on order 10^{-6}. These differences are natural since the precision of the phase space factor β_{II} is of the order $O(m_n^{-1})$.

Furthermore, to account for the contributions at order 10^{-5}, the parameters f_6, f_7, and f_8 may also have significant effects. To show these contributions, we define the difference as

$$\delta_3 X = X_{\beta,\mathrm{B}}^{\mathrm{our}} - X_{\beta,\mathrm{A}}^{\mathrm{our}}, \tag{20}$$

where the subindex "B" refers to the case with nonzero $f_{6,7,8}$ while keeping the same values of $f_{1,2,3,4,5}$ as in case "A". The values of $f_{6,7,8}$ are taken from Ref. [27] as

$$f_6 = 228,\;\; f_7 = 2.5 f_1,\;\; f_8 = 1.92 f_4. \tag{21}$$

TABLE V. Analytical results for $\mathcal{C}_{ij}^{\bar{Q}}$, where $L_{ij}^{\bar{Q}}$ refers to the factor from the lepton part. Certain contributions, such as $\mathcal{C}_{17}^{\bar{Q}}$ and $\mathcal{C}_{26}^{\bar{Q}}$, are zero and have been excluded in the table.

	$L_{ij}^{\bar{Q}}$	$O(1)$	$O(m_n^{-1})$	$O(m_n^{-2})$
$\mathcal{C}_{11}^{\bar{Q}}$	$g_{eV}^2 + g_{eA}^2$	0	$\frac{E_0-E_e}{3m_n}$	$\frac{(3E_0^2+8E_0E_e-8E_e^2)+(-2E_0+8E_e+3m_e)m_e}{12m_n^2}$
$\mathcal{C}_{12}^{\bar{Q}}$	$g_{eV}^2 + g_{eA}^2$	0	$\frac{2(E_0-E_e)}{3(m_n+m_p)}$	$\frac{2(E_0^2+2E_0E_e)+(-6E_0+15E_e+3m_e)m_e}{6m_n(m_n+m_p)}$
$\mathcal{C}_{13}^{\bar{Q}}$	$g_{eV}^2 + g_{eA}^2$	0	0	$-\frac{(2E_0-5E_e-3m_e)m_e}{6m_n(m_n+m_p)}$
$\mathcal{C}_{14}^{\bar{Q}}$	$g_{eV}g_{eA}$	2	$\frac{2(E_0+2E_e-3m_e)}{3m_n}$	$\frac{16(E_0E_e-E_e^2)+2(4E_0-16E_e-3m_e)m_e}{3m_n^2}$
$\mathcal{C}_{15}^{\bar{Q}}$	$g_{eV}g_{eA}$	0	$\frac{2(-2E_0+2E_e+3m_e)}{3(m_n+m_p)}$	$-\frac{(2E_0^2+8E_0E_e-16E_e^2)+(E_0+17E_e)m_e}{3m_n(m_n+m_p)}$
$\mathcal{C}_{16}^{\bar{Q}}$	$g_{eV}g_{eA}$	0	0	$\frac{(E_0-E_e)m_e}{3m_n(m_n+m_p)}$
$\mathcal{C}_{18}^{\bar{Q}}$	$g_{eV}g_{eA}$	0	0	$\frac{8(E_0E_e-E_e^2)+2(-2E_0+2E_e+3m_e)m_e}{3m_n^2}$
$\mathcal{C}_{22}^{\bar{Q}}$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{(E_0^2-6E_0E_e+8E_e^2)-(4E_0-7E_e)m_e}{6m_n(m_n+m_p)}$
$\mathcal{C}_{23}^{\bar{Q}}$	$g_{eV}^2 + g_{eA}^2$	0	0	$-\frac{(2E_0-5E_e-3m_e)m_e}{6m_n(m_n+m_p)}$
$\mathcal{C}_{24}^{\bar{Q}}$	$g_{eV}g_{eA}$	0	$\frac{4(2E_0-5E_e-3m_e)}{3(m_n+m_p)}$	$\frac{-2(E_0^2-28E_0E_e+48E_e^2)+(21E_0-57E_e+6m_e)m_e}{3m_n(m_n+m_p)}$
$\mathcal{C}_{25}^{\bar{Q}}$	$g_{eV}g_{eA}$	0	0	$\frac{-4(E_0^2-2E_0E_e-2E_e^2)+2(4E_0-E_e-3m_e)m_e}{3(m_n+m_p)^2}$
$\mathcal{C}_{34}^{\bar{Q}}$	$g_{eV}g_{eA}$	0	$\frac{-2m_e}{m_n+m_p}$	$\frac{(5E_0-23E_e-6m_e)m_e}{3m_n(m_n+m_p)}$
$\mathcal{C}_{35}^{\bar{Q}}$	$g_{eV}g_{eA}$	0	0	$\frac{2(2E_0+E_e)m_e}{3(m_n+m_p)^2}$
$\mathcal{C}_{44}^{\bar{Q}}$	$g_{eV}^2 + g_{eA}^2$	1	$\frac{-2E_0+11E_e+3m_e}{3m_n}$	$-\frac{(3E_0^2+40E_0E_e-136E_e^2)+(14E_0-56E_e+15m_e)m_e}{12m_n^2}$
$\mathcal{C}_{45}^{\bar{Q}}$	$g_{eV}^2 + g_{eA}^2$	0	$-\frac{2(2E_0+E_e)}{3(m_n+m_p)}$	$\frac{2(E_0^2-14E_0E_e-8E_e^2)+(-10E_0+E_e+9m_e)m_e}{6m_n(m_n+m_p)}$
$\mathcal{C}_{46}^{\bar{Q}}$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{(2E_0-5E_e-3m_e)m_e}{6m_n(m_n+m_p)}$
$\mathcal{C}_{47}^{\bar{Q}}$	$g_{eV}g_{eA}$	0	0	$\frac{8(E_0E_e-E_e^2)+2(-2E_0+2E_e+3m_e)m_e}{3m_n^2}$
$\mathcal{C}_{48}^{\bar{Q}}$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{16(E_0E_e-E_e^2)+2(2E_0-2E_e+3m_e)m_e}{3m_n^2}$
$\mathcal{C}_{55}^{\bar{Q}}$	$g_{eV}^2 + g_{eA}^2$	0	0	$\frac{E_0^2+2E_0E_e}{3(m_n+m_p)^2}$

It is worth mentioning that our f_6 value is twice that in Ref. [27] due to different definitions of the form factors in Γ_{Wnp}.

In Fig. 5, we present the E_e dependence of $\delta_3 X$, where the definitions of the curves are the same as those in Fig. 3. It can be observed that the absolute magnitudes of $\delta_3\zeta$, $\delta_3\bar{A}$, $\delta_3\bar{G}$,

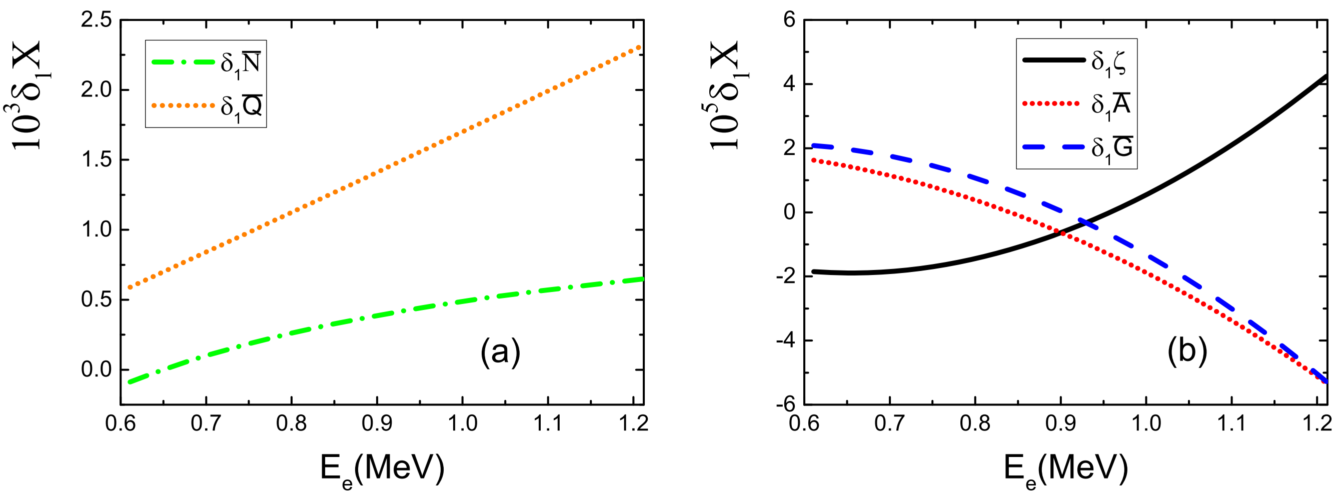

FIG. 3. Numeric results for $\delta_1 X$ vs. E_e, where the left panel displays the result for $X = \bar{N}, \bar{Q}$ and the right panel displays the result for $X = \zeta, \bar{A}, \bar{G}$.

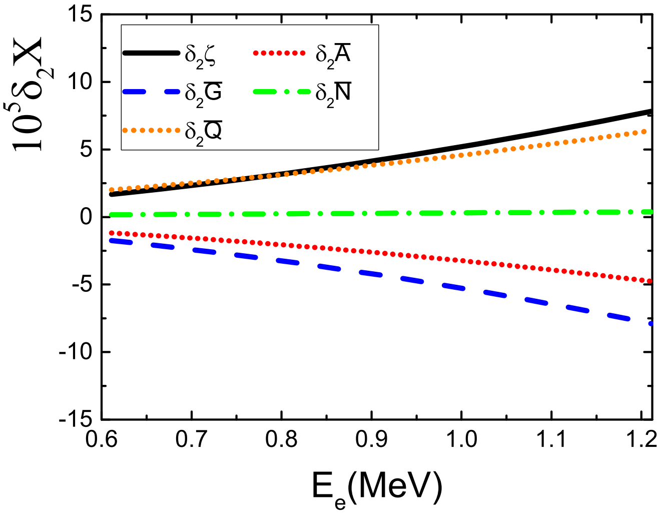

FIG. 4. Numeric results for $\delta_2 X$ vs. E_e where the index X refers to ζ, $\bar{A}$, $\bar{G}$, $\bar{N}$, $\bar{Q}$, respectively.

and $\delta_3\bar{N}$ are on the order 10^{-5}, and the magnitude of $\delta_3\bar{Q}$ even reaches 10^{-4} for large E_e. These properties show that the contributions from f_6, f_7, and f_8 should also be taken into account when aiming for a precision of 10^{-5}.

In practical calculations, we have also checked our results to higher orders, such as $O(m_n^{-4})$, and have found that the results remain consistent with negligible differences.

In summary, the recoil corrections to the correlation coefficients are calculated up to order $O(m_n^{-2})$ when considering a general form of Wnp interactions. Both the analytical and numeric results indicate that our analytic results for $\zeta(E_e)$, $\bar{A}(E_e)$, and $\bar{G}(E_e)$ at the order $O(m_n^{-1})$ are consistent with those presented in Eqs. (6), (7) of Ref. [20], except for differences at the order $O(m_n^{-2})$. However, our analytic results for $\bar{N}(E_e)$ and $\bar{Q}(E_e)$ are slightly different from their results at the order $O(m_n^{-1})$. Numerical results show that contributions from order $O(m_n^{-2})$ and additional form factors of Wnp interactions are required to achieve a precision of 10^{-5}.

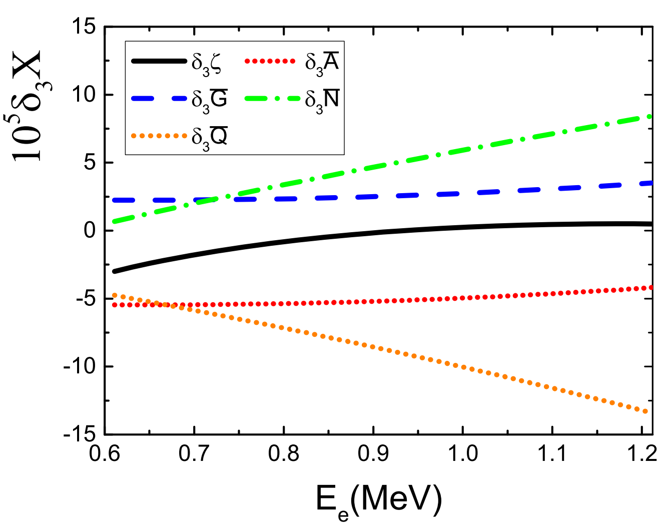

FIG. 5. Numeric results for $\delta_3 X$ vs. E_e where the index X refers to ζ, $\bar{A}$, $\bar{G}$, $\bar{N}$, $\bar{Q}$, respectively.

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APPENDIX A: KINEMATICS

In this Appendix, we present the explicit expressions for the momenta used in the calculation. In the rest frame of the neutron, we take the momenta and corresponding spin vectors of the polarized neutron and polarized electron as follows:

$$\begin{aligned} p_n &\equiv (m_n, \vec{0}), \\ p_e &\equiv (E_e, \vec{p}_e), \\ p_\nu &\equiv (E_\nu, \vec{p}_\nu), \\ p_p &= p_n - p_e - p_\nu = (m_n - E_e - E_\nu, -\vec{p}_e - \vec{p}_\nu), \\ S_n &= (0, \vec{\xi}_n), \\ S_e &= \left(\frac{\vec{p}_e \cdot \vec{\xi}_e}{m_e}, \vec{\xi}_e + \frac{\vec{p}_e(\vec{p}_e \cdot \vec{\xi}_e)}{m_e(E_e + m_e)} \right). \end{aligned} \tag{A1}$$

Furthermore, by utilizing the on-shell condition $p_p^2 = m_p^2$, we can derive the following expression:

$$E_\nu = \frac{m_n(E_0 - E_e)}{m_n - E_e + \vec{p}_e \cdot \vec{n}_\nu}, \tag{A2}$$

where the unit vector $\vec{n}_\nu$ is directed along the antineutrino's three-momentum $\vec{p}_\nu$. This is just the first replacement rule in Eq. (13).

APPENDIX B: THE INTEGRATION OVER THE SOLID ANGLE OF ANTINEUTRINO

After expanding the differential cross section on m_n^{-1}, the integration of $d\Omega_\nu$ can be carried out using the following results:

$$\begin{aligned}
\int(\vec{n}_\nu\cdot\vec{a}_1)d\Omega_\nu &= 0,\\
\int(\vec{n}_\nu\cdot\vec{a}_1)(\vec{n}_\nu\cdot\vec{a}_2)d\Omega_\nu &= \frac{4\pi}{3}\vec{a}_1\cdot\vec{a}_2,\\
\int(\vec{n}_\nu\cdot\vec{a}_1)(\vec{n}_\nu\cdot\vec{a}_2)(\vec{n}_\nu\cdot\vec{a}_3)d\Omega_\nu &= 0,\\
\int(\vec{n}_\nu\cdot\vec{a}_1)(\vec{n}_\nu\cdot\vec{a}_2)(\vec{n}_\nu\cdot\vec{a}_3)(\vec{n}_\nu\cdot\vec{a}_4)d\Omega_\nu &= \frac{4\pi}{15}[(\vec{a}_1\cdot\vec{a}_2)(\vec{a}_3\cdot\vec{a}_4)+(\vec{a}_1\cdot\vec{a}_3)(\vec{a}_2\cdot\vec{a}_4)+(\vec{a}_1\cdot\vec{a}_4)(\vec{a}_2\cdot\vec{a}_3)],\\
\int(\vec{n}_\nu\cdot\vec{a}_1)(\vec{n}_\nu\cdot\vec{a}_2)(\vec{n}_\nu\cdot\vec{a}_3)(\vec{n}_\nu\cdot\vec{a}_4)(\vec{n}_\nu\cdot\vec{a}_5)d\Omega_\nu &= 0,
\end{aligned}\tag{B1}$$

and

$$\begin{aligned}
&\int(\vec{n}_\nu\cdot\vec{a}_1)(\vec{n}_\nu\cdot\vec{a}_2)(\vec{n}_\nu\cdot\vec{a}_3)(\vec{n}_\nu\cdot\vec{a}_4)(\vec{n}_\nu\cdot\vec{a}_5)(\vec{n}_\nu\cdot\vec{a}_6)d\Omega_\nu\\
&= \frac{4\pi}{105}[(\vec{a}_1\cdot\vec{a}_2)(\vec{a}_3\cdot\vec{a}_4)(\vec{a}_5\cdot\vec{a}_6)+(\vec{a}_1\cdot\vec{a}_3)(\vec{a}_2\cdot\vec{a}_4)(\vec{a}_5\cdot\vec{a}_6)+(\vec{a}_1\cdot\vec{a}_4)(\vec{a}_2\cdot\vec{a}_3)(\vec{a}_5\cdot\vec{a}_6)\\
&\quad+(\vec{a}_1\cdot\vec{a}_2)(\vec{a}_3\cdot\vec{a}_5)(\vec{a}_4\cdot\vec{a}_6)+(\vec{a}_1\cdot\vec{a}_3)(\vec{a}_2\cdot\vec{a}_5)(\vec{a}_4\cdot\vec{a}_6)+(\vec{a}_1\cdot\vec{a}_5)(\vec{a}_2\cdot\vec{a}_3)(\vec{a}_4\cdot\vec{a}_6)\\
&\quad+(\vec{a}_1\cdot\vec{a}_2)(\vec{a}_3\cdot\vec{a}_6)(\vec{a}_4\cdot\vec{a}_5)+(\vec{a}_1\cdot\vec{a}_3)(\vec{a}_2\cdot\vec{a}_6)(\vec{a}_4\cdot\vec{a}_5)+(\vec{a}_1\cdot\vec{a}_6)(\vec{a}_2\cdot\vec{a}_3)(\vec{a}_4\cdot\vec{a}_5)\\
&\quad+(\vec{a}_1\cdot\vec{a}_4)(\vec{a}_2\cdot\vec{a}_5)(\vec{a}_3\cdot\vec{a}_6)+(\vec{a}_1\cdot\vec{a}_5)(\vec{a}_2\cdot\vec{a}_4)(\vec{a}_3\cdot\vec{a}_6)+(\vec{a}_1\cdot\vec{a}_4)(\vec{a}_2\cdot\vec{a}_6)(\vec{a}_3\cdot\vec{a}_5)\\
&\quad+(\vec{a}_1\cdot\vec{a}_6)(\vec{a}_2\cdot\vec{a}_4)(\vec{a}_3\cdot\vec{a}_5)+(\vec{a}_1\cdot\vec{a}_5)(\vec{a}_2\cdot\vec{a}_6)(\vec{a}_3\cdot\vec{a}_4)+(\vec{a}_1\cdot\vec{a}_6)(\vec{a}_2\cdot\vec{a}_5)(\vec{a}_3\cdot\vec{a}_4)],
\end{aligned}\tag{B2}$$

where $\vec{a}_i$ are vectors that are independent of Ω_ν.

[1] H. Abele, Prog. Part. Nucl. Phys. **60**, 1 (2008).
[2] H. Abele, M. A. Hoffmann, S. Baessler, D. Dubbers, F. Gluck, U. Muller, V. Nesvizhevsky, J. Reich, and O. Zimmer, Phys. Rev. Lett. **88**, 211801 (2002).
[3] M. Gell-Mann, Phys. Rev. **111**, 362 (1958).
[4] S. Weinberg, Phys. Rev. **112**, 1375 (1958).
[5] D. Dubbers and M. G. Schmidt, Rev. Mod. Phys. **83**, 1111 (2011); F. E. Wietfeldt and G. L. Greene, *ibid.* **83**, 1173 (2011); N. Severijns and O. Naviliat-Cuncic, Annu. Rev. Nucl. Part. Sci. **61**, 23 (2011).
[6] J. D. Jackson, S. B. Treiman, and H. W. Wyld, Phys. Rev. **106**, 517 (1957).
[7] G. Darius *et al.* (CORN Collaboration), Phys. Rev. Lett. **119**, 042502 (2017).
[8] D. Mund, B. Markisch, M. Deissenroth, J. Krempel, M. Schumann, H. Abele, A. Petoukhov, and T. Soldner, Phys. Rev. Lett. **110**, 172502 (2013); B. Märkisch *et al.*, *ibid.* **122**, 242501 (2019); A. Czarnecki, W. J. Marciano, and A. Sirlin, *ibid.* **120**, 202002 (2018).
[9] A. Kozela *et al.*, Phys. Rev. Lett. **102**, 172301 (2009); Phys. Rev. C **85**, 045501 (2012); K. Bodek *et al.*, EPJ Web Conf. **219**, 04001 (2019).
[10] A. Sirlin, Phys. Rev. **164**, 1767 (1967).
[11] A. Sirlin, Phys. Rev. D **35**, 3423 (1987); A. Sirlin and R. Zucchini, Phys. Rev. Lett. **57**, 1994 (1986); W. Jaus and G. Rasche, Nucl. Phys. A **143**, 202 (1970).
[12] W. Jaus and G. Rasche, Phys. Rev. D **35**, 3420 (1987).
[13] W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **56**, 22 (1986).
[14] W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **96**, 032002 (2006).
[15] S. Ando *et al.*, Phys. Lett. B **595**, 250 (2004).
[16] C. Y. Seng, M. Gorchtein, H. H. Patel, and M. J. Ramsey-Musolf, Phys. Rev. Lett. **121**, 241804 (2018).
[17] B. R. Holstein, Rev. Mod. Phys. **46**, 789 (1974); **48**, 673 (1976); D. H. Wilkinson, Nucl. Phys. A **377**, 474 (1982).
[18] V. Cirgiliano *et al.*, arXiv:1907.02164.
[19] A. N. Ivanov *et al.*, Phys. Rev. D **88**, 073002 (2013).
[20] A. N. Ivanov, R. Hollwieser, N. I. Troitskaya, M. Wellenzohn, and Y. A. Berdnikov, Phys. Rev. C **95**, 055502 (2017); **104**, 069901(E) (2021).
[21] A. N. Ivanov *et al.*, Results Phys. **21**, 103806 (2021).
[22] A. N. Ivanov, R. Hollwieser, N. I. Troitskaya, M. Wellenzohn, and Y. A. Berdnikov, Phys. Rev. D **104**, 033006 (2021).
[23] P. A. Zyla *et al.* (Particle Data Group), PTEP **2020**, 083C01 (2020).
[24] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).
[25] J. D. Jackson, S. B. Treiman, and H. W. Wyld, Jr., Z. Phys. **150**, 640 (1958); E. J. Konopinski, *The Theory of Beta Radioactivity* (Clarendon, Oxford, 1966).
[26] V. Shtabovenko, R. Mertig, and F. Orellana, Comput. Phys. Commun. **256**, 107478 (2020).
[27] J. L. García-Luna and A. Garcia, J. Phys. G **32**, 333 (2006).
[28] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevC.109.025204 for our Mathematica code used to calculate the correlation coefficients.