# Descent of the nucleus from the fission barrier in the presence of long-range memory effects

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The peculiarities of descent of the nucleus from the fission barrier are investigated within the generalized Langevin equation with the power-law memory function $f(t-t')=(|t-t'|/\tau)^{-\alpha}$. It is observed that there is much stronger slowing down of the descent in the presence of long-range memory effects, caused by the power-law memory function at $0<\alpha<1$, than of short-range memory effects, generated by the exponential memory function $f(t-t')=\exp(-|t-t'|/\tau)$. At a specific value of the exponent $\alpha=1/2$ of the power-law memory function, it becomes possible to find analytically the trajectory of descent and demonstrate that the long-range memory effects give rise to complex time oscillations of nuclear shape. One found fairly long ($>10^{-20}$ s) times of descent of $^{236}$U at the correlation times $\tau\in[10^{-24},10^{-22}]$ s.

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## I. INTRODUCTION

The Langevin equation [1–3] is a powerful tool under the transport description of different dynamical processes in systems of many interacting particles. In the case of nuclear many-body systems, the Langevin approaches have been used to describe fission [4], fusion [5], and deep-inelastic processes [6]. All these Langevin approaches are based on the separation of nuclear degrees of freedom into several macroscopic (collective) $q(t)$ and a large number of microscopic (nucleonic) modes of motion [7,8]. The latter constitutes a heat bath with temperature $T$, exerting a friction $\kappa_0\int_0^t f(t-t')[dq/dt](t')dt'$ and a random $\xi(t)$ forces on collective motion. The friction and random forces are related to each other through the fluctuation-dissipation theorem [8] and are defined by the time-spread memory function $f(t-t')$, representing the complex energy flow between collective and nucleonic degrees of freedom [9,10]. In the literature, there is a controversial opinion on the importance of such non-Markovian (memory) effects in nuclear large-amplitude collective dynamics; see, for example, Refs. [8,11–18]. Importantly, all these non-Markovian studies of the nuclear collective dynamics use different versions of exponential memory function $f(t-t')=\exp(-|t-t'|/\tau)$, where $\tau$ is the relaxation (correlation) time. It has been demonstrated [13,16] that non-Markovian descent from the fission barrier may be accompanied by characteristic oscillations of nuclear shape. Memory effects show nonmonotonic dependence on the correlation time $\tau$: at the limits of quite small and fairly large values of the correlation time, nuclear collective dynamics becomes Markovian [13,19]. Such memory effects can be considered to be of short-range type as far as they are only prominent in a sufficiently short interval of the correlation times, which are comparable to the reciprocal of the characteristic frequency of nuclear collective motion [12,13].

In the present study I investigate the nuclear fission dynamics on a descent from the top of the fission barrier to scission by the help of the generalized Langevin equation with a power-law memory function $f(t-t')=(|t-t'|/\tau)^{-\alpha}$. Such a memory function has been successfully used under the generalized Langevin description of many dynamical systems, exhibiting anomalous diffusion behavior [20–23]. The anomalous character of diffusion there is reflected in the fractional time dependence of the mean square displacement of the system and in the power-law decay of its velocity autocorrelation function [24,25]. These remarkable features of the anomalous diffusion process are caused by the presence of long-range memory effects [26]: memory effects existing over a broad range of timescales of the system's dynamics [27].

The plan of the paper is as follows. In Sec. II, the generalized Langevin equation of motion for a nuclear shape variable is established. Section III is devoted to the discussion of the features of the generalized Langevin dynamics, subject to an exponential memory function. In Sec. IV, an analytical solution to the Langevin equation of motion, governed by a power-law memory function, is presented. In Sec. V, distribution of times of descent from the fission barrier is discussed. A summary and conclusions are given in Sec. VI.

## II. THE GENERALIZED LANGEVIN EQUATION OF MOTION

I start by postulating the generalized Langevin equation of motion for a single nuclear shape variable $q(t)$,

$$M(q)\frac{d^2q(t)}{dt^2}=-\frac{1}{2}\frac{\partial M(q)}{\partial q}\left(\frac{dq(t)}{dt}\right)^2-\frac{\partial E_{\rm pot}(q)}{\partial q}-\kappa_0\int_0^t f(t-t')\frac{dq(t')}{dt'}dt'+\xi(t), \quad (1)$$

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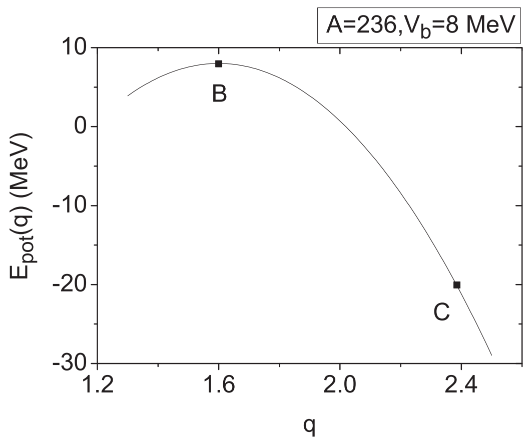


FIG. 1. Schematic representation of the parabolic fission barrier [Eq. (4)], characterized by a saddle point B and a scission point C.

In Eq. (1), $M(q)$ is a collective mass parameter and $\kappa_0$ is the strength of the retarded friction force, related to the random force $\xi(t)$ through the fluctuation-dissipation theorem,

$$\langle\xi(t)\xi(t')\rangle = T\kappa_0 f(t-t'). \tag{2}$$

Here, $T$ is temperature of a nucleus and the ensemble averaging $\langle\cdots\rangle$ is performed over all random realizations of a Gaussian stationary process $\xi(t)$. The memory function $f(t-t')$ in Eqs. (1) and (2) is assumed to be a decaying function of its argument $x \equiv |t-t'|/\tau$,

$$f(x) \to 0, \quad x \to \infty, \tag{3}$$

where the correlation time $\tau$ measures both the time spread of the retarded friction force and the timescale over which the values of the random force $\xi(t)$ and $\xi(t')$ correlate significantly.

On a descent part from the top $q = q_b$ of the fission barrier (point B in Fig. 1) to scission (point C in Fig. 1), the collective potential energy of a nucleus $E_{\rm pot}(q)$ in Eq. (1) can be approximated by the inverted parabolic dependence on $q$,

$$E_{\rm pot}(q) = V_b - \left(M_b\omega_b^2/2\right)(q-q_b)^2. \tag{4}$$

Here, $V_b$ is the height of the fission barrier and $\omega_b$ is the frequency parameter, defining the nuclear stiffness at $q = q_b$,

$$\omega_b = \sqrt{\frac{1}{M_b}\left|\frac{d^2E_{\rm pot}(q)}{dq^2}\right|_{q=q_b}}, \quad M_b = M(q=q_b). \tag{5}$$

The mass parameter $M(q)$ in Eq. (1) is taken as

$$M(q) = \frac{1}{5}Am_0R_0^2\left(1+\frac{1}{2q^3}\right), \tag{6}$$

where $A$ is a mass number of the nucleus, $m_0$ is the nucleon mass, and $R_0 = r_0A^{1/3}$ is a radius of the equal volume spherical nucleus. In Fig. 1, the parabolic fission barrier $E_{\rm pot}(q)$ (4) is shown for the following set of the parameters [28]:

$$A = 236, \quad V_b = 8\ {\rm MeV}, \quad q_b = 1.6, \quad \hbar\omega_b = 1.16\ {\rm MeV}. \tag{7}$$

### General solution of the generalized Langevin equation of motion

In attempt to find an analytical solution of the generalized Langevin equation of motion for the nuclear shape variable $q(t)$, I linearized Eq. (1) in the vicinity of the saddle point $q_b$ and obtained for a displacement, $\Delta q(t) \equiv q(t) - q_b$, the following equation:

$$\frac{d^2\Delta q(t)}{dt^2} = \omega_b^2\Delta q(t) - (\kappa_0/M_b)\int_0^t f(t-t')\frac{d\Delta q(t')}{dt'}dt' + (1/M_b)\xi(t), \tag{8}$$

where $M_b$ is given by Eq. (5). The general solution of Eq. (8), subject to the initial conditions

$$\Delta q(t=0) = 0, \quad [d\Delta q/dt](t=0) = v_0 > 0, \tag{9}$$

can be represented as

$$\Delta q(t) = B(t)v_0 + (1/M_b)\int_0^t B(t-t')\xi(t')dt', \tag{10}$$

where $v_0$ is the initial velocity of the nuclear system, and $B(t)$ is a solution of the homogeneous equation,

$$\frac{d^2B(t)}{dt^2} = \omega_b^2B(t) - (\kappa_0/M_b)\int_0^t f(t-t')\frac{dB(t')}{dt'}dt', \quad B(t=0) = 0, \quad [dB/dt](t=0) = 1. \tag{11}$$

This linear integrodifferential equation can be solved via the Laplace transformation method as

$$\tilde{B}(s) = \frac{1}{s^2 + (\kappa_0/M_b)s\tilde{f}(s) - \omega_b^2}, \tag{12}$$

where $\tilde{B}(s)$ and $\tilde{f}(s)$ are the Laplace transforms of the solution function $B(t)$ and of the memory function $f(t)$, respectively. If the denominator of the rational expression in Eq. (12) has in total $N$ ordered zeros,

$${\rm Re}(s_1) \geqslant {\rm Re}(s_2) \geqslant \cdots \geqslant {\rm Re}(s_N), \tag{13}$$

then the first zero $s_1$ will define a behavior of the solution function in the long-time limit:

$$B(t) \sim {\rm e}^{s_1t}, \quad t \to \infty. \tag{14}$$

## III. EXPONENTIAL MEMORY FUNCTION

One first considers the features of non-Markovian dynamics of descent from the fission barrier, governed by an exponential memory function,

$$f(t-t') = \exp\left(-\frac{|t-t'|}{\tau}\right). \tag{15}$$

This memory function recovers two Markovian limits of Eq. (11). In the limit $\tau \ll 1/\omega_b$, the time integral in Eq. (11) can be evaluated by parts,

$$-\kappa_0\int_0^t \exp\left(-\frac{|t-t'|}{\tau}\right)\frac{dB(t')}{dt'}dt' \approx -\kappa_0\tau\frac{dB(t)}{dt}, \quad \omega_b\tau \ll 1, \tag{16}$$

giving rise to the appearance of a time local (Markovian) friction with the $\tau$-dependent friction coefficient $\kappa_0\tau$. The corresponding value of the solution function $B(t)$ is given by

$$B(t) = \frac{1}{s_1 - s_2}\left(e^{s_1 t} - e^{s_2 t}\right),$$
$$s_{1,2} = -\frac{\kappa_0\tau}{2M_b} \pm \sqrt{\left(\frac{\kappa_0\tau}{2M_b}\right)^2 + \omega_b^2}. \quad (17)$$

In the opposite limit $\tau \gg 1/\omega_b$, the time integral in Eq. (11) is reduced to a Markovian restorative force,

$$-\kappa_0 \int_0^t \exp\left(-\frac{|t-t'|}{\tau}\right)\frac{dB(t')}{dt'}dt' \approx -\kappa_0 B(t), \quad \omega_b\tau \gg 1. \quad (18)$$

The corresponding solution function $B(t)$ is either exponentially growing with time,

$$B(t) = \frac{1}{s_1 - s_2}\left(e^{s_1 t} - e^{s_2 t}\right),$$
$$s_{1,2} = \pm\sqrt{\omega_b^2 - \kappa_0/M_b}, \quad \frac{\kappa_0}{M_b\omega_b^2} < 1, \quad (19)$$

or oscillates with time,

$$B(t) = \frac{1}{\Omega}\sin(\Omega t), \quad \Omega \equiv |\mathrm{Im}(s_1, s_2)| = \sqrt{\kappa_0/M_b - \omega_b^2},$$
$$\frac{\kappa_0}{M_b\omega_b^2} \geqslant 1, \quad (20)$$

depending on the value of $\kappa_0/(M_b\omega_b^2)$. In the Markovian restorative limit given by Eq. (20), the nuclear system $q(t)$ remains in the close vicinity of the saddle point $q_b$ infinitely long.

In general, the time-retarded force in Eq. (11) contains both a time-irreversible (friction) and time-reversible (restorative) components,

$$-\kappa_0 \int_0^t \exp\left(-\frac{|t-t'|}{\tau}\right)\frac{dB(t')}{dt'}dt'$$
$$= -\gamma(t,\tau)dB(t)/dt - \kappa(t,\tau)B(t), \quad (21)$$

where $\gamma(t,\tau)$ is the time-dependent friction coefficient and $\kappa(t,\tau)$ is the time-dependent spring coefficient. The corresponding solution function $B(t)$ can be found by substituting the Laplace transform $\tilde{f}(s) = 1/(s - 1/\tau)$ of the exponential memory function (15) into Eq. (12):

$$\tilde{B}(s) = \frac{s - 1/\tau}{s^3 + (1/\tau)s^2 + (\kappa_0/M_b - \omega_b^2)s - 1/\tau} = \frac{C_1}{s - s_1}$$
$$+ \frac{C_2}{s - s_2} + \frac{C_3}{s - s_3}. \quad (22)$$

By performing the inverse Laplace transform of Eq. (22), one obtains the solution function:

$$B(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + C_3 e^{s_3 t}, \quad (23)$$

where the constants $C_1$, $C_2$, and $C_3$ are defined by the initial conditions,

$$C_i = (s_i + 1/\tau)\prod_{j=1(j\neq i)}^{3}\frac{1}{(s_i - s_j)}, \quad i = \overline{1,3}, \quad (24)$$

and $s_1, s_2, s_3$ are the three roots of the cubic secular equation:

$$(s/\omega_b)^3 + \frac{1}{\omega_b\tau}(s/\omega_b)^2 + \left(\frac{\kappa_0}{M_b\omega_b^2} - 1\right)(s/\omega_b) - \frac{1}{\omega_b\tau} = 0. \quad (25)$$

This equation always has one real positive root, $s_1 > 0$, while the other two roots $s_2$ and $s_3$ may be either both real and negative or complex conjugated. In the latter case, memory effects in the dynamics of descent [Eq. (11)] are quite prominent and Eq. (23) for the corresponding solution function $B(t)$ can be rewritten as

$$B(t) = C_1 e^{s_1 t} + [C_+\cos(\Omega t) + C_-\sin(\Omega t)]e^{-\Gamma t}, \quad (26)$$

where

$$\Omega = |\mathrm{Im}(s_2, s_3)|, \quad \Gamma = |\mathrm{Re}(s_2, s_3)|, \quad C_\pm = C_2 \pm C_3. \quad (27)$$

### IV. POWER-LAW MEMORY FUNCTION

Now, one can investigate the peculiarities of the non-Markovian dynamics of descent, subject to a power-law memory function,

$$f(t - t') = (|t - t'|/\tau)^{-\alpha}. \quad (28)$$

It is determined by the exponent $\alpha$ of the power-law memory function and by the dimensionless strength of the retarded friction force

$$\rho_\alpha = \frac{\kappa_0}{M_b\omega_b^2}(\omega_b\tau)^\alpha. \quad (29)$$

In the following, it is assumed that $\kappa_0/(M_b\omega_b^2) = 42$, taken from the nuclear Fermi-liquid model study [18] of nuclear fission dynamics.

For the power-law memory function $f(t-t') = (|t - t'|/\tau)^{-\alpha}$ [$\tilde{f}(s) = \tau^\alpha\Gamma(1-\alpha)/s^{1-\alpha}$], the long-time behavior $e^{s_1 t}$ of the solution function $B(t)$ is defined by the largest positive root $s_1$ of the secular equation

$$(s/\omega_b)^2 + \rho_\alpha\Gamma(1-\alpha)(s/\omega_b)^\alpha - 1 = 0, \quad (30)$$

where $\Gamma(x)$ is the gamma function. In Fig. 2, the values of $s_1$ [Eq. (30)] are plotted as functions of the correlation time $\tau$ at different values $\alpha = 1/4$, $1/2$, and $3/4$ of the exponent $\alpha$ of the power-law memory function $f(x) = x^{-\alpha}$. The largest positive root $s_1$ of the cubic secular equation (25), corresponding to the exponential memory function $f(x) = \exp(-x)$, is shown in Fig. 2 by the dashed line. As can be seen from Fig. 2, the non-Markovian dynamics of descent is significantly decelerated in the long-time limit [Eq. (14)]. The deceleration is much stronger in the case of the long-range $f(t-t') = (|t-t'|/\tau)^{-\alpha}$ than the short-range $f(t - t') = \exp(-|t-t'|/\tau)$ memory effects in Eq. (11).

Due to significant suppression of the exponentially unstable component $e^{s_1 t}$ of the solution function $B(t)$, one would

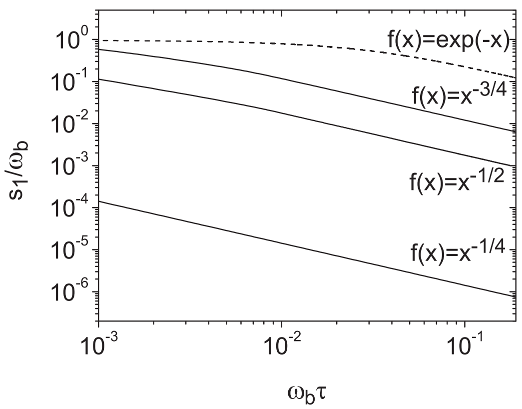


FIG. 2. The largest positive root $s_1$ of the secular equation (30) versus the correlation time $\tau$ at different values $\alpha = 1/4$, $1/2$, and $3/4$ of the exponent $\alpha$ of the power-law memory function $f(x) = x^{-\alpha}$. The dashed line represents the largest positive root $s_1$ of the cubic secular equation (25), corresponding to the exponential memory function $f(x) = \exp(-x)$.

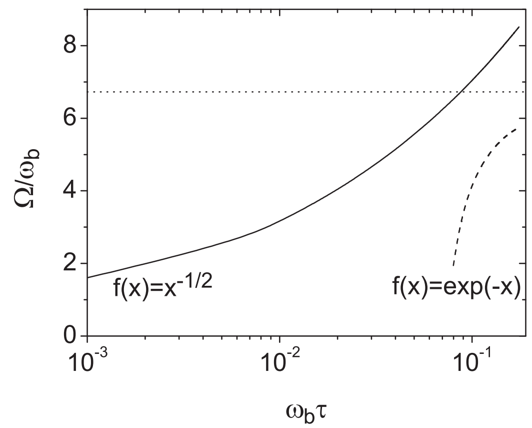


FIG. 3. Frequency $\Omega$ of memory-induced oscillations, produced either by the power-law $f(x) = x^{-1/2}$ (solid line) or the exponential $f(x) = \exp(-x)$ (dashed line) memory function in the deterministic non-Markovian dynamics of descent [Eq. (11)]. The dotted line is the frequency of oscillations [Eq. (20)], corresponding to the Markovian restorative limit [Eq. (18)] of Eq. (11).

like to know the entire time evolution of $B(t)$. With that purpose, I only considered a particular case of $\alpha = 1/2$ in Eq. (28). At this value of the exponent $\alpha$ of the power-law memory function, Eq. (12) reads

$$\tilde{B}(s) = \frac{1}{s^2 + (\kappa_0\sqrt{\pi\tau}/M_b)\sqrt{s} - \omega_b^2}. \tag{31}$$

The denominator of this expression can be factorized as

$$s^2 + (\kappa_0\sqrt{\pi\tau}/M_b)\sqrt{s} - \omega_b^2 = (\sqrt{s} - \mu_1)(\sqrt{s} - \mu_2) \times (\sqrt{s} - \mu_3)(\sqrt{s} - \mu_4). \tag{32}$$

This allows one to make a decomposition in Eq. (31),

$$\tilde{B}(s) = \frac{C_1}{\sqrt{s} - \mu_1} + \frac{C_2}{\sqrt{s} - \mu_2} + \frac{C_3}{\sqrt{s} - \mu_3} + \frac{C_4}{\sqrt{s} - \mu_4}, \tag{33}$$

which, in turn, gives rise to the following solution function:

$$B(t) = \sum_{i=1}^{4} C_i\mu_i\big(1 + \mathrm{erf}(\mu_i\sqrt{t})\big)e^{\mu_i^2 t}, \tag{34}$$

Here, erf$(x)$ is the error function,

$$C_i = \prod_{j\neq i}^{4} \frac{1}{\mu_i - \mu_j}, \quad i = \overline{1, 4}, \tag{35}$$

and $\mu_1, \mu_2, \mu_3, \mu_4$ are four roots of the quartic secular equation:

$$(\mu/\sqrt{\omega_b})^4 + \rho_{1/2}\sqrt{\pi}(\mu/\sqrt{\omega_b}) - 1 = 0, \tag{36}$$

where $\rho_{1/2}$ is given by Eq. (29) at $\alpha = 1/2$. According to the Viet theorem, the last equation always has two real roots (one is positive, another is negative) and two complex conjugated roots such that

$$\mu_1 > 0, \quad \mu_2 < 0, \quad \mu_3 = \mu_4^*, \quad 0 < |\mathrm{Re}(\mu_3, \mu_4)| < |\mathrm{Im}(\mu_3, \mu_4)|. \tag{37}$$

Therefore, one obtains for the first term in the right-hand side of Eq. (34),

$$C_1\mu_1(1 + \mathrm{erf}(\mu_1\sqrt{t}))e^{\mu_1^2 t} \to 2C_1\mu_1 e^{\mu_1^2 t}, \quad \mu_1^2 t \to \infty. \tag{38}$$

As far as $\mu_2 < 0$, the second term in the right-hand side of Eq. (34),

$$C_2\mu_2(1 + \mathrm{erf}(\mu_2\sqrt{t}))e^{\mu_2^2 t} \to \frac{C_2}{|\mu_2|\sqrt{t}}, \quad \mu_2^2 t \to \infty, \tag{39}$$

shows an algebraic decay with time. The last two terms in Eq. (34) can be rewritten as

$$[C_+(t)\cos(\Omega t) + C_-(t)\sin(\Omega t)]e^{-\Gamma t}, \tag{40}$$

where

$$\Omega = \big|\mathrm{Im}\big(\mu_3^2, \mu_4^2\big)\big|, \quad \Gamma = \big|\mathrm{Re}\big(\mu_3^2, \mu_4^2\big)\big|,$$
$$C_\pm(t) = C_3(1 + \mathrm{erf}(\mu_3\sqrt{t})) \pm C_4(1 + \mathrm{erf}(\mu_4\sqrt{t})). \tag{41}$$

In Figs. 3 and 4, the frequency $\Omega$ and damping rate $\Gamma$ of memory-induced time oscillations are compared for the power-law $f(x) = x^{-1/2}$ [Eq. (41)] (solid lines) and the exponential $f(x) = \exp(-x)$ [Eq. (27)] (dashed lines) memory functions in Eq. (11). The dotted line in Fig. 3 is the frequency [Eq. (20)] of oscillations of the nuclear coordinate $q(t)$ in the Markovian restorative limit [Eq. (18)].

Both characteristics, the frequency $\Omega$ and damping rate $\Gamma$ of the memory-induced oscillations, turned out to be not very sensitive to the choice of the memory function $f(t - t')$ in the deterministic non-Markovian equation (11), determining the descent of the nucleus from the fission barrier. There is only a difference in tendency of the damping rate $\Gamma$ with the correlation time $\tau$. The long-range memory effects

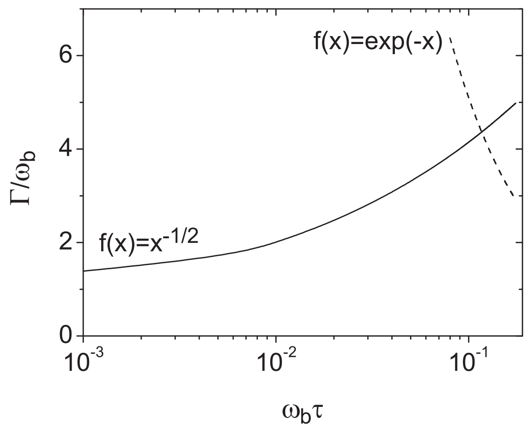


FIG. 4. The same as in Fig. 3 but for damping rate Γ of memory-induced time oscillations.

$f(t-t') = (|t-t'|/\tau)^{-1/2}$ induce more damped oscillations, while the short-range ones $f(t-t') = \exp(-|t-t'|/\tau)$ give rise to undamped time oscillations in the limit $\tau \to \infty$.

Figure 5 shows the entire time evolution of the solution function $B(t)$ [Eq. (34)] at the correlation times $\omega_b\tau = 10^{-3}$ [panel (a)] and $\omega_b\tau = 10^{-2}$ [panel (b)]. At both values of the correlation time, the deterministic non-Markovian dynamics of descent [Eq. (11)], subject to the exponential memory function $f(t-t') = \exp(-|t-t'|/\tau)$, remains Markovian; see Eqs. (16) and (17). At the same time, the character of the dynamics of descent [Eq. (11)], governed by the power-law memory function $f(t-t') = (|t-t'|/\tau)^{-1/2}$, changes drastically if one goes from $\omega_b\tau = 10^{-3}$ to $\omega_b\tau = 10^{-2}$. In the first case [Fig. 5(a)] the deterministic trajectory of descent exponentially grows with time, while in the second case [Fig. 5(b)] it becomes an oscillation with time. In other words, the long-range memory effect $f(t-t') = (|t-t'|/\tau)^{-1/2}$ on the dynamics of descent [Eq. (11)] is quite weak at $\omega_b\tau = 10^{-3}$ and fairly large at $\omega_b\tau = 10^{-2}$. The transition between these two regimes occurs at $\omega_b\tau \approx 2\times 10^{-3}$, which differs significantly from the condition $\omega_b\tau \sim 1$, defining the onset of the short-range memory effects $f(t-t') = \exp(-|t-t'|/\tau)$ in Eq. (11).

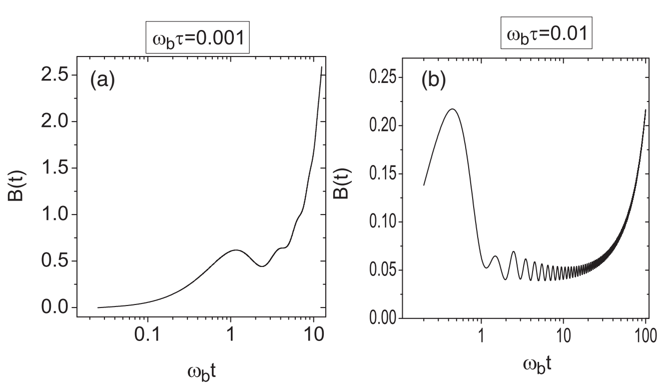


FIG. 5. Analytical solution $B(t)$ [Eq. (34)] to the deterministic non-Markovian dynamics of descent [Eq. (11)], governed by the power-law memory function $f(x) = x^{-1/2}$. In (a) $B(t)$ is shown for the correlation time $\omega_b\tau = 10^{-3}$, when the long-range memory effects are of a minor importance. In (b) $B(t)$ is shown for the correlation time $\omega_b\tau = 10^{-2}$, when the long-range memory effects are well pronounced.

## V. TIME OF DESCENT

One of the fission's characteristics is the time of motion of the nuclear system from the top $q_b$ of the fission barrier to a scission point $q_{sc}$ [point $C$ in Fig. 1]. The position of $q_{sc}$ can be determined through the condition [28]:

$$E_{\rm pot}(q_b) - E_{\rm pot}(q_{sc}) = -20 \text{ MeV}. \tag{42}$$

Here, it is assumed that all trajectories of descent $q(t)$ start at the top $q_b$ of the fission barrier with some positive velocity $v_0$, defined by the initial kinetic energy of a nuclear system [28]:

$$\frac{1}{2}M_b v_0^2 = \frac{\pi T}{4}. \tag{43}$$

### A. Simulation of the trajectories of descent

To simulate possible trajectories of descent $q(t)$ according to Eq. (10), one has to model the Gaussian distributed random force term $\xi(t)$ with the zero mean value and the autocorrelation function $\langle\xi(t)\xi(t')\rangle \sim (|t-t'|/\tau)^{-\alpha}$ [Eq. (2)]. $\xi(t)$ is treated as a fractional Gaussian noise [29]:

$$\xi(t) = \sqrt{\frac{2T\kappa_0\tau^\alpha}{\mathbb{H}(2\mathbb{H}-1)}}\frac{d\mathcal{B}_{\mathbb{H}}(t)}{dt}. \tag{44}$$

In the last equation, $\mathcal{B}_{\mathbb{H}}(t)$ is a process of fractional Brownian motion [29], whose statistical properties are given by

$$\langle\mathcal{B}_{\mathbb{H}}(t)\rangle = 0,$$
$$\langle\mathcal{B}_{\mathbb{H}}(t)\mathcal{B}_{\mathbb{H}}(t')\rangle = (|t|^{2\mathbb{H}} - |t-t'|^{2\mathbb{H}} + |t'|^{2\mathbb{H}})/2, \tag{45}$$

where $\mathbb{H}$ is the Hurst index $(1/2 < \mathbb{H} < 1)$. Since $\langle[d\mathcal{B}_{\mathbb{H}}(t)/dt][d\mathcal{B}_{\mathbb{H}}(t')/dt']\rangle \sim |t-t'|^{2\mathbb{H}-2}$, one can relate the Hurst index to the exponent $\alpha$ of the power-law memory function $f(t-t') = (|t-t'|/\tau)^{-\alpha}$,

$$\mathbb{H} = 1 - \alpha/2. \tag{46}$$

With the help of Eq. (44), one can define the descent trajectories [Eq. (10)] as a stochastic integral over the fractional Brownian motion $\mathcal{B}_{\mathbb{H}}(t)$ [Eq. (45)],

$$q(t) = q_b + B(t)v_0 + \sqrt{\frac{1}{\mathbb{H}(2\mathbb{H}-1)}\frac{2T}{M_b\omega_b^2}\frac{\kappa_0}{M_b\omega_b^2}}\tau^{\alpha/2} \times \int_0^t B(t-t')d\mathcal{B}_{\mathbb{H}}(t'). \tag{47}$$

The stochastic integral here is considered within the Stratonovich definition [30]. The integration over continuous time variable $t$ in Eq. (47) can approximately be replaced by a summation over discrete moments of time $t_i = i\Delta t$,

$i = \overline{0, L-1}$:

$$\int_0^t B(t-t')d\mathcal{B}_{\mathbb{H}}(t') \approx \sum_{i=0}^{L-1} B(t-[t_{i+1}+t_i]/2)[\mathcal{B}_{\mathbb{H}}(t_{i+1}) - \mathcal{B}_{\mathbb{H}}(t_i)] = (\Delta t)^{\mathbb{H}} \sum_{i=0}^{L-1} B(t-[t_{i+1}+t_i]/2)(Y_{i+1}-Y_i). \quad (48)$$

In Eq. (48), $\mathcal{B}_{\mathbb{H}}(t_i) = (\Delta t)^{\mathbb{H}} Y_i$, and $Y_i$ are Gaussian distributed random numbers with

$$\langle Y_i \rangle = 0, \ \langle Y_i Y_k \rangle = (i^{2\mathbb{H}} - |i-k|^{2\mathbb{H}} + k^{2\mathbb{H}})/2, \quad i, k = \overline{0, L}. \quad (49)$$

In the calculations, $Y_i$ were simulated with the help of the circulant matrix method [31]. Thus, for each realization of the random numbers $Y_i$, $i = \overline{0, L}$ [Eq. (49)], the trajectory of descent $q(t)$ can be found as

$$q(t) = q_b + B(t)v_0 + \sqrt{\frac{16T}{3M_b\omega_b^2}\frac{\kappa_0}{M_b\omega_b^2}}(\omega_b\tau)^{1/4}(\omega_b\Delta t)^{3/4} \times \sum_{i=0}^{L-1} B(t-[t_{i+1}+t_i]/2)(Y_{i+1}-Y_i), \quad (50)$$

where $B(t)$ is given by Eq. (34) and $\mathbb{H} = 3/4$ [$\alpha = 1/2$].

In the case of the random force term $\xi(t)$ in Eq. (10) with an exponentially decaying autocorrelation function $\langle \xi(t)\xi(t')\rangle \sim \exp(-|t-t'|/\tau)$, $\xi(t)$ can be modeled as a colored Gaussian noise [32]:

$$\xi(t) = \sqrt{2T\kappa_0/\tau}\int_0^t \exp\left(-\frac{|t-t'|}{\tau}\right)d\mathcal{B}_{1/2}(t'), \quad (51)$$

where $\mathcal{B}_{1/2}(t)$ is the fractional Brownian motion $\mathcal{B}_{\mathbb{H}}(t)$ [Eq. (46)] at the Hurst index $\mathbb{H} = 1/2$.

Integrating by parts in Eq. (51), one can rewrite Eq. (10) in the following form:

$$q(t) = q_b + B(t)v_0 + \sqrt{\frac{2T}{M_b\omega_b^2}\frac{\kappa_0}{M_b\omega_b^2}\frac{1}{\tau}}\int_0^t A(t-t')d\mathcal{B}_{1/2}(t'), \quad (52)$$

where

$$A(t) = \omega_b^2 \int B(t)dt = \omega_b^2((C_1/s_1)e^{s_1 t} + (C_2/s_2)e^{s_2 t} + (C_3/s_3)e^{s_3 t}), \quad (53)$$

and the constants $C_1$, $C_2$. and $C_3$ are given by Eq. (24). If one further replaces the time integration in Eq. (52) by a summation over discrete moments of time $t_i = i\Delta t$, $i = \overline{0, L-1}$, then one obtains the following expression for the trajectories of descent:

$$q(t) = q_b + B(t)v_0 + \sqrt{\frac{2T}{M_b\omega_b^2}\frac{\kappa_0}{M_b\omega_b^2}}(\omega_b\tau)^{-1/2}(\omega_b\Delta t)^{1/2} \times \sum_{i=0}^{L-1} A(t-[t_{i+1}+t_i]/2)(X_{i+1}-X_i), \quad (54)$$

where $X_i$ are independent normally distributed random numbers with the zero mean value and unit variance.

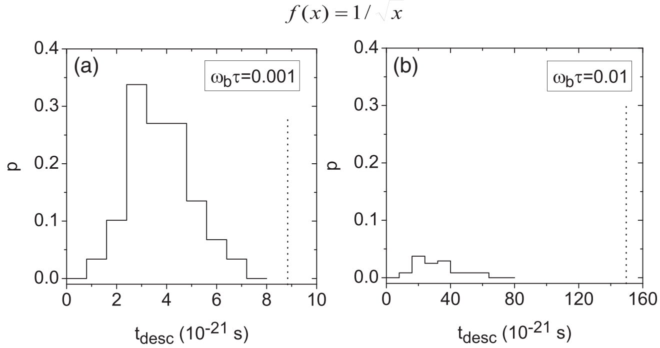


FIG. 6. Histogram of the probability density distribution, $p$, of times of descent $t_{\rm desc}$ for a nucleus $^{236}$U at constant temperature $T = 2$ MeV. $t_{\rm desc}$ were obtained as times of the first hit of the trajectories of descent $q(t)$ [Eq. (50)] with the scission point $q_{sc}$ [Eq. (42)]. In (a) $p$ is plotted for the correlation time $\omega_b\tau = 10^{-3}$, at which the long-range memory effects, determined by the power-law memory function $f(x) = x^{-1/2}$ and the fractional Gaussian noise [Eq. (44)] in the generalized Langevin equation of motion (1), are relatively weak. In (b) $p$ is plotted for the correlation time $\omega_b\tau = 10^{-2}$, at which the long-range memory effects are sufficiently strong. Vertical dotted lines are times of the descent, calculated in the absence of the random force term in Eq. (1).

### B. Distribution of times of descent

I performed calculations of times of descent $t_{\rm desc}$ for a nucleus $^{236}$U at constant temperature $T = 2$ MeV. The results of the calculations are presented in Fig. 6 for the trajectories of descent [Eq. (50)] which are the solutions of the generalized Langevin equation of motion (1) with the power-law memory function $f(x) = x^{-1/2}$ and the fractional Gaussian noise [Eq. (44)], and in Fig. 7 for the trajectories of descent [Eq. (54)] which are the solutions of Eq. (1) with the exponential memory function $f(x) = \exp(-x)$ and the colored Gaussian noise [Eq. (51)]. Panels (a) of the figures correspond to the values of the correlation times $\tau$ at which the long-range and short-range memory effects are of a minor importance, and panels (b) correspond to the values of $\tau$ at which the memory effects of both types are well pronounced. Vertical dotted lines in the Figures represent times of descent, calculated in the absence of the random force in the Langevin equation of motion (1).

As can be seen in Figs. 6 and 7, the distributions of times of descent become more spread with the increase of the correlation time $\tau$. This indicates the growing role of the fluctuating part $\int_0^t B(t-t')\xi(t')dt'$ of the descent trajectories $q(t)$ [Eq. (10)] with the strength of the memory effects $\tau$. As

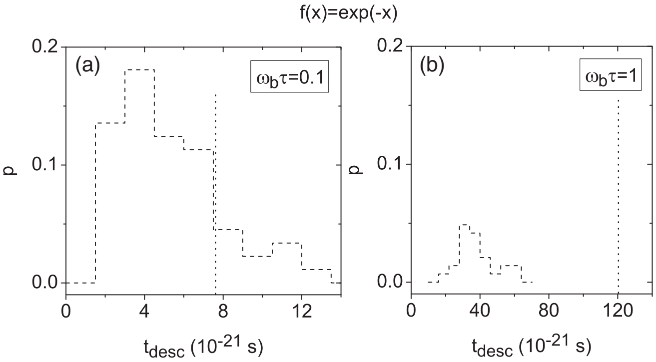


FIG. 7. The same as in Fig. 6 but for the trajectories of descent $q(t)$ [Eq. (54)], defined by the exponential memory function $f(x) = \exp(-x)$ and the colored Gaussian noise [Eq. (51)] in the generalized Langevin equation of motion (1).

was previously observed in Fig. 5, the memory effects drastically suppress the deterministic part $B(t)v_0$ of the descent trajectories $q(t)$, resulting in a fairly long trapping of a nuclear system in the close vicinity of the top $q_b$ of the fission barrier. In such a situation, the movement of the nuclear system to the scission $q_{sc}$ occurs mainly due to random changes of $q(t)$. The system is stochastically accelerated and this effect is much more prominent for the fractional [Eqs. (44)–(46)] than for the colored [Eq. (51)] Gaussian noise in the Langevin equation of motion (1). The latter is caused by a persistent character of the fractional Brownian motion [Eq. (46)] at the Hurst indices $\mathbb{H} > 1/2$, i.e., by the existence of relatively large positive correlation between random changes of the coordinate $q(t)$ toward the scission point $q_{sc}$.

Figure 8 presents the mean value $\langle t_{\rm desc}\rangle$ of times of descent. The calculation of $\langle t_{\rm desc}\rangle$ with the trajectories of descent [Eq. (50)], corresponding to the power-law memory function $f(x) = x^{-1/2}$ and the fractional Gaussian noise [Eq. (44)] in the generalized Langevin equation of motion (1), is shown by the solid line. The dashed line is for the trajectories

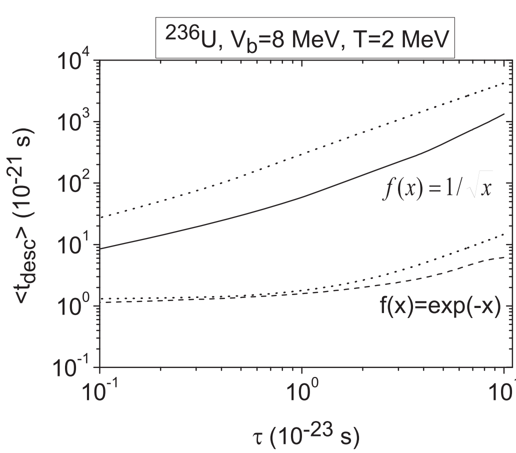


FIG. 8. Mean time $\langle t_{\rm des}\rangle$ of descent of the nucleus $^{236}$U at constant temperature $T = 2$ MeV from the parabolic fission barrier [Eq. (4)] to the scission [Eq. (42)] is shown as a function of the correlation time $\tau$ in the generalized Langevin equation of motion (1). Other notations are the same as in Figs. 6 and 7.

[Eq. (54)], corresponding to the exponential memory function $f(x) = \exp(-x)$ and the colored Gaussian noise [Eq. (51)] in Eq. (1). Two dotted lines in the figure are the times of descent, calculated in the absence of the random force in Eq. (1). Very large values of $\langle t_{\rm desc}\rangle$, seen in Fig. 8, imply extremely slow character of the descent of the nucleus in the presence of the long-range memory effects $f(t-t') = (|t-t'|/\tau)^{-1/2}$ in the generalized Langevin equation of motion (1).

## VI. SUMMARY

In the present study the peculiarities of descent of the nucleus from the fission barrier have been investigated within the one-dimensional generalized Langevin equation of motion (1) with a power-law memory function $f(t-t') = (|t-t'|/\tau)^{-\alpha}$. Such a memory function produces long-range memory effects in nuclear collective motion $q(t)$, caused by a slow decay of $f(t-t') = (|t-t'|/\tau)^{-\alpha}$ at the exponents $0 < \alpha < 1$. The memory effects of this type are remarkably different from short-range ones, generated by an exponential memory function $f(t-t') = \exp(-|t-t'|/\tau)$ and which have been studied extensively in many earlier publications [8,11–18]. At a fixed value of the exponent $\alpha$, the strength of the long-range memory effects is only regulated by the correlation time $\tau$ [Eq. (29)], defining the time spread of the retarded friction force and the correlation properties of the random force in the Langevin equation of motion (1).

Having linearized the Langevin equation of motion (1) in the vicinity of the top of the parabolic fission barrier [Eq. (4)], I found enormous suppression of the exponential growth $\sim e^{s_1 t}$ of the trajectories of descent $q(t)$ in the long-time limit $t \to \infty$ (Fig. 2). The growth rate $s_1$ gets much smaller values [Eq. (30)] in the presence of the long-range $f(t-t') = (|t-t'|/\tau)^{-\alpha}$ than the short-range $f(t-t') = \exp(-|t-t'|/\tau)$ [Eq. (25)] memory effects in the Langevin equation of motion (1). It should be also pointed out that there is a strong dependence of the growth rate $s_1$ on the value of the exponent $\alpha$ of the power-law memory function $f(t-t') = (|t-t'|/\tau)^{-\alpha}$. The $s_1$ decreases with $\alpha$ such that in the limit $\alpha \to 0$ the nuclear system $q(t)$ remains blocked ($s_1 = 0$) near the top $q_b$ of the fission barrier (Fig. 1).

Except for the exponentially unstable $\sim e^{s_1 t}$ mode of motion, the trajectory of descent $q(t)$ [Eq. (10)] is defined by several other modes of motion. Thus, at the specific value $\alpha = 1/2$ of the exponent of the power-law memory function $f(t-t') = (|t-t'|/\tau)^{-\alpha}$, I have determined analytically an algebraically decaying $\sim 1/\sqrt{t}$ component and an oscillating $\sim e^{\pm i\Omega t - \Gamma t}$ component of $q(t)$; see Eq. (34). The frequency $\Omega$ and the damping rate $\Gamma$ of the oscillatory component [Eq. (40)] are increasing functions of the correlation time $\tau$ (Figs. 3 and 4), which implies more oscillatory and damped character of descent of the nucleus with the growth of the long-range memory effects [Eq. (29)].

To define properly the trajectories of descent $q(t)$ [Eq. (10)] in the presence of the Gaussian distributed random force $\xi(t)$ with the autocorrelation function $\langle \xi(t)\xi(t')\rangle \sim (|t-t'|/\tau)^{-\alpha}$ [Eq. (2)], $\xi(t)$ was treated as fractional Gaussian noise [Eqs. (44)–(46)]. The stochastic paths [Eq. (50)] of the nuclear system $q(t)$ were then used to get statistics of times $t_{desc}$ of the first hit of $q(t)$ with a given (scission) point $q_{sc}$ [Eq. (42)]. The obtained $t_{desc}$ distributions [Fig. 6] are shifted to the left with respect to delta peaks, corresponding to times of descent, calculated in the absence of the random force $\xi(t)$ in the generalized Langevin equation of motion (1).

The calculations of the mean value of times $t_{desc}$ (Fig. 8) reveal stronger—by orders of magnitude—slowing down of the descent of the nucleus from the fission barrier, occurring in the presence of the power-law memory function $f(t-t') = (|t-t'|/\tau)^{-1/2}$ and the fractional Gaussian noise [Eqs. (44)–(46)], than in the presence of the exponential memory function $f(t-t') = \exp(-|t-t'|/\tau)$ and the colored Gaussian noise [Eq. (51)] in the Langevin equation of motion (1). Thus, the mean time of descent $\langle t_{desc}\rangle$ of the nucleus $^{236}$U at temperature $T = 2$ MeV is fairly large, $\langle t_{desc}\rangle > 10^{-20}$ s, even at extremely small values of the correlation time $\tau \approx 10^{-24}$–$10^{-23}$ s. This may serve as an estimation of the correlation time under the fractional Langevin description of nuclear fission dynamics, governed by the power-law memory function $f(t-t') = (|t-t'|/\tau)^{-\alpha}$. Interestingly, at the correlation times $\tau > 3 \times 10^{-23}$ s, the mean time of descent of $^{236}$U becomes comparable to the typical fission timescale of actinide nuclei, $(17\text{–}40) \times 10^{-20}$ s [33].

Of course, relatively large values of time of descent may be explained even within pure Markovian Langevin approaches to nuclear fission dynamics. Thus, in Refs. [34–36], fairly long saddle-to-scission times are obtained as a result of sufficiently long trapping of a nuclear system in one of the local potential wells. In order to model such a complex structure of the nuclear potential energy landscape, one has to introduce several model parameters, which should be estimated. On the other hand, the fractional Langevin approach, presented here, is fully determined by the single parameter $\rho_\alpha$ [Eq. (29)], which is a dimensionless combination of different characteristics of nuclear fission dynamics. In my opinion, the non-Markovianity is an essential and important feature of nuclear fission dynamics and it has to be a necessary ingredient of more realistic Langevin approaches to fission of heavy nuclei.


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