Exact expressions of the distributions of total magnetic quantum number and angular momentum in single-j orbits: A general technique for any number of fermions

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A general method is proposed to obtain the distribution of the total quantum number M for a set of N identical fermions with momentum j, which is a cornerstone of the nuclear shell model. This can be performed using a recursive procedure on N, yielding closed-form expressions, which are found to be linear combinations of piecewise polynomials. We also highlight and implement in that framework two three-term recurrence relations over N, more convenient than Talmi's five-term recurrence which has nevertheless already proved its worth in the past. In addition, the current approach allows one to consider both integer and half-integer values of j on the same footing. The technique is illustrated by detailed examples, corresponding to $N = 3$ to 6 fermions.

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I. INTRODUCTION

The knowledge of the distributions of magnetic quantum number M and angular momentum J for N identical fermions with half-integer spins j is a cornerstone of nuclear-physics studies [1–3]. Such quantities, which are obviously inseparable from the structural foundations of the nuclear shell model, are encountered in various physical studies, for instance about the J-pairing interaction or nuclear transitions [4,5]. The problem is rather complicated, since the possible occurrence of a specific value of J is governed by the Pauli exclusion principle. Several methods, such as generating functions, recurrence relations or algebraic number theory—for instance via Gaussian polynomials [6,7]—have proven effective in addressing that issue. Classification and numbering of states can also be tackled with the seniority scheme [3,8] or Molien functions applied to supergroups [9]. It is possible to compute the number of states $P(M; j, N)$ with a given projection M on the quantization axis and the number of levels $Q(J; j, N)$ with spin J in j^N configurations by a procedure dealing with integer partitions and Young diagrams [10]. In addition, the latter quantities can be determined using concepts from (or related to) group theory, such as irreducible representations, Schur functions, unitary and symplectic groups, as well as seniority [11–13]. However, exact analytical expressions for $P(M; j, N)$ and $Q(J; j, N)$ are known only in very simple cases, for small numbers of fermions. In a work about the quantum Hall effect [14], Ginocchio and Haxton obtained a simple formula for $Q(0; j, 4)$, which is also equal to $Q(j; j, 3)$. Zamick and Escuderos interpreted the Ginocchio-Haxton formula in the framework of a combinatorial approach for $J = j$ with $N = 3$ [15]. Zhao and Arima exhibited patterns in $Q(J; j, 3)$ and $Q(J; j, 4)$, and obtained empirical formulas [16]. They also showed that $Q(J; j, N)$ could be enumerated by the reduction from $SU(N+1)$ to $SO(3)$ and obtained analytical expressions of $Q(J; j, 4)$ [17]. Talmi suggested to express $Q(J; j, N)$ in terms of $Q(J; j-1, N)$, $Q(J; j-1, N-1)$, and $Q(J; j-1, N-2)$. Using the corresponding recurrence relation, he proved some results found empirically by Zhao and Arima [16]. Zhang *et al.* extended Talmi's recursion relation to boson systems and proved empirical formulas for five bosons. They also obtained the number of states with given spin for three and four bosons by using sum rules of six-j and nine-j symbols [18,19]. Five years later, Jiang *et al.* derived the analytical formulas for three fermions in a single-j shell $Q(J; j, 3)$ and three bosons with spin ℓ, by using a reduction rule from the $U(4)$ to the $O(3)$ group chain, $U(4) \supset \mathrm{Sp}(4) \supset O(3)$ [20,21]. In Refs. [22,23], the authors extended the studies for three and four fermions to the number of states with given spin J and isospin T.

Recently, using a five-term recurrence relation established by Talmi [24] we have been able to derive explicit expressions for the the distributions $P(M; j, 3)$, $Q(J; j, 3)$, $P(M; j, 4)$ and $Q(J; j, 4)$ [25]. This led us to deduce exact formulas for the total number of levels in single-j orbits for three, four, and five fermions. In the $N = 3$ case, an alternative derivation much simpler than the one previously published and relying on the use of fractional parentage coefficients [26] was obtained. In a similar approach, Yin and Zhao generalized Talmi's recursion formulas by further considering the isospin couplings and derived explicit formulas of the number of states with a given

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total spin I and isospin T for N nucleons in a single-j shell [27].

The aim of this paper is to show that closed-form expressions, explicitly as linear combinations of piecewise polynomials, may be given for the fermion distribution $P(M; j, N)$ for any value of N. This can be performed using a recursive procedure on N. As mentioned above, for $N \leqslant 4$ we proposed in our previous work [25] to use Talmi's recursion relation involving five P values [24]. In the present paper we show that a better option may be provided by using a simpler recurrence relation involving only three values $P(M; j, N)$, $P(M'; j-1/2, N)$, $P(M''; j-1/2, N-1)$. We first outline in Sec. II the method implementing the three-term recurrence to get the distribution $P(M; j, N)$ for the lowest N values. One interest of the current method is that it allows us to consider the half-integer as well as integer momenta j on the same footing. This method is applied to various values of the fermion number N (up to 6), which illustrates its efficiency. We also provide the corresponding expressions for the number of total angular momentum $Q(J; j, N)$.

II. RECURSIVE GENERAL ALGORITHM

A. Definitions and recurrence properties

For a subshell of N fermions with individual momentum j, we study the number of possible values of the total magnetic quantum number M, written as $P(M; j, N)$. Because of Pauli principle, if the magnetic quantum numbers of each fermion are written $m_1, m_2, \cdots m_N$, then one has $M \leqslant J_{\max} = j + (j-1) + \cdots (j-N+1) = Nj - N(N-1)/2$. The number of levels with angular momentum J may be derived from the well-known relations $Q(J; j, N) = P(J; j, N) - P(J+1; j, N)$ if $J < J_{\max}$, and $Q(J_{\max}; j, N) = P(J_{\max}; j, N)$. In this paper we will assume $M \geqslant 0$ unless otherwise mentioned.

The distribution P obeys several useful recurrence properties. In our previous work, we used extensively a five-term recurrence equation derived by Talmi [24]. However, an interesting alternative is provided by two three-term relations derived in Ref. [28] [labeled (B.8) and (B.4), respectively]:

$$P(M; j, N) = P(M + N/2; j - 1/2, N) + P(M - j + (N-1)/2; j - 1/2, N - 1), \quad \text{(1a)}$$

$$P(M; j, N) = P(M - N/2; j - 1/2, N) + P(M + j - (N-1)/2; j - 1/2, N - 1). \quad \text{(1b)}$$

An analogous relation on the number of levels with given total momentum J has been published by Bao *et al.* [29].

B. Piecewise polynomial decomposition

From the analysis performed in our previous work [25], we assume that the magnetic moment M distribution may be written as

$$P(M; j, N) = \sum_{k\geqslant 0} (-1)^k H(Nj - M - 2kj) \times P_{N,k}(Nj - M - 2kj) \quad \text{for } M \geqslant 0, \quad (2)$$

where $H(m)$ is the Heaviside function, $H(m) = 1$ if $m \geqslant 0$, 0 otherwise. Negative M are not considered here, since $P(-M; j, N) = P(M; j, N)$. For a given M, because of the Heaviside functions, the maximum index in the sum is defined be the constraint $M - Nj - 2kj \geqslant 0$, so that

$$k_{\max}(M, j, N) = \lfloor (Nj - M)/(2j) \rfloor, \quad (3)$$

where $\lfloor x \rfloor$ is the integer part of x. If $0 \leqslant M < j$ and N is odd (one may show that $-j < M < j$ is then allowed by our formalism), then one has $k_{\max} = (N-1)/2$. If $0 \leqslant M < 2j$ and N even, then one has $k_{\max} = (N-2)/2$. Therefore, the general expression is $k_{\max} = \lfloor (N-1)/2 \rfloor$ for any N.

The determination of the functions $P_{N,k}(X)$ is the purpose of the present work. Furthermore, we add the constraints on the $P_{N,k}$ functions

$$P_{N,k}(X) = 0 \quad \text{for } X = 0, -1, \cdots - N + k + 1. \quad (4)$$

The property (2) is established here by recurrence on N. We have checked its validity for $N \leqslant 4$ [25], where the $P_{N,k}$ functions are piecewise polynomials. Explicit values for the first P_N, k are given in the next subsection.

An interesting property in using the recurrence (1a) is that the $k_{\max}$ of the three involved P values are identical. More precisely, as analyzed in the Appendix B, one can show that $k_{\max}(M, j, N) = k_{\max}(M + N/2, j - 1/2, N) = k_{\max}(M - j + (N-1)/2, j - 1/2, N - 1)$. Moreover, using Eq. (1b), one shows in the same Appendix that $k_{\max}(M, j, N) = k_{\max}(M - N/2, j - 1/2, N) = k_{\max}(M + j - (N-1)/2, j - 1/2, N - 1) + 1$. A finer analysis, detailed in Appendix B, shows that the above identities may be violated but one has always has $k_1 \geqslant k_3 \geqslant k_2$ in the first case using the notations (B1), and $k'_2 \geqslant k'_3 + 1 \geqslant k'_1$ in the second case, with notations (B5). Furthermore, using Eq. (1a), the cases where inequalities occur corresponds to zero $P_{N,k}$ value: for instance, if $k_1 = k_2 + 1 = k_3 + 1$, then one can check that $P_{N,k_1}(X)$ vanishes.

C. Examples of $P_{N,k}$ values

If $N = 2$, from the well-known value $P(M; j, 2) = \lfloor (2j + 1 - M)/2 \rfloor$, then the sum contains indeed one term,

$$P_{2,0}(X) = X/2 + \pi(X)/2, \quad \text{(5a)}$$

where $X = 2j - M$ and $\pi(X) = 1$ if X is odd, 0 if X is even. As shown in Ref. [25] and derived in Appendix A using three-term recurrence, one has

$$P_{3,0}(X) = X^2/12 + \alpha(\mathrm{mod}(X, 6)),$$
$$P_{3,1}(X) = X^2/4 - \pi(X)/4, \quad \text{(5b)}$$

with α defined in Appendix A. Throughout this paper $\mathrm{mod}(m, n)$ is the modulo function, defined for $n > 0$ as the remainder in the Euclidean division of m by n—notice that such remainder is nonnegative. The parity of m may be expressed as a modulo function

$$\pi(m) = \mathrm{mod}(m, 2). \quad (6)$$

The next sections are devoted to the derivation of the expression of $P_{N,k}$ polynomials for N between 4 and 6.

D. Implementation of the recurrence if $M \geqslant (N-2)j$

Let us first consider the case $Nj - 2j \leqslant M \leqslant Nj$, for which $k_{\max} = 0$. Substituting the expansion (2) for each of the three P in the recurrence (1a), one easily checks that each of the three sums involves only one term $k = 0$. The arguments of the $P_{N,0}$ functions are, respectively, $X = Nj - M$, $X' = N(j - 1/2) - M - N/2 = X - N$, and $X'' = (N-1)(j-1/2) - M + j - (N-1)/2 = X - N + 1$. The resulting recurrence on $P_{N,0}$ is

$$P_{N,0}(X) = P_{N,0}(X - N) + P_{N-1,0}(X - N + 1). \qquad (7)$$

One notices that we ignored the Heaviside functions in this derivation. Though $X > 0$, one may have $X' < 0$ if $X < N$. In this case from the constraints (4), one has $P_{N,0}(X) = P_{N,0}(X - N) = P_{N-1,0}(X - N + 1) = 0$ so that the cancellation of P allows us to omit the Heaviside functions. Writing $X = Nj - M = N\nu + n$, with $\nu = \lfloor X/N \rfloor$, $n = \mathrm{mod}(X, N)$ and applying the recurrence (7) ν times, one gets, noting that $P_{N,0}(n) = 0$ from Eq. (4),

$$P_{N,0}(X) = \sum_{i=1}^{\nu} P_{N-1,0}(X - (i-1)\nu + 1). \qquad (8)$$

This formula bears several consequences. Since the known values for $P_{N,0}$ if $1 \leqslant N \leqslant 3$ are indeed piecewise polynomials of degree $N-1$ of the argument $X = Nj - M$, it proves that for any N $P_{N,0}$ is also a polynomial in $X = Nj - M$ of degree $N-1$. This provides an efficient way to get the analytical expression of $P_{N,0}$. An alternate derivation for the $P_{N,0}$ computation is developed in Appendix C.

E. Implementation of the recurrence if $0 \leqslant M \leqslant (N-2)j$

We now consider the case $(N-4)j \leqslant M \leqslant (N-2)j$. The $P_{N,k}$ expansion contains two terms, and the recurrence (1a) is written for $X = (N-2)j - M \geqslant 0$,

$$\begin{aligned} &P_{N,0}(X + 2j) - P_{N,1}(X) \\ &\quad = P_{N,0}(X - N + 2j) - P_{N,1}(X - N) \\ &\qquad + P_{N-1,0}(X - N + 2j + 1) - P_{N-1,1}(X - N + 1). \end{aligned} \qquad (9)$$

The key point of our method lies in the fact that, since the identity (7) has been established for $0 \leqslant X \leqslant 2j$ and that the $P_{N,0}$ are piecewise polynomials with coefficient obeying simple congruence properties, one also has $P_{N,0}(X + 2j) = P_{N,0}(X - N + 2j) + P_{N-1,0}(X - N + 2j + 1)$. Therefore, the above equations implies

$$\begin{aligned} P_{N,1}(X) &= P_{N,1}(X - N) - P_{N-1,1}(X - N + 1) \\ &\quad \text{for } N \geqslant 3 \text{ and } 0 \leqslant X \leqslant 2j. \end{aligned} \qquad (10)$$

The known value of $P_{3,1}$ shows that $P_{N,1}(X)$ is also a piecewise polynomial of degree $N-1$, that can be efficiently derived in a closed form using the above recurrence. In a similar way as in the $k = 0$ case, one may demonstrate that $P_{N,1}(X) = 0$ if $X = 0, 1, \cdots N-2$.

The same procedure may be used repeatedly for each k value, using the above cancellation property if $k = 2$, etc. Writing $X = Nj - M - 2kj$ the argument of the $P_{N,k}$ function in the left side of Eq. (1a), the arguments of $P(M + N/2; j - 1/2, N)$ and $P(M - j + (N-1)/2; j - 1/2, N-1)$ in the expansion (2) are, respectively, $X' = N(j - 1/2) - M - N/2 - k(2j-1) = X - N + k$ and $X'' = (N-1)(j-1/2) - M + j - (N-1)/2 - k(2j-1) = X - N + k + 1$. The fact that the k indices are taken equal in X, X', X'' is justified by the analysis done in Appendix B. We therefore obtain

$$P_{N,k}(X) = P_{N,k}(X - N + k) + P_{N-1,k}(X - N + k + 1). \qquad (11)$$

The above equation applies for $k \leqslant (N-2)/2$ (respectively, $k \leqslant (N-3)/2$) if N is even (respectively, odd). Using Eq. (11) and the constraints (4), one may also demonstrate that

$$P_{N,k}(X) = 0 \quad \text{if } X = 0, 1, \cdots N - k - 1. \qquad (12)$$

F. Case $M \leqslant j$ and N odd

The above method does not work for the maximum value $k = (N-1)/2$ if $N = 2\nu + 1$ is odd. Equation (11) would involve as its last term $P_{N-1,(N-1)/2}$ which does not exist. The most direct method relies on the expansion (2), where the last term $k = \nu$ is singled out—since $M \leqslant j$ Heaviside functions are set equal to 1—

$$\begin{aligned} &P(M; j, 2\nu + 1) \\ &\quad = O(M; j, 2\nu + 1) \\ &\qquad + (-1)^{\nu} P_{2\nu+1,\nu}((2\nu + 1)j - M - 2\nu j), \end{aligned} \qquad (13a)$$

$$\begin{aligned} &O(M; j, 2\nu + 1) \\ &\quad = \sum_{k=0}^{\nu-1} (-1)^k P_{2\nu+1,k}((2\nu + 1)j - M - 2kj). \end{aligned} \qquad (13b)$$

Substituting this sum in the recurrence (1a) provides the equation

$$\begin{aligned} &(-1)^{\nu} P_{2\nu+1,\nu}(j - M) + O(M; j, 2\nu + 1) \\ &\quad = (-1)^{\nu} P_{2\nu+1,\nu}(j - M - \nu - 1) + O(M + N/2; \\ &\qquad j - 1/2, 2\nu + 1) + P(M + \nu - j; j - 1/2, 2\nu). \end{aligned} \qquad (14)$$

The last term in this equation may be rewritten $P(j - M - \nu; j - 1/2, 2\nu)$ to get a positive first argument. Here we assume $M \leqslant j$ and not $M \leqslant j - \nu$ but the marginal case $j - \nu \leqslant M \leqslant j$ could be treated by a similar procedure; it has been checked that separating both cases $M + \nu - j$ positive versus negative does not lead to different expressions. To sum up this discussion, according to the above equation, one may express the difference $P_{2\nu+1,\nu}(j - M) - P_{2\nu+1,\nu}(j - M - \nu - 1)$ as a linear combination of the $O(M; j, 2\nu + 1)$ and $P(M, j, 2\nu)$ which are all known since they involve either $P_{2\nu+1,k}$ with $k < \nu$ or $P_{2\nu,k}$. As an illustration, this procedure has been used in the cases $N = 3$ (Appendix A) and $N = 5$ (Sec. IV). The main drawback of this method, in addition to the rather tedious computation needed, is that it does not obviously shows that $P_{2\nu+1,\nu}$ is only a function of the argument $Nj - M - 2\nu j = j - M$. To overcome this drawback, an alternative method is now proposed.

G. A variant of the three-term recurrence

The purpose of this section is to obtain the relation corresponding to Eq. (11) when the recurrence (1b) is used instead of Eq. (1a). We proceed similarly as in Sec. II E. If we insert the form (2) into the equation (1b), then the argument of the kth term in the expansion of $P(M; j, N)$ is $X = Nj - N - 2kj$. Accordingly, this argument is $X' = N(j - 1/2) - M + N/2 - (2j - 1)k' = X + k' + 2(k - k')j$ for $P(M - N/2; j - 1/2, N)$, and $X'' = (N - 1)(j - 1/2) - M + j + (N - 1)/2 - (2j - 1)k'' = X + k'' + 2(k - k'' - 1)j$ for $P(M - j - (N - 1)/2; j - 1/2, N - 1)$. As shown by the analysis of Appendix B, the k indices must then be chosen so that $k = k' = k'' + 1$. A similar procedure as above allows us to obtain separately an equation for each interval k. One then obtains, using $k = k' = k'' + 1$,

$$P_{N,k}(X) = P_{N,k}(X + k) - P_{N-1,k-1}(X + k - 1) \quad \text{for } k \geqslant 1. \tag{15}$$

The minus sign in front of the last term comes from the $(-1)^k$ factor in the expansion (2), namely $(-1)^{k''} = -(-1)^k$. This can be rewritten as $P_{N,k}(X) = P_{N,k}(X - k) + P_{N-1,k-1}(X - 1)$. Such relation may be applied, for instance, if $N = 2\nu + 1$ is odd and $M < j$, for which the relation (11) was inefficient. One has then $k = \nu$ and the function $P_{2\nu+1,\nu}$ obeys

$$P_{2\nu+1,\nu}(X) = P_{2\nu+1,\nu}(X + \nu) + P_{2\nu,\nu-1}(X) \quad \text{for } \nu \geqslant 1. \tag{16}$$

III. APPLICATION OF THE THREE-TERM RECURRENCE TO THE FOUR-FERMION CASE

The aim of this section is to use the three-term recurrence relation (1a) to compute $P(2j - p; j, 4)$ with $-2j + 6 \leqslant p \leqslant 2j$, so that $M = 2j - p$ is nonnegative and below or equal to $J_{\max}(j, 4) = 4j - 6$. As in the three-fermion case, we split the discussion according to the sign of p.

A. Case M greater than or equal to $2j$

The general formula (C4) writes in the four-fermion case

$$P(2j - p; j, 4) = \sum_{s=1}^{t} P\left(j - p + \frac{5}{2}s - 1; j - \frac{s}{2}, 3\right) \quad \text{with } t = \left\lfloor \frac{2j + p - 2}{4} \right\rfloor. \tag{17}$$

The sum (17) involves in its generic term the three-fermion distribution $P(\bar{j} - \bar{p}; \bar{j}, 3)$ with $\bar{j} = j - s/2$, $\bar{p} = p - 3s + 1$. If $p \leqslant 0$, then one has also $\bar{p} \leqslant 0$ so that the relation (A11) applies with only the first two terms included. One has $P(2j - p; j, 4) = S_1 + S_2$, with

$$S_1 = \sum_{s=1}^{t} \frac{(2j + p - 4s + 1)^2}{12}, \quad S_2 = \sum_{s=1}^{t} \alpha(\mathrm{mod}(2j + p - 4s + 1, 6)), \tag{18}$$

with $p \leqslant 0$, and t given by Eq. (17). We define

$$2j + p = 12\nu + m, \tag{19}$$

with ν and m integers. The S_1 sum is obtained by standard algebra and evaluates to

$$S_1 = \frac{(2j + p - 1)^3}{144} - \frac{2j + p - 1}{36} + \left(-\frac{5}{48}, -\frac{1}{3}, \frac{1}{48}, 0\right) \tag{20}$$

for $\mathrm{mod}(2j + p, 4) = 0, 1, 2, 3$, respectively. The sum S_2, rewritten as $= \sum_{s=1}^{t} \alpha(\mathrm{mod}(m + 2s + 1, 6))$ evaluates as follows. We notice that the number of terms t is $3\nu - 1, 3\nu, 3\nu + 1, 3\nu + 2$ for $0 \leqslant m \leqslant 1, 2 \leqslant m \leqslant 5, 6 \leqslant m \leqslant 9, 10 \leqslant m \leqslant 11$, respectively. Since one needs to evaluate $\alpha(\mathrm{mod}(m + 2s + 1, 6))$, this quantity has a period 3 in s and it is natural to collect s terms by groups of three and to put the number of terms into the form $t = 3\lfloor t/3 \rfloor + \mathrm{mod}(t, 3)$. For instance, in the $m = 0$ case one has $t = 3(\nu - 1) + 2$ terms. Gathering the various terms in S_2 as $\nu - 1$ times the three-term partial sum $\alpha(3) + \alpha(5) + \alpha(1)$ plus two additional terms $\alpha(3) + \alpha(5)$, and using the explicit list (A5d), one obtains

$$S_2 = (\nu - 1)(\alpha(3) + \alpha(5) + \alpha(1)) + \alpha(3) + \alpha(5) = \frac{\nu + 1}{12} = \frac{2j + p + 12}{144} \quad \text{if } \mathrm{mod}(2j + p, 12) = 0. \tag{21}$$

The 11 other cases $m = 1, \cdots 11$ are dealt with accordingly. One gets

$$S_2 = \left(\frac{\nu + 1}{12}, \frac{(1 - 2\nu)}{3}, \frac{\nu}{12}, -\frac{2\nu}{3}, \frac{\nu}{12}, -\frac{2\nu}{3}, \frac{\nu + 3}{12}, -\frac{(2\nu + 1)}{3}, \frac{\nu - 1}{12}, -\frac{2\nu}{3}, \frac{\nu + 2}{12}, -\frac{2(\nu + 1)}{3}\right) \tag{22}$$

for $m = 0 \cdots 11$, respectively. One notices that the ν-dependent part in this list is $\nu/12$ (respectively, $-2\nu/3$) if m is even (respectively, odd). Making the substitution $\nu = (2j + p - m)/12$, and adding the S_2 value to S_1 given by Eq. (20) one gets the result, valid if $p \leqslant 0$,

$$P(2j-p;j,4) = \frac{(2j+p-1)^3}{144} - \left(\frac{1}{12} - \frac{\pi(2j+p-1)}{16}\right)(2j+p-1) + \omega(\mathrm{mod}(2j+p-1,12)), \tag{23a}$$

$$\text{with}\quad \omega(m) = \left(0, \frac{1}{72}, \frac{1}{9}, -\frac{1}{8}, -\frac{1}{9}, \frac{17}{72}, 0, -\frac{17}{72}, \frac{1}{9}, \frac{1}{8}, -\frac{1}{9}, -\frac{1}{72}\right) \quad \text{if } m = (0, 1, \cdots 11), \text{ respectively.} \tag{23b}$$

The above list may also be expressed as a sum of modulo functions

$$\omega(m) = \frac{\mathrm{mod}(m,3) - \mathrm{mod}(-m,3)}{9} - \frac{\mathrm{mod}(m,4) - \mathrm{mod}(-m,4)}{16}. \tag{23c}$$

This agrees with Eq. (3.51) of Ref. [25]. However, the present derivation is simpler and applies to integer values of j.

B. Case M less than $2j$

Writing the total magnetic quantum number as $M = 2j - p$, from Eq. (11) one has

$$P_{4,1}(p) - P_{4,1}(p-3) = P_{3,1}(p-2) = \frac{(p-2)^2}{4} - \frac{\mathrm{mod}(p,2)}{4} \quad \text{for } p > 0. \tag{24}$$

We write $p = 3\nu + n$ with $n = 0$, 1 or 2. One may notice that the difference (24) vanishes if $p = 1, 2, 3$, so that to get $P_{4,1}(p)$ one has to iterate the equation ν times. One gets after such iteration

$$(P_{4,1}(p) - P_{4,1}(p-3)) + (P_{4,1}(p-3) - P_{4,1}(p-6)) + \cdots + (P_{4,1}(n+3) - P_{4,1}(n)) = P_{4,1}(p) - P_{4,1}(n) = T_1 + T_2, \tag{25a}$$

$$T_1 = \sum_{i=0}^{\nu-1} (p-2-3i)^2/4, \tag{25b}$$

$$T_2 = -\sum_{i=0}^{\nu-1} \mathrm{mod}(p-3i,2)/4. \tag{25c}$$

These sums are easy to evaluate through basic algebraic manipulations, e.g., with Mathematica software. The result depends on mod$(p, 3)$ and mod$(p, 6)$ for T_1 and T_2, respectively. One has

$$T_1 = \frac{p^3}{36} - \frac{p^2}{24} - \frac{p}{24} + \left(0, \frac{1}{18}, \frac{1}{36}\right) \quad \text{for mod}(p,3) = (0,1,2), \tag{26a}$$

$$T_2 = -\frac{p}{24} + \left(0, \frac{1}{24}, \frac{1}{12}, -\frac{1}{8}, \frac{1}{6}, -\frac{1}{24}\right) \quad \text{for mod}(p,6) = (0,1,2,3,4,5), \tag{26b}$$

$$P_{4,1}(p) = T_1 + T_2 = \frac{p^3}{36} - \frac{p^2}{24} - \frac{p}{12} + \left(0, \frac{7}{72}, \frac{1}{9}, -\frac{1}{8}, \frac{2}{9}, -\frac{1}{72}\right) \quad \text{for mod}(p,6) = (0,1,2,3,4,5). \tag{26c}$$

As mentioned above, one easily verifies that $P_{4,1}(p) = 0$ for $-2 \leqslant p \leqslant 3$. The recurrence hypothesis is confirmed by this $P_{4,1}$ value which is indeed a function of p only. One notices that the list of constants appearing between parentheses in the $P_{4,1}(p)$ expression is identical to the quantity $\xi(p) + 1/9$ defined in Ref. [25]. The complete result agrees with Eq. (3.51) in this paper and may be written as

$$\begin{aligned} P(2j-p;j,4) = {} & \frac{(2j+p-1)^3}{144} - \left(\frac{1}{12} - \frac{\pi(2j+p-1)}{16}\right)(2j+p-1) + \omega(\mathrm{mod}(2j+p-1,12)) \\ & -H(p)\left[\frac{p^3}{36} - \frac{p^2}{24} - \frac{p}{12} + \frac{1}{9} + \xi(\mathrm{mod}(p,6))\right] \quad \text{for } -2j+6 \leqslant p \leqslant 2j, \end{aligned} \tag{27a}$$

$$\text{with}\quad \xi(m) = \left(-\frac{1}{9}, -\frac{1}{72}, 0, -\frac{17}{72}, \frac{1}{9}, -\frac{1}{8}\right) \quad \text{if } m = (0,1,2,3,4,5), \text{ respectively.} \tag{27b}$$

Solving a simple linear system, one can easily show that

$$\xi(m) = -\frac{1}{9} - \frac{\mathrm{mod}(m,2)}{8} + \frac{\mathrm{mod}(-m,3)}{9}. \tag{27c}$$

The interest of this derivation, in addition to its conciseness, is that it holds even for integer j. It must be noticed that this formula do not apply for negative M, i.e., if $p > 2j$. From the relation (27), setting $p = 2l$, one gets the total number of levels for four fermions with integer j,

$$Q_{\text{tot}}(l^4) = P(0;l,4) = \sum_L Q(L;l,4) = \frac{2l^3}{9} - \frac{l^2}{6} + \frac{l}{6} + \begin{cases} 0 & \text{if } \text{mod}(l,3)=0 \\ -\frac{2}{9} & \text{if } \text{mod}(l,3)=1 \\ -\frac{4}{9} & \text{if } \text{mod}(l,3)=2 \end{cases} \tag{28a}$$

$$= \frac{2l^3}{9} - \frac{l^2}{6} + \frac{l}{6} - \frac{2}{9}\text{mod}(l,3). \tag{28b}$$

The corresponding expression for half-integer j has been derived previously, see Eq. (3.54) in Ref. [25].

IV. APPLICATION TO THE FIVE-FERMION CASE

We now turn to the computation of $P(M;j,5)$ with $M = 3j - p$, $-2j + 10 \leqslant p \leqslant 3j$, so that M is such that $0 \leqslant M \leqslant J_{\max}(j,5) = 5j - 10$. The discussion is split by considering successively the cases $p \leqslant 0$, $0 \leqslant p \leqslant 2j$, and $2j \leqslant p \leqslant 3j$. The case $p > 3j$ does not need to be considered since $P(-M;j,N) = P(M;j,N)$.

A. Case *M* greater than or equal to 3*j*

The general formula (C4) writes in the five-fermion case

$$P(3j-p;j,5) = \sum_{s=1}^{t} P\left(2j - p + 3s - 1; j - \frac{s}{2}, 4\right) \quad \text{with} \quad t = \left\lfloor \frac{2j+p-5}{5} \right\rfloor. \tag{29}$$

The sum (29) involves in its generic term the four-fermion distribution $P(\bar{j} - \bar{p}; \bar{j}, 4)$ with $\bar{j} = j - s/2$, $\bar{p} = p - 4s + 1$. We may use the expression (27). If $p \leqslant 0$, then one has also $\bar{p} \leqslant 0$ so that the relation (27) applies, ignoring the part factored by $H(p)$. One has $P(3j - p; j, 5) = S_1 + S_2$, with

$$S_1 = \sum_{s=1}^{t} \frac{(2j+p-5s)^3}{144} - \left(\frac{1}{12} - \frac{\pi(2j+p-5s)}{16}\right)(2j+p-5s), \quad S_2 = \sum_{s=1}^{t} \omega(\text{mod}(2j+p-5s,12)), \tag{30}$$

with $p \leqslant 0$, and t given by Eq. (29). Since the index t involves the ratio $(2j + p)/5$ and the array ω involves $\text{mod}(2j + p, 12)$, we are led to consider values of $2j + p$ modulo the lowest common multiple of 5 and 12, i.e., we define

$$2j + p = 60\nu + m, \tag{31}$$

with ν and m integers.

The S_1 sum is obtained by standard algebra and evaluates to

$$S_1 = \frac{(2j+p)^4}{2880} - \frac{(2j+p)^3}{288} + \frac{(2j+p)^2}{288} + \frac{2j+p}{24} - \frac{\text{mod}(2j+p,2)}{32}(2j+p) + c_1(\text{mod}(2j+p,10)), \tag{32}$$

with

$$c_1(i) = \left(0, \frac{1}{320}, \frac{13}{360}, -\frac{39}{320}, -\frac{1}{5}, \frac{5}{64}, -\frac{3}{40}, \frac{329}{2880}, -\frac{1}{5}, -\frac{39}{320}\right), \tag{33}$$

for $i = 0, 1, \cdots 9$, respectively.

The sum S_2, rewritten as $\sum_{s=1}^{t} \omega(\text{mod}(m - 5s + 1, 12))$ is a quantity that is periodic in $2j + p$ with period 60. At variance with the sum of α computed in Eq. (21), there is no term proportional to $(2j + p)$ appearing in S_2 because one has $\sum_{i=0}^{11} \alpha(i) = 0$. Here we do not give the list of the 60 values of S_2 according to m, because the computation is straightforward and because the useful value is indeed $S_2 + c_1(\text{mod}(2j + p, 10))$. The computation performed in the 60 cases according to $m = \text{mod}(2j + p, 60)$ provides the value of the "constant" term of the polynomial $P_{5,0}(2j + p)$

$$\phi_5(m) = S_2(m) + c_1(\text{mod}(m,10)) = \varphi(\text{mod}(m,12)) + \frac{\delta(\text{mod}(m,5))}{5}, \quad \text{with } 0 \leqslant m \leqslant 59. \tag{34}$$

For $m = 0, 1 \cdots 11$, the $\varphi(m)$ term may be expressed as a list, or identified to a simple linear combination of modulo functions

$$\varphi(m) = \left(-\frac{1}{5}, -\frac{31}{2880}, -\frac{3}{40}, \frac{1}{320}, -\frac{4}{45}, -\frac{39}{320}, -\frac{3}{40}, \frac{329}{2880}, -\frac{1}{5}, -\frac{39}{320}, \frac{13}{360}, \frac{1}{320}\right) \tag{35a}$$

$$= -\frac{1}{5} + \frac{\text{mod}(m,2)}{64} + \frac{-\text{mod}(m,3) + 2\text{mod}(-m,3)}{27} + \frac{\text{mod}(m,4)}{16}. \tag{35b}$$

Therefore, the constant term is, for $0 \leqslant m \leqslant 59$,

$$\phi_5(m) = \frac{\text{mod}(m,2)}{64} + \frac{-\text{mod}(m,3) + 2\text{mod}(-m,3)}{27} + \frac{\text{mod}(m,4)}{16} - \frac{\text{mod}(m,5) + \text{mod}(-m,5)}{25}, \tag{36}$$

and the complete result is

$$P_{5,0}(X) = \frac{X^4}{2880} - \frac{X^3}{288} + \frac{X^2}{288} + \frac{X}{24} - \pi(X)\frac{X}{32} + \phi_5(\text{mod}(X,60)), \tag{37}$$

for $X = p + 2j = 5j - M$.

B. Case M greater than j and below $3j$

Using the general relation (11) and the assumption (4), one may write

$$P_{5,1}(p) - P_{5,1}(p-4) = P_{4,1}(p-3), \tag{38}$$

$$P_{5,1}(p) - P_{5,1}(p_0) = \sum_{i=1}^{\nu} P_{4,1}(p_0 + 4i - 3), \tag{39}$$

with the modulo 4 definition $p = 4\nu + p_0$, ν, p_0 integers. Using the known definition for $P_{4,1}(p)$, given by Eq. (3.51) of Ref. [25] or by Eq. (27), the above sum is easy to obtain. One gets

$$P_{5,1}(p) - P_{5,1}(p_0) = (P_{5,1}(p_0 + 4\nu) - P_{5,1}(p_0 + 4\nu - 4)) + \cdots + (P_{5,1}(p_0 + 4) - P_{5,1}(p_0)) = T_1 + T_2, \tag{40a}$$

$$T_1 = \sum_{i=1}^{\nu} f_1((p_0 + 4i - 3)/2), \quad T_2 = \sum_{i=1}^{\nu} \xi(\text{mod}(p_0 + 4i - 3, 6)). \tag{40b}$$

With the value $f_1(p/2) = p^3/96 - p^2/24 - p/12 + 1/9$, one gets after some basic algebra

$$T_1 = \frac{p^4}{576} - \frac{p^3}{96} - \frac{p^2}{288} + \frac{7p}{96} + \left(0, -\frac{35}{576}, -\frac{11}{144}, -\frac{3}{64}\right) \quad \text{for mod}(p,4) = (0,1,2,3), \tag{41a}$$

$$T_2 = -(1 - \pi(p))\frac{p}{32} + \left(0, 0, \frac{1}{16}, 0, \frac{1}{9}, 0, -\frac{7}{144}, \frac{1}{9}, \frac{1}{9}, -\frac{1}{9}, \frac{1}{16}, \frac{1}{9}\right) \quad \text{for mod}(p,12) = (0, 1, \cdots 11). \tag{41b}$$

The contribution $P_{5,1}(p)$ is given by the sum $T_1 + T_2$, and one easily gets

$$P_{5,1}(p) = \frac{p^4}{576} - \frac{p^3}{96} - \frac{p^2}{288} + \frac{p}{24} + \pi(p)\frac{p}{32} + \eta(\text{mod}(p,12)), \tag{42}$$

with $\eta(m)$ given, for $m = 0, 1 \cdots 11$, by

$$\eta(m) = \left(0, -\frac{35}{576}, -\frac{1}{72}, -\frac{3}{64}, \frac{1}{9}, -\frac{35}{576}, -\frac{1}{8}, \frac{37}{576}, \frac{1}{9}, -\frac{11}{64}, -\frac{1}{72}, \frac{37}{576}\right) \tag{43a}$$

$$= \frac{\text{mod}(m,2)}{64} + \frac{\text{mod}(m,3) + \text{mod}(-m,3)}{27} - \frac{\text{mod}(-m,4)}{16}. \tag{43b}$$

C. Case $M < j$

In this case we apply the method sketched in Sec. II F. The three-term recurrence (1a) is for $N = 5$, $p \geqslant 2j$, substituting the form (2) and dropping the Heaviside functions which are equal to 1,

$$P_{5,0}(2j + p) - P_{5,1}(p) + P_{5,2}(p - 2j) = P_{5,0}(2j + p - 5) - P_{5,1}(p - 4) + P_{5,2}(p - 2j - 3) + P(2j - p + 2; j - 1/2, 4). \tag{44}$$

All functions in this equation are known except $P_{5,2}$. Since we know from the general theory of Sec. II that

$$P_{5,0}(2j + p) - P_{5,0}(2j + p - 5) = P_{4,0}(2j + p - 4), \tag{45a}$$

$$P_{5,1}(p) - P_{5,1}(p - 4) = P_{4,1}(p - 3), \tag{45b}$$

the difference $P_{5,2}(p - 2j) - P_{5,2}(p - 2j - 3)$ depends only on four-fermion functions $P_{4,0}, P_{4,1}, P(2j - p + 2; j - 1/2, 4)$. When evaluating this last element, one must take care that $2j - p + 2$ is negative and that the expression (27) does not hold for negative M. We use the $M \to -M$ symmetry and the known value for the four-fermion distribution,

$$P(2j - p + 2; j - 1/2, 4) = P(p - 2j - 2; j - 1/2, 4) = P_{4,0}(6j - p) - P_{4,1}(4j + 1 - p). \tag{46}$$

In the special cases $p = 2j - 2$ and $p = 2j - 1$, one could assume that since $p - 2j - 2 < 0$, one has $M = -2$ or -1 and that the above equation does not apply. In fact, one can check by direct substitution that the analytical form (27) does hold for $p = 2j + 1$ ($M = -1$) and $p = 2j + 2$ ($M = -2$), so that the above equation is valid in the interval $2j \leqslant p \leqslant 4j$.

Gathering the above equations, we obtain after basic algebra, with the substitution $p \to X = p - 2j$,

$$P_{5,2}(p - 2j) - P_{5,2}(p - 2j - 3) = \frac{X^3}{24} - \frac{X^2}{4} + \frac{11X}{24} - \frac{1}{4} + \pi(X - 1)\left(\frac{1}{4} - \frac{X}{8}\right) + \Delta(X, 2j), \tag{47a}$$

$$\Delta(X, 2j) = -\xi(\mathrm{mod}((2j - X + 1, 6)) + \xi(\mathrm{mod}(2j + X - 3, 6)) + \omega(\mathrm{mod}(4j - X - 1, 12)) - \omega(\mathrm{mod}(4j + X - 5, 12)). \tag{47b}$$

Using the known values of the arrays ω and ξ, and performing the computation for all the values of $\mathrm{mod}(X, 12)$ and $\mathrm{mod}(2j, 6)$, one easily verifies that $\Delta(X, 2j) = 0$ for any pair $(X, 2j)$. We may therefore write, using the relation $\pi(X - 1) = 1 - \pi(X)$,

$$P_{5,2}(X) - P_{5,2}(X - 3) = f_{52}(X) = \frac{X^3}{24} - \frac{X^2}{4} + \frac{X}{3} + \pi(X)\left(\frac{X}{8} - \frac{1}{4}\right), \tag{48}$$

which holds for $X \geqslant 3$. The value for $P_{5,2}(X)$ is obtained as above, setting the modulo 3 definition $X = 3\nu + X_0$. One has

$$P_{5,2}(X) - P_{5,2}(X_0) = \sum_{i=1}^{\nu} f_{52}(X_0 + 3i). \tag{49}$$

Once again this sum is easily performed using the above definition of $f_{52}(X)$. Since it contains a $\mathrm{mod}(X, 2)$ term, and X_0 is 0, 1, or 2, six cases must be considered. One gets

$$P_{5,2}(X) = \frac{X^4}{288} - \frac{X^3}{144} - \frac{X^2}{36} + \pi(X)\frac{X}{16} + \gamma_5(\mathrm{mod}(X, 6)), \tag{50a}$$

$$\gamma_5(m) = \left(0, -\frac{1}{32}, \frac{1}{9}, -\frac{1}{32}, 0, \frac{23}{288}\right) \quad \text{for } m = (0, 1, 2, 3, 4, 5) \tag{50b}$$

$$= -\frac{\mathrm{mod}(m, 2)}{32} + \frac{2\mathrm{mod}(m, 3) - \mathrm{mod}(-m, 3)}{27}. \tag{50c}$$

D. Summary and examples for the five-fermion case

Collecting expressions (37), (42), and (50), the complete value $P(3j - p; j, 5) = P_{5,0}(p + 2j) - H(p)P_{5,1}(p) + H(p - 2j)P_{5,2}(p - 2j)$ is therefore given, after changing $3j - p$ into M for a better readability, by

$$\begin{aligned}
P(M; j, 5) = {} & \frac{(5j - M)^4}{2880} - \frac{(5j - M)^3}{288} + \frac{(5j - M)^2}{288} + \frac{(5j - M)}{24} - \pi(5j - M)\frac{(5j - M)}{32} \\
& + \varphi(\mathrm{mod}(5j - M, 12)) + \frac{\delta(\mathrm{mod}(5j - M, 5))}{5} \\
& - H(3j - M)\left[\frac{(3j - M)^4}{576} - \frac{(3j - M)^3}{96} - \frac{(3j - M)^2}{288} + \frac{(3j - M)}{24}\right. \\
& \left. + \pi(3j - M)\frac{(3j - M)}{32} + \eta(\mathrm{mod}(3j - M, 12))\right] \\
& + H(j - M)\left[\frac{(j - M)^4}{288} - \frac{(j - M)^3}{144} - \frac{(j - M)^2}{36} + \pi(j - M)\frac{(j - M)}{16} + \gamma_5(\mathrm{mod}(j - M, 6))\right].
\end{aligned} \tag{51}$$

The values for φ, η, and γ_5 are given by Eqs. (35b), (43), and (50c), respectively.

A careful inspection of the above derivation shows that it holds not only for $2j \leqslant p \leqslant 3j$ but also for $2j \leqslant p \leqslant 4j$, or, in terms of the total magnetic quantum number $-j \leqslant M \leqslant j$. This property is important if one needs to use this value to get the expression for $P(M; j, 6)$ for $0 \leqslant M \leqslant 2j$. It is worth mentioning that we have been able to obtain again the formula (51)

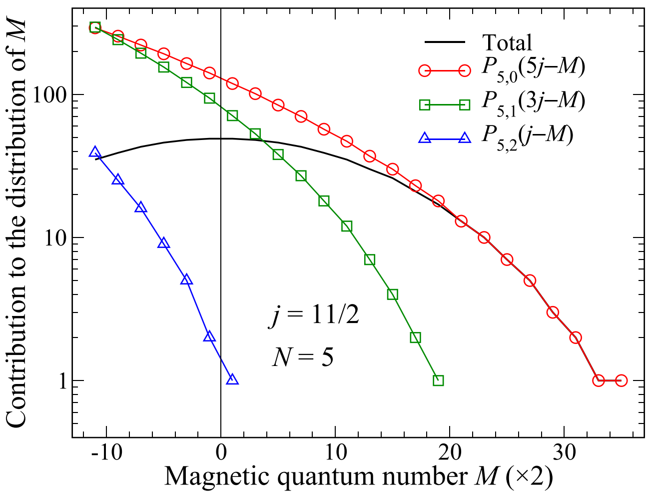

FIG. 1. Polynomials $P_{5,0}$, $P_{5,1}$, and $P_{5,2}$ and total distribution function $P(M; j, 5)$ for $j = 11/2$. The plot is done in the interval from $-j$ to $J_{\max}$ $(-11/2 \leqslant M \leqslant 35/2)$.

using Talmi's five-term recurrence. However, the latter derivation is more cumbersome and is not detailed here.

Figures 1 and 2 display the three latter polynomials together with the total $P(M; j, 5)$ distribution, respectively, for $j = 11/2$ and $j = 23/2$. If one ignores the integer values of $X = 3j - M$ in the interval 0–6 for which $P_{5,1}(X)$ vanishes ($P_{5,1}(0) = \cdots = P_{5,1}(6) = 0$), and the integer values of $X = j - M$ in the interval 0–4 for which $P_{5,2}(X)$ vanishes, then one notices that the ratios $P_{51}(3j - M)/P_{50}(j - M)$ and $P_{52}(j - M)/P_{50}(j - M)/$ are positive and rapidly decreasing functions of M. As mentioned in Sec. VI of our previous paper [25], one has for any j, N the property $P(J_{\max} - 1; j, N) = P(J_{\max}; j, N) = 1$. This is visible on the black and red curves of Figs. 1 and 2, which level off at $P = 1$ for $M \geqslant J_{\max} - 1$.

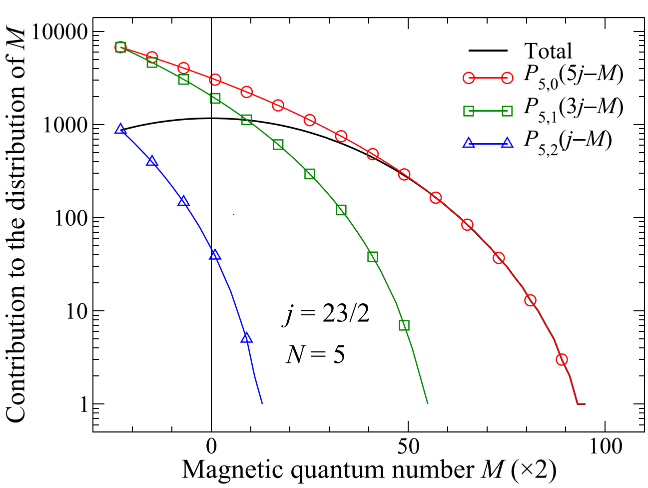

FIG. 2. Polynomials $P_{5,0}$, $P_{5,1}$, and $P_{5,2}$ and total distribution function $P(M; j, 5)$ for $j = 23/2$. The abscissa ranges from $-j$ to $J_{\max}$ $(-23/2 \leqslant M \leqslant 95/2)$.

Using expressions (37), (42), and (50), and keeping only the fourth-degree term in the polynomials—which is correct for $j \gtrsim 5$ one easily obtains for $M \simeq 0$ the ratio $P_{5,2}(j)/P_{5,1}(3j) \simeq 2/81$ or $P_{5,2}(j)/P_{5,0}(5j) \simeq 2/125$. This is confirmed by the plots displayed here. More generally from the above formulas one gets $P_{5,2}(j - M)/P_{5,1}(3j - M) \simeq 2(j - M)^4/(3j - M)^4$ which indeed decreases with M. An interesting point is that $P_{5,0}(5j - M)$, $P_{5,1}(3j - M)$ and $P_{5,2}(j - M)$ are not even functions of M, while $P(M; j, 5)$ is (i.e., $P(-M; j, 5) = P(M; j, 5)$). As seen on these plots, for $M = -j$ one observes the approximate—though not exact—cancellation of the difference $P_{5,0}(5j - M) - P_{5,1}(3j - M)$, so that $P(-j; j, 5) \simeq P_{5,2}(2j)$.

Since the piecewise polynomial forms for $P_{5,k}(5j - M - 2kj)$ are the same for integer and half-integer j, the plot of these quantities for integer j do not differ significantly from the plot for half-integer j. This why we only consider in the above figures the half-integer case. Finally, we could check that the general aspect of the plots do not change much when considering large-j value.

E. Number of levels with given total angular momentum

Using the relation $Q(J; j, N) = P(J; j, N) - P(J + 1; j, N)$ (for $J < J_{\max}$) [30], one can derive the total number of levels with a given total moment $J = 3j - p$ for five fermions. We thus need to evaluate $P(3j - p; j, 5) - P(3j - (p - 1); j, 5)$, and when computing the contribution of $-H(p)P_{5,1}(p)$, i.e., the second part of Eq. (51), one may notice that $P_{5,1}(-1) = 0$, so that the contribution of this part to $Q(J; j, 5)$ is simply $-H(p)P_{5,1}(p) + H(p - 1)P_{5,1}(p - 1) = -H(p)(P_{5,1}(p) - P_{5,1}(p - 1))$ even if $p = 0$. A similar consideration applies to the term in factor of $H(p - 2j)$. Using the relation (51) for P one gets, after some basic simplifications,

$$
\begin{aligned}
Q(J; j, 5) = {} & \frac{(5j - J)^3}{720} - \frac{(5j - J)^2}{80} + \frac{(5j - J)}{20} \\
& - \pi(5j - J)\frac{(5j - J)}{16} + \overline{\gamma_{5,0}}(\mathrm{mod}(5j - J, 60)) \\
& - H(3j - J)\bigg[\frac{(3j - J)^3}{144} - \frac{(3j - J)^2}{24} \\
& + \pi(3j - J)\frac{(3j - J)}{16} + \overline{\gamma_{5,1}}(\mathrm{mod}(3j - J, 12))\bigg] \\
& + H(j - J)\bigg[\frac{(j - J)^3}{72} - \frac{(j - J)^2}{24} - \frac{(j - J)}{12} \\
& + \pi(j - J)\frac{(j - J)}{8} + \overline{\gamma_{5,2}}(\mathrm{mod}(j - J, 6))\bigg].
\end{aligned}
\tag{52a}
$$

In the above equation, one has

$$\overline{\gamma_{5,0}}(m) = -\frac{1}{5} + \frac{\text{mod}(m,2)}{16} - \frac{\text{mod}(m,3) - \text{mod}(-m,3)}{9} + \frac{\text{mod}(m,4) + \text{mod}(-m,4)}{16} + \frac{\delta(\text{mod}(m,5))}{5} - \frac{\delta(\text{mod}(m-1,5))}{5} \tag{52b}$$

$$\overline{\gamma_{5,1}}(m) = -\frac{\text{mod}(m,2)}{8} + \frac{\text{mod}(-m,3)}{9} + \frac{\text{mod}(m,4) - \text{mod}(-m,4)}{16} \tag{52c}$$

$$\overline{\gamma_{5,2}}(m) = -\frac{\text{mod}(m,2)}{8} + \frac{\text{mod}(m,3)}{9}, \tag{52d}$$

with m ranging from 0 to 59 for $\overline{\gamma_{5,0}}(m)$, from 0 to 11 for $\overline{\gamma_{5,1}}(m)$, and from 0 to 5 for $\overline{\gamma_{5,2}}(m)$.

The expression (51) allows us to obtain the total number of levels for $j = l$ integer. One has

$$Q_{\text{tot}}(l^5) = P(0;l,5) = \sum_L Q(L;l,5) = \frac{23l^4}{288} - \frac{23l^3}{144} + \frac{13l^2}{144} + \frac{l}{12} - \pi(l)\frac{3l}{16} + s_0, \tag{53a}$$

$$\text{with} \quad s_0 = \left(0, \frac{3}{32}, \frac{17}{36}, \frac{11}{32}, 0, \frac{91}{288}, \frac{1}{4}, \frac{11}{32}, \frac{2}{9}, \frac{3}{32}, \frac{1}{4}, \frac{163}{288}\right) \quad \text{for } \text{mod}(l,12) = 0, 1 \cdots 11 \tag{53b}$$

$$= -\frac{\text{mod}(l,2)}{32} + \frac{4\text{mod}(l,3) - 2\text{mod}(-l,3)}{27} + \frac{\text{mod}(l,4)}{8}. \tag{53c}$$

The corresponding number of levels for half-integer j has already been published, see Eq. (4.11) of Ref. [25].

V. APPLICATION TO THE SIX-FERMION CASE

To demonstrate the efficiency of the recurrence (1a), we provide as a last example its application to the derivation of expressions for $P(4j - p; j, 6)$.

A. Case M greater than or equal to $4j$

The general formula (C4) becomes in the case $N = 6$ and $p \leqslant 0$

$$P(4j - p; j, 6) = P_{6,0}(p + 2j) \tag{54a}$$

$$= \sum_{s=1}^{t} P(4j - p + 3s - j + (s-2)/2; j - s/2, 5), \tag{54b}$$

where $t = \lfloor (2j + p - 9)/6 \rfloor$. With the substitutions $j \to j - s/2$, $p \to p - 5s + 2$ the five-fermion expression for P (51) provides, after basic algebraic manipulations, setting $\pi(X) = \text{mod}(X, 2)$,

$$P_{6,0}(X) = \frac{X^5}{86400} - \frac{X^4}{3840} + \frac{19X^3}{12960} + \pi(X)\frac{X^2}{384} + \psi_{6,0}(\text{mod}(X,6))X + \varphi_{6,0}(\text{mod}(X,60)) \quad \text{with } X = p + 2j. \tag{55}$$

The coefficient factoring $X = p + 2j$ is

$$\psi_{6,0}(m) = \left(\frac{1}{180}, -\frac{629}{17280}, -\frac{7}{540}, -\frac{103}{5760}, -\frac{7}{540}, -\frac{629}{17280}\right) \quad \text{for } m = (0, 1, \dots 5) \tag{56a}$$

$$= \frac{1}{180} - \frac{3\text{mod}(m,2)}{128} - \frac{\text{mod}(m,3) + \text{mod}(-m,3)}{162} \tag{56b}$$

$$= -\frac{7}{540} - \frac{3\text{mod}(m,2)}{128} + \frac{\delta(\text{mod}(m,3))}{54}, \tag{56c}$$

where we have used the Kronecker symbol $\delta(n) = 1$ if $n = 0$, $\delta(n) = 0$ otherwise. The term $\varphi_{6,0}(m)$ can be given explicitly as a list of 60 values $(0, 16889/518400, 583/32400, \dots, -313/32400, -19319/518400)$. A simpler formulation is provided by identifying this list to a linear combination of modulo functions. Solving a simple linear system leads to

$$\varphi_{6,0}(m) = -\frac{5\,\text{mod}(m,2)}{768} - \frac{\text{mod}(m,3) - \text{mod}(-m,3)}{162} - \frac{\text{mod}(-m,4)}{32} + \frac{1 - \delta\big(\text{mod}(m,6)\big)}{6} - \frac{\text{mod}(m,5)}{25}. \tag{56d}$$

One notices that $\varphi_{6,0}(m)$ is a function of $\text{mod}(p + 2j, 60)$. As a rule the zeroth-order term in the polynomial $P_{N,k}(X)$ is a function of $\text{mod}(X, L(N - k))$ where $L(N - k)$ is the least common multiple of the integers $(2, 3, \dots, N - k)$.

B. Case M between $2j$ and $4j$

The general formalism set in Sec. II leads us to write $P_{6,1}(p) = P_{6,1}(p-5) + P_{5,1}(p-4)$. Iterating this equation, with the modulo 5 definition $p = p_0 + 5\nu$ we get

$$P_{6,1}(p) - P_{6,1}(p_0) = P_{6,1}(p) \tag{57a}$$

$$= (P_{6,1}(p_0+5\nu) - P_{6,1}(p_0+5\nu-5)) + \cdots + (P_{6,1}(p_0+5) - P_{6,1}(p_0)) \tag{57b}$$

$$= \sum_{i=1}^{\nu} P_{5,1}(p_0+5i-4). \tag{57c}$$

Using the known piecewise expression for $P_{5,1}(p)$ (42) we get, with the assumption $P_{6,1}(p_0) = 0$ for $0 \leqslant p_0 \leqslant 4$,

$$P_{6,1}(p) = \frac{p^5}{14400} - \frac{p^4}{960} + \frac{13p^3}{4320} + \frac{p^2}{96} - \frac{p}{24} + \pi(p)\frac{p}{64} + \varphi_{6,1}(\mathrm{mod}(p,60)). \tag{58a}$$

The zeroth-order term $\varphi_{6,1}(m)$ is, separating the contribution with period 12 and the contribution with period 5,

$$\varphi_{6,1}(m) = \alpha_{6,1}(\mathrm{mod}(m,12)) + \frac{\mathrm{mod}(-m,5)}{25}, \tag{58b}$$

$$\alpha_{6,1}(n) = \left(0, -\frac{253}{1728}, -\frac{19}{216}, -\frac{7}{64}, -\frac{1}{27}, -\frac{125}{1728}, \frac{1}{8}, -\frac{253}{1728}, \frac{1}{27}, -\frac{7}{64}, -\frac{35}{216}, -\frac{125}{1728}\right) \quad \text{for } n = 0, 1 \cdots 11, \tag{58c}$$

$$\varphi_{6,1}(m) = \frac{\mathrm{mod}(m,2)}{64} + \frac{\mathrm{mod}(m,3) - \mathrm{mod}(-m,3)}{27} - \frac{1-\delta(\mathrm{mod}(m,4))}{8} + \frac{\mathrm{mod}(-m,5)}{25}. \tag{58d}$$

Using the above formulas one checks that $P_{6,1}(p) = 0$ if $-4 \leqslant p \leqslant 10$, in agreement with the assumption (57a).

C. Case $0 \leqslant M \leqslant 2j$

The general formula (11) implies for $N = 6$, $k = 2$, and $p - 2j \geqslant 4$,

$$P_{6,2}(p-2j) = P_{6,2}(p-2j-4) + P_{5,2}(p-2j-3). \tag{59}$$

Setting $X = p - 2j$, and $X = X_0 + 4\nu$ with ν integer, we obtain by repeated application of the above relation

$$P_{6,2}(X) - P_{6,2}(X_0) = (P_{6,2}(X_0+4\nu) - P_{6,2}(X_0+4\nu-4)) + \cdots + (P_{6,2}(X_0+4) - P_{6,2}(X_0)) \tag{60a}$$

$$= \sum_{i=1}^{\nu} P_{5,2}(X_0+4i-3). \tag{60b}$$

The known expression for $P_{5,2}(p)$ (50) provides, after basic algebra, and assuming $P_{6,2}(X_0) = 0$ for $0 \leqslant X_0 \leqslant 3$,

$$P_{6,2}(X) = \frac{X^5}{5760} - \frac{X^4}{768} - \frac{X^3}{864} + \frac{X^2}{48} - \frac{X}{60} - \pi(X)\frac{X^2-3X}{128} + \gamma_{6,2}(\mathrm{mod}(X,12)). \tag{61a}$$

The last term of this expression is, for $m = 0, 1 \cdots 11$, respectively,

$$\gamma_{6,2}(m) = \left(0, -\frac{121}{6912}, -\frac{11}{432}, -\frac{11}{256}, -\frac{1}{27}, \frac{391}{6912}, -\frac{1}{16}, -\frac{553}{6912}, \frac{1}{27}, \frac{5}{256}, -\frac{43}{432}, -\frac{41}{6912}\right) \tag{61b}$$

$$= \frac{13\,\mathrm{mod}(m,2)}{256} + \frac{\mathrm{mod}(m,3) - \mathrm{mod}(-m,3)}{27} - \frac{\mathrm{mod}(m,4)}{32}. \tag{61c}$$

With the above formulas one may easily check that $P_{6,2}(X) = 0$ for $-4 \leqslant X \leqslant 7$, in agreement with the recurrence hypothesis.

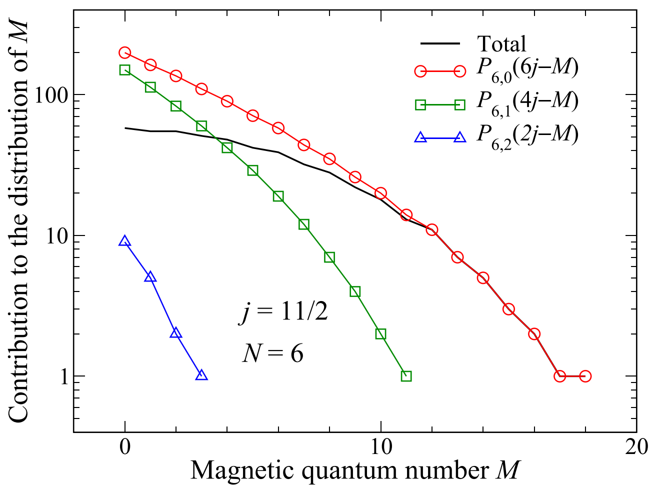

FIG. 3. Polynomials $P_{6,0}$, $P_{6,1}$, and $P_{6,2}$ and total distribution function $P(M; j, 6)$ for $j = 11/2$. The total magnetic quantum number M ranges from 0 to 18.

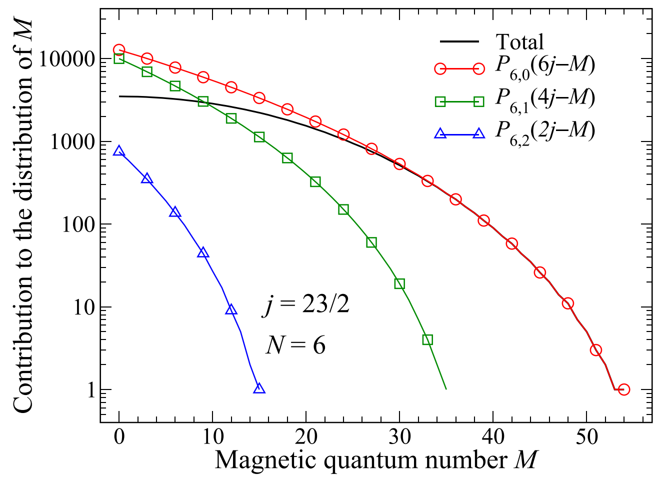

FIG. 4. Polynomials $P_{6,0}$, $P_{6,1}$, and $P_{6,2}$ and total distribution function $P(M; j, 6)$ for $j = 23/2$. The quantum number M varies from 0 to 54.

D. Summary and examples for the six-fermion case

Collecting the polynomial expressions (55), (58), and (61), one has, in the six-fermion case,

$$\begin{aligned}
&P(M; j, 6)\\
&\quad= \frac{(6j-M)^5}{86400} - \frac{(6j-M)^4}{3840} + \frac{19(6j-M)^3}{12960}\\
&\qquad+ \pi(6j-M)\frac{(6j-M)^2}{384}\\
&\qquad+ \psi_{6,0}(\mathrm{mod}(6j-M,6))(6j-M)\\
&\qquad+ \varphi_{6,0}(\mathrm{mod}(6j-M,60))\\
&\qquad+ H(4j-M)\bigg[\frac{(4j-M)^5}{14400} - \frac{(4j-M)^4}{960}\\
&\qquad+ \frac{13(4j-M)^3}{4320} + \frac{(4j-M)^2}{96} - \frac{(4j-M)}{24}\\
&\qquad+ \pi(4j-M)\frac{(4j-M)}{64} + \varphi_{6,1}(\mathrm{mod}(4j-M,60))\bigg]\\
&\qquad+ H(2j-M)\bigg[\frac{(2j-M)^5}{5760} - \frac{(2j-M)^4}{768}\\
&\qquad- \frac{(2j-M)^3}{864} + \frac{(2j-M)^2}{48} - \frac{(2j-M)}{60}\\
&\qquad- \pi(2j-M)\frac{(2j-M)^2-3(2j-M)}{128}\\
&\qquad+ \gamma_{6,2}(\mathrm{mod}(2j-M,12))\bigg],
\end{aligned} \tag{62}$$

where M varies from 0 to $6(2j-5)/2$. The various phase factors are provided by Eqs. (56), (58d), and (61c).

Of course as in the five-fermion case, such expressions allows one to obtain the total number of levels of a given angular momentum J, with the help of the usual relation $Q(J; j, 6) = P(J; j, 6) - P(J+1; j, 6)$. We can also derive the total number of levels from the expression $P(0; j, 6)$.

Figures 3 and 4 display the three polynomials $P_{6,k}$ together with the total $P(M; j, 6)$ distribution, respectively, for $j = 11/2$ and $j = 23/2$. Contrary to the $N = 5$ case, the distribution are plotted only for $M \geqslant 0$. For $M < 0$, the symmetry property $P(-M; j, 6) = P(M; j, 6)$ must be used. One also notices that the distributions undergo moderate changes when j increases from $11/2$ to $23/2$. The ratios $P_{6,1}(4j-M)/P_{6,0}(6j-M)$ and $P_{6,2}(2j-M)/P_{6,0}(6j-M)$ decreasing functions of M. For $M \simeq 0$, the former ratio is about 1, while the latter amounts to roughly one order of magnitude, as can be checked on the above-mentioned analytical forms.

E. Number of levels with given total angular momentum for six fermions

The number of levels with total angular momentum J is given by the usual formula $Q(J; j, 6) = P(J; j, 6) - P(J+1; j, 6)$. From the expansion (2), one may write

$$\begin{aligned}
Q(J; j, 6) &= Q_{6,0}(6j-J) - H(4j-J)Q_{6,1}(4j-J)\\
&\quad + H(2j-J)Q_{6,2}(2j-J).
\end{aligned} \tag{63}$$

For $J \geqslant 4j$, using the value $P_{60}(X)$ given by Eq. (55), simple algebraic manipulations allows us to write, if $X = p + 2j = 6j - J$, then

$$Q_{6,0}(X) = \frac{X^4}{17280} - \frac{X^3}{864} + \frac{X^2}{288} + \pi(X)\frac{X^2}{192} + \overline{\psi_{60}}(\text{mod}(X,6))X + \overline{\phi_{60}}(\text{mod}(X,60)), \tag{64a}$$

$$\overline{\psi_{60}}(m) = \frac{1}{24} - \frac{5\text{mod}(m,2)}{96} - \frac{\text{mod}(-m,3)}{54}, \tag{64b}$$

$$\overline{\phi_{60}}(m) = \frac{13\text{mod}(m,2)}{384} - \frac{\text{mod}(m,3) - 4\text{mod}(-m,3)}{54} + \frac{\text{mod}(m,4) - \text{mod}(-m,4)}{32} - \frac{\text{mod}(m,5) + \text{mod}(-m,5)}{25} + \frac{\text{mod}(-m,6)}{36}. \tag{64c}$$

For $2j \leqslant J \leqslant 4j$, setting $X = 4j - M$, the evaluation of $P_{61}(X) - P_{61}(X-1)$ from Eq. (58) leads to

$$Q_{61}(X) = \frac{X^4}{2880} - \frac{7X^3}{1440} + \frac{23X^2}{1440} - \frac{X}{120} + \pi(X)\frac{X}{32} + \overline{\phi_{61}}(\text{mod}(X,60)). \tag{65a}$$

The constant term can be expressed as

$$\overline{\phi_{61}}(m) = -\frac{3\ \text{mod}(m,2)}{64} + \frac{\delta(\text{mod}(m-2,3))}{9} - \frac{\text{mod}(-m,4)}{16} + \frac{\delta(\text{mod}(m-1,5))}{5}. \tag{65b}$$

For $J \leqslant 2j$, we get, using $P_{62}(X)$ provided by Eq. (61) and $X = 2j - J$,

$$Q_{6,2}(X) = \frac{X^4}{1152} - \frac{X^3}{144} + \frac{X^2}{72} - \pi(X)\frac{X^2 - 4X}{64} + \overline{\phi_{62}}(\text{mod}(X,12)) \tag{66a}$$

$$\overline{\phi_{62}}(m) = \frac{9\ \text{mod}(m,2)}{128} + \frac{\delta(\text{mod}(m-2,3))}{9} + \frac{\delta(\text{mod}(m,4)) - 1}{8}. \tag{66b}$$

To get the total number of levels $Q_{\text{tot}}(j^6) = \sum_J Q(J; j, 6)$, we use

$$Q_{\text{tot}}(j^6) = P(0; j, 6) = P_{6,0}(6j) - P_{6,1}(4j) + P_{6,2}(2j) \tag{67}$$

and expressions (55), (58), and (61) for the $P_{6,k}(X)$. After elementary algebra, we obtain

$$Q_{\text{tot}}(j^6) = P(0; j, 6) = \frac{11j^5}{450} - \frac{11j^4}{120} + \frac{31j^3}{270} - \frac{j^2}{12} + \frac{j}{6} + \pi(2j)\left(\frac{j^2}{16} - \frac{3j}{32}\right) + \Gamma_6(\text{mod}(2j,60)), \tag{68a}$$

$$\text{where}\quad \Gamma_6(m) = \frac{43}{128}\text{mod}(m,2) + \frac{2}{27}(\text{mod}(m,3) - \text{mod}(-m,3)) - \frac{\text{mod}(m,4)}{16} - \frac{2}{25}\text{mod}(3m,5). \tag{68b}$$

If $j = l$ is integer, then Γ_6 receives a somewhat simpler expression. Knowing that $\text{mod}(2l,3) - \text{mod}(-2l,3) = \text{mod}(l+1,3) - 1$, $\text{mod}(2l,4) = 2\text{mod}(l,2)$, and $\text{mod}(6l,5) = \text{mod}(l,5)$ we finally get

$$\Gamma_6(\text{mod}(2l,60)) = -\frac{\text{mod}(l,2)}{8} + \frac{2}{27}(\text{mod}(l+1,3) - 1) - \frac{2}{25}\text{mod}(l,5). \tag{69}$$

VI. CONCLUSION

In this work, we presented a method to determine the distributions of the total magnetic quantum number M and of the total angular momentum J without any restriction, apart from the fact that the complexity of the calculation increases with the number of fermions N. The method boils down to closed-form expressions as piecewise polynomials which obeys simple recurrence relations. An interesting fact is that the closed-form expressions obtained here or in Ref. [25] can be formulated such that they apply for *half-integer as well as integer momenta* j. Explicit expressions for the M-distribution are provided by Eqs. (27), (51), and (62) for $N = 4$, 5, and 6, respectively. Formulas were also provided for the distribution of the total angular momentum J.

In a general way, the relations established for the piecewise polynomials can be implemented in an algorithm that would enable one to determine the distribution of angular momentum M whatever the number of fermions. The techniques presented here can be generalized to the case of two angular momenta, for instance to determine the number of states with a given total spin I and isospin T for N nucleons in a single-j shell [27].

It is worth mentioning that Zhao and Arima [16] proved that the number of states with a given value of $J_{\max} - J$ of N fermions in a j orbit is equal to the number of states of N spin-ℓ bosons with total spin L, the value of $L_{\max} - L$ ($L_{\max} = N\ell$) being equal to $J_{\max} - J$. In the present work we choose, for simplicity, to carry out the derivations for fermions, but the results can also be extended to bosons.

TABLE I. Elements for computing $P(j-p;j,3)$ if $p \leqslant 0$ using sum (A3). See main text for details.

m	0	1	2	3	4	5
Lowest term	1	1	2	1	1	2
Highest term	$3\nu-1$	3ν	3ν	$3\nu+1$	$3\nu+1$	$3\nu+2$
Missing terms	$3i$	$3i+2$	$3i+1$	$3i$	$3i+2$	$3i+1$
Sum	$3\nu^2$	$3\nu^2+\nu$	$3\nu^2+2\nu$	$3\nu^2+3\nu+1$	$3\nu^2+4\nu+1$	$3\nu^2+5\nu+2$

A priori, it is possible to relate the total number of levels $Q_{\text{tot}}(j^N)$ in a j^N configuration to sums involving the coefficients of fractional parentage, which is likely to yield sum rules for $3nj$ symbols or recoupling coefficients. We already applied that idea [31] previously to the three- and four-fermion cases, and it turns out that the corresponding sum rules involve $3nj$ coefficients up to $n=2$ and $n=3$, respectively [25,26]. For the five- and six-fermion cases, one may expect higher $3nj$ coefficients ($12j$, $15j$, etc.), which suggests rather tedious calculations, but would definitely be worth investigating.

APPENDIX A: THREE-TERM RECURRENCE APPLIED TO THE THREE-FERMION CASE

The aim of this Appendix is twofold. First, it illustrates the general method sketched in Sec. II. Then it provides the values of the polynomials $P_{3,0}(X)$, $P_{3,1}(X)$ required to initiate the recursion on N.

If $N=3$, then one has from the relations (C4) and (C3)

$$P(M;j,3) = \sum_{s=1}^{t} P(M+2s-j-1, j-s/2, 2). \tag{A1}$$

As in Ref. [25] we use the notation $M=j-p$, so that

$$t = \left\lfloor j - \frac{M}{3} \right\rfloor = \left\lfloor \frac{2j+p}{3} \right\rfloor. \tag{A2}$$

Noting that $\overline{M} = M+2s-j-1 = 2s-p-1$ is positive, the general term of the above sum is obtained from the well-known value $P(M;j,2) = \lfloor j+1/2-|M|/2 \rfloor$. We consider first the $p \leqslant 0$ case (or $M \geqslant j$), so that $\overline{M} \geqslant 0$. We write $2j+p = 6\nu+m$ with ν, m integers and get

$$P(2s-p-1, j-s/2, 2) = \left\lfloor \frac{2j+p}{2} - \frac{3s}{2} + 1 \right\rfloor = \left\lfloor 3\nu + \frac{m}{2} - \frac{3s}{2} + 1 \right\rfloor \tag{A3}$$

to be summed from $s=1$ to $t=2\nu+\lfloor m/3 \rfloor$. One must separate six cases according to m. For instance, if $m=0$, then the sum (A1) involves the terms $3\nu-1, 3\nu-2, 3\nu-4, \cdots 4, 2, 1$. It involves all integers i from 1 to $3\nu-1$, except those verifying $\text{mod}(i,3)=0$. This sum is easily computed as

$$P(j-p;j,3)|_{m=0} = \sum_{i=1}^{3\nu-1} i - \sum_{i=1}^{\nu-1} 3i = 3\nu^2 = \frac{(2j+p)^2}{12}. \tag{A4}$$

The various cases $m=0$ to 5 are summed up in Table I. One obtains from this table

$$P(j-p;j,3) = \frac{(2j+p)^2}{12} + \alpha(\text{mod}(2j+p,6)), \tag{A5a}$$

$$\text{with} \quad \alpha(m) = \left(0, -\frac{1}{12}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{3}, -\frac{1}{12}\right) \quad \text{if } m=(0,1,2,3,4,5) \tag{A5b}$$

$$= -\frac{1}{12} - \frac{\delta(\text{mod}(m,2))}{4} + \frac{\delta(\text{mod}(m,3))}{3} \tag{A5c}$$

$$= \frac{\text{mod}(m,2)}{4} - \frac{\text{mod}(m,3)+\text{mod}(-m,3)}{9}, \tag{A5d}$$

where $\delta(n)$ is the Kronecker symbol. This agrees with Ref. [25] but here we do not assume j half-integer.

In the case $p \geqslant 0$ as in Ref. [25] we write the assumption, to be verified by recurrence,

$$P(j-p;j,3) = \frac{(2j+p)^2}{12} + \alpha(2j+p) - H(p)P_{3,1}(p), \tag{A6}$$

where $H(p)$ is the Heaviside function ($H(p) = 1$ if $p \geqslant 0$, 0 otherwise) and $P_{3,1}(p)$ a function to be determined. Using the value $P(j+1; j, 3)$ mentioned in Ref. [25] one notices that $P_{3,1}(1) = 0$. Putting such form in the basic relation (1a), we get, for $p \geqslant 2$ (which allows us to drop the Heaviside functions),

$$\frac{(2j+p)^2}{12} + \alpha(2j+p) - P_{3,1}(p) = \frac{(2j+p-3)^2}{12} + \alpha(2j+p-3) - P_{3,1}(p-2) + P(1-p; j-1/2, 2). \tag{A7}$$

From the value $P(M; j, 2) = \lfloor j + 1/2 - |M|/2 \rfloor$ one easily checks

$$P(1-p; j-1/2, 2) = P(p-1; j-1/2, 2) = \left\lfloor j - \frac{p-1}{2} \right\rfloor = j - \frac{p}{2} + \frac{\pi(2j+p)}{2}. \tag{A8}$$

It is interesting to note that the above relation holds not only for $p \leqslant j$ but also for $p \leqslant 2j$. Using the property $\alpha(m) - \alpha(\text{mod}(m-3, 6)) = -(-1)^m/4$, one obtains from Eq. (A7), if $2 \leqslant p \leqslant 2j$,

$$P_{3,1}(p) - P_{3,1}(p-2) = p - 1, \tag{A9}$$

which does not depend on j as expected from the general analysis of Sec. II. Separating the cases p even and odd, we get

$$P_{3,1}(p) = \begin{cases} P_{3,1}(2q) - P_{3,1}(0) = \sum_{i=1}^{q}(2i-1) = q^2 = p^2/4 & \text{if } p = 2q, \\ P_{3,1}(2q+1) - P_{3,1}(1) = \sum_{i=1}^{q}(2i) = q(q+1) = p^2/4 - 1/4 & \text{if } p = 2q+1. \end{cases} \tag{A10}$$

One has thus obtained

$$P(j-p; j, 3) = \frac{(2j+p)^2}{12} + \alpha(2j+p) - H(p)\left(\frac{p^2}{4} - \frac{\pi(p)}{4}\right), \tag{A11}$$

with $-2j + 3 \leqslant p \leqslant 2j$ and α given by Eq. (A5d). This expression agrees with Eq. (2.23) of Ref. [25]. The interest of the present derivation is that it applies even if j is integer. In addition we have established the formula remains valid for $-j \leqslant M = j - p \leqslant 3j - 3$, though it must not be applied in the full M-range, namely for $p > 2j$ or $M < -j$.

Such formula with $j = p = l$ integer allows one to obtain the total number of levels for three fermions. One has

$$P(0; l, 3) = \sum_L Q(L; l, 3) = \begin{cases} l^2/2 & \text{if } l \text{ is even} \\ (l^2+1)/2 & \text{if } l \text{ is odd} \end{cases} = \left\lfloor \frac{l^2+1}{2} \right\rfloor. \tag{A12}$$

The corresponding formula for half-integer j has already been obtained, using fractional parentage coefficients [26] or recurrence relations [25].

APPENDIX B: MAXIMUM VALUES OF THE INTERVAL INDICES IN THREE-TERM RECURRENCES

The aim of this Appendix is the study of the maximum k indices according to the sum representation (2) for each P function involved in the recurrence (1a). Using the function $k_{\max}$ defined by Eq. (3), we set

$$k_1 = k_{\max}(M, j, N), \quad k_2 = k_{\max}(M + N/2, j - 1/2, N), \quad k_3 = k_{\max}(M - j + (N-1)2, j - 1/2, N - 1). \tag{B1}$$

Let us show that the three indices k_1, k_2, and k_3 for relation (1a) are equal or differ by 1. Here also we limit ourselves to $M \geqslant 0$. Let us first set $a_1 = (Nj - M)/(2j)$, $a_2 = (Nj - M - N)/(2j-1)$ and $a_3 = (N(j-1) + 1 - M)/(2j-1)$, so that $k_1 = \lfloor a_1 \rfloor$, $k_2 = \lfloor a_2 \rfloor$ as well as $k_3 = \lfloor a_3 \rfloor$. We have

$$a_1 - a_2 = \frac{M + Nj}{2j(2j-1)}. \tag{B2}$$

Since $M \leqslant (2j + 1 - N)N/2$, we get

$$M + Nj \leqslant 2Nj - \frac{N(N-1)}{2} = N(2j - (N-1)/2) \leqslant N(2j-1) \quad \text{if } N \geqslant 3. \tag{B3}$$

Therefore, $a_1 - a_2 \leqslant N/(2j) \leqslant 1$ if $N \geqslant 3$ and $N \leqslant 2j$ (we omit the trivial full-shell case). As a consequence, $k_1 \geqslant k_2$ and $k_1 - k_2$ is equal to 0 or 1. In the same way $a_3 - a_2 = 1/(2j-1)$, so that $0 \leqslant a_3 - a_2 \leqslant 1$, and taking integer parts, $k_3 \geqslant k_2$ and $k_3 - k_2$ is equal to 0 or 1. We have also, using the same upper value for M as above

$$a_1 - a_3 = \frac{M + (N-2)j}{2j(2j-1)} \leqslant \frac{(N-1)(2j - N/2)}{2j(2j-1)} \leqslant \frac{N-1}{2j} \leqslant 1 \quad \text{if } N \geqslant 2. \tag{B4}$$

Furthermore, assuming again $N \geqslant 2$, one has $M + (N-2) \geqslant 0$, so that $0 \leqslant a_1 - a_3 \leqslant 1$, and $k_1 - k_3$ is equal to 0 or 1. Therefore, the relation $k_1 \geqslant k_3 \geqslant k_2$ always holds.

If $k_1 = k_2 + 1$, then for the arguments of the Heaviside functions, one has $X_1 = X(k_1) = Nj - M - 2k_1.j \geqslant 0$ and $N(j - 1/2) - M - N/2 - (2j-1)k_1 < 0$. This means that $X'(k_2) \geqslant 0$ but $X'(k_1) < 0$, or equivalently $Nj - M \geqslant 2k_1.j > Nj - M - N + k_1$. This corresponds to $0 \leqslant X_1 \leqslant N - k_1 - 1$. From Eq. (12) we know that in such conditions $P_{N,k_1}(X_1) = 0$.

The relations on the maximum indices in the recurrence (1b) are obtained in a quite similar way. Defining now

$$k'_1 = k_{\max}(M, j, N), \quad k'_2 = k_{\max}(M - N/2, j - 1/2, N), \quad k'_3 = k_{\max}(M + j - (N-1)2, j - 1/2, N - 1), \tag{B5}$$

one has then $k'_1 \leqslant k'_3 + 1 \leqslant k'_2$, and $k'_2 - k'_1$ does not exceed 1. In the "general" case, one has $k'_1 = k'_3 + 1 = k'_2$, and if $k'_2 - k'_1 = 1$ one checks that the corresponding $P_{N,k}$ vanishes.

APPENDIX C: ALTERNATE METHOD FOR DERIVING THE M DISTRIBUTION IF $M \geqslant (N-2)j$

This alternate method also relies on the use of Eq. (1a). The general procedure to get P values from this recurrence is to apply it again to the $P(M + N/2; j - 1/2, N)$ quantity in the second member. Iterating Eq. (1a) n times one gets

$$P\left(M + \frac{n-1}{2}N; j - \frac{n-1}{2}, N\right) = P\left(M + \frac{n}{2}N; j - \frac{n}{2}, N\right) + P\left(M + \frac{n}{2}N - j + \frac{n-2}{2}; j - \frac{n}{2}, N - 1\right). \tag{C1}$$

This allows us to express $P(M; j, N)$ as the sum

$$P(M; j, N) = P\left(M + \frac{\tau}{2}N; j - \frac{\tau}{2}, N\right) + \sum_{s=1}^{\tau} P\left(M + \frac{s}{2}N - j + \frac{s-2}{2}; j - \frac{s}{2}, N - 1\right). \tag{C2}$$

Here we choose $\tau = t$, t being such that $P(M + tN/2; j - t/2, N) = 0$ while $P(M + (t-1)/2N; j - (t-1)/2, N) \neq 0$. In other words, one imposes $M + tN/2 > J_{\max}(j - t/2, N) = N(j - t/2) - N(N-1)/2$, and $M + (t-1)N/2 \leqslant J_{\max}(j - (t-1)/2, N) = N(j - (t-1)/2) - N(N-1)/2$. Noting the integer part of x as $\lfloor x \rfloor$, one gets

$$t = \left\lfloor j - \frac{N-1}{2} - \frac{M}{N} \right\rfloor + 1 = \left\lfloor \frac{Nj - M - (N-3)N/2}{N} \right\rfloor. \tag{C3}$$

Therefore, if $M \geqslant (N-2)j$, then the $P(M; j, N)$ distribution is given by

$$P(M; j, N) = \sum_{s=1}^{t} P\left(M + \frac{s}{2}N - j + \frac{s-2}{2}; j - \frac{s}{2}, N - 1\right). \tag{C4}$$

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