Application of a universal reaction function to the description of heavy-ion reaction cross sections

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Approximating the angular momentum dependence of the reaction probability at a given bombarding energy by shifting it by the centrifugal energy and using the analytical formula for the elastic scattering probability, new analytical formulas for heavy-ion reaction cross sections and the universal reaction function are derived. It has been found that these new formulas describe the experimental data well and can be used for the analysis and predictions of heavy-ion reaction cross sections.

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I. INTRODUCTION

To analyze the fusion (capture) cross sections $\sigma_f(E)$ in heavy-ion reactions with different Coulomb barrier heights V_b and radii R_b calculated in the case of spherical nuclei, it is useful to compare not the excitation functions, but the dependence of the dimensionless quantities $F_0(x) = 2E\sigma_f(E)/(\hbar\omega_b R_b^2)$ versus $x = (E - V_b)/(\hbar\omega_b)$ [1–3]. Here, $E = E_{\text{c.m.}}$, ω_b, and μ are the bombarding energy in the center-of-mass system, the frequency of an inverted oscillator approximating the barrier, and the reduced mass parameter of the system, respectively. In this way the geometrical and barrier hight effects can be eliminated. This reduction method is suggested by Wong's formula [4]

$$\sigma_f(E) = \frac{\hbar\omega_b R_b^2}{2E} \ln\{1 + \exp\left(2\pi[E - V_b]/(\hbar\omega_b)\right)\}$$

for the fusion cross section [1–3]. This analytic expression is derived by approximating the barrier as an inverse parabola and neglecting the variation of the barrier radius with angular momentum. In this case

$$F_0(x) = \ln[1 + \exp(2\pi x)].$$

This is expected to be the universal fusion function (UFF) for any fusion (capture) reaction. In reactions with heavy nuclei ($Z_1Z_2 > 1600$), fusion does not always occur after the projectile is captured by the target nucleus because of competition with quasifission [5]. If the capture and fusion cross sections coincide, the comparison of experimental data with the UFF [1–3] allows us to conclude about the contributions of static deformations of the colliding nuclei and the nucleon transfer between them to the fusion cross section. Indeed, the UFF disregards these effects, which are indicated by its deviation from experiments.

In the present paper, by analogy to UFF we suggest the universal reaction function (URF) for the heavy-ion reaction cross section, which is the total interaction cross section minus the cross section of elastic scattering. So, the heavy-ion reaction cross section is larger than σ_f because it contains the contribution of other (inelastic, breakup, transfer) channels. Approximating the angular momentum dependence of the reaction probability at a given bombarding energy by shifting it by the centrifugal energy, we derive the analytical formula for the reaction cross section. This formula provides the universal trends of heavy-ion reaction cross section and will allow us to conclude about the contributions of static deformations of the colliding nuclei to the reaction cross section. The deformation effects are disregarded by the URF and well manifested in the comparison of the URF with experiments.

Thus, the challenge is to a have universal function to reduce the experimental efforts on the measurement of heavy-ion reaction cross sections. There is a nice suggestion for the UFF. Our aim is to suggest similarly for the reaction function. In Sec. II, we describe our method. In Sec. III, we derive the URF and the universal reaction probability. The results of calculations are presented in Sec. IV and summarized in Sec. V.

II. REACTION CROSS SECTION

If the elastic backscattering probability $P_{\text{el}}(E, J = 0)$ at backward angle ($\theta = 180^{\text{o}}$) or zero angular momentum (J=0) is known, the reaction probability $P_R(E, J = 0)$ can be found [6–9]:

$$P_R(E, 0) = 1 - P_{\text{el}}(E, 0). \quad (1)$$

The value of $P_{\text{el}}(E, 0)$ is defined by the ratio of the elastic scattering differential cross section and the Rutherford differential cross section $d\sigma_C$ at $\theta = 180^{\text{o}}$, that is,

$$P_{\text{el}}(E, 0) = d\sigma_{\text{el}}/d\sigma_C. \quad (2)$$

The value of σ_{el} deviates from σ_C because of the action of nuclear forces. At backward angle, the nuclei converge at a minimum distance $R_{\min}(\theta = 180^{\circ})$ where the nuclear forces are rather large and, thus, this deviation is maximal [10].

Furthermore, one can approximate the J dependence of the reaction probability $P_R(E, J)$ at a given bombarding energy E by shifting [6–9] it by the Coulomb scattering rotational energy $E_R(J)$,

$$\epsilon_J = E - E_R(J), \tag{3}$$

such that one can write

$$P_R(E, J) \approx P_R(\epsilon_J, 0). \tag{4}$$

Using the Coulomb scattering unique relations

$$J = \eta \cot\left[\frac{\theta}{2}\right] \tag{5}$$

and

$$R_{\min}(\theta) = \frac{Z}{2E}\left(1 + \sin^{-1}\left[\frac{\theta}{2}\right]\right) \tag{6}$$

between the entrance-channel angular momentum J and exit-channel scattering angle θ in the center-of-mass system, and between the distance $R_{\min}$ of closest approach on a classical Coulomb trajectory and θ, respectively, we obtain

$$E_R(J) = \frac{\hbar^2 J^2}{2\mu R^2_{\min}(\theta)} = E\frac{(\eta^2 + J^2)^{1/2} - \eta}{(\eta^2 + J^2)^{1/2} + \eta}. \tag{7}$$

Here,

$$\eta = Z\sqrt{\frac{\mu}{2\hbar^2 E}} \tag{8}$$

is the Sommerfeld parameter [11] for Coulomb collision with $Z = Z_1 Z_2 e^2$, the charge numbers $Z_{1,2}$ of interacting nuclei, and the reduced mass $\mu = m_0\frac{A_1A_2}{A_1+A_2}$ (m_0 is the nucleon mass and $A_{1,2}$ are the mass numbers of the interacting nuclei).

Employing Eq. (4), we obtain the partial reaction cross section

$$\sigma_R(E, J) \approx \pi\lambda\!\!\!^{-2}(2J + 1)P_R(\epsilon_J, 0), \tag{9}$$

where, $\lambda\!\!\!^{-2} = \hbar^2/(2\mu E)$ is the reduced de Broglie wavelength. Then, the reaction cross section is

$$\sigma_R(E) = \sum_{J=0}^{\infty} \sigma_R(E, J). \tag{10}$$

Using Eqs. (3) and (7), converting the sum over the partial waves J into an integral in Eq. (10), and expressing J by the variable $\epsilon = \epsilon_J$, one can derive the following simple expression for the reaction cross section:

$$\sigma_R(E) = \frac{\pi Z^2}{E}\int_0^E d\epsilon \frac{2E - \epsilon}{\epsilon^3} P_R(\epsilon, 0). \tag{11}$$

In Ref. [10], the general perturbation treatment of elastic scattering of heavy ions with a complex optical potential of the Woods-Saxon type was introduced, and a simple analytical expression for the elastic scattering probability was derived in the first order of perturbation theory. This leads to an easy estimate of the nuclear effects, if the energy of the projectile is in the neighborhood of the Coulomb barrier. The derived expression is quite accurate compared to the numerical solution of the Schrödinger equation, as long as the elastic cross section deviates less than about 50% from the pure Rutherford cross section [10]. Using the analytical formula

$$P_{\mathrm{el}}(E, J = 0) = 1 - 2\mathrm{Re}\left\{\frac{V_R}{E}e^{R_R/a_R}D\left(a_R^{-1}\right) + \frac{iV_I}{E}e^{R_I/a_I}D\left(a_I^{-1}\right)\right\} \tag{12}$$

of Ref. [10], where

$$D(a^{-1}) = \frac{1}{2}\left(\frac{\pi Z}{aE}\right)^{1/2}[1 - 2ika]e^{-Z/(aE)},$$

$$k = (2\mu E/\hbar^2)^{1/2},$$

and $V_{R,I} > 0$, $R_{R,I}$, and $a_{R,I}$ are the depths, radii, and diffuseness parameters, respectively, of the spherical complex nuclear optical potential $U(R) = -\tilde{V}_R(R) - i\tilde{V}_I(R)$ [$\tilde{V}_R(R)$ and $\tilde{V}_I(R)$ are the real and imaginary parts parametrized by the Woods-Saxon forms], we obtain

$$P_R(E, J = 0) = V_R\left(\frac{\pi Z}{a_R}\right)^{1/2}\frac{e^{(R_R - Z/E)/a_R}}{E^{3/2}} + 2a_I V_I\left(\frac{2\pi\mu Z}{\hbar^2 a_I}\right)^{1/2}\frac{e^{(R_I - Z/E)/a_I}}{E}. \tag{13}$$

Employing Eqs. (11) and (13), we finally derive

$$\begin{aligned}\sigma_R(E) &= \frac{\pi^{3/2}Z^{3/2}}{E^2}\left[a_R^{1/2}V_R\frac{e^{(R_R - Z/E)/a_R}}{E^{1/2}} + 2a_I^{3/2}V_I\left(\frac{2\mu}{\hbar^2}\right)^{1/2}e^{(R_I - Z/E)/a_I}\right]\\ &= \frac{\pi^{3/2}Z^{3/2}}{E^{5/2}}\left[a_R^{1/2}V_R e^{(R_R - Z/E)/a_R} + 2a_I^{3/2}V_I\left(\frac{2\mu E}{\hbar^2}\right)^{1/2}e^{(R_I - Z/E)/a_I}\right]\\ &= \frac{\pi^{3/2}R_m^{5/2}}{Z}\left[a_R^{1/2}V_R e^{(R_R - R_m)/a_R} + 2a_I^{3/2}V_I\left(\frac{2\mu Z}{\hbar^2 R_m}\right)^{1/2}e^{(R_I - R_m)/a_I}\right],\end{aligned} \tag{14}$$

where

$$R_m = Z/E$$

is the closest distance at the angle $\theta = 180^{\circ}$. As one can see, Eqs. (13) and (14) are valid at sub-barrier energies to ensure R_m is larger than R_I and R_R. If the values of R_I and a_I are comparable, respectively, to the values of R_R and a_R, the factor

$\psi_0 = 2a_I(\frac{2\mu E}{\hbar^2})^{1/2}V_I/V_R$ defines the relative contribution of the imaginary absorbing part of the optical potential to the reaction cross section. Usually the depth V_I of the imaginary potential is smaller than the value of V_R. Setting $V_I/V_R = 0.5$, for the considered reactions ^{6}He, ^{9}Be + ^{27}Al, ^{6}Li + ^{209}Bi, ^{9}Be + ^{208}Pb, ^{11}B + ^{209}Bi, and ^{12}C + ^{209}Bi, the values of ψ_0 are estimated as 0.79–0.92, 1.06–2.2, 1.94–2.42, 2.2–2.9, 2.6–3, and 2.7–3.1, respectively. Thus, in reactions with heavy nuclei the main contribution to $P_R(E, J = 0)$ comes from the imaginary potential with V_I [10]. The role of the real potential V_R can be taken effectively into account through the replacements $Z \to Z' = Z(1 - \frac{a_R}{R_b})$ (R_b is the position of the Coulomb barrier—see the Appendix—where the value of V_b is defined) and $R_m \to R'_m = Z'/E$ in the second term of Eq. (14). In this case, we effectively multiply ψ_0 by the factor $\exp[Z/(R_bE)]$, which is larger than e at sub-barrier energies and leads to $\psi_0 > 2$ for all reactions considered. Thus, instead of Eq. (14) we can employ the simpler formula

$$\sigma_R(E) = 2[\pi a_I]^{3/2}V_I R_m'^2\left(\frac{2\mu}{\hbar^2 Z'}\right)^{1/2} e^{(R_I - R'_m)/a_I}$$
$$= \frac{2[\pi a_I Z']^{3/2}V_I}{E^2}\left(\frac{2\mu}{\hbar^2}\right)^{1/2} e^{(R_I - Z'/E)/a_I}, \qquad (15)$$

which seems to be justified for all reactions considered, with the exception of the ^{6}He + ^{27}Al reaction. As shown below, this reaction is satisfactorily described using Eq. (15).

The simple expression (15) can also be extended to any value of E if the energy dependence is introduced in R_I as

$$R_I = \left(0.5 - 0.6\exp(-\chi/14.09) + \frac{0.717}{1 + \exp[(E - V_b - 7.3)/15.7]}\right)R_R,$$

where $\chi = Z_1Z_2/(A_1^{1/3} + A_2^{1/3})$. So, the lighter the interacting nuclei are, the larger the deviation of R_I is from $R_R = (R_1 + R_2)$. This expression was obtained to describe the experimental values of σ_R using the Akyüz-Winther optical potential [12] with the following parameters:

$$V_R = 15.2\pi\left(1 - 1.8\frac{N_1 - Z_1}{A_1}\frac{N_2 - Z_2}{A_2}\right)a_R\frac{R_1R_2}{R_1 + R_2},$$
$$V_I = 0.5V_R,$$
$$a_R = 1\big/\big[1.17\big\{1 + 0.53\big(A_1^{-1/3} + A_2^{-1/3}\big)\big\}\big]\ \text{fm},$$

where $R_i = 1.2A_i^{1/3} - 0.09$ fm. For a better description of the experimental functions $\sigma_R(E)$, we set $a_I = a_R$ in the ^{6}He + ^{27}Al reaction, $a_I = 1.05a_R$ in the reactions ^{9}Be + ^{27}Al and ^{6}Li + ^{209}Bi, $a_I = 0.85a_R$ in the ^{9}Be + ^{208}Pb reaction, and $a_I = 0.75a_R$ in the reactions ^{11}B + ^{209}Bi and ^{12}C + ^{209}Bi. The value of R_I decreases with respect to R_R with increasing E, which allows us to use Eq. (15) at $E > V_b$.

III. REDUCED REACTION CROSS SECTION AND PROBABILITY

To analyze the reaction cross sections $\sigma_R(E)$ in the collisions with different Coulomb barrier heights V_b and radii R_b calculated in the case of spherical nuclei, it is useful to consider the dimensionless quantities

$$\sigma_R(E) \to \sigma_R^{\text{red}}(x) = \frac{1}{R_m'^2}\sigma_R(E)\left[\frac{\hbar^2 Z'}{(2\pi a_I)^3 V_I^2\mu}\right]^{1/2}$$
$$= E^2\sigma_R(E)\left[\frac{\hbar^2}{(2\pi a_I Z')^3 V_I^2\mu}\right]^{1/2}$$
$$= e^{-x} \qquad (16)$$

versus

$$E \to x = (R'_m - R_I)/a_I = (Z'/E - R_I)/a_I.$$

Thus, $\sigma_R^{\text{red}}(x)$ is the same function for any reaction and can be called the URF. The URF allows us to conclude about the role of static deformations of the colliding nuclei in the reaction cross section, particularly at sub-barrier energies. Indeed, the URF disregards this effect.

Analogously, one can suggest the universal reaction probability (URP) at backward angle ($J = 0$ or $\theta = 180^{\circ}$):

$$P_R(E, J = 0) \to P_R^{\text{red}}(x) = \frac{1}{R'_m}P_R(E, J = 0)\left[\frac{\hbar^2 Z'}{2\pi a_I\mu V_I^2}\right]^{1/2}$$
$$= EP_R(E, J = 0)\left[\frac{\hbar^2}{2\pi a_I\mu Z' V_I^2}\right]^{1/2}$$
$$= e^{-x} \qquad (17)$$

versus $E \to x$. In the case of the reaction probability at any angle θ, one can generalize Eq. (17):

$$P_R(E, \theta) \to P_R^{\text{red}}(y) = \frac{E}{1+\cot(\theta/2)}P_R(E, \theta)\left[\frac{2\hbar^2}{\pi a_I\mu Z' V_I^2}\right]^{1/2}$$
$$= e^{-y} \qquad (18)$$

versus

$$(E, \theta) \to y = \{Z'[1 + 1/\sin(\theta/2)]/(2E) - R_I\}/a_I.$$

Here, $R''_m(\theta) = Z'[1 + 1/\sin(\theta/2)]/(2E)$ is the closest distance at the angle θ. As follows from our analysis of the experimental data, the reduced probability $P_R^{\text{red}}(y)$ has also universal behavior on y.

IV. RESULTS OF THE CALCULATIONS

Employing the formulas (16), we compare the URF

$$\sigma_R^{\text{red}}(x) = e^{-x} \qquad (19)$$

and reduced experimental reaction cross sections

$$\sigma_{R\,\text{exp}}^{\text{red}}(x) = E^2\sigma_{R\,\text{exp}}(E)\left[\frac{\hbar^2}{(2\pi a_I Z')^3 V_I^2\mu}\right]^{1/2}. \qquad (20)$$

In Fig. 1, the results are presented for the reactions ^{6}He + ^{27}Al; ^{6}Li, ^{12}C + ^{209}Bi; ^{9}Be + ^{27}Al, ^{208}Pb; and ^{11}B + ^{209}Bi. As one can see, the reduced experimental cross sections and URF results are rather close. Thus, for all reactions, the suggested URF reproduces well the

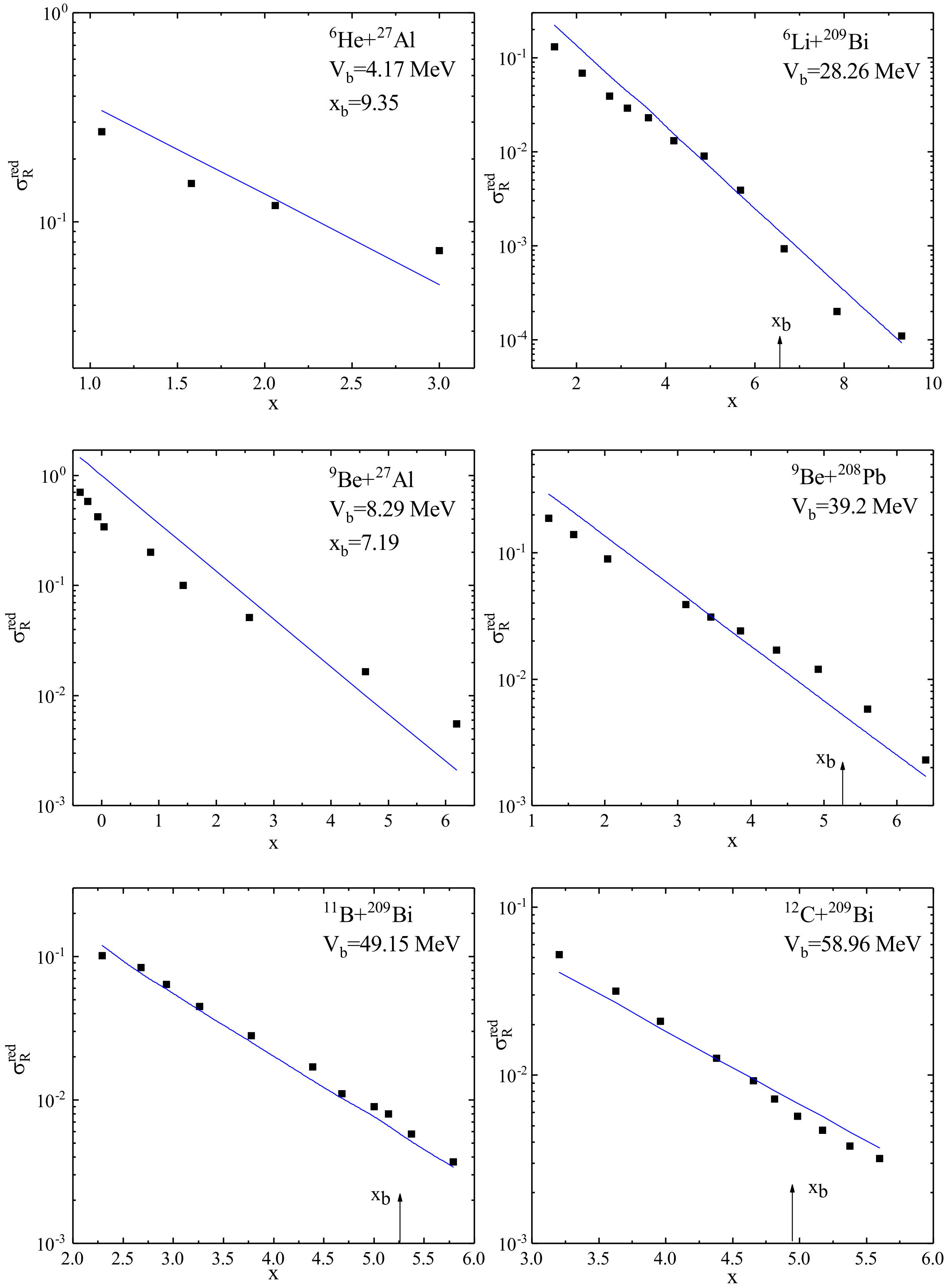

FIG. 1. The comparison of the reduced experimental reaction cross sections (symbols) with the URF (lines) versus x for the specified reactions. The experimental reaction cross sections are from Refs. [3,13–19]. The heights of the Coulomb barriers defined in the Appendix and corresponding values of x_b are indicated.

energy dependence and, correspondingly, the formula (16) is justified. The URF is quite sensitive to the value of a_I. For the reactions under consideration, the found values of a_I are in the range (0.60–0.65) fm, which is consistent with the value of a_R and parameters of the known optical potentials. So, the parameters of the optical model can be tested in the comparison of the reduced experimental reaction cross sections and URF.

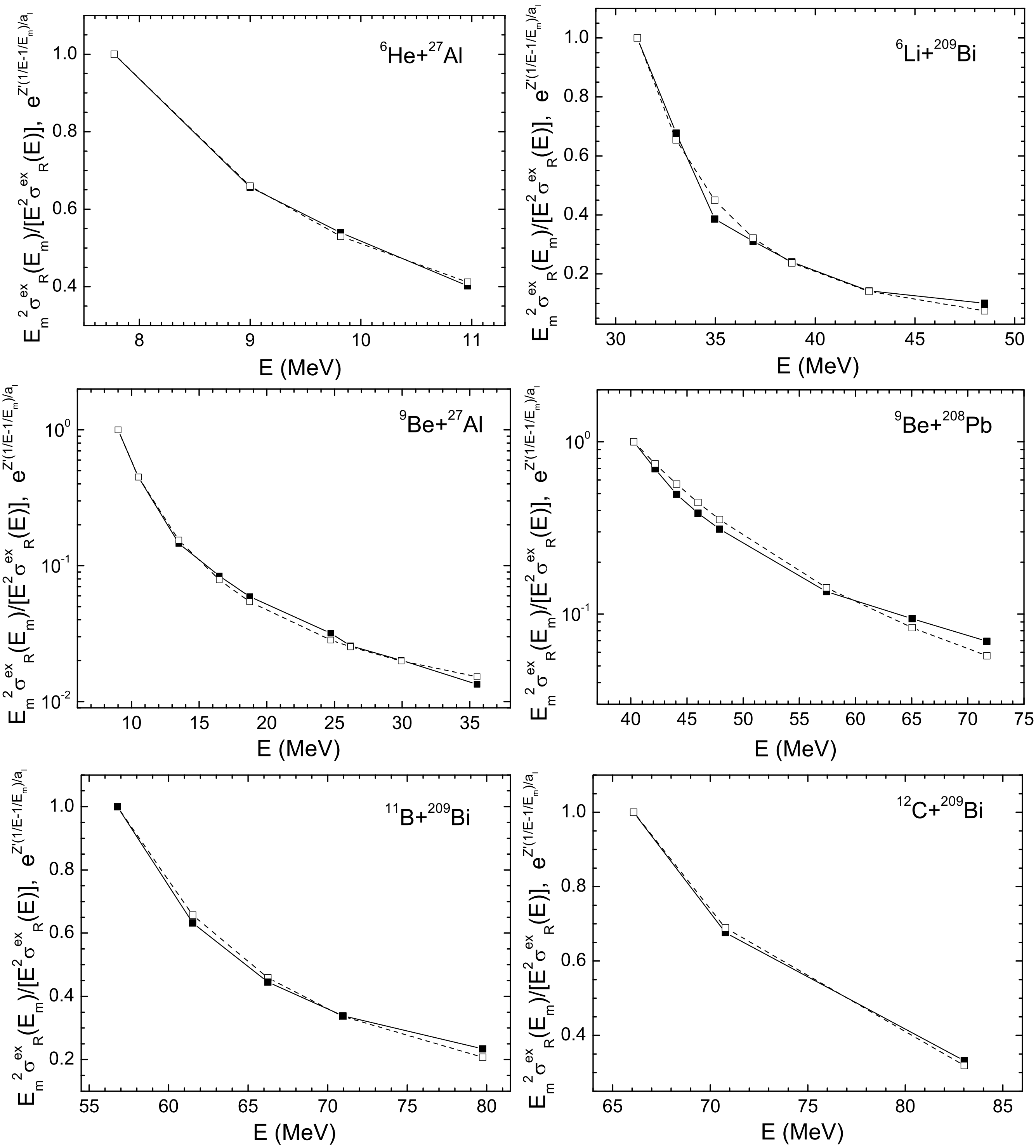

FIG. 2. The dependencies of the experimental ratio $\frac{E_m^2\sigma_R^{\rm ex}(E_m)}{E^2\sigma_R^{\rm ex}(E)}$ (closed squares connected by solid lines) and the calculated value of $e^{\frac{Z'}{a_I}(\frac{1}{E}-\frac{1}{E_m})}$ (open squares connected by dashed lines) on the bombarding energy E for the specified reactions. The experimental reaction cross sections are from Refs. [3,13–19].

If we disregard the energy dependence of R_I at $E > V_b$, the ratio

$$\frac{\sigma_R^{\rm red}(E_n)}{\sigma_R^{\rm red}(E)} = \frac{E_n^2\sigma_R(E_n)}{E^2\sigma_R(E)} = e^{\frac{Z'}{a_I}(\frac{1}{E}-\frac{1}{E_n})} \tag{21}$$

can be used to estimate the cross section at any $E > E_n$, if the cross sections at two energies, including $E = E_n$, are known to fix the parameter a_I. Here the value of E_n can be taken as a minimum bombarding energy E under consideration. As seen in Fig. 2, the experimental and calculated results are close or coincide, and the energy dependence is well reproduced. The calculated value of $e^{\frac{Z'}{a_I}(\frac{1}{E}-\frac{1}{E_m})}$ is quite sensitive to a_I. For the reactions under consideration, the found values of a_I are in the range (1.3–1.7) fm, and are larger than those in Eq. (16) to take effectively into account the role of $R_I(E)$. So, Eqs. (16) and (21) with corresponding adjustment of a_I can be used to estimate the reaction cross sections.

V. SUMMARY

Approximating the angular momentum dependence of the reaction probabilities at a given bombarding energy by shifting it by the centrifugal energy and using the analytical formula for the elastic scattering probability, we derived a new analytical formula for heavy-ion reaction cross sections at any value of the bombarding energy. As a result we transformed the reaction cross section in a such a way as to obtain an exponential dependence on the dimensionless value x, which is used as the URF. We presented two variants. The first one includes the energy dependence of the radius R_I of the imaginary potential. The second variant is the exponential dependence with the exponent defined from two experimental points. So, the heavy-ion reaction cross section can be defined at any energy if it is known for as few as two energy values. As shown for several systems, the suggested URF works well for the reaction cross section and provides global trends for it. The URF can be used along with the UFF when we are also interested in the reaction cross section. We suggest the URF for the analysis of experimental data and predictions of heavy-ion reaction cross sections.

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APPENDIX: EFFECTIVE SOMMERFELD PARAMETER

Near the Coulomb barrier at $R = R_b$ the nuclear part of the nucleus-nucleus interaction potential $V(R) = V_N(R) + V_C(R)$ [$V_N(R)$ and $V_C(R)$ are the nuclear and Coulomb parts] can be approximated by the exponential function $V_N(R) \sim \exp[-R/a_b]$, with the diffuseness parameter a_b. From

$$\frac{d}{dR}V(R)|_{R=R_b} = \frac{d}{dR}V_C(R)|_{R=R_b} + \frac{d}{dR}V_N(R)|_{R=R_b} = 0,$$

we obtain the following expressions:

$$V_N(R_b) = -\frac{a_b Z_1 Z_2 e^2}{R_b^2} = -\frac{a_b Z}{R_b^2}$$

and

$$V_b = V(R_b) = \frac{Z}{R_b}\left(1 - \frac{a_b}{R_b}\right) = \frac{Z'}{R_b},$$

where

$$Z' = Z\left(1 - \frac{a_b}{R_b}\right).$$

Thus, the nuclear part of interaction mainly reduces the height of the Coulomb barrier and very weakly influences the barrier shape at $R > R_b$. This effect of the reduction of the Coulomb barrier is taken effectively into consideration in the parameter Z' or η'. Then, the Coulomb scattering relation is

$$J = \eta \cot\left[\frac{\theta}{2}\right],$$

where $\eta = Z\sqrt{\frac{\mu}{2\hbar^2 E}}$ can be replaced by

$$J = \eta' \cot\left[\frac{\theta}{2}\right]$$

with the effective Sommerfeld parameter

$$\eta' = Z'\sqrt{\frac{\mu}{2\hbar^2 E}} = Z\left(1 - \frac{a_b}{R_b}\right)\sqrt{\frac{\mu}{2\hbar^2 E}}$$

in the presence of the nuclear part of the nucleus-nucleus interaction potential. Using this relation between the entrance-channel angular momentum and exit-channel angle, one can calculate the reaction cross sections [6–9].

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