Hydrodynamic attractor of noisy plasmas

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(Received 12 July 2022; revised 16 June 2023; accepted 21 November 2023; published 22 December 2023)

We provide a generalized formulation of fluctuating hydrodynamics for the far-from-equilibrium noisy medium. As an example, we consider a noisy plasma experiencing Bjorken expansion, for which the leading order evolution is captured by the hydrodynamic attractor of classical hydrodynamics, while the quadratic couplings of fluctuations are solved effectively via a generalized version of the hydrodynamic kinetic equation. In the far-from-equilibrium plasma, backreaction of hydrodynamic fluctuations results in renormalization of transport properties, as well as long-time tails, of high orders. In particular, corresponding to a renormalized hydrodynamic attractor, evolution in a noisy plasma towards equilibrium becomes nonmonotonic.

DOI: 10.1103/PhysRevC.108.064911

I. INTRODUCTION

Hydrodynamics is an effective theory that by construction applies to thermal systems close to local equilibrium. In hydrodynamics, departures from ideal fluids are captured by gradients of hydrodynamic fields as well as hydrodynamic fluctuations. Hydrodynamic fluctuations are in general suppressed in systems with a large amount of constituents, it is therefore not surprising that theoretical formulations without hydrodynamic fluctuations (frameworks sometimes referred to as the *classical* hydrodynamics [1]) have achieved remarkable successes. Such examples include in particular the hydrodynamic modeling of quark-gluon plasma (QGP) in high-energy nuclear physics [2].

Nonetheless, hydrodynamic fluctuations cannot be neglected when they are significant in system dynamical evolution. For instance, when a thermal system evolves close to a critical point, correlations among fluctuations of order parameters diverge, resulting in novel hydrodynamic modes [3]. Hydrodynamic fluctuations are amplified in small systems, such as the QGP droplet created in high-energy proton-lead collisions [4], owing to the fact that correlations among thermal fluctuations are inversely proportional to the system volume. More importantly, the nonlinear nature of hydrodynamics allows for corrections from couplings of fluctuations [5]. Backreaction of the coupled modes renormalizes transport properties [6–9], generates nonanalytical long-time tail structures [10], and even influences evolution history in a fluid [11].

The equation of motion of fluctuating hydrodynamics follows the conservation of energy and momentum,

$$\partial_\mu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = T^{\mu\nu}_{\rm cl} + \delta T^{\mu\nu} + S^{\mu\nu}, \tag{1}$$

where the classical energy-momentum tensor, $T^{\mu\nu}_{\rm cl}$, consists of energy density e, pressure P, fluid four-velocity u^μ, and expansion in terms of their gradients [12],

$$T^{\mu\nu}_{\rm cl} = eu^\mu u^\nu + P\Delta^{\mu\nu} + \pi^{\mu\nu}. \tag{2}$$

The expansion can be characterized by the Knudsen number Kn. Up to second order in gradients, the constitutive equation is often formulated via the stress tensor $\pi^{\mu\nu}$ relaxing to its Navier-Stokes correspondence, i.e., the Israel-Stewart hydrodynamics [13]. In the classical constitutive equation (2), variables are thermal averaged quantities without corrections from thermal fluctuations; namely, they are *bare* variables to be distinguished later from the renormalized ones. Fluctuations of energy-momentum tensor $\delta T^{\mu\nu}$ are constructed accordingly in terms of thermal fluctuations of hydrodynamic variables, which are further induced through the random noise $S^{\mu\nu}$, subject to the condition $\langle S^{\mu\nu}(x)\rangle = 0$ and the fluctuation-dissipation relation $\langle S^{\mu\nu}(x)S^{\alpha\beta}(y)\rangle = 2T\eta\Delta^{\mu\nu\alpha\beta}\delta^4(x-y)$ [14]. Here, η is the shear viscosity and the brackets indicate an average over the ensemble of thermal fluctuations. With an equation of state, $P = P(e)$, Eqs. (1) and (2) are closed.

Equation (1) is stochastic, hence the resulting system evolution fluctuates in space and time. However, the averaged evolution is deterministic, which can be obtained, for instance, through an ensemble average over numerical simulations of the stochastic processes. Alternatively, with respect to an effective field theory approach of fluctuating hydrodynamics [15–17], by treating thermal fluctuations as perturbations, averaged quantities can be solved order by order. This is a strategy analogous to the loop expansion in quantum field theory. The tree-level analysis corresponds to solving the

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classical hydrodynamics, $\partial_\mu T^{\mu\nu}_{\rm cl} = 0$. The effect of thermal fluctuations then arises when hydrodynamic fluctuations are included and constrained by $\partial_\mu \delta T^{\mu\nu} = -\partial_\mu S^{\mu\nu}$, which accordingly determines multipoint correlations [7,8,18,19]. The two-point correlations, $\langle \delta T^{\mu\nu}\delta T^{\alpha\beta}\rangle$, in particular, contain already the information of quadratic couplings of modes that contribute to $\langle T^{\mu\nu}\rangle$, and the renormalization of transport properties and the long-time tails.

Recently, extensive studies have been devoted to the generalization of *classical* hydrodynamics to far-from-equilibrium systems. These works are motivated in part by exploring the applicability condition of hydrodynamics through the convergence of the gradient expansion [20,21], and in part by the experimental observations of collectivity in QGP created from colliding nuclei of small sizes (cf. Ref. [4]). From either aspect, it was acknowledged that classical hydrodynamics admits the so-called attractor solutions in some comoving flows [22–32], owing to the expected hydrodynamic fixed points in these systems [33].

Bjorken flow, for instance, applies approximately to the very early stages of high-energy heavy-ion collisions, where the system expands dominantly along the beam axis (z axis). In the Milne coordinates, $\tau = \sqrt{t^2 - z^2}$ and $\zeta = \tanh^{-1}(z/t)$, with respect to the Israel-Stewart formulation, *classical* hydrodynamics reduces to coupled equations,

$$\frac{de}{d\tau} = -\frac{1}{\tau}(e + P + \pi), \tag{3a}$$

$$\pi = -\frac{4}{3}\frac{\eta}{\tau} - \tau_\pi\left[\frac{d\pi}{d\tau} + \frac{4}{3}\frac{\pi}{\tau}\right], \tag{3b}$$

where $\pi = \pi^\zeta_\zeta$ is the $\zeta\zeta$ component of the stress tensor. For later convenience, we introduce dimensionless constants,

$$\eta = C_\eta s, \quad \tau_\pi = C_\tau C_\eta / T, \quad e = C_e T^4, \tag{4}$$

to parametrize shear viscosity η, shear relaxation time τ_π, and energy density. We also consider the system conformal, so that $P = c_s^2 e$.

Defining the inverse Knudsen number $\mathrm{Kn}^{-1} = w \equiv \tau/\tau_\pi$, Bjorken expansion of QGP is fully captured by the relative decay rate of energy density: $g(w) \equiv d\ln e / d\ln\tau$. Especially, isotropization of the system is related to $g(w)$ through

$$\frac{P_L}{e} = \frac{\tau^2 T^{\zeta\zeta}_{\rm cl}}{T^{\tau\tau}_{\rm cl}} = -1 - g(w). \tag{5}$$

As will be clear later, this relation gets renormalized by hydrodynamic fluctuations. In terms of $g(w)$, the hydrodynamic attractor behaves as a smooth and monotonic connection between the free streaming fixed point at early times, $g(w) \approx -1$, and the ideal hydrodynamic fixed point at late times, $g(w) = -4/3$, while evolution with arbitrary initial conditions merge swiftly towards the attractor. The hydrodynamic attractor can be solved numerically, as well as analytically upon approximations [34]. In the leading order adiabatic approximation [35] (or the leading order slow-roll approximation [36]), the hydrodynamic attractor can be

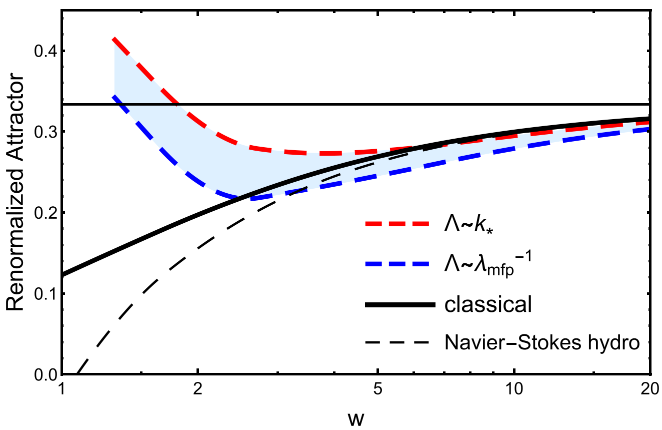

FIG. 1. Attractive isotropization of the medium captured in terms of the ratio P_L/e without contributions from hydrodynamic fluctuations (black solid line) and with hydrodynamic fluctuations (colored band). For comparison, first-order gradient expansion (Navier-Stokes hydrodynamics) is shown as the black dashed line, and the thin black line at $1/3$ indicates isotropization.

written as,

$$g(w) = -\frac{1}{2}\left[\frac{22}{7} + w - \sqrt{\left(\frac{10}{21} + w\right)^2 + \frac{64}{45}}\right]. \tag{6}$$

In the region $w \gtrsim 1$, Eq. (6) is not sensitive to second order transport coefficients [35].

In Fig. 1, the istropization of medium corresponding to the *classical* hydrodynamic attractor, Eq. (6), is shown as the black solid line, which evolves monotonically from free streaming towards the ideal hydrodynamic fixed point, $1/3$, at late times. Note in particular, deviations from $1/3$, at late times, are proportional to the bare shear viscosity η.

The hydrodynamic attractor conceptually extends the applicability of classical hydrodynamics to systems with large Kn. This is not only because the attractor universally describes system evolution irrespective of initial conditions, but also a consequence of the attractor accounting for a systematic resummation of gradients in terms of trans-series, including nonanalytical transient modes $\propto w^{-C_\tau/2C_\eta}e^{-3/2w}$, in the far-from-equilibrium medium [34,37]. The transient modes imply an exponential decay of fluctuations at late times (i.e., $w > 1$) towards the attractor solution, hence a separation of timescales of the fluctuations and the background flow. It is analogous to the nonhydro mode decay in the Isreal-Stewart hydrodynamics [38]. The separation of timescales is essential to the application of fluctuation-dissipation relation, so that fluctuations are guaranteed to be equilibrated towards the background flow. We shall therefore consider an ansatz of the fluctuation-dissipation relation out of equilibrium, with temperature and shear viscosity determined with respect to the background attractor.

It is our purpose of this paper to investigate the effects of hydrodynamic fluctuations in a plasma out of equilibrium. Without loss of generality, we take Eq. (6) as the decay rate of the bare energy density, as the input for the next leading order analysis of fluctuating hydrodynamics.

II. HYDRODYNAMIC KINETIC EQUATION IN THE FAR-FROM-EQUILIBRIUM REGIME

Bjorken symmetry is broken by fluctuations. In terms of Fourier modes of fluctuations of energy and momentum densities,

$$\delta e(\tau, \mathbf{k}) = \int d\zeta d^2\vec{x}_\perp e^{i\vec{k}_\perp\cdot\vec{x}_\perp + i\tau k_\zeta \zeta}\delta T^{\tau\tau}(\tau, \vec{x}_\perp, \zeta), \quad (7a)$$

$$g^i(\tau, \mathbf{k}) = \int d\zeta d^2\vec{x}_\perp e^{i\vec{k}_\perp\cdot\vec{x}_\perp + i\tau k_\zeta \zeta}\delta T^{\tau i}(\tau, \vec{x}_\perp, \zeta), \quad (7b)$$

the equation $\partial_\mu \delta T^{\mu\nu} = -\partial_\mu S^{\mu\nu}$ leads to coupled stochastic differential equations for $\phi_a = (c_s\delta e, g^x, g^y, \tau g^\eta)$. Here k_ζ is dimensionful and conjugate to $\tau\zeta$. These differential equations are equivalent to a hierarchy of equations for multipoint correlators [19]. Especially, the equal-time two-point correlators, for which we define as N_{ab} through

$$\langle \phi_a(\tau, \vec{k})\phi_b(\tau, -\vec{k}')\rangle \equiv N_{ab}(\tau, \vec{k})(2\pi)^3\delta^3(\vec{k} - \vec{k}'), \quad (8)$$

satisfy effectively hydrodynamic kinetic equations [7]. Of course, when applies to out-of-equilibrium systems with large Kn, the background fluid evolution should be accounted for by the hydrodynamic attractor.

To facilitate analyses, following Ref. [7], it is convenient to rotate in $\vec{k}$ space, which accordingly defines two longitudinal modes and two transverse modes, ϕ_A with $A = (\pm, T_1, T_2)$. After the rotation, the hydrodynamic kinetic equation is dominated by the diagonal components. We therefore find formally

$$\left(1 + \frac{g(w)}{4}\right)\frac{\partial R_A}{\partial \ln w} = -\alpha_A w\tilde{k}^2(R_A - 1) - \beta_A(w)R_A, \quad (9)$$

where $R_A \equiv \tau N_{AA}/T(e+P)$ is the normalized correlator and $\tilde{k}^2 = \tau_\pi^2(|\vec{k}_\perp|^2 + k_\zeta^2)$. The coefficients are

$$\alpha_\pm = \frac{4}{3C_\tau}, \quad \alpha_{T_1} = \alpha_{T_2} = \frac{2}{C_\tau}, \quad (10)$$

and

$$\beta_\pm = 1 + \tfrac{5}{4}g(w) + c_s^2 + \cos^2\theta_k,$$

$$\beta_{T_1} = 1 + \tfrac{5}{4}g(w), \quad \beta_{T_2} = 1 + \tfrac{5}{4}g(w) + 2\sin^2\theta_k, \quad (11)$$

where $\cos\theta_k = k_\zeta/|\vec{k}|$. The left-hand side of Eq. (9) represents time derivative on top of the background attractor. The first term ($\propto \alpha_A$) and the second term ($\propto \beta_A$) on the right-hand side play the roles of collision and longitudinal expansion, respectively. System evolution towards equilibrium relies then on these two competing effects. Note that in the ideal hydrodynamics limit, i.e., when $g(w) \to -4/3$, Eq. (9) reduces to the original form of hydrokinetic theory in Ref. [7]. Once Eq. (9) is solved, the equal-time two-point correlators N_{aa} in the original basis can be obtained respectively via an inverse rotation.

Hydrodynamic kinetic equation applies between separated scales: $c_s^{-1}\nabla \sim (c_s\tau)^{-1} \ll k \ll \lambda_{\rm mfp}^{-1}$ [7]. In the large-k limit, or more precisely when $w\tilde{k}^2 \gg 1$ according to Eq. (9), the two-point correlators approach $T(e+P)/\tau$, which can be understood as the "equilibrium" expectation in an out-of-equilibrium system defined according to the background attractor. One thereby introduces a critical scale $k_* = \sqrt{w}/\tau$, above which the two-point correlators are well captured by the background attractor. In the out-of-equilibrium system, the separated scales, as well as k_*, are time dependent. In particular, because these scales merge around $w \approx 1$, a reliable description from Eq. (9) for the out-of-equilibrium plasma should only apply when $w \in [1, +\infty)$.

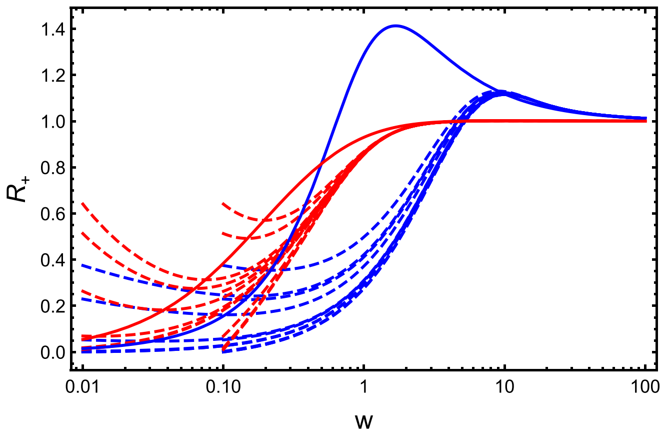

FIG. 2. Evolution of R_+ with respect to various initial conditions. Dashed blue and dashed red lines are obtained with $\tilde{k} = 1, 3$, and $\cos\theta_k = 0.1, 0.5$, respectively. Solid lines are the slow-roll approximation of the corresponding attractor, $R_+^{\text{slow-roll}} = \alpha_+ w\tilde{k}^2/(\alpha_+ w\tilde{k}^2 + \beta_+)$.

As a consequence of fixed points, Eq. (9) itself possesses attractor solutions. To see this, we first notice that $R_A = 0$ is a fixed point solution in the small $w\tilde{k}^2$ extreme, which is stable only when $\beta_A > 0$. Nevertheless, the stability of this fixed point does not affect the two-point correlators at late times. In the large-$w\tilde{k}^2$ extreme, Eq. (9) allows for solution in terms of a double expansion,

$$R_A(w, w\tilde{k}^2) = 1 - \frac{\beta_A(w)}{\alpha_A w\tilde{k}^2} + \ldots = \sum_{n,m} \frac{F_{n,m}^{(A)}}{w^n(w\tilde{k}^2)^m}. \quad (12)$$

Note that the correlator depends explicitly on $w\tilde{k}^2$, as dictated by Eq. (9). The expansion in $1/w$ is the hydrodynamic gradient expansion of the two-point correlators, which is asymptotic. The expansion in $1/\tilde{k}^2$ is asymptotic as well, which, however, differs from the usual hydrodynamic gradient expansion in wave numbers [39]. In the large-$w\tilde{k}^2$ extreme, the solution is well captured by the first several terms in the expansion. In analogy to the hydrodynamic fixed point in classical hydrodynamics at large w, the large-$w\tilde{k}^2$ behavior represents a stable hydrodynamic fixed point in the two-point correlators.

In Fig. 2, for illustrative purposes, the evolution of R_+ is shown with two sets of $\tilde{k}$ and $\cos\theta_k$ values. Irrespective of initial conditions, R_+ tends to follow universal curves at late times (dashed lines), which is exactly the feature that one expects in an attractor solution. The universal curves stand for the attractors, which with the slow-roll approximation ($\partial_w R_A = 0$) can be approximated as

$R_A^{\text{slow-roll}} = \alpha_A w\tilde{k}^2/(\alpha_A w\tilde{k}^2 + \beta_A)$ (solid lines in Fig. 2). Similar behavior can be found in other modes as well. Note that the behavior of the solutions in the region $w < 1$ in Fig. 2 demonstrate the properties of an "early-time" attractor according to the structure of Eq. (9). However, it is only the region $w > 1$ that we shall consider for realistic analyses.

III. RENORMALIZATION IN FAR-FROM-EQUILIBRIUM NOISY FLUIDS

The resulting equal-time two-point correlators N_{aa} suffice to determine thermal corrections to the averaged energy-momentum tensor out of equilibrium. With respect to an integral in $\vec{k}$-space, the thermal corrections can be classified as a cutoff dependent correction $T_\Lambda^{\mu\nu}$ and a long-time tail correction $\Delta T^{\mu\nu}$. For instance, the averaged value of $\tau\tau$ component of the energy-momentum tensor contains contribution at the quadratic order, $\langle \delta T^{0i}\delta T^{0i}\rangle/(e+P)$. In terms of N_{aa}, one has [7,18]

$$\langle T^{\tau\tau}\rangle - T_{\text{cl}}^{\tau\tau} = \frac{1}{e+P}\int \frac{d^3\vec{k}}{(2\pi)^3}\sum_{a=\vec{x}_\perp,\zeta} N_{aa}(w, w\tilde{k}^2)$$
$$= T_\Lambda^{\tau\tau} + \Delta T^{\tau\tau}, \tag{13}$$

where the explicit dependence on $w\tilde{k}^2$ is rooted in Eq. (9). As indicated in Eq. (12), the integral contains a piece of cubic order in Λ and a linearly divergent piece, which can be regulated by introducing a cutoff scale Λ. These regulated integrals then give rise to the cutoff dependent corrections. For $T_\Lambda^{\tau\tau}$ and $\tau^2 T_\Lambda^{\zeta\zeta}$, one finds,

$$T_\Lambda^{\tau\tau} = \frac{T\Lambda^3}{2\pi^2} - \frac{\Lambda T^3}{4\pi^2}\frac{C_\tau}{(C_\tau C_\eta)^2}\frac{35}{8w}\left(\frac{4}{3} + g(w)\right), \tag{14a}$$

$$\tau^2 T_\Lambda^{\zeta\zeta} = \frac{T\Lambda^3}{6\pi^2} - \frac{\Lambda T^3}{4\pi^2}\frac{C_\tau}{(C_\tau C_\eta)^2}\frac{1}{w}\left(\frac{27}{10} + \frac{35}{24}g(w)\right). \tag{14b}$$

In the limit $w \gg 1$, these cutoff dependent corrections can be absorbed into the energy-momentum tensor, so that energy density $[O(w^0)]$, pressure $[O(w^0)]$ [7] and shear viscosity $[O(w^{-1})]$ [6] get renormalized, respectively. Note that, since the cutoff dependent correction at $1/w$ is negative ($\propto -\Lambda T^3$), the resulting renormalized shear viscosity is actually enhanced. Again, when $g(w) \to -4/3$, i.e., in the ideal hydrodynamics limit, the renormalization reduces to that of the original hydrokinetic theory in Ref. [7]. In the far-from-equilibrium regime, with respect to the trans-series expansion in $g(w)$, higher order transport coefficients ($O(w^{-2})$ and beyond) are renormalized as well.

Renormalization in fluctuating hydrodynamics reflects the fact that hydrodynamic fluctuations stay in equilibrium above the critical scale. In the out-of-equilibrium medium with large Kn, the cutoff scale can be taken according to $k_* \ll \Lambda \ll \lambda_{\text{mfp}}^{-1}$. In practice, given the information of the physically measured quantities at a certain scale in the expanding system, Λ is w dependent and well constrained.

Out-of-equilibrium long-time tails. After renormalization, a finite piece in the thermal corrections remains. As shown in Eq. (13), the explicit dependence on $w\tilde{k}^2$ implies an overall factor $w^{-3/2}$ in the finite integral, which leads to the nonanalytical structure in the well-known long-time tails,

$$\frac{\Delta T^{\tau\tau}}{e} = \frac{w^{-3/2}}{C_e(C_\tau C_\eta)^3}\sum_{n=0}\frac{f_n^{\tau\tau}}{w^n}, \tag{15a}$$

$$\frac{\tau^2\Delta T^{\zeta\zeta}}{e} = \frac{w^{-3/2}}{C_e(C_\tau C_\eta)^3}\sum_{n=0}\frac{f_n^{\zeta\zeta}}{w^n}. \tag{15b}$$

Note that the long-time tails are cutoff independent. The coefficients f_n can be solved in principle by a summation of $F_{n,m}$ in Eq. (12). Via a polynomial fit with respect to the numerical solutions of Eq. (9), we are allowed to identify $f_0^{\tau\tau} = 0.45 \pm 0.1$, while $f_0^{\zeta\zeta} = 0.15032 \pm 0.00002$ and $f_1^{\zeta\zeta} = -0.53 \pm 0.05$.

IV. RENORMALIZED ATTRACTOR

In fluctuating hydrodynamics, the effective out-of-equilibrium system evolution should be monitored by the thermal averaged components in the energy-momentum tensor. In particular, the renormalized ratio,

$$\frac{\langle\tau^2 T^{\zeta\zeta}\rangle}{\langle T^{\tau\tau}\rangle} = \frac{\tau^2 T_{\text{cl}}^{\zeta\zeta} + \tau^2 T_\Lambda^{\zeta\zeta} + \tau^2\Delta T^{\zeta\zeta}}{T_{\text{cl}}^{\tau\tau} + T_\Lambda^{\tau\tau} + \Delta T^{\tau\tau}}$$
$$= [-1 - g(w)]\left(1 + \frac{3\tau^2 T_\Lambda^{\zeta\zeta}}{e} + \frac{3\tau^2\Delta T^{\zeta\zeta}}{e} - \frac{T_\Lambda^{\tau}}{e} - \frac{\Delta T^{\tau\tau}}{e} + \cdots\right)$$
$$\equiv [-1 - g(w)]Z_{\text{att}}^{-1}(w), \tag{16}$$

captures the observed system isotropization in the presence of hydrodynamic fluctuations. In Eq. (16), a multiplicative renormalization factor Z_{att}^{-1} is introduced, which contains expansion in $1/w$ from the cutoff dependent corrections, and nonanalytical corrections starting from $w^{-3/2}$ from the long-time tails.

With respect to a *noisy* gluonic plasma [20], with $C_\eta = 1/4\pi, C_\tau = 2(2 - \ln 2)$, and $C_e = 16\pi^2/30$, the renormalized attractor is solved and shown as the colored band in Fig. 1. The upper and lower boundaries are determined according to the two extreme cutoff scales, $\Lambda \sim k_*$ and $\Lambda \sim \lambda_{\text{mfp}}^{-1}$, respectively. In both cases, when Λ is explicitly taken into account, both the cutoff dependent corrections and the long-time tails are constrained by an overall factor $1/C_e(C_\tau C_\eta)^3$. This factor is roughly the inverse of the number degrees of freedom in a unit volume, consistent with the physical expectation of quadratic couplings of hydrodynamic fluctuations [7].

When $w \gg 1$, the system is close to an ideal fluid, hence effects of hydrodynamic fluctuations are expected to be suppressed. Indeed, at large w, the renormalized attractor follows the trend of classical hydrodynamics, approaching 1/3 irrespective of hydrodynamic fluctuations. However, a closer look reveals that the renormalized attractor is actually below the classical result. This feature is expected, since the effective shear viscosity, which quantifies the reduction from 1/3, is enhanced by the renormalization due to hydrodynamic fluctuations [6,7]. Moreover, taking into account the fact that the renormalized correction to shear viscosity is proportional to Λ, a larger reduction in the renormalized attractor is expected with respect to a larger cutoff scale, as manifested in Fig. 1.

Unlike the classical attractor, which increases monotonically from far from equilibrium towards ideal fluids, the renormalized attractor becomes nonmonotonic in the far-from-equilibrium region, as shown in Fig. 1. This nonmonotonic behavior qualitatively reflects the long-time tail contribution to $Z_{\rm att}^{-1}$. More precisely, the leading order long-time tails give rise to positive corrections in $Z_{\rm att}^{-1}$, which compensates the reduction from the renormalized shear viscosity at large w [6], but dominates when the system is far from equilibrium. Parametrically, by comparing the leading order long-time tails and the cutoff corrections in $Z_{\rm att}^{-1}$ in the $\zeta\zeta$ sector, the minimum point of the renormalized attractor can be found around $\sqrt{w} \sim 4\pi^2 f_0^{\zeta\zeta}/C_\tau$.

V. SUMMARY AND DISCUSSION

With the help of the hydrodynamic attractor, fluctuating hydrodynamics can be applied to far-from-equilibrium noisy systems. As an example, the hydrodynamic kinetic equation, which was developed previously for a noisy fluid close to equilibrium, can be generalized to far-from-equilibrium noisy plasmas. In the far-from-equilibrium medium, although hydrodynamic fluctuations lead to qualitatively similar contributions from the coupled modes, higher order contributions, such as terms of order $O(1/w^2)$ and $O(w^{-5/2})$, and even some more complicated trans-series structures in $g(w)$, are involved. Moreover, backreactions of the hydrodynamic fluctuations already become significant in the out-of-equilibrium region with $w > 1$, which modifies the system evolution towards equilibrium. In particular, due to the long-time tails, in the noisy plasma the effective isotropization is nonmonotonic.

Although the current framework considers background flow with both early-time and late-time attractors, it is actually the late-time dynamics with $w > 1$ that really matters. As long as the background system approaches an ideal fluid at rather late times, an ideal hydrodynamics fixed point exists which leads to late-time attractor behavior. Such a late-time attractor suffices to provide the separation of timescales in the region $w > 1$. For cases even without early-time attractors in the background flow, such as the Bjorken expansion solved with respect to AdS-CFT or nonconformal fluid dynamics, the current framework should not be affected.

The current analysis could be improved systematically by including more ingredients in a noisy fluid system, such as the nonlinear couplings of modes beyond the quadratic order and nonconformal corrections. With respect to realistic high-energy nuclear collisions, initial state fluctuations, which differ conceptually from hydrodynamic fluctuations, should be taken into account as well.

ACKNOWLEDGMENTS

L.Y. is supported by the National Natural Science Foundation of China through Grant No. 11975079. L.Y. thanks Yi Yin for helpful discussions.

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