Ambiguity of Siegert transformations for isospin-forbidden electric dipole transitions

Daniel Baye [*]
Physique Quantique and Physique Nucléaire Théorique et Physique Mathématique, C.P. 229, Université Libre de Bruxelles (ULB), B 1050 Brussels, Belgium

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An infinity of extensions of the Siegert transformation exist in the Coulomb gauge. All resulting transition multipoles of the electric field are identical at the long-wavelength approximation but differ beyond. In the particular case of isospin forbidden electric dipole transitions, they introduce an ambiguity about the form of the transition operator. Various expressions of the electric dipole operator are presented and compared, without and with Siegert transformation. Their influence on calculations of radiative capture reactions of isospin-zero nuclei is discussed.

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I. INTRODUCTION

Two expressions for the electric multipoles of the vector potential of the electromagnetic field have been derived in the Coulomb gauge. The oldest one appears in textbooks [1,2] and is sometimes called after Partovi [3]. A more recent expression implicitly used by Friar and Fallieros [4,5] was explicitly derived in Ref. [6]. Although apparently different, these expressions are strictly equivalent as they correspond to the same gauge.

The electric multipoles of the electromagnetic field derived from these expressions are thus also equivalent. As they have a different structure, they have different gradient parts. Hence, when applying the Siegert transformation [7,8], different multipole operators are obtained. As expected, differences occur only beyond the long-wavelength approximation (LWA). They have thus little importance in most allowed electric transitions.

In $N = Z$ nuclei or in capture reactions between such nuclei, however, these differences do have an importance. Indeed, in isospin-zero systems, the electric dipole transitions are approximately forbidden by an isospin selection rule. These transitions remain possible thanks to various mechanisms: (i) through the beyond-LWA scalar part of the $E1$ operator, (ii) from or to small isospin $T = 1$ components in the wave functions, and (iii) through the spin part of the $E1$ operator. The corresponding amplitudes add coherently. Their relative order of magnitude cannot be guessed from simple arguments. They may all be small but for different reasons: (i) because operators beyond the LWA contain higher powers of the photon wave number, (ii) because of the smallness of $T = 1$ components in $N = Z$ systems, and (iii) because of unfavorable spin structures of the initial and final states.

In complement to Ref. [6], the aim of this paper is to clarify and extend equivalent expressions of the electric multipoles before the Siegert transformation and study their differences after this transformation. I start with a simple direct proof of the equivalence between both forms of the vector potential multipoles discussed in Ref. [6]. In fact, since two such expressions exist, an infinity of ways of performing the Siegert transformation are possible. The various expressions differ in the scalar part of the transition operators. A relation is established between scalar parts before and after transformation at the next order beyond the LWA.

In Sec. II, a simple direct proof of the equivalence between the Partovi and Friar-Fallieros expressions is given. Different forms of the electric field multipoles are presented in Sec. III without and with Siegert transformation. The case of the electric dipole operator is discussed in Sec. IV with emphasis on the $N = Z$ case. Properties of the magnetic dipole operator are briefly summarized in Sec. V. A discussion concludes in Sec. VI.

II. EQUIVALENCE OF EXPRESSIONS OF ELECTRIC MULTIPOLES OF THE VECTOR POTENTIAL

The multipoles of the vector potential are solutions of the Helmholz equation. They are eigenfunctions of $\boldsymbol{L}^2$ and L_z with respective eigenvalues $\lambda(\lambda+1)$ and μ. Here, and in the rest of this paper, the momentum operator is $\boldsymbol{p} = -i\boldsymbol{\nabla}$, and the orbital momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ and spin $\boldsymbol{S}$ operators are dimensionless. The discussion below in this section concerns the vector potential expressed in the Coulomb gauge. Multipoles of the vector potential are defined with various normalizations and phases in the literature [1,2,9]. Those below follow Ref. [2].

The traditional expression for the $\lambda\mu$ electric multipoles of the vector potential in the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A}^E_{\lambda\mu} = 0$, sometimes called the Partovi expression, is given by [2]

$$\boldsymbol{A}^E_{\lambda\mu} = k^{-1}[\lambda(\lambda+1)]^{-1/2}\boldsymbol{\nabla} \times \boldsymbol{L}\phi_{\lambda\mu} = \frac{i}{k[\lambda(\lambda+1)]^{1/2}}\left\{\boldsymbol{\nabla}\left[\frac{\partial}{\partial r}(r\phi_{\lambda\mu})\right] + k^2\boldsymbol{r}\phi_{\lambda\mu}\right\}, \quad (1)$$

[*] daniel.baye@ulb.be

where

$$\phi_{\lambda\mu}(k\boldsymbol{r}) = j_\lambda(kr)Y_{\lambda\mu}(\Omega) \tag{2}$$

and k is the photon wave number. The spherical Bessel function $j_\lambda(x)$ satisfies an equation conveniently written for later use as

$$(xj_\lambda)'' + xj_\lambda = \lambda(\lambda+1)x^{-1}j_\lambda, \tag{3}$$

where primes denote derivatives with respect to the argument.

Another expression is derived in Ref. [6] as

$$\boldsymbol{A}^E_{\lambda\mu} = \frac{i}{k}[\lambda(\lambda+1)]^{1/2}\boldsymbol{\nabla}[Y_{\lambda\mu}(\Omega)G_\lambda(kr)] - \frac{k}{[\lambda(\lambda+1)]^{1/2}}\boldsymbol{r}\times\boldsymbol{L}Y_{\lambda\mu}(\Omega)H_\lambda(kr), \tag{4}$$

where

$$G_\lambda(x) = \int_0^x \frac{j_\lambda(u)}{u}du, \quad H_\lambda(x) = \frac{1}{x^2}\int_0^x uj_\lambda(u)du. \tag{5}$$

With Eq. (3), these functions satisfy the relation

$$\lambda(\lambda+1)G_\lambda - x^2H_\lambda = (xj_\lambda)'. \tag{6}$$

Expression (4) is implicit in the multipole operators of the electric field derived by Friar and Fallieros [4] with slightly different notations, but not explicitly given in their paper.

Let me stress that these two apparently different operators are strictly identical. They are two writings of the same operator defined within the same Coulomb gauge. In the following, I sometimes use the argument P or FF to distinguish the writing used but this does not mean that the operators are different. A direct derivation of the second expression is given in Ref. [6]. Here, I give a simple proof of their equivalence.

Let me denote the angular part of the gradient operator as

$$\boldsymbol{\nabla}_\Omega = -i(\boldsymbol{e}_r\times\boldsymbol{L}) = r\boldsymbol{\nabla} - \boldsymbol{r}\frac{\partial}{\partial r}, \tag{7}$$

where $\boldsymbol{e}_r$ is the radial unit vector. Denoting kr as x, the Partovi expression can be written with Eq. (3) as

$$\begin{aligned} -i\sqrt{\lambda(\lambda+1)}\boldsymbol{A}^E_{\lambda\mu}(\mathrm{P}) &= Y_{\lambda\mu}\boldsymbol{e}_r[(xj_\lambda)'' + xj_\lambda] + (\boldsymbol{\nabla}_\Omega Y_{\lambda\mu})x^{-1}(xj_\lambda)' \\ &= Y_{\lambda\mu}\boldsymbol{e}_r\lambda(\lambda+1)x^{-1}j_\lambda + (\boldsymbol{\nabla}_\Omega Y_{\lambda\mu})x^{-1}(xj_\lambda)'. \end{aligned} \tag{8}$$

Similarly, starting from the Friar-Fallieros expression and rewriting Eq. (7) as

$$\boldsymbol{r}\times\boldsymbol{L} = \boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{p}) - r^2\boldsymbol{p} = ir\boldsymbol{\nabla}_\Omega, \tag{9}$$

one obtains with Eq. (6)

$$\begin{aligned} -i\sqrt{\lambda(\lambda+1)}\boldsymbol{A}^E_{\lambda\mu}(\mathrm{FF}) &= Y_{\lambda\mu}\boldsymbol{e}_r\lambda(\lambda+1)G'_\lambda + (\boldsymbol{\nabla}_\Omega Y_{\lambda\mu})x^{-1}[\lambda(\lambda+1)G_\lambda - x^2H_\lambda] \\ &= Y_{\lambda\mu}\boldsymbol{e}_r\lambda(\lambda+1)x^{-1}j_\lambda + (\boldsymbol{\nabla}_\Omega Y_{\lambda\mu})x^{-1}(xj_\lambda)', \end{aligned} \tag{10}$$

establishing the equivalence.

Hence, the most general form of the Coulomb gauge electric multipoles reads

$$\boldsymbol{A}^E_{\lambda\mu}(\alpha) = (1-\alpha)\boldsymbol{A}^E_{\lambda\mu}(\mathrm{P}) + \alpha\boldsymbol{A}^E_{\lambda\mu}(\mathrm{FF}), \tag{11}$$

where $\boldsymbol{A}^E_{\lambda\mu}(\mathrm{P})$ and $\boldsymbol{A}^E_{\lambda\mu}(\mathrm{FF})$ are given by Eqs. (1) and (4), respectively, and coefficient α is arbitrary.

The magnetic multipoles are given by [1,2,9]

$$\boldsymbol{A}^M_{\lambda\mu} = [\lambda(\lambda+1)]^{-1/2}\boldsymbol{L}\phi_{\lambda\mu}. \tag{12}$$

They are related to the electric multipoles by [2]

$$\boldsymbol{\nabla}\times\boldsymbol{A}^{E(M)}_{\lambda\mu} = k\boldsymbol{A}^{M(E)}_{\lambda\mu}. \tag{13}$$

In the Coulomb gauge, all multipoles commute with the momentum operator,

$$\boldsymbol{p}\cdot\boldsymbol{A}^{E(M)}_{\lambda\mu} = \boldsymbol{A}^{E(M)}_{\lambda\mu}\cdot\boldsymbol{p}. \tag{14}$$

III. ELECTRIC TRANSITION OPERATORS

The electric transition operators are defined as [2,10]

$$\mathcal{M}^{E\lambda}_\mu = \frac{1}{c}\sqrt{\frac{\lambda}{\lambda+1}}\frac{(2\lambda+1)!!}{k^\lambda}\int \boldsymbol{J}\cdot\boldsymbol{A}^E_{\lambda\mu}d\boldsymbol{r}, \tag{15}$$

where

$$\boldsymbol{J}(\boldsymbol{r}) = \boldsymbol{J}_c(\boldsymbol{r}) + \boldsymbol{J}_m(\boldsymbol{r}) \tag{16}$$

is the sum of the convection current density $\boldsymbol{J}_c(\boldsymbol{r})$ and the magnetization current density $\boldsymbol{J}_m(\boldsymbol{r})$. For pointlike nucleons, these current densities read

$$\boldsymbol{J}_c(\boldsymbol{r}) = \frac{e\hbar}{m_p}\sum_{j=1}^{A} g_{lj}\frac{1}{2}[\boldsymbol{p}_j\delta(\boldsymbol{r}_j-\boldsymbol{r}) + \delta(\boldsymbol{r}_j-\boldsymbol{r})\boldsymbol{p}_j] \tag{17}$$

and

$$\boldsymbol{J}_m(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{\mu}_m(\boldsymbol{r}) \tag{18}$$

with the density of intrinsic magnetic moment

$$\boldsymbol{\mu}_m(\boldsymbol{r}) = \mu_N\sum_{j=1}^{A} g_{sj}\delta(\boldsymbol{r}_j-\boldsymbol{r})\boldsymbol{S}_j. \tag{19}$$

In these expressions, m_p is the proton mass, $\mu_N = e\hbar/2m_p$ is the nuclear magneton, and $\boldsymbol{r}_j$, $\boldsymbol{p}_j$, and $\boldsymbol{S}_j$ are the coordinate, momentum, and spin operators of nucleon j. In terms of third component t_{j3} of the isospin of nucleon j, the coefficients read $g_{lj} = \frac{1}{2} - t_{j3}$ and $g_{sj} = g_p(\frac{1}{2} - t_{j3}) + g_n(\frac{1}{2} + t_{j3})$ as a function of the proton g_p and neutron g_n gyromagnetic factors.

The adjoints of these operators are given by

$$\mathcal{M}^{E\lambda\dagger}_\mu = (-1)^{\mu+1}\mathcal{M}^{E\lambda}_{-\mu}. \tag{20}$$

Since two equivalent expressions exist for the multipoles of the vector potential, two equivalent expressions also exist for the transition operator. Only the writing of the convection part differs. Indeed, the integral involving the magnetization current simplifies as

$$\int \boldsymbol{J}_m\cdot\boldsymbol{A}^{E(M)}_{\lambda\mu}d\boldsymbol{r} = k\int \boldsymbol{\mu}_m\cdot\boldsymbol{A}^{M(E)}_{\lambda\mu}d\boldsymbol{r}. \tag{21}$$

For electric transitions, it only depends on the magnetic multipoles $\boldsymbol{A}^M_{\lambda\mu}$. With the commutation relation (14), the convection part reads

$$\int \boldsymbol{J}_c\cdot\boldsymbol{A}^{E(M)}_{\lambda\mu}d\boldsymbol{r} = -\frac{ie\hbar}{m_p}\sum_{j=1}^{A} g_{lj}\boldsymbol{A}^{E(M)}_{\lambda\mu}(\boldsymbol{r}_j)\cdot\boldsymbol{\nabla}_j. \tag{22}$$

Equation (1) corresponds to [11]

$$\mathcal{M}_\mu^{E\lambda} = \frac{e\hbar}{m_p c}\frac{(2\lambda+1)!!}{(\lambda+1)k^\lambda}\sum_{j=1}^{A}\left\{\frac{g_{lj}}{k}\left[\left(k^2\boldsymbol{r}+\nabla\frac{\partial}{\partial r}r\right)\phi_{\lambda\mu}\right]_j \cdot\nabla_j + \frac{1}{2}kg_{sj}[\boldsymbol{L}\phi_{\lambda\mu}]_j\cdot\boldsymbol{S}_j\right\}. \quad (23)$$

Equation (4) corresponds to [4,6]

$$\mathcal{M}_\mu^{E\lambda} = \frac{e\hbar}{m_p c}\frac{(2\lambda+1)!!}{(\lambda+1)k^\lambda}\sum_{j=1}^{A}\left\{\frac{g_{lj}}{k}[\lambda(\lambda+1)\nabla Y_{\lambda\mu}(\Omega)G_\lambda(kr) + ik^2\boldsymbol{r}\times\boldsymbol{L}Y_{\lambda\mu}(\Omega)H_\lambda(kr)]_j\cdot\nabla_j + \frac{1}{2}kg_{sj}[\boldsymbol{L}\phi_{\lambda\mu}]_j\cdot\boldsymbol{S}_j\right\}. \quad (24)$$

Using one or the other equivalent expression is just a question of convenience.

A part of the nuclear current can be eliminated from electric transition operators with the Siegert transformation [7]. Operators derived with a Siegert transformation will be denoted by a tilde. They are related to the original operators through the matrix elements

$$\langle f|\mathcal{M}_\mu^{E\lambda}|i\rangle = \langle f|\widetilde{\mathcal{M}}_\mu^{E\lambda}|i\rangle, \quad (25)$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states of the transition between energies E_i and E_f. Both expressions (1) and (4) of the multipoles of the vector potential and all expressions (11) have the structure

$$\boldsymbol{A}_{\lambda\mu}^E(\boldsymbol{r}) = \nabla f_{\lambda\mu}(\boldsymbol{r}) + \boldsymbol{A}_{\lambda\mu}^{\prime E}(\boldsymbol{r}). \quad (26)$$

The convection integral can be transformed with relation

$$\int \boldsymbol{J}_c(\boldsymbol{r})\cdot\nabla f_{\lambda\mu}(\boldsymbol{r})d\boldsymbol{r} = \frac{i}{\hbar}\int f_{\lambda\mu}(\boldsymbol{r})[H,\rho(\boldsymbol{r})]d\boldsymbol{r}, \quad (27)$$

where H is the Hamiltonian of the nuclear system and $\rho(\boldsymbol{r})$ is the charge density for pointlike nucleons,

$$\rho(\boldsymbol{r}) = e\sum_{j=1}^{A} g_{lj}\delta(\boldsymbol{r}_j-\boldsymbol{r}). \quad (28)$$

Hence the convection integral in the right-hand side of Eq. (25) becomes

$$\int \boldsymbol{J}_c\cdot\boldsymbol{A}_{\lambda\mu}^E d\boldsymbol{r} = -\frac{i(E_i-E_f)}{\hbar}\int f_{\lambda\mu}(\boldsymbol{r})\rho(\boldsymbol{r})d\boldsymbol{r} + \int \boldsymbol{J}_c\cdot\boldsymbol{A}_{\lambda\mu}^{\prime E}d\boldsymbol{r}. \quad (29)$$

This means that, by using the Siegert theorem, the equivalent expressions (23) and (24) lead to different results since $f_{\lambda\mu}$ and $\boldsymbol{A}_{\lambda\mu}^{\prime E}$ differ.

With

$$\epsilon = \mathrm{sgn}(E_i - E_f), \quad (30)$$

the Siegert form of the transition operator derived from Eq. (1) is [11,12]

$$\widetilde{\mathcal{M}}_\mu^{E\lambda}(\mathrm{P}) = \frac{(2\lambda+1)!!}{(\lambda+1)k^\lambda}\sum_{j=1}^{A}\left\{\epsilon e g_{lj}\left(\phi_{\lambda\mu}+r\frac{\partial\phi_{\lambda\mu}}{\partial r}\right)_j + \frac{e\hbar k}{2m_pc}\left[g_{lj}\left(3\phi_{\lambda\mu}+r\frac{\partial\phi_{\lambda\mu}}{\partial r}+2\phi_{\lambda\mu}r\frac{\partial}{\partial r}\right)_j + g_{sj}(\boldsymbol{L}\phi_{\lambda\mu})_j\cdot\boldsymbol{S}_j\right]\right\}. \quad (31)$$

As $\boldsymbol{A}_\mu^{\prime E\lambda}$ does not commute with the momentum, the full symmetrized expression (17) of the convection current has been used.

Similarly, with

$$\nabla_\Omega^2 = -\boldsymbol{L}^2, \quad (32)$$

one obtains, from Eq. (4) [4,6],

$$\widetilde{\mathcal{M}}_\mu^{E\lambda}(\mathrm{FF}) = \frac{(2\lambda+1)!!}{(\lambda+1)k^\lambda}\sum_{j=1}^{A}\left\{\epsilon e g_{lj}\lambda(\lambda+1)G_\lambda(kr_j)Y_{\lambda\mu}(\Omega_j) + \frac{e\hbar k}{2m_pc}\left(g_{lj}H_\lambda(kr_j)[\lambda(\lambda+1)Y_{\lambda\mu}(\Omega_j)+2(\boldsymbol{L}Y_{\lambda\mu})_j\cdot\boldsymbol{L}_j] + g_{sj}(\boldsymbol{L}\phi_{\lambda\mu})_j\cdot\boldsymbol{S}_j\right)\right\}. \quad (33)$$

More generally, with Eq. (11), one has

$$\widetilde{\mathcal{M}}_\mu^{E\lambda}(\alpha) = (1-\alpha)\widetilde{\mathcal{M}}_\mu^{E\lambda}(\mathrm{P}) + \alpha\widetilde{\mathcal{M}}_\mu^{E\lambda}(\mathrm{FF}). \quad (34)$$

All these operators are different but their long-wavelength approximations are identical. Differences occur at higher orders in k. The existence of this ambiguity is usually of minor importance except for 'forbidden' $E1$ transitions between isospin zero states.

Notice that these operators do not verify the adjoint property (23). The transformed part contains the ϵ sign which is necessary to compute transposed matrix elements. In Ref. [6], equations (23) and (24) implicitly assume $E_i > E_f$.

IV. ELECTRIC DIPOLE OPERATOR

For discussing the electric dipole operator, one must use Galilean invariant expressions of the coordinates [13,14],

$$\boldsymbol{r}'_j = \boldsymbol{r}_j - \boldsymbol{R}, \quad (35)$$

$$\boldsymbol{p}'_j = \boldsymbol{p}_j - A^{-1}\boldsymbol{p}, \quad (36)$$

where $\boldsymbol{R}$ and $\boldsymbol{P}$ are the coordinate and momentum of the center of mass. The corresponding orbital momentum is $\boldsymbol{L}'_j = \boldsymbol{r}'_j \times \boldsymbol{p}'_j$.

The $E1$ operators are given at leading order by

$$\mathcal{M}_\mu^{E1} = \mathcal{M}_\mu^{E1,\mathrm{S}} + \mathcal{M}_\mu^{E1,\mathrm{IV}} + \mathcal{M}_\mu^{E1,\mathrm{SV}}. \tag{37}$$

The successive terms are scalar (S), isovector (IV), and spin-vector (SV) parts of the operator. The scalar term vanishes at the LWA. The dominant term is then usually the isovector term but it also vanishes in matrix elements of $N = Z$ systems if pure $T = 0$ wave functions are used. This term contributes because of small $T = 1$ components in the wave functions. In $N = Z$ systems, the three types of terms shown in Eq. (37) may compete and their relative importance is poorly known (see Sec. VI). The spin-dependent last term common to all expressions of the $E1$ operator also contains an isovector component which should contribute weakly and is not considered here.

Let me first discuss the LWA part of the isovector term. Since nonzero isospin components are small in $N = Z$ systems, higher order terms can safely be neglected. When the Siegert transform is not performed, the isovector term is given from either Eq. (23) or (24) by

$$\begin{aligned}\mathcal{M}_\mu^{E1,\mathrm{IV}} &= \frac{ie\hbar}{m_pck}\sum_{j=1}^{A}\left(\frac{1}{2} - t_{j3}\right)(\nabla r Y_{1\mu})_j \cdot \boldsymbol{p}'_j \\ &= -\frac{ie\hbar}{m_pck}\sqrt{\frac{3}{4\pi}}\sum_{j=1}^{A} t_{j3} p'_{j\mu},\end{aligned}$$

where $p'_{j\mu}$ are the tensor components corresponding to vector operator $\boldsymbol{p}'$. After the Siegert transform, it takes the well-known form

$$\widetilde{\mathcal{M}}_\mu^{E1,\mathrm{IV}}(P, FF) = -e\sum_{j=1}^{A} t_{j3} r'_j Y_{1\mu}(\Omega'_j). \tag{38}$$

The isoscalar spin-vector term reads

$$\mathcal{M}_\mu^{E1,\mathrm{SV}} = \widetilde{\mathcal{M}}_\mu^{E1,\mathrm{SV}} = \frac{e\hbar k}{8m_pc}(g_p + g_n)\sum_{j=1}^{A} r'_j(\boldsymbol{L}Y_{1\mu})'_j \cdot \boldsymbol{S}_j. \tag{39}$$

This part is identical in all expressions of the $E1$ operator. Let $\boldsymbol{V}$ be a vector operator and V_μ^1 with $\mu = -1, 0, +1$ be the associated tensor of rank 1. A useful relation is

$$(\boldsymbol{L}Y_{1\mu}) \cdot \boldsymbol{V} = \sqrt{2}\,[Y^1 \otimes V^1]_\mu^1 = i\sqrt{\frac{3}{4\pi}}(\boldsymbol{e}_r \times \boldsymbol{V})_\mu^1. \tag{40}$$

In particular, for the orbital momentum, this relation becomes, with Eqs. (7) and (9),

$$(\boldsymbol{L}Y_{1\mu}) \cdot \boldsymbol{L} = -\sqrt{\frac{3}{4\pi}}\nabla_{\Omega\mu} = -i\left(Y_{1\mu}\boldsymbol{r} \cdot \boldsymbol{p} - \sqrt{\frac{3}{4\pi}} r p_\mu\right), \tag{41}$$

where $\nabla_{\Omega\mu}$ is the tensor operator associated to the vector operator defined by Eq. (7).

Now let us consider the scalar part. Without Siegert transform, it reads from either Eq. (23) or (24),

$$\mathcal{M}_\mu^{E1,\mathrm{S}} = \frac{ie\hbar k}{20m_pc}\sum_{j=1}^{A}\left[r'_j Y_{1\mu}(\Omega'_j)\boldsymbol{r}'_j \cdot \boldsymbol{p}'_j - 2\sqrt{\frac{3}{4\pi}} r'^2_j p'_{j\mu}\right]. \tag{42}$$

With Siegert transform, one obtains, from Eq. (31) or (33) (see Eqs. (29) and (33) in Ref. [6]),

$$\begin{aligned}\widetilde{\mathcal{M}}_\mu^{E1,\mathrm{S}}(\mathrm{P}) = &-\epsilon e\frac{k^2}{10}\sum_{j=1}^{A} r'^3_j Y_{1\mu}(\Omega'_j) \\ &+ \frac{ie\hbar k}{4m_pc}\sum_{j=1}^{A} r'_j Y_{1\mu}(\Omega'_j)\boldsymbol{r}'_j \cdot \boldsymbol{p}'_j\end{aligned} \tag{43}$$

and

$$\begin{aligned}\widetilde{\mathcal{M}}_\mu^{E1,\mathrm{S}}(\mathrm{FF}) = &-\epsilon e\frac{k^2}{60}\sum_{j=1}^{A} r'^3_j Y_{1\mu}(\Omega'_j) \\ &+ \frac{e\hbar k}{12m_pc}\sum_{j=1}^{A} r'_j(\boldsymbol{L}Y_{1\mu})'_j \cdot \boldsymbol{L}'_j.\end{aligned} \tag{44}$$

The latter expression can be rewritten with Eq. (41) as

$$\begin{aligned}&\widetilde{\mathcal{M}}_\mu^{E1,\mathrm{S}}(\mathrm{FF}) \\ &= -\epsilon e\frac{k^2}{60}\sum_{j=1}^{A} r'^3_j Y_{1\mu}(\Omega'_j) + \frac{ie\hbar k}{12m_pc}\sum_{j=1}^{A}\Bigg[r'_j Y_{1\mu}(\Omega'_j)\boldsymbol{r}'_j \cdot \boldsymbol{p}'_j \\ &\quad - \sqrt{\frac{3}{4\pi}} r'^2_j p'_{j\mu}\Bigg].\end{aligned} \tag{45}$$

These expressions are linked with a simple relation

$$\widetilde{\mathcal{M}}_\mu^{E1,\mathrm{S}}(\mathrm{FF}) = \tfrac{5}{6}\mathcal{M}_\mu^{E1,\mathrm{S}} + \tfrac{1}{6}\widetilde{\mathcal{M}}_\mu^{E1,\mathrm{S}}(\mathrm{P}). \tag{46}$$

Operator (42) is thus equivalent to a member of the scalar part of the family (34) of transformed operators with $\alpha = 6/5$,

$$\mathcal{M}_\mu^{E1,\mathrm{S}} = -\tfrac{1}{5}\widetilde{\mathcal{M}}_\mu^{E1,\mathrm{S}}(\mathrm{P}) + \tfrac{6}{5}\widetilde{\mathcal{M}}_\mu^{E1,\mathrm{S}}(\mathrm{FF}). \tag{47}$$

An infinity of other expressions are obtained with Eq. (34). The combination in the right-hand side of Eq. (47) is the only one in this family to verify property (20) of the adjoint.

V. MAGNETIC DIPOLE OPERATOR

For completeness, let me briefly recall the expressions for magnetic transition operators. The vector potential is given by Eq. (12). With the definition [2,10]

$$\mathcal{M}_\mu^{M\lambda} = -\frac{i}{c}\sqrt{\frac{\lambda}{\lambda+1}}\frac{(2\lambda+1)!!}{k^\lambda}\int \boldsymbol{J} \cdot \boldsymbol{A}_{\lambda\mu}^M d\boldsymbol{r}, \tag{48}$$

the magnetic operators can be written with Eqs. (22), (12), (21), and (1) as [1,6]

$$\mathcal{M}_{\mu}^{M\lambda} = \frac{2\mu_N}{c}\frac{(2\lambda+1)!!}{(\lambda+1)k^{\lambda}}\sum_{j=1}^{A}\left\{g_{lj}[\boldsymbol{\nabla}\phi_{\lambda\mu}]_j\cdot\boldsymbol{L}_j + \frac{1}{2}g_{sj}\left[\left(k^2\boldsymbol{r}+\boldsymbol{\nabla}\frac{\partial}{\partial r}r\right)\phi_{\lambda\mu}\right]_j\cdot\boldsymbol{S}_j\right\}. \quad (49)$$

An equivalent expression can be obtained with the Friar-Fallieros form (4) of the electric multipole in the magnetization term (21).

At the LWA, the magnetic dipole operator is given in translation-invariant form by

$$\mathcal{M}_{\mu}^{M1} = \frac{\mu_{\rm N}}{c}\sqrt{\frac{3}{4\pi}}\sum_{j=1}^{A}(g_{lj}L'_{j\mu}+g_{sj}S_{j\mu}). \quad (50)$$

The initial and final states of a transition are eigenstates of the total angular momentum of the system. Because of their orthogonality, an equivalent operator in the sense of Eq. (25) is

$$\widetilde{\mathcal{M}}_{\mu}^{M1} = \frac{\mu_{\rm N}}{c}\sqrt{\frac{3}{4\pi}}\left\{\frac{1}{2}(g_p+g_n-1)S_{\mu} - \sum_{j=1}^{A}t_{j3}[L'_{j\mu}+(g_p-g_n)S_{j\mu}]\right\}, \quad (51)$$

where $\boldsymbol{S}$ is the total spin operator of the nucleons.

VI. DISCUSSION

In summary, let me again emphasize that the two apparently different expressions (23) and (24) of the electric transition operators in the Coulomb gauge are identical. When used in numerical calculations, they must give the same results up to rounding errors. On the contrary, after the Siegert transformation (29), these operators lead to different expressions and their use should give different results. The differences occur only beyond the long-wavelength approximation and do not much concern the majority of electromagnetic transitions in nuclei or reactions. On the contrary, they might affect the physics of reactions involving $T = 0$ nuclei where dipole transitions are strongly hindered and are in competition with other multipoles.

A consequence unnoticed in Ref. [6] is that a continuous infinity of transformed electric multipoles exists which introduces an additional ambiguity when the usually dominant long-wavelength term vanishes. This problem is significant for dipole operators when $N = Z$. Determining the most realistic form of the Siegert transformed $E1$ multipole in this case or even whether the transformation is useful remains an open question requiring numerical studies on physical examples.

Among the infinite family of transformed dipole operators, one of them has the same leading scalar and spin parts as the non transformed operator [see Eq. (47)]. It has the same adjoint property (20) as the unique non-transformed operator and allows a comparison of the isovector parts of these operators.

In reactions between $T = 0$ nuclei, the leading part of the $E1$ operator contains three components with different physical natures: scalar, isovector, and spin vector. Without explicit numerical calculations, it is difficult to estimate the relative orders of magnitude of these parts because some results depend on small isospin and spin mixings. The total $E1$ transition strength and thus the relative importance of $E1$ with respect to $E2$ and $M1$ remain uncertain.

Partial information is available in the literature for the α + deuteron radiative capture reaction, but is not conclusive. In the nonmicroscopic three-body model of Ref. [15], the isovector term is important at low energies but the isospin component of the wave function might be overestimated by the technique of elimination of forbidden states [16]. The scalar part computed with the Friar-Fallieros form (44) is very small. The isovector part is found negligible in the *ab initio* calculation of Ref. [17] but the scalar component is not considered. In the microscopic cluster model of Ref. [18], the $E1$ spin term is found dominant with respect to the beyond-LWA $E1$ scalar term but isospin mixing cannot be considered in this simple one-center model of the deuteron. A full evaluation of the different terms remains to be done in this $\alpha + d$ case. Knowing the accuracy of the $E1$ treatment for this reaction is important for future computational studies of other capture reactions between $N = Z$ nuclei. In particular, the $E1$ component of the $\alpha + {}^{12}$C radiative capture reaction which is crucial for astrophysics remains insufficiently known in spite of many decades of experimental and theoretical efforts. The $\alpha + d$ reaction is the simplest testing ground to fix uncertainties on the treatment of isospin-forbidden dipole transitions.

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